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PP-Wave Light-Cone Superstring Field Theory

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Abstract

We construct the cubic interaction vertex and dynamically generated supercharges in light-cone superstring field theory in the pp-wave background. We show that these satisfy the pp-wave superalgebra at first order in string coupling. The cubic interaction vertex and dynamical supercharges presented here differ from the expressions previously given in the literature. Using this vertex we compute various string theory three-point functions and comment on their relation to gauge theory in the BMN limit.

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1 Introduction and Summary

Following earlier work on maximally supersymmetric solutions of eleven-dimensional supergravity [1, 2, 3], a new maximally supersymmetric solution of Type IIB supergravity [4] was found. This paved the way for quantisation in light-cone gauge of superstring theory in a constant R-R flux [5]. The spectrum consists of a unique massless groundstate on which a tower of massive states is constructed [5, 6] using creation operators whose masses are of order $\omega_n/(\alpha' p^+)$ where

$$\omega_n \equiv \sqrt{n^2 + (\alpha' \mu p^+)^2}\,, \qquad n \in \mathbb{N}\,, \tag{1.1}$$

p^+ is momentum along the x^- light-cone direction and μ is the R-R flux. This new string background merits further investigation.

Since the string spectrum in the pp-wave background is now known, string interactions are the next step in the study of the pp-wave background. In this background ten-dimensional Lorentz invariance is broken by the non-zero R-R flux, and hence it is no longer possible to set p^+ to zero in general scattering amplitudes. This obstruction significantly hinders the vertex operator approach to string interactions. There is only one other known way of studying string interactions in light-cone gauge pioneered by Mandelstam [7] for the bosonic string. In this approach an interaction vertex for the scattering of three strings can be constructed by requiring continuity of string fields on the worldsheet depicted in Figure 1. This continuity is enforced by a delta functional on string fields

$$\Delta(X^1 + X^2 - X^3)\,. \tag{1.2}$$

For computational purposes it is essential to express the delta functional in Fourier modes [8].

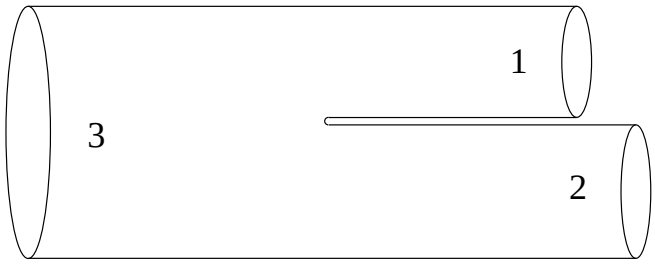

Figure 1: The worldsheet of the three string interaction vertex.

The interaction vertex may then be written as an exponential of creation operators which enforce the delta functional conditions mode by mode.

The functional approach [7, 9] can be extended to the superstring [10, 11, 12]. The cubic interaction vertex is a first order in string coupling g_s correction to the free Hamiltonian. In order to satisfy the superalgebra with this new Hamiltonian, dynamical supercharges also receive corrections. This essential difference from the bosonic string modifies the form of the vertex [11, 12]. In particular [12], in the oscillator basis, the superstring interaction vertex as well as the dynamical supercharges not only have a part exponential in creation operators, but also have a prefactor which is polynomial in creation operators. The exact form of the prefactors is determined by two requirements. Firstly, they should not destroy worldsheet continuity enforced by the exponential part of the vertex. Secondly, the superalgebra should be satisfied order by order in string coupling. In [12] an oscillator expression for the flat spacetime superstring vertex and dynamical supercharges was constructed and shown to satisfy all said consistency conditions.

In this paper we apply the formalism of [8, 11, 12] to the pp-wave background in order to construct an interaction vertex and dynamical supercharges in the oscillator basis. We prove explicitly that our expressions satisfy all the aforementioned consistency conditions. We use this vertex to compute various three-point functions in string theory.

The pp-wave interaction vertex has also been recently investigated in [13, 14]. Our results differ from the expressions presented in [13, 14] in several ways. Firstly, our bosonic prefactor contains a piece, not present in [13, 14], proportional to $\mu\delta^{IJ}$ (*cf.* equation (4.19)). This piece is crucial in proving (see section 4) that the dynamically generated supercharges and interaction Hamiltonian satisfy the pp-wave superalgebra at first order in string coupling. Further, without it, three-point functions of states containing only fermionic creation operators would vanish (see section 5.2). Secondly, we also present the oscillator-basis expression for the fermionic prefactor; in particular, the μ-dependent normalisation (*cf.* equation (4.22)) is essential in proving that the dynamical superalgebra is satisfied. The exponential part of our vertex also contains a piece linear in fermionic zero-modes [15] (*cf.* equation (3.17)) and our expression for the coefficients of the negatively moded bosonic creation operators in the prefactor (*cf.* equation (4.11)) differs by a factor of i [15] from the original one in [14].[1] Since we prove that our expressions for the dynamical supercharges and Hamiltonian satisfy the superalgebra, we believe that the vertex constructed here is the correct one.

The pp-wave background can be obtained as a Penrose limit [16] of $AdS_5 \times S^5$ [4]. Through the AdS/CFT correspondence [17, 18, 19], this has led to a conjectured duality between string theory in the pp-wave background and a double scaling limit of SU(N), $\mathcal{N} = 4$ Super Yang-Mills theory [20]. Explicitly one considers a sector of (BMN) operators with $U(1)$ R-charge

[1] Recently a new version of [14] has appeared correcting the factor of i discrepancy.

J and conformal dimension Δ such that the difference $\Delta - J$ remains fixed in the limit J, $N \to \infty$, J^2/N and $g_{\rm YM}$ fixed [20]. The BMN operators are field theory duals of perturbative string states in the pp-wave background. The SYM theory is believed to be expandable in a double series in the effective genus counting parameter g_2^2 and effective coupling λ' [21, 22]

$$g_2^2 = \frac{J^4}{N^2}\,, \qquad \lambda' = \frac{g_{\rm YM}^2 N}{J^2}\,. \tag{1.3}$$

Moreover, the finite quantity $\Delta - J$ is a function of λ' [20, 23, 24] and g_2^2 [21, 22]. The expansion parameters g_2 and λ' are related to string theory parameters by [20]

$$\frac{1}{(\mu\alpha' p^+)^2} = \lambda'\,, \qquad 4\pi g_{\rm s}(\mu\alpha' p^+)^2 = g_2\,. \tag{1.4}$$

One of the most appealing aspects of this duality is that both string theory and the BMN sector of gauge theory are simultaneously perturbatively accessible.

While some further investigation seems necessary to clarify the exact form of this correspondence, at the planar (interacting) field theory two-point function/free string level, there is increasing evidence [20, 23, 24] that the BMN operators do correspond to the free string theory. The extension of the duality beyond this limit, by considering string interactions and the non-planar sector of (interacting) gauge theory respectively, needs further clarification. Indeed, the exact notion of the Penrose limit in a CFT needs to be better understood [25]. By focusing on a certain subclass of the BMN operators, several proposals for such an extension of the BMN correspondence have been made and were studied in [21, 22, 26, 27, 28, 29, 30, 31, 32, 33]. In this paper we use our interaction vertex to compute three-point functions for various string states. In particular, for the class of amplitudes studied so far in the literature the term in the cubic vertex proportional to $\mu\delta^{IJ}$ does not contribute and we recover the results of [14, 32, 33]. We show that so-called impurity preserving three-point functions of states dual to a class of protected operators vanish in string theory, as would be expected by the extension of the duality recently proposed in [32, 33]. Further, we compute three-point functions for string states having only fermionic creation operators. Here only the $\mu\delta^{IJ}$ term contributes and these amplitudes may provide a check of any extension of the string-bit dynamical supercharges to terms bilinear in fermions, which are not known at present [30]. We hope our results can shed some light on the BMN correspondence and that the three-string vertex can be used to investigate the duality further. Finally if the BMN duality is to be believed, D-branes in the pp-wave background [34, 35] should play an important role in gauge theory.

This paper is organized as follows. In section 2 we briefly review the free string in the pp-wave background and set our notation. In sections 3 and 4 we construct the superstring vertex in the

pp-wave background. Section 3 focuses on the exponential part of the vertex. Most of the results of this section have appeared in [13, 14, 15] and are included here for completeness. In section 4 we construct the prefactors and show that the superalgebra is satisfied with these modified generators. Using the vertex, we compute some three-point functions and compare them to field theory in the BMN limit in section 5. Several appendices, containing computational details are also included.

2 Review of free string theory on the pp-wave

The free string in the maximally supersymmetric pp-wave background in light-cone gauge is described by $x_r^I(\sigma_r)$ and $\vartheta_r^a(\sigma_r)$ in position space or by $p_r^I(\sigma_r)$ and $\lambda_r^a(\sigma_r)$ in momentum space, where I is a transverse SO(8) vector index, a is a SO(8) spinor index.[2] The index $r = 1, 2, 3$ denotes the rth string (see Figure 1). The bosonic part of the light-cone action in the pp-wave background is [5]

$$S_{\text{bos.}(r)} = \frac{e(\alpha_r)}{4\pi\alpha'} \int d\tau \int_0^{2\pi|\alpha_r|} d\sigma_r \big[\dot{x}_r^2 - x_r'^2 - \mu^2 x_r^2\big] , \tag{2.1}$$

where

$$\dot{x}_r \equiv \partial_\tau x_r , \qquad x_r' \equiv \partial_{\sigma_r} x_r , \qquad \alpha_r \equiv \alpha' p_r^+ , \qquad e(\alpha_r) \equiv \frac{\alpha_r}{|\alpha_r|} . \tag{2.2}$$

In a collision process p_r^+ will be positive for an incoming string and negative for an outgoing one. The mode expansions of the fields $x_r^I(\sigma_r, \tau)$ and $p_r^I(\sigma_r, \tau)$ at $\tau = 0$ are

$$\begin{aligned} x_r^I(\sigma_r) &= x_{0(r)}^I + \sqrt{2}\sum_{n=1}^{\infty}\big(x_{n(r)}^I \cos\frac{n\sigma_r}{|\alpha_r|} + x_{-n(r)}^I \sin\frac{n\sigma_r}{|\alpha_r|}\big) , \\ p_r^I(\sigma_r) &= \frac{1}{2\pi|\alpha_r|}\big[p_{0(r)}^I + \sqrt{2}\sum_{n=1}^{\infty}\big(p_{n(r)}^I \cos\frac{n\sigma_r}{|\alpha_r|} + p_{-n(r)}^I \sin\frac{n\sigma_r}{|\alpha_r|}\big)\big] . \end{aligned} \tag{2.3}$$

The Fourier modes can be re-expressed in terms of creation and annihilation operators as

$$x_{n(r)}^I = i\sqrt{\frac{\alpha'}{2\omega_{n(r)}}}\big(a_{n(r)}^I - a_{n(r)}^{I\,\dagger}\big) , \qquad p_{n(r)}^I = \sqrt{\frac{\omega_{n(r)}}{2\alpha'}}\big(a_{n(r)}^I + a_{n(r)}^{I\,\dagger}\big) . \tag{2.4}$$

Canonical quantization of the bosonic coordinates

$$[x_r^I(\sigma_r), p_s^J(\sigma_s)] = i\delta^{IJ}\delta_{rs}\delta(\sigma_r - \sigma_s) \tag{2.5}$$

[2] We will often suppress these indices in what follows.

yields the usual commutation relations

$$[a^I_{n(r)}, a^{J\,\dagger}_{m(s)}] = \delta^{IJ}\delta_{nm}\delta_{rs}\,. \tag{2.6}$$

The fermionic part of the light-cone action in the pp-wave background is [5]

$$S_{\text{ferm.}(r)} = \frac{1}{8\pi}\int d\tau \int_0^{2\pi|\alpha_r|} d\sigma_r[i(\bar\vartheta_r\dot\vartheta_r + \vartheta_r\dot{\bar\vartheta}_r) - \vartheta_r\vartheta_r' + \bar\vartheta_r\bar\vartheta_r' - 2\mu\bar\vartheta_r\Pi\vartheta_r]\,, \tag{2.7}$$

where ϑ^a_r is a complex, positive chirality SO(8) spinor and

$$\Pi_{ab} \equiv (\gamma^1\gamma^2\gamma^3\gamma^4)_{ab} \tag{2.8}$$

is symmetric, traceless and squares to one.[3] The mode expansion of ϑ^a_r and its conjugate momentum

$$i\lambda^a_r \equiv \frac{\delta S_{\text{ferm.}(r)}}{\delta\dot\vartheta_{a(r)}} = i\frac{1}{4\pi}\bar\vartheta^a_r\,, \tag{2.9}$$

at $\tau = 0$ is

$$\begin{aligned}
\vartheta^a_r(\sigma_r) &= \vartheta^a_{0(r)} + \sqrt{2}\sum_{n=1}^{\infty}\big(\vartheta^a_{n(r)}\cos\frac{n\sigma_r}{|\alpha_r|} + \vartheta^a_{-n(r)}\sin\frac{n\sigma_r}{|\alpha_r|}\big)\,,\\
\lambda^a_r(\sigma_r) &= \frac{1}{2\pi|\alpha_r|}\big[\lambda^a_{0(r)} + \sqrt{2}\sum_{n=1}^{\infty}\big(\lambda^a_{n(r)}\cos\frac{n\sigma_r}{|\alpha_r|} + \lambda^a_{-n(r)}\sin\frac{n\sigma_r}{|\alpha_r|}\big)\big]\,.
\end{aligned} \tag{2.10}$$

The Fourier modes satisfy the reality condition

$$\lambda^a_{n(r)} = \frac{|\alpha_r|}{2}\bar\vartheta^a_{n(r)}\,, \tag{2.11}$$

and, due to the canonical anti-commutation relations for the fermionic coordinates

$$\{\vartheta^a_r(\sigma_r), \lambda^b_s(\sigma_s)\} = \delta^{ab}\delta_{rs}\delta(\sigma_r - \sigma_s)\,, \tag{2.12}$$

they obey the following anti-commutation rules

$$\{\vartheta^a_{n(r)}, \lambda^b_{m(s)}\} = \delta^{ab}\delta_{nm}\delta_{rs}\,. \tag{2.13}$$

It is convenient to define a new set of fermionic operators [13]

$$\vartheta_{n(r)} = \frac{c_{n(r)}}{\sqrt{|\alpha_r|}}\left[(1+\rho_{n(r)}\Pi)b_{n(r)} + e(\alpha_r)e(n)(1-\rho_{n(r)}\Pi)b^\dagger_{-n(r)}\right]\,, \tag{2.14}$$

[3] Throughout this paper we use the gamma matrix conventions of [12]. It is convenient to use a representation of gamma matrices in which $\Pi_{ab} = \text{diag}(\mathbf{1}_4\,, -\mathbf{1}_4)$.

which break the $SO(8)$ symmetry to $SO(4) \times SO(4)$. Here

$$\rho_{n(r)} = \rho_{-n(r)} = \frac{\omega_{n(r)} - |n|}{\mu \alpha_r}\,, \qquad c_{n(r)} = c_{-n(r)} = \frac{1}{\sqrt{1+\rho^2_{n(r)}}}\,. \tag{2.15}$$

These modes satisfy

$$\{b^a_{n(r)}, b^{b\,\dagger}_{m(s)}\} = \delta^{ab}\delta_{nm}\delta_{rs}\,. \tag{2.16}$$

The free string light-cone Hamiltonian is

$$H_{2(r)} = \frac{1}{\alpha_r}\sum_{n\in\mathbb{Z}} \omega_{n(r)}\big(a^\dagger_{n(r)}a_{n(r)} + b^\dagger_{n(r)}b_{n(r)}\big)\,. \tag{2.17}$$

In the above the zero-point energies cancel between bosons and fermions. Since the Hamiltonian only depends on two dimensionful quantities μ and α_r, α' and p^+_r should not be thought of as separate parameters. The vacuum $|v\rangle_r$ is defined as

$$a_{n(r)}|v\rangle_r = 0\,, \qquad b_{n(r)}|v\rangle_r = 0\,, \quad n \in \mathbb{N}\,. \tag{2.18}$$

The isometries of the pp-wave superalgebra are generated by H, P^+, P^I, J^{+I}, J^{ij} and $J^{i'j'}$. The latter two are angular momentum generators of the transverse $SO(4)\times SO(4)$ symmetry of the pp-wave background. The 32 supersymmetries split into kinematical supercharges Q^+, $\bar{Q}^+$ and dynamical ones Q^-, $\bar{Q}^-$. The latter square to the light-cone Hamiltonian and hence get corrected by interactions. A subset of the superalgebra that will be of importance throughout this paper is [5]

$$\begin{aligned}
[H, P^I] &= -i\mu^2 J^{+I}\,, \qquad [H, Q^+] = -\mu\Pi Q^+\,,\\
\{Q^-_{\dot a}, \bar{Q}^-_{\dot b}\} &= 2\delta_{\dot a\dot b}H - i\mu\big(\gamma_{ij}\Pi\big)_{\dot a\dot b}J^{ij} + i\mu\big(\gamma_{i'j'}\Pi\big)_{\dot a\dot b}J^{i'j'}\,.
\end{aligned} \tag{2.19}$$

The dynamical supercharges can be expressed in terms of world-sheet fields as

$$Q^-_{(r)} = \sqrt{\frac{2}{\alpha'}}\int_0^{2\pi|\alpha_r|} d\sigma_r\, \left[2\pi\alpha' e(\alpha_r)p_r\gamma\lambda_r - ix'_r\gamma\bar\lambda_r - i\mu x_r\gamma\Pi\lambda_r\right] \tag{2.20}$$

and $\bar{Q}^-_{(r)} = e(\alpha_r)\big[Q^-_{(r)}\big]^\dagger$. Expanding in modes one finds

$$\begin{aligned}
Q^-_{(r)} = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\gamma\Big(&\sqrt{\mu}\left[a_{0(r)}(1+e(\alpha_r)\Pi) + a^\dagger_{0(r)}(1-e(\alpha_r)\Pi)\right]\lambda_{0(r)}\\
&+\sum_{n\neq 0}\sqrt{|n|}\left[a_{n(r)}P^{-1}_{n(r)}b^\dagger_{n(r)} + e(\alpha_r n)a^\dagger_{n(r)}P_{n(r)}b_{-n(r)}\right]\Big)\,,
\end{aligned} \tag{2.21}$$

where

$$P_{n(r)} = \frac{1}{\sqrt{1-\rho^2_{n(r)}}}(1-\rho_{n(r)}\Pi)\,. \tag{2.22}$$

3 Interacting string theory

In interacting light-cone string field theory some generators of the supersymmetry algebra, such as J, P and Q^+, are quadratic in fields. These kinematical generators correspond to symmetries which are manifest in the theory and do not receive $g_{\rm s}$ corrections. On the other hand, dynamical generators such as H, Q^- and $\bar{Q}^-$, depend on the interactions and hence involve higher powers of fields. For example H, the full Hamiltonian of the interacting theory, has an expansion in string coupling

$$H = H_2 + g_{\rm s} H_3 + \cdots , \tag{3.1}$$

where H_3 creates or destroys a single string, while the H_2 conserves the number of strings.

Despite such corrections to the generators, the supersymmetry algebra has to be satisfied order by order in $g_{\rm s}$. This consistency condition gives rise to two types of constraints: the kinematic constraints arise from demanding that (anti)commutation relations of kinematical with dynamical generators are realized at order $\mathcal{O}(g_{\rm s})$; the dynamic constraints arise from the (anti)commutation relations of dynamical generators alone. The former are relatively easy to satisfy, and it is the latter relations that severely constrain the form of the dynamical generators and will be used to determine the interaction vertex below. In this section we review the exponential part of the cubic vertex which enforces the kinematic constraints.[4] In the following section we construct the prefactor of the interaction vertex by imposing the dynamic constraints.

The bosonic contribution $|E_a\rangle$ to the exponential part of the three-string interaction vertex has to satisfy the kinematic constraints [11, 12]

$$\sum_{r=1}^{3} p_r(\sigma_r)|E_a\rangle = 0\,, \qquad \sum_{r=1}^{3} e(\alpha_r) x_r(\sigma_r)|E_a\rangle = 0\,. \tag{3.2}$$

These are the same as in flat space and arise from the commutation relations of H with P^I (*cf.* equation (2.19)) and J^{+I}. They guarantee momentum conservation and continuity of the string worldsheet in the interaction. The coordinates of the three strings are parameterized by

$$\begin{aligned} \sigma_1 &= \sigma && -\pi\alpha_1 \le \sigma \le \pi\alpha_1\,, \\ \sigma_2 &= \begin{cases} \sigma - \pi\alpha_1 & \pi\alpha_1 \le \sigma \le \pi(\alpha_1+\alpha_2)\,, \\ \sigma + \pi\alpha_1 & -\pi(\alpha_1+\alpha_2) \le \sigma \le -\pi\alpha_1\,, \end{cases} \qquad (3.3) \\ \sigma_3 &= -\sigma && -\pi(\alpha_1+\alpha_2) \le \sigma \le \pi(\alpha_1+\alpha_2) \qquad (3.4) \end{aligned}$$

[4] Apart from the differences mentioned in the introduction, our expression for the exponential part of the vertex agrees with the construction first presented in [13].

and $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $\alpha_3 < 0$. The solution to the constraints in equation (3.2) can be constructed as a functional integral and is given by [13]

$$|E_a\rangle \sim \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n\in\mathbb{Z}} a^\dagger_{m(r)}\bar{N}^{rs}_{mn}a^\dagger_{n(s)}\right)|0\rangle_{123}\,, \tag{3.5}$$

where $|0\rangle_{123} = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$ is annihilated by $a_{n(r)}$, $n \in \mathbb{Z}$. The determinant factor coming from the functional integral will be cancelled by the fermionic determinant except for the zero-mode part which is $\left(\frac{2}{\alpha'}\frac{2\mu}{\pi^3}\frac{\alpha_1\alpha_2}{\alpha_3}\right)^2$. In the equation above the non-vanishing elements of the bosonic Neumann matrices for m, $n > 0$ are [13]

$$\bar{N}^{rs}_{mn} = \delta^{rs}\delta_{mn} - 2\sqrt{\frac{\omega_{m(r)}\omega_{n(s)}}{mn}}\left(A^{(r)\,T}\Gamma^{-1}A^{(s)}\right)_{mn}\,, \tag{3.6}$$

$$\bar{N}^{rs}_{m0} = -\sqrt{2\mu\alpha_s\omega_{m(r)}}\varepsilon^{st}\alpha_t\bar{N}^r_m\,, \qquad s\in\{1,2\}\,, \tag{3.7}$$

$$\bar{N}^{rs}_{00} = (1-4\mu\alpha K)\left(\delta^{rs} + \frac{\sqrt{\alpha_r\alpha_s}}{\alpha_3}\right)\,, \qquad r,s\in\{1,2\}\,, \tag{3.8}$$

$$\bar{N}^{r3}_{00} = -\sqrt{-\frac{\alpha_r}{\alpha_3}}\,, \qquad r\in\{1,2\}\,. \tag{3.9}$$

Here

$$\alpha \equiv \alpha_1\alpha_2\alpha_3 \tag{3.10}$$

and

$$\Gamma \equiv \sum_{r=1}^{3} A^{(r)}U_{(r)}A^{(r)\,T}\,, \tag{3.11}$$

where

$$U_{(r)} \equiv C^{-1}\big(C_{(r)} - \mu\alpha_r\big)\,, \qquad C_{mn} \equiv m\delta_{mn}\,, \quad \big(C_{(r)}\big)_{mn} \equiv \omega_{m(r)}\delta_{mn}\,. \tag{3.12}$$

The matrices $A^{(r)}$ arise from the Fourier transformation of the constraints in equation (3.2) and are given in appendix E. Notice that in the Neumann matrices the inverse of the infinite dimensional matrix Γ appears. In contrast to flat space [7], in the pp-wave explicit expressions for the Neumann matrices valid for all μ, are not known. It is however possible to obtain their large μ behavior using the approach of [36] (see also appendix E). We also define

$$\bar{N}^r \equiv -C^{-1/2}A^{(r)\,T}\Gamma^{-1}B\,, \qquad K \equiv -\frac{1}{4}B^T\Gamma^{-1}B\,. \tag{3.13}$$

An explicit expression for the vector B is given in appendix E. The quantities Γ, $\bar{N}^r$ and K manifestly reduce to their flat space counterparts, defined in [11, 12], as $\mu \to 0$. The only non-vanishing matrix elements with negative indices are $\bar{N}^{rs}_{-m,-n}$. They are related to $\bar{N}^{rs}_{mn}$ via [13]

$$\bar{N}^{rs}_{-m,-n} = -\left(U_{(r)}\bar{N}^{rs}U_{(s)}\right)_{mn}\,, \qquad m,\,n>0\,. \tag{3.14}$$

A very useful formula relating $\bar{N}^{rs}_{mn}$ to $\bar{N}^{r}_{m}\bar{N}^{s}_{n}$ is [36, 15]

$$\bar{N}^{rs}_{mn} = -(1-4\mu\alpha K)^{-1}\frac{\alpha}{\alpha_r\omega_{n(s)}+\alpha_s\omega_{m(r)}}\left[U^{-1}_{(r)}C^{1/2}_{(r)}C\bar{N}^{r}\right]_m\left[U^{-1}_{(s)}C^{1/2}_{(s)}C\bar{N}^{s}\right]_n . \tag{3.15}$$

This factorization theorem can be used to verify [15] that $|E_a\rangle$ satisfies equation (3.2). It will also prove essential throughout the next section.

Analogously to the bosonic case, the fermionic exponential part of the interaction vertex has to satisfy [11, 12]

$$\sum_{r=1}^{3}\lambda_{(r)}(\sigma_r)|E_b\rangle = 0\,, \qquad \sum_{r=1}^{3}e(\alpha_r)\vartheta_{(r)}(\sigma_r)|E_b\rangle = 0\,. \tag{3.16}$$

These constraints arise from the commutation relations of H with Q^+ and $\bar{Q}^+$, *cf.* equation (2.19). The solution is [15]

$$|E_b\rangle \sim \exp\left[\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty}b^{\dagger}_{-m(r)}Q^{rs}_{mn}b^{\dagger}_{n(s)} - \sqrt{2}\Lambda\sum_{r=1}^{3}\sum_{m=1}^{\infty}Q^{r}_{m}b^{\dagger}_{-m(r)}\right]|E^0_b\rangle\,, \tag{3.17}$$

where $\Lambda \equiv \alpha_1\lambda_{0(2)} - \alpha_2\lambda_{0(1)}$ and $|E^0_b\rangle$ is the pure zero-mode part of the fermionic vertex

$$|E^0_b\rangle = \prod_{a=1}^{8}\left[\sum_{r=1}^{3}\lambda^{a}_{0(r)}\right]|0\rangle_{123}\,. \tag{3.18}$$

Here $|0\rangle_r$ is not the vacuum defined to be annihilated by the $b_{0(r)}$. Rather, it satisfies $\vartheta_{0(r)}|0\rangle_r = 0$ and $H_{2(r)}|0\rangle_r = 4\mu e(\alpha_r)|0\rangle_r$. In the limit $\mu \to 0$ it coincides with the flat space state that generates the massless multiplet by acting with $\lambda^{a}_{0(r)}$ on it. The fermionic Neumann matrices can be expressed in terms of the bosonic ones as [15]

$$Q^{rs}_{mn} = e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}\left[P^{-1}_{(r)}U_{(r)}C^{1/2}\bar{N}^{rs}C^{-1/2}U_{(s)}P^{-1}_{(s)}\right]_{mn}\,, \tag{3.19}$$

$$Q^{r}_{n} = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\left[P_{(r)}C^{1/2}_{(r)}C^{1/2}\bar{N}^{r}\right]_n\,. \tag{3.20}$$

In summary, the part of the cubic contribution to the dynamical generators satisfying the kinematic constraints is

$$|V\rangle \equiv |E_a\rangle|E_b\rangle\delta\left(\sum_{r=1}^{3}\alpha_r\right)\,. \tag{3.21}$$

4 The Complete $\mathcal{O}(g_{\rm s})$ Superstring Vertex

In the previous section we reviewed the exponential part of the vertex, which solves the kinematic constraints. The remaining dynamic constraints are much more restrictive and are solved by introducing a prefactor [11, 12], polynomial in creation operators, in front of $|V\rangle$.[5] Within the functional formalism, the prefactor can be re-interpreted as an insertions of local operators at the interaction point [10, 11] (see also appendix D). In this section we present expressions for the dynamical generators and prove that they satisfy the superalgebra up to order $\mathcal{O}(g_{\rm s})$.

4.1 The Superalgebra and the Constituents of the Prefactors

Define the linear combinations of the free supercharges ($\eta = e^{i\pi/4}$)

$$\sqrt{2}\eta\, Q \equiv Q^- + i\bar{Q}^- \,, \qquad \text{and} \qquad \sqrt{2}\bar{\eta}\,\widetilde{Q} = Q^- - i\bar{Q}^- \tag{4.1}$$

which satisfy

$$\begin{aligned} \{Q_{\dot{a}}, \widetilde{Q}_{\dot{b}}\} &= -\mu\big(\gamma_{ij}\Pi\big)_{\dot{a}\dot{b}} J^{ij} + \mu\big(\gamma_{i'j'}\Pi\big)_{\dot{a}\dot{b}} J^{i'j'} \,, \\ \{Q_{\dot{a}}, Q_{\dot{b}}\} &= \{\widetilde{Q}_{\dot{a}}, \widetilde{Q}_{\dot{b}}\} = 2\delta_{\dot{a}\dot{b}} H \,. \end{aligned} \tag{4.2}$$

Since J^{ij} and $J^{i'j'}$ are not corrected by the interaction, it follows that at order $\mathcal{O}(g_{\rm s})$ the dynamical generators have to satisfy

$$\sum_{r=1}^{3} Q_{\dot{a}(r)} |Q_{3\,\dot{b}}\rangle + \sum_{r=1}^{3} Q_{\dot{b}(r)} |Q_{3\,\dot{a}}\rangle = 2|H_3\rangle\delta_{\dot{a}\dot{b}} \,, \tag{4.3}$$

$$\sum_{r=1}^{3} \widetilde{Q}_{\dot{a}(r)} |\widetilde{Q}_{3\,\dot{b}}\rangle + \sum_{r=1}^{3} \widetilde{Q}_{\dot{b}(r)} |\widetilde{Q}_{3\,\dot{a}}\rangle = 2|H_3\rangle\delta_{\dot{a}\dot{b}} \,, \tag{4.4}$$

$$\sum_{r=1}^{3} Q_{\dot{a}(r)} |\widetilde{Q}_{3\,\dot{b}}\rangle + \sum_{r=1}^{3} \widetilde{Q}_{\dot{b}(r)} |Q_{3\,\dot{a}}\rangle = 0 \,. \tag{4.5}$$

In order to derive equations that determine the full expressions for the dynamical generators one has to compute (anti)commutators of $Q_{\dot{a}(r)}$ and $\widetilde{Q}_{\dot{a}(r)}$ with the constituents of the prefactors. Moreover, the action of the supercharges on $|V\rangle$ has to be known in terms of these constituents. Here the factorization theorem (3.15) for the bosonic Neumann matrices and

[5] The prefactors are polynomials of $\mathcal{K}$, $\widetilde{\mathcal{K}}$ and $\mathcal{Y}$, defined below, where $\mathcal{K}$, $\widetilde{\mathcal{K}}$ and $\mathcal{Y}$ are linear in creation operators. Apart from an important factor of i (see discussion in the introduction), the oscillator expressions for $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ were first derived in [14]. The corresponding expression for $\mathcal{Y}$ was first derived in [15].

the relation between the bosonic and fermionic Neumann matrices given in equations (3.19) and (3.20) prove to be essential.

So as not to destroy the constraints in equation (3.2), the bosonic part of the prefactors (that we will collectively denote by $\mathcal{P}$) has to satisfy [11, 12]

$$\big[\sum_{r=1}^{3} p_r(\sigma_r), \mathcal{P}\big] = 0 = \big[\sum_{r=1}^{3} e(\alpha_r) x_r(\sigma_r), \mathcal{P}\big] \,. \tag{4.6}$$

Making an ansatz linear in creation oscillators and taking the Fourier transform of the above equations one finds that negative and non-negative modes decouple from each other. The resulting constraint equations on the infinite component vectors appearing in the ansatz can be solved using flat space identities [11]; for details, see [14, 15]. The solution of these constraints involving non-negatively moded bosonic creation oscillators is [14]

$$\mathcal{K}_0 + \mathcal{K}_+ \equiv \mathbb{P} - i\mu\frac{\alpha}{\alpha'}\mathbb{R} + \sum_{r=1}^{3}\sum_{n=1}^{\infty} F_{n(r)} a^{\dagger}_{n(r)} \,, \tag{4.7}$$

where

$$\mathbb{P} \equiv \alpha_1 p_{0(2)} - \alpha_2 p_{0(1)} \,, \qquad \alpha_3 \mathbb{R} \equiv x_{0(1)} - x_{0(2)} \,, \qquad [\mathbb{R}, \mathbb{P}] = i \tag{4.8}$$

and in terms of the zero-mode creation oscillators

$$\sum_{r=1}^{3} F_{0(r)} a^{\dagger}_{0(r)} \equiv \mathbb{P} - i\mu\frac{\alpha}{\alpha'}\mathbb{R} = \sqrt{\frac{2}{\alpha'}}\sqrt{\mu\alpha_1\alpha_2}\big(\sqrt{\alpha_1} a^{\dagger}_{0(2)} - \sqrt{\alpha_2} a^{\dagger}_{0(1)}\big) \,. \tag{4.9}$$

The explicit expression for $F_{n(r)}$ is given in appendix A, equation (A.1). The solution involving negatively moded creation oscillators is

$$\mathcal{K}_- \equiv \sum_{r=1}^{3}\sum_{n=1}^{\infty} F_{-n(r)} a^{\dagger}_{-n(r)} \,, \tag{4.10}$$

where

$$F_{-n(r)} = iU_{n(r)} F_{n(r)} \,. \tag{4.11}$$

The fermionic parts of the prefactor have to satisfy the conditions

$$\big\{\sum_{r=1}^{3} \lambda_r(\sigma), \mathcal{P}\big\} = 0 = \big\{\sum_{r=1}^{3} e(\alpha_r)\vartheta_r(\sigma), \mathcal{P}\big\} \tag{4.12}$$

that are solved in a similar way to the bosonic case. The solution involving non-negatively moded creation oscillators is

$$\mathcal{Y} \equiv \sqrt{\frac{2}{\alpha'}}\Lambda + \sum_{r=1}^{3}\sum_{n=1}^{\infty} G_{n(r)} b^{\dagger}_{n(r)} \,, \tag{4.13}$$

where the infinite component vector $G_{n(r)}$ can be expressed in terms of the bosonic one $F_{n(r)}$ [15] (see appendix A, equation (A.2)). As in flat space [11, 12], it turns out that the prefactors do not involve negatively moded fermionic creation oscillators.

Below we present the results necessary to verify equations (4.3) and (4.4), given the ansatz (4.19)-(4.21) for the cubic vertex and dynamical supercharges. We need

$$\sqrt{2}\eta\sum_{r=1}^{3}[Q_{(r)},\widetilde{\mathcal{K}}]\,|V\rangle=\sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)},\mathcal{K}]\,|V\rangle=\mu\gamma(1+\Pi)\mathcal{Y}|V\rangle\,, \tag{4.14}$$

where

$$\mathcal{K}\equiv\mathcal{K}_0+\mathcal{K}_++\mathcal{K}_-\,,\qquad \widetilde{\mathcal{K}}\equiv\mathcal{K}_0+\mathcal{K}_+-\mathcal{K}_- \tag{4.15}$$

and

$$\begin{aligned}\sqrt{2}\eta\sum_{r=1}^{3}\{Q_{(r)},\mathcal{Y}\}\widetilde{\mathcal{K}}^I|V\rangle&=i\gamma^J(1-2\mu\alpha K(1-\Pi))\mathcal{K}^J\widetilde{\mathcal{K}}^I|V\rangle-i\mu\frac{\alpha}{\alpha'}\gamma^I(1-\Pi)|V\rangle\,,\\ \sqrt{2}\bar{\eta}\sum_{r=1}^{3}\{\widetilde{Q}_{(r)},\mathcal{Y}\}\mathcal{K}^I|V\rangle&=-i\gamma^J(1-2\mu\alpha K(1-\Pi))\widetilde{\mathcal{K}}^J\mathcal{K}^I|V\rangle+i\mu\frac{\alpha}{\alpha'}\gamma^I(1-\Pi)|V\rangle\,.\end{aligned} \tag{4.16}$$

Notice, that the above identities are only valid when both sides of the equation act on $|V\rangle$. The action of the supercharges on $|V\rangle$ is

$$\sqrt{2}\eta\sum_{r=1}^{3}Q_{(r)}|V\rangle=-\frac{\alpha'}{\alpha}\mathcal{K}\gamma(1-2\mu\alpha K(1+\Pi))\mathcal{Y}|V\rangle\,, \tag{4.17}$$

$$\sqrt{2}\bar{\eta}\sum_{r=1}^{3}\widetilde{Q}_{(r)}|V\rangle=-\frac{\alpha'}{\alpha}\widetilde{\mathcal{K}}\gamma(1-2\mu\alpha K(1+\Pi))\mathcal{Y}|V\rangle\,. \tag{4.18}$$

The proof of equations (4.14)-(4.18) is given in appendix A.

4.2 The dynamical generators at order $\mathcal{O}(g_{\rm s})$

The results of the previous subsection motivate the following ansatz for the explicit form of the dynamical supercharges and the three-string interaction vertex

$$|H_3\rangle=\Big((1-4\mu\alpha K)\widetilde{\mathcal{K}}^I\mathcal{K}^J-\mu\frac{\alpha}{\alpha'}\delta^{IJ}\Big)\,v_{IJ}(Y)|V\rangle\,, \tag{4.19}$$

$$|Q_{3\,\dot{a}}\rangle=(1-4\mu\alpha K)^{1/2}\widetilde{\mathcal{K}}^I s^I_{\dot{a}}(Y)|V\rangle\,, \tag{4.20}$$

$$|\widetilde{Q}_{3\,\dot{a}}\rangle=(1-4\mu\alpha K)^{1/2}\mathcal{K}^I\tilde{s}^I_{\dot{a}}(Y)|V\rangle\,. \tag{4.21}$$

Here

$$Y \equiv (1-4\mu\alpha K)^{-1/2}(1-2\mu\alpha K(1+\Pi))\mathcal{Y}\,. \tag{4.22}$$

Substituting this into equations (4.3) and (4.4) and using equations (4.14)-(4.18), we get the following equations for v^{IJ}, $s^I_{\dot a}$ and $\tilde s^I_{\dot a}$ [6]

$$\delta_{\dot a\dot b}v^{IJ} = \frac{i}{\sqrt{2}}\frac{\alpha'}{\alpha}\gamma^J_{a(\dot a}D^a s^I_{\dot b)}\,, \qquad \delta_{\dot a\dot b}v^{IJ} = -\frac{i}{\sqrt{2}}\frac{\alpha'}{\alpha}\gamma^I_{a(\dot a}\bar D^a \tilde s^J_{\dot b)}\,, \tag{4.23}$$

which originate from terms proportional to $\widetilde{\mathcal{K}}_I\mathcal{K}_J$ and are identical to the flat space equations of [12]. Two additional equations, arising from terms proportional to $\mu\delta_{IJ}$ are

$$\begin{aligned} -\delta_{\dot a\dot b}v^{II} &= \frac{i}{\sqrt{2}}\frac{\alpha'}{\alpha}\gamma^I_{a(\dot a}\Big(D^a + i\big[\Pi\bar D\big]^a\Big)s^I_{\dot b)}\,,\\ -\delta_{\dot a\dot b}v^{II} &= -\frac{i}{\sqrt{2}}\frac{\alpha'}{\alpha}\gamma^I_{a(\dot a}\Big(\bar D^a - i\big[\Pi D\big]^a\Big)\tilde s^I_{\dot b)}\,. \end{aligned} \tag{4.24}$$

As in flat space [12] we define

$$D^a \equiv \eta Y^a + \bar\eta\frac{\alpha}{\alpha'}\frac{\partial}{\partial Y_a}\,, \qquad \bar D^a \equiv \bar\eta Y^a + \eta\frac{\alpha}{\alpha'}\frac{\partial}{\partial Y_a}\,. \tag{4.25}$$

Recall first the solution of the flat space equations (4.23) [12]. One introduces the following functions of Y^a

$$w^{IJ} = \delta^{IJ} + \left(\frac{\alpha'}{\alpha}\right)^2\frac{1}{4!}t^{IJ}_{abcd}Y^aY^bY^cY^d + \left(\frac{\alpha'}{\alpha}\right)^4\frac{1}{8!}\delta^{IJ}\varepsilon_{abcdefgh}Y^a\cdots Y^h\,, \tag{4.26}$$

$$iy^{IJ} = \frac{\alpha'}{\alpha}\frac{1}{2!}\gamma^{IJ}_{ab}Y^aY^b + \left(\frac{\alpha'}{\alpha}\right)^3\frac{1}{2\cdot 6!}\gamma^{IJ}_{ab}\varepsilon^{ab}{}_{cdefgh}Y^c\cdots Y^h\,, \tag{4.27}$$

$$\frac{1}{2}s^I_{1\,\dot a} = \gamma^I_{a\dot a}Y^a + \left(\frac{\alpha'}{\alpha}\right)^2\frac{1}{6!}u^I_{abc\dot a}\varepsilon^{abc}{}_{defgh}Y^d\cdots Y^h\,, \tag{4.28}$$

$$\frac{1}{2}s^I_{2\,\dot a} = -\frac{\alpha'}{\alpha}\frac{1}{3!}u^I_{abc\dot a}Y^aY^bY^c + \left(\frac{\alpha'}{\alpha}\right)^3\frac{1}{7!}\gamma^I_{a\dot a}\varepsilon^a{}_{bcdefgh}Y^b\cdots Y^h\,. \tag{4.29}$$

Here

$$t^{IJ}_{abcd} \equiv \gamma^{IK}_{[ab}\gamma^{JK}_{cd]}\,, \qquad u^I_{abc\dot a} \equiv -\gamma^{IJ}_{[ab}\gamma^J_{c]\dot a}\,. \tag{4.30}$$

Notice that t^{IJ}_{abcd} is traceless and symmetric in I, J, hence w^{IJ} is a symmetric tensor of $SO(8)$, whereas y^{IJ} is antisymmetric. The solution of equations (4.23) is [12]

$$v^{IJ} \equiv w^{IJ} + y^{IJ}\,, \qquad s^I_{\dot a} \equiv -\frac{2}{\alpha'}\frac{i}{\sqrt{2}}\big(\eta s^I_{1\,\dot a} + \bar\eta s^I_{2\,\dot a}\big)\,, \qquad \tilde s^I_{\dot a} \equiv \frac{2}{\alpha'}\frac{i}{\sqrt{2}}\big(\bar\eta s^I_{1\,\dot a} + \eta s^I_{2\,\dot a}\big)\,. \tag{4.31}$$

[6] Here $(\dot a\dot b)$ denotes symmetrization in $\dot a$, $\dot b$.

Next consider the additional equations (4.24). Using the flat space solution, these can be rewritten as

$$0 = \gamma^I_{a(\dot{a}} \big[\Pi \bar{D}\big]^a s^I_{\dot{b})} \qquad 0 = \gamma^I_{a(\dot{a}} \big[\Pi D\big]^a \tilde{s}^I_{\dot{b})} \,. \tag{4.32}$$

We prove in appendix B that these equations are also satisfied by equation (4.31).

The proof of equation (4.5) is more involved and provides an important consistency check of the ansatz (4.19)–(4.21). We show in appendix C that it leads to the equations

$$\delta^{IJ} m_{\dot{a}\dot{b}} - \frac{1}{\sqrt{2}} \frac{\alpha'}{\alpha} \gamma^{(I}_{a\dot{a}} D^a \tilde{s}^{J)}_{\dot{b}} = 0 \,, \tag{4.33}$$

$$\delta^{IJ} m_{\dot{a}\dot{b}} - \frac{1}{\sqrt{2}} \frac{\alpha'}{\alpha} \gamma^{(I}_{a\dot{b}} \bar{D}^a s^{J)}_{\dot{a}} = 0 \,, \tag{4.34}$$

$$\sqrt{2}\big(\gamma^I_{a\dot{a}} \eta \tilde{s}^I_{\dot{b}} - \gamma^I_{a\dot{b}} \bar{\eta} s^I_{\dot{a}}\big) - 4i m_{\dot{a}\dot{b}} Y_a = 0 \,, \tag{4.35}$$

$$\big(\gamma^I_{a\dot{a}} \bar{D}_b \tilde{s}^I_{\dot{b}} + \gamma^I_{a\dot{b}} D_b s^I_{\dot{a}}\big)(1-\Pi)^{ab} = 0 \,. \tag{4.36}$$

Here

$$\begin{aligned} m_{\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}} &+ \frac{i}{4} \frac{\alpha'}{2\alpha} \gamma^{IJ}_{\dot{a}\dot{b}} \gamma^{IJ}_{ab} Y^a Y^b - \frac{1}{4 \cdot 4!} \left(\frac{\alpha'}{2\alpha}\right)^2 \gamma^{IJKL}_{\dot{a}\dot{b}} t^{IJKL}_{abcd} Y^a Y^b Y^c Y^d \\ &- \frac{i}{6!} \left(\frac{\alpha'}{2\alpha}\right)^3 \gamma^{IJ}_{\dot{a}\dot{b}} \gamma^{IJ}_{ab} \varepsilon^{ab}{}_{cdefgh} Y^c \cdots Y^h - \frac{2}{7!} \left(\frac{\alpha'}{2\alpha}\right)^4 \delta_{\dot{a}\dot{b}} \varepsilon_{abcdefgh} Y^a \cdots Y^h \end{aligned} \tag{4.37}$$

and

$$t^{IJKL}_{abcd} \equiv \gamma^{[IJ}_{[ab} \gamma^{KL]}_{cd]} \,. \tag{4.38}$$

The first three equations are identical to those in flat space[7] and were proven in [12]. The additional equation (4.36) is proved in appendix B.

We have not yet fixed the overall normalisation of the dynamical generators which can depend on μ and the α_r's. This is more difficult to do than in flat space since there is no J^{-I} generator in the pp-wave background. A comparison with a supergravity calculation fixes the normalisation for small μ to be $\sim (\alpha' \mu^2)/(\alpha_3^4)$ [37]. However, this does not completely fix the normalisation; one may still multiply by a function of μ and α_r that goes to one as $\mu \to 0$. On the other hand, the definition of Y in equation (4.22) and the fact that the terms $\widetilde{\mathcal{K}}^I \mathcal{K}^J$ and $\mu \delta^{IJ}$ in equation (4.19) are multiplied by different powers of $1 - 4\mu\alpha K$ are fixed by demanding that one recovers the flat space equations (4.23), (4.33)-(4.35). In order to obtain the supergravity expressions for the dynamical generators from equations (4.19)–(4.21), one should set K to zero, since it originates from massive string modes. Together with $[\mathbb{R}^I, \mathbb{P}^J] = i\delta^{IJ}$, one can

[7] In appendix C we correct some minor typos present in [12].

check that the supergravity vertex obtained in this way agrees with the supergravity vertex presented in section 4 of [13].

We would like to stress that the nontrivial relation between Y and $\mathcal{Y}$ in equation (4.22), as well as the part in the cubic interaction vertex proportional to $\mu\delta^{IJ}$, were not present in [13, 14] and played an essential role in the above proof of the superalgebra in the interacting string field theory. In [13], functional expressions for the constituents of the prefactors were used to argue that the vertex of [13] satisfied the superalgebra. It is well known [11, 12] that these functional expressions do not in general agree with the oscillator basis expressions given in equations (4.7), (4.10) and (4.13). As explained in appendix D, it would appear that this subtlety is the origin for the absence of the $\mu\delta^{IJ}$ term in [13]. In flat space the functional expression agrees with the one in the oscillator basis only when acting directly on $|V\rangle$ [11, 12]. In the pp-wave however, they agree only up to non-trivial functions of μ [15] (see also appendix D). This leads to equation (4.22). In summary the oscillator expression (4.19), which is used in the following section to compute three-point functions, has been shown explicitly to satisfy the superalgebra at $\mathcal{O}(g_{\rm s})$.

5 3-string amplitudes

In this section we use H_3 to compute three-string amplitudes. For three general string states $|\Phi_i\rangle$ in the number-basis representation the amplitude is

$$g_{\rm s}\langle\Phi_3|H_3|\Phi_1\rangle|\Phi_2\rangle = g_{\rm s}\langle\Phi_3|\langle\Phi_2|\langle\Phi_1|H_3\rangle\,. \tag{5.1}$$

We restrict ourselves to the case, where either all states have bosonic excitations or fermionic ones. We compare our expressions to gauge theory results in the BMN limit. Recall that the full interacting Hamiltonian H/μ in string theory is identified with the operator $\Delta - J$ in field theory [20, 23, 26]. A comparison of matrix elements of the two operators requires an identification of the free string basis used in string theory with a dual basis in field theory. At the planar level, multi-trace BMN operators are identified with multi-string states [20]. However, non-planar corrections lead to a mixing of single and multi trace BMN operators [28, 29] and this effect has to be taken into account when comparing matrix elements of H_3 to field theory. The dual basis was proposed [30, 32, 33] to be the one, in which the matrix of two-point functions of redefined BMN operators (*i.e.* now mixtures of single and multi-trace) is diagonal in free field theory. As emphasized in [32, 33], the required basis transformation is not unique.[8]

[8]We are grateful to Jan Plefka for discussions on this point.

5.1 Amplitudes of bosonic states

To facilitate the comparison with field theory, we change our oscillator basis to the BMN basis. For $n > 0$ the transformation is

$$\sqrt{2}a_n = \alpha_n + \alpha_{-n}\,, \qquad i\sqrt{2}a_{-n} = \alpha_n - \alpha_{-n}\,, \qquad a_0 = \alpha_0\,. \tag{5.2}$$

Using equation (4.11) we have

$$\mathcal{K} = \sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} \mathcal{K}_{n(r)}\alpha^\dagger_{n(r)}\,, \qquad \widetilde{\mathcal{K}} = \sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} \widetilde{\mathcal{K}}_{n(r)}\alpha^\dagger_{n(r)}\,, \tag{5.3}$$

where

$$\mathcal{K}_{n(r)} = \begin{cases} F_{0(r)} & , n = 0 \\ \frac{1}{\sqrt{2}}F_{|n|(r)}\big(1 - U_{n(r)}\big) & , n \neq 0 \end{cases}, \qquad \widetilde{\mathcal{K}}_{n(r)} = \begin{cases} F_{0(r)} & , n = 0 \\ \frac{1}{\sqrt{2}}F_{|n|(r)}\big(1 + U_{n(r)}\big) & , n \neq 0 \end{cases}. \tag{5.4}$$

In this basis the Neumann matrices are

$$\widetilde{N}^{rs}_{mn} = \begin{cases} \frac{1}{2}\bar{N}^{rs}_{|m||n|}\big(1 + U_{m(r)}U_{n(s)}\big) & , m\,, n \neq 0 \\ \frac{1}{\sqrt{2}}\bar{N}^{rs}_{|m|0} & , m \neq 0 \\ \bar{N}^{rs}_{00}\,. \end{cases} \tag{5.5}$$

We compute amplitudes involving the string states

$$|v\rangle_r\,, \qquad |0_i\rangle_r \equiv \alpha^{i\,\dagger}_{0(r)}|v\rangle_r\,, \qquad |n_{i,j}\rangle_r \equiv \alpha^{i\,\dagger}_{n(r)}\alpha^{j\,\dagger}_{-n(r)}|v\rangle_r\,. \tag{5.6}$$

The fermionic contribution to these amplitudes is simple to determine. The pp-wave vacua ${}_r\langle v|$ are related to ${}_r\langle 0|$ via

$${}_r\langle v| = {}_r\langle 0|\left(\frac{\alpha_r}{2}\right)^2\prod_{a=5}^{8}\vartheta^a_{0(r)}\,, \quad r \in \{1,2\}\,, \qquad {}_3\langle v| = -{}_3\langle 0|\left(\frac{\alpha_3}{2}\right)^2\prod_{a=1}^{4}\vartheta^a_{0(r)}\,. \tag{5.7}$$

Eight of the zero-modes in equation (5.7), namely $\vartheta^a_{0(3)}$, $a = 1,\ldots,4$ and, say, $\vartheta^a_{0(2)}$, $a = 5,\ldots,8$ are saturated by $|E^0_b\rangle$, so, to give a non-zero contribution the remaining four zero-modes must be contracted with the $\mathcal{O}(Y^4)$ term in $v_{IJ}(Y)$. Hence, the fermionic contribution is

$$-\left(\frac{\alpha_3}{2}\right)^4(1 - 4\mu\alpha K)^{-2}\Pi_{IJ}\,, \tag{5.8}$$

where $\Pi^{IJ} \equiv t^{IJ}_{5678} = \mathrm{diag}(\mathbf{1}_4\,, -\mathbf{1}_4)$; the factor $(1 - 4\mu\alpha K)^{-2}$ is due to equation (4.22). The bosonic zero-mode determinant can be computed using results in appendix B of [13]

$$\left(\frac{2}{\alpha'}\frac{2\mu}{\pi^3}\frac{\alpha_1\alpha_2}{\alpha_3}\right)^2. \tag{5.9}$$

As discussed at the end of section 4 the overall normalisation of the cubic interaction vertex is still not completely fixed. We will take it to be

$$\frac{\pi^7\alpha'^3}{\alpha^2}(1-4\mu\alpha K)^2 \tag{5.10}$$

by comparing a three-string amplitude (*e.g.* equation (5.13)) in the limit $\mu\to\infty$ to a known field theory result. Strictly speaking, from the comparison with supergravity and field theory we only know that equation (5.10) is correct for $\mu=0$ and $\mu\to\infty$. Since the function $(1-4\mu\alpha K)$ plays such a prominent role in pp-wave string field theory, we conjecture that it is the correct choice for all μ. Combining the various contributions we conclude that for amplitudes of string states build out of bosonic oscillators acting on the $|v\rangle_r$, the interaction vertex is[9]

$$g_{\rm s}|H_3\rangle = -\pi g_{\rm s}\alpha'\mu^2(1-4\mu\alpha K)\mathcal{K}^I\widetilde{\mathcal{K}}^J\Pi_{IJ}|E_a\rangle\delta\left(\sum_{r=1}^3\alpha_r\right)\,. \tag{5.11}$$

In particular, the term proportional to $\mu\delta^{IJ}$ does not contribute, since Π_{IJ} is traceless. Using the factorization theorem (3.15) this can be further simplified to [32, 38]

$$g_{\rm s}|H_3\rangle = \frac{1}{2}g_2\beta(\beta+1)\sum_{r=1}^3\sum_{n\in\mathbb{Z}}\frac{\omega_{n(r)}}{\alpha_r}\alpha^{I\,\dagger}_{n(r)}\alpha^J_{-n(r)}\Pi_{IJ}|E_a\rangle|\alpha_3|\delta\left(\sum_{r=1}^3\alpha_r\right)\,. \tag{5.12}$$

Recall that $g_{\rm s}=g_2/(4\pi\mu^2\alpha_3^2)$. Here $\beta=\alpha_1/\alpha_3$ is related to $r=J_1/J$, the ratio of R-charges conventionally used in SYM, by $\beta=-r$. For amplitudes involving two zero-mode oscillators and one massive string state we find[10]

$$\begin{aligned}
g_{\rm s}\langle n_{1,2}|H_3|0_1\rangle|0_2\rangle &= -g_2\mu\alpha_1\alpha_2(-\beta(\beta+1))^{3/2}\left[CU^{-1}_{(3)}C_{(3)}\right]_n\left(\bar N^3_{|n|}\right)^2\\
&= -\mu g_2\lambda'\sqrt{-\beta(\beta+1)}\frac{\sin^2 n\pi\beta}{2\pi^2}\left[1-\frac{1}{4}(3-16a_R)\lambda' n^2+\mathcal{O}(\lambda'^2)\right]\,, \qquad (5.13)\\
g_{\rm s}\langle n_{1,1}|H_3|0_1\rangle|0_1\rangle &= -2g_2\mu\alpha_1\alpha_2(-\beta(\beta+1))^{3/2}\left[CU^{-1}_{(3)}C_{(3)}\right]_n\left(\bar N^3_{|n|}\right)^2\left[1+\frac{1}{4\mu\alpha_3}\left[CU^{-1}_{(3)}\right]_n\right]\\
&= -\mu g_2\lambda'\sqrt{-\beta(\beta+1)}\frac{\sin^2 n\pi\beta}{\pi^2}\left[1-\frac{1}{4}(7-16a_R)\lambda' n^2+\mathcal{O}(\lambda'^2)\right]\,. \qquad (5.14)
\end{aligned}$$

The leading order result given in equation (5.13) has also been computed in [32, 33] and is in agreement with field theory [28, 29]. The subleading terms are predictions for the 2-loop results in field theory from which one should be able to fix the yet unknown constant a_R (*cf.*

[9] For general amplitudes it is $g_{\rm s}|H_3\rangle = 16\pi g_{\rm s}\alpha'\mu^2\alpha_3^{-4}(1-4\mu\alpha K)^2\left((1-4\mu\alpha K)\mathcal{K}^I\widetilde{\mathcal{K}}^J-\mu\frac{\alpha}{\alpha'}\delta^{IJ}\right)v_{IJ}(Y)|V\rangle\,.$

[10] To avoid clutter of notation we suppress the index r of the rth string and the factor of $|\alpha_3|\delta\big(\sum_r\alpha_r\big)$.

appendix E). An amplitude of two massive string states and the pp-wave vacuum is

$$g_{\rm s}\langle n_{1,2}|H_3|m_{1,2}\rangle|v\rangle = g_2\frac{\beta+1}{4\alpha_3^2}\big(\alpha_1\omega_{n(3)}+\alpha_3\omega_{m(1)}\big)\left(\bar{N}^{13}_{|m||n|}\right)^2\left(1-U^2_{m(1)}U^2_{n(3)}\right)$$
$$= \mu g_2\lambda'(\beta+1)\frac{\sin^2 n\pi\beta}{2\pi^2}+\mathcal{O}(\lambda'^2)\,. \tag{5.15}$$

This amplitude has also been computed in [32, 33] and agrees with the field theory result [28, 29]. Finally, consider amplitudes involving only zero-mode oscillators. From equation (5.12) it follows, that all amplitudes with, say, l oscillators for the first string, m oscillators for the second string, n oscillators for the third string and $l+m=n$ vanish for all μ. This is in agreement with the proposal of [30, 32, 33], since the operators in field theory dual to single and double string states at order $\mathcal{O}(g_2)$ are constructed in such a way that they do not mix in free field theory. Since the operator dual of the single string is protected this is an exact result for all λ', thereby matching the string theory result.

5.2 Amplitudes of fermionic states

As in the bosonic case, we change the oscillator basis to the BMN basis ($n>0$)

$$\sqrt{2}b_n=\beta_n+\beta_{-n}\,,\qquad i\sqrt{2}b_{-n}=\beta_n-\beta_{-n}\,,\qquad b_0=\beta_0\,. \tag{5.16}$$

and compute amplitudes of the states

$$|v\rangle_r\,,\qquad |0_a\rangle_r\equiv\beta^{a\,\dagger}_{0(r)}|v\rangle_r\,,\qquad |n_{a,b}\rangle_r\equiv\beta^{a\,\dagger}_{n(r)}\beta^{b\,\dagger}_{-n(r)}|v\rangle\,. \tag{5.17}$$

In this case the cubic interaction vertex reduces to

$$g_{\rm s}|H_3\rangle=-4\mu g_2\frac{\beta(\beta+1)}{\alpha_3^4}(1-4\mu\alpha K)^2{\rm tr}(v)(Y)|E_b\rangle|\alpha_3|\delta\left(\sum_r\alpha_r\right)\,. \tag{5.18}$$

Due to the trace over v_{IJ} only terms of order $\mathcal{O}(Y^0)$ and $\mathcal{O}(Y^8)$ contribute. Note that these amplitudes are identically zero for the vertex constructed in [13], because the trace over v_{IJ} comes from the term proportional to $\mu\delta^{IJ}$. As an example, we consider

$$g_{\rm s}\langle n_{5,6}|H_3|0_7\rangle|0_8\rangle\,,\qquad g_{\rm s}\langle n_{5,6}|H_3|m_{7,8}\rangle|v\rangle\,, \tag{5.19}$$

which are fermionic analogues to the bosonic amplitudes in equations (5.13) and (5.15). For the above amplitudes to be non-zero it is essential that the fermionic exponential part of the

vertex contains a term linear in zero-modes. Counting fermionic zero-modes shows that in this case only the $\mathcal{O}(Y^0)$ term in $\mathrm{tr}(v)$ gives a contribution. We find

$$\begin{aligned} g_{\mathrm{s}}\langle n_{5,6}|H_3|0_7\rangle|0_8\rangle &= -\mu\alpha_1\alpha_2 g_2\big(-\beta(\beta+1)\big)^{3/2}\big[CU_{(3)}^{-1}C_{(3)}\big]_n\left(\bar{N}^3_{|n|}\right)^2 \\ &= \mu g_2\lambda'\sqrt{-\beta(\beta+1)}\frac{\sin^2 n\pi\beta}{2\pi^2}\left[1-\frac{1}{4}(3-16a_R)\lambda' n^2+\mathcal{O}(\lambda'^2)\right], \qquad (5.20) \\ g_{\mathrm{s}}\langle n_{5,6}|H_3|m_{7,8}\rangle|v\rangle &= -\frac{1}{2}\mu g_2\alpha_1^2\alpha_2^2(\beta+1)(1-4\mu\alpha K)^{-2}\big[CU_{(1)}^{-1}C_{(1)}\big]_m\big[CU_{(3)}^{-1}C_{(3)}\big]_n\left(\bar{N}^1_{|m|}\bar{N}^3_{|n|}\right)^2 \\ &= \mu g_2\lambda'(\beta+1)\frac{\sin^2 n\pi\beta}{2\pi^2}+\mathcal{O}(\lambda'^2)\,. \qquad (5.21) \end{aligned}$$

Up to a sign, the first amplitude in equation (5.20) is identical to its bosonic analogue (5.13) for all μ, whereas the second one, equation (5.21), is the same at leading order. These results reflect that the vertex constructed in section 4 satisfies the superalgebra. One can check that equations (5.20) and (5.21) are unchanged if all spinor indices are in $\{1,2,3,4\}$ and (up to permutations) vanish otherwise.

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A Some computational details

Here we give the details leading to equations (4.14), (4.16), (4.17) and (4.18). The explicit expression for the infinite component vector $F_{(r)}$ appearing in the bosonic constituent of the prefactors is

$$F_{(r)} = -\frac{1}{\sqrt{\alpha'}}\frac{\alpha}{1-4\mu\alpha K}\frac{1}{\alpha_r}U_{(r)}^{-1}C_{(r)}^{1/2}C\bar{N}^r\,. \qquad (A.1)$$

The fermionic quantity $G_{(r)}$ can be expressed in terms of $F_{(r)}$ as [15]

$$G_{(r)} = \big(1 - 2\mu\alpha K(1-\Pi)\big)\sqrt{|\alpha_r|}P^{-1}_{(r)}U_{(r)}C^{-1/2}F_{(r)}\,. \tag{A.2}$$

The following identities prove very useful ($\alpha_3\Theta \equiv \vartheta_{0(1)} - \vartheta_{0(2)}$)

$$\mathbb{R}|V\rangle = i\sqrt{\alpha'}\left[2K\sqrt{\alpha'}\left(\mathbb{P} - i\frac{\mu\alpha}{\alpha'}\mathbb{R}\right) + \sum_{r,n>0} C^{1/2}_{n(r)}\bar{N}^r_n a^\dagger_{n(r)}\right]|V\rangle\,, \tag{A.3}$$

$$\Theta|V\rangle = -\sqrt{2}\sum_{r,n} Q^r_n b^\dagger_{-n(r)}|V\rangle\,. \tag{A.4}$$

Using the mode expansions of $Q^-_{(r)}$, $\bar{Q}^-_{(r)}$, $\mathcal{K}_0+\mathcal{K}_+$, $\mathcal{K}_-$ and $\mathcal{Y}$ given in equations (2.21), (4.7), (4.10) and (4.13) one gets

$$\sum_{r=1}^3\{Q^-_{(r)},\mathcal{Y}\} = -\gamma\sum_{r=1}^3\frac{1}{\sqrt{|\alpha_r|}}\sum_{n=1}^\infty\big[P_{(r)}C^{1/2}G_{(r)}\big]_n a^\dagger_{-n(r)}\,, \tag{A.5}$$

$$\sum_{r=1}^3\{\bar{Q}^-_{(r)},\mathcal{Y}\} = \mathbb{P}\gamma - i\frac{\mu\alpha}{\alpha'}\mathbb{R}\gamma\Pi + \gamma\sum_{r=1}^3\frac{1}{\sqrt{|\alpha_r|}}\sum_{n=1}^\infty\big[P^{-1}_{(r)}C^{1/2}G_{(r)}\big]_n a^\dagger_{n(r)}\,, \tag{A.6}$$

$$\sum_{r=1}^3[Q^-_{(r)},\mathcal{K}_0+\mathcal{K}_+] = \mu\gamma(1+\Pi)\sqrt{\frac{2}{\alpha'}}\Lambda + \gamma\sum_{r=1}^3\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^\infty\big[P^{-1}_{(r)}C^{1/2}F_{(r)}\big]_n b^\dagger_{n(r)}\,, \tag{A.7}$$

$$\sum_{r=1}^3[Q^-_{(r)},\mathcal{K}_-] = i\gamma\sum_{r=1}^3\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^\infty\big[P^{-1}_{(r)}C^{1/2}U_{(r)}F_{(r)}\big]_n b^\dagger_{-n(r)}\,, \tag{A.8}$$

$$\sum_{r=1}^3[\bar{Q}^-_{(r)},\mathcal{K}_0+\mathcal{K}_+] = -\frac{\mu\alpha}{\sqrt{2\alpha'}}\gamma(1-\Pi)\Theta + \gamma\sum_{r=1}^3\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^\infty\big[P_{(r)}C^{1/2}F_{(r)}\big]_n b^\dagger_{-n(r)}\,, \tag{A.9}$$

$$\sum_{r=1}^3[\bar{Q}^-_{(r)},\mathcal{K}_-] = -i\gamma\sum_{r=1}^3\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^\infty\big[P_{(r)}C^{1/2}U_{(r)}F_{(r)}\big]_n b^\dagger_{n(r)}\,. \tag{A.10}$$

Using equations (A.2), (A.3) and (A.4) we find equations equations (4.14) and (4.16). The action of the supercharges on $|V\rangle$ given in equations (4.17) and (4.18) can be proven similarly. One needs

$$\begin{aligned}
&\bar{N}^{rs}_{nm} + e(\alpha_s)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{3/2}P_{n(r)}P_{m(s)}Q^{rs}_{nm} = -\frac{\alpha}{\alpha_s}(1-4\mu\alpha K)^{-1}\big[C^{1/2}_{(r)}\bar{N}^r\big]_n\big[U^{-1}_{(s)}C^{1/2}_{(s)}C\bar{N}^s\big]_m\,,\\
&\bar{N}^{rs}_{-n,-m} + e(\alpha_r)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{1/2}P_{n(r)}P_{m(s)}Q^{rs}_{nm} = 0\,,
\end{aligned} \tag{A.11}$$

$$\bar{N}^{rs}_{nm} - e(\alpha_r)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{1/2} P^{-1}_{n(r)}P^{-1}_{m(s)}Q^{rs}_{nm} = -\mu\alpha(1-4\mu\alpha K)^{-1}(1-\Pi)\big[C^{1/2}_{(r)}\bar{N}^r\big]_n\big[C^{1/2}_{(s)}\bar{N}^s\big]_m\,,$$
$$\bar{N}^{rs}_{-n,-m} - e(\alpha_s)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{3/2} P^{-1}_{n(r)}P^{-1}_{m(s)}Q^{rs}_{nm} = \frac{\alpha}{\alpha_s}(1-4\mu\alpha K)^{-1}\big[P^{-2}_{(r)}C^{1/2}_{(r)}\bar{N}^r\big]_n\big[C^{1/2}_{(s)}C\bar{N}^s\big]_m \tag{A.12}$$

which follow from (3.15) and (3.19).

B Proof of equations (4.24) and (4.36)

In this appendix we prove that

$$\gamma^I_{a(\dot{a}}\big[\Pi\bar{D}\big]^a s^I_{\dot{b})} = 0\,, \tag{B.1}$$
$$\gamma^I_{a(\dot{a}}\big[\Pi D\big]^a \widetilde{s}^I_{\dot{b})} = 0\,, \tag{B.2}$$
$$\Big(\gamma^I_{a\dot{a}}\bar{D}_b\widetilde{s}^I_b + \gamma^I_{a\dot{b}}D_b s^I_{\dot{a}}\Big)(1-\Pi)^{ab} = 0\,. \tag{B.3}$$

Equations (B.1) and (B.2) are equivalent to

$$\Big(\gamma^I_{a(\dot{a}}Y_b s^I_{1\,\dot{b})} + \frac{\alpha}{\alpha'}\gamma^I_{a(\dot{a}}\frac{\partial}{\partial Y^b}s^I_{2\,\dot{b})}\Big)\Pi^{ab} = 0\,, \tag{B.4}$$
$$\Big(\gamma^I_{a(\dot{a}}Y_b s^I_{2\,\dot{b})} - \frac{\alpha}{\alpha'}\gamma^I_{a(\dot{a}}\frac{\partial}{\partial Y^b}s^I_{1\,\dot{b})}\Big)\Pi^{ab} = 0\,, \tag{B.5}$$

The first equation has terms of order $\mathcal{O}(Y^2)$ and $\mathcal{O}(Y^6)$, whereas the second one has terms of order $\mathcal{O}(Y^0)$, $\mathcal{O}(Y^4)$ and $\mathcal{O}(Y^8)$. There are two contributions to the order $\mathcal{O}(Y^2)$ in equation (B.4), both vanish separately. The first one is

$$\gamma^I_{a(\dot{a}}Y_b s^I_{1\,\dot{b})}\Pi^{ab} = 2\gamma^I_{a(\dot{a}}\gamma^I_{c\dot{b})}Y^bY^c\Pi^{ab} = -2\delta_{\dot{a}\dot{b}}\Pi_{ab}Y^aY^b = 0\,, \tag{B.6}$$

whereas the second one is

$$\begin{aligned}&\frac{\alpha}{\alpha'}\gamma^I_{a(\dot{a}}\frac{\partial}{\partial Y^b}s^I_{2\,\dot{b})}\Pi^{ab} = -\gamma^I_{a(\dot{a}}u^I_{bcd\dot{b})}Y^cY^d\Pi^{ab} = \\ &\frac{1}{16}\left(\gamma^{IJ}\gamma^{KL}\right)_{(\dot{a}\dot{b})}\gamma^{IJ}_{a[b}\gamma^{KL}_{cd]}\Pi^{ab}Y^cY^d = \frac{1}{24}\left(\gamma^{IJ}\gamma^{KL}\right)_{(\dot{a}\dot{b})}\left(\gamma^{IJ}\Pi\gamma^{KL}\right)_{cd}Y^cY^d = 0\,.\end{aligned} \tag{B.7}$$

Here we have used equations (B.19) and (B.22). From the Fourier identities [12]

$$\begin{aligned}s_{1\,\dot{a}}(\phi) &= \left(\frac{\alpha}{\alpha'}\right)^4\int d^8Y\, s^I_{2\,\dot{a}}(Y)e^{\frac{\alpha'}{\alpha}\phi Y}\,,\\ s_{2\,\dot{a}}(\phi) &= \left(\frac{\alpha}{\alpha'}\right)^4\int d^8Y\, s^I_{1\,\dot{a}}(Y)e^{\frac{\alpha'}{\alpha}\phi Y}\,,\end{aligned} \tag{B.8}$$

it follows that the terms of order $\mathcal{O}(Y^6)$ vanish as well. This proves equation (B.1). The $\mathcal{O}(Y^0)$ term in equation (B.5) is

$$\gamma^I_{a(\dot a}\gamma^I_{b\dot b)}\Pi^{ab} = \delta_{\dot a\dot b}\mathrm{tr}(\Pi) = 0\,, \tag{B.9}$$

and the order $\mathcal{O}(Y^8)$ term vanishes by (B.8). The terms of order $\mathcal{O}(Y^4)$ in equation (B.5) are

$$\begin{aligned}
&\Pi^a{}_b\gamma^I_{a(\dot a}u^I_{cde\dot b)}\left(Y^bY^cY^dY^e + \frac{1}{24}\varepsilon^{cdeb}{}_{ghij}Y^gY^hY^iY^j\right)\\
&= -\frac{1}{16}\Pi^a{}_b\left(\gamma^{IJ}\gamma^{KL}\right)_{(\dot a\dot b)}\gamma^{IJ}_{a[c}\gamma^{KL}_{de]}\left(Y^bY^cY^dY^e + \frac{1}{24}\varepsilon^{cdeb}{}_{ghij}Y^gY^hY^iY^j\right)\\
&= -\frac{1}{16}\Pi^a{}_b\left(\gamma^{IJKL}_{\dot a\dot b} - 2\delta_{\dot a\dot b}\delta^{IK}\delta^{JL}\right)\gamma^{IJ}_{a[c}\gamma^{KL}_{de]}\left(Y^bY^cY^dY^e + \frac{1}{24}\varepsilon^{cdeb}{}_{ghij}Y^gY^hY^iY^j\right)\\
&= -\frac{1}{16}\Pi^a{}_b\gamma^{IJKL}_{\dot a\dot b}t^{IJKL}_{acde}\left(Y^bY^cY^dY^e + \frac{1}{24}\varepsilon^{cdeb}{}_{ghij}Y^gY^hY^iY^j\right) = 0\,.
\end{aligned} \tag{B.10}$$

In the last step we used that Π is symmetric and traceless and

$$t^{IJKL}_{abcd} = -\frac{1}{24}\varepsilon_{abcd}{}^{efgh}t^{IJKL}_{efgh}\,. \tag{B.11}$$

This proves equation (B.2). Finally, equation (B.3) is equivalent to

$$\left(\gamma^I_{a(\dot a}Y_bs^I_{1\,\dot b)} - \frac{\alpha}{\alpha'}\gamma^I_{a(\dot a}\frac{\partial}{\partial Y^b}s^I_{2\,\dot b)}\right)(1-\Pi)^{ab} = 0\,, \tag{B.12}$$

$$\left(\gamma^I_{a[\dot a}Y_bs^I_{2\,\dot b]} + \frac{\alpha}{\alpha'}\gamma^I_{a[\dot a}\frac{\partial}{\partial Y^b}s^I_{1\,\dot b]}\right)(1-\Pi)^{ab} = 0\,. \tag{B.13}$$

The first equation is symmetric in $\dot a$, $\dot b$ and contains terms of order $\mathcal{O}(Y^2)$ and $\mathcal{O}(Y^6)$. These vanish for the same reason as those in equation (B.1). The second equation is antisymmetric in $\dot a$, $\dot b$ and contains terms of order $\mathcal{O}(Y^0)$, $\mathcal{O}(Y^4)$ and $\mathcal{O}(Y^8)$. The $\mathcal{O}(Y^0)$ contribution to equation (B.13) is

$$\gamma^I_{a[\dot a}\gamma^I_{b\dot b]}(1-\Pi)^{ab} = \frac{1}{4}\gamma^{IJ}_{\dot a\dot b}\gamma^I_{ab}(1-\Pi)^{ab} = 0\,. \tag{B.14}$$

From equation (B.8) it follows that the term of order $\mathcal{O}(Y^8)$ vanishes as well. Finally, there are two contributions to the terms of order $\mathcal{O}(Y^4)$, both of them vanish separately. The first one is

$$\begin{aligned}
&\frac{\alpha}{\alpha'}\gamma^I_{a[\dot a}Y_bs^I_{2\,\dot b]}(1-\Pi)^{ab} = -\frac{1}{3}\gamma^I_{a[\dot a}u^I_{cde\dot b]}(1-\Pi)^a{}_bY^bY^cY^dY^e =\\
&\frac{1}{12}\left(\gamma^{IJ}_{\dot a\dot b}\delta_{a[c}\gamma^{IJ}_{de]} + \frac{1}{4}\left(\gamma^{IJ}\gamma^{KL}\right)_{[\dot a\dot b]}\gamma^{IJ}_{a[c}\gamma^{KL}_{de]}\right)(1-\Pi)^a{}_bY^bY^cY^dY^e =\\
&\frac{1}{12}\gamma^{IJ}_{\dot a\dot b}\gamma^{IK}_{a[c}\gamma^{KJ}_{de]}(1-\Pi)^a{}_bY^bY^cY^dY^e = \frac{1}{6}\gamma^{IJ}_{\dot a\dot b}\gamma^{IJ}_{bc}(1-\Pi)_{de}Y^bY^cY^dY^e = 0\,.
\end{aligned} \tag{B.15}$$

In the last step we have used equation (B.20). The second contribution of order $\mathcal{O}(Y^4)$ then vanishes by equation (B.8). This concludes the proof of equation (B.13).

Apart from symmetry and tracelessness of Π we have used the following identities

$$\gamma^{IJ}_{ab} = -\gamma^{IJ}_{ba}\,, \tag{B.16}$$

$$\gamma^{I}_{a\dot{a}}\gamma^{I}_{b\dot{b}} = \delta_{ab}\delta_{\dot{a}\dot{b}} + \frac{1}{4}\gamma^{IJ}_{ab}\gamma^{IJ}_{\dot{a}\dot{b}}\,, \tag{B.17}$$

$$\begin{aligned}\left(\gamma^{IJ}\gamma^{KL}\right)_{ab} &= \gamma^{IJKL}_{ab} + \delta^{IL}\gamma^{JK}_{ab} + \delta^{JK}\gamma^{IL}_{ab} \\ &\quad - \delta^{IK}\gamma^{JL}_{ab} - \delta^{JL}\gamma^{IK}_{ab} + \left(\delta^{JK}\delta^{IL} - \delta^{JL}\delta^{IK}\right)\delta_{ab}\,,\end{aligned} \tag{B.18}$$

$$\gamma^{I}_{a\dot{a}}u^{I}_{bcd\dot{b}} = -\frac{1}{4}\gamma^{IJ}_{\dot{a}\dot{b}}\delta_{a[b}\gamma^{IJ}_{cd]} - \frac{1}{16}\left(\gamma^{IJ}\gamma^{KL}\right)_{\dot{a}\dot{b}}\gamma^{IJ}_{a[b}\gamma^{KL}_{cd]}\,, \tag{B.19}$$

$$\gamma^{IK}_{a[b}\gamma^{JK}_{cd]} = t^{IJ}_{abcd} - 2\delta_{a[b}\gamma^{IJ}_{cd]}\,, \tag{B.20}$$

$$\gamma^{IJ}_{ab}\gamma^{IJ}_{cd} = 8\big(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\big)\,, \tag{B.21}$$

$$\gamma^{IJKL}_{\dot{a}\dot{b}}\left(\gamma^{KL}\Pi\gamma^{IJ}\right)_{[ab]} = 0\,. \tag{B.22}$$

C $\{Q, \widetilde{Q}\}$ at order $\mathcal{O}(g_{\mathbf{s}})$

Here we demonstrate that equation (4.5) leads to the constraints (4.33)–(4.36) given in section 4. To this end, we adopt a trick introduced in [12]. Namely, we associate the world-sheet coordinate dependence with the oscillators as

$$a_{n(r)} \to e^{-i\omega_{n(r)}\tau/\alpha_r}\left(a_{n(r)}\cos\frac{n\sigma_r}{\alpha_r} - a_{-n(r)}\sin\frac{n\sigma_r}{\alpha_r}\right)\,, \tag{C.1}$$

$$a_{-n(r)} \to e^{-i\omega_{n(r)}\tau/\alpha_r}\left(a_{-n(r)}\cos\frac{n\sigma_r}{\alpha_r} + a_{n(r)}\sin\frac{n\sigma_r}{\alpha_r}\right)\,, \tag{C.2}$$

$$b_{n(r)} \to e^{-i\omega_{n(r)}\tau/\alpha_r}\left(b_{n(r)}\cos\frac{n\sigma_r}{\alpha_r} - b_{-n(r)}\sin\frac{n\sigma_r}{\alpha_r}\right)\,, \tag{C.3}$$

$$b_{-n(r)} \to e^{-i\omega_{n(r)}\tau/\alpha_r}\left(b_{-n(r)}\cos\frac{n\sigma_r}{\alpha_r} + b_{n(r)}\sin\frac{n\sigma_r}{\alpha_r}\right)\,. \tag{C.4}$$

Then we integrate the constraint equation (4.5) over the σ_r. In dealing with the resulting expressions we can integrate by parts since the integrand is periodic. In addition to the identities in equations (4.16),[11] (4.17) and (4.18) we have to calculate the commutator of $\sum_r Q_{(r)}$ with

[11] In fact, here we need the analogue of equation (4.16) with $\mathcal{K} \leftrightarrow \widetilde{\mathcal{K}}$.

$\mathcal{K}$ and and its tilded counterpart. We get

$$\begin{aligned}
\sqrt{2}\eta\sum_{r=1}^{3}[Q_{(r)},\mathcal{K}]\,|V\rangle &= -2i(1-4\mu\alpha K)^{-1/2}\gamma\left[\dot{Y}+Y'+\frac{i}{2}\mu(1-\Pi)\,(Y-2Y_0)\right]|V\rangle\,,\\
\sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)},\widetilde{\mathcal{K}}]\,|V\rangle &= -2i(1-4\mu\alpha K)^{-1/2}\gamma\left[\dot{Y}-Y'+\frac{i}{2}\mu(1-\Pi)\,(Y-2Y_0)\right]|V\rangle\,.
\end{aligned} \tag{C.5}$$

Here Y_0 is the zero-mode part of Y, we suppressed the τ, σ_r dependence and

$$\dot{Y}\equiv\partial_\tau Y\,,\qquad Y'\equiv\sum_{r=1}^{3}\partial_{\sigma_r}Y\,. \tag{C.6}$$

The fact that the above equations have a term which only depends on the zero mode Y_0 is important. Combining the various contributions to equation (4.5) yields

$$\begin{aligned}
\int\prod_{r=1}^{3}d\sigma_r\,\Bigg(&\sqrt{2}(1-4\mu\alpha K)^{-1}\left[\left(\gamma^I_{a\dot{a}}\eta\tilde{s}^I_b-\gamma^I_{a\dot{b}}\bar{\eta}s^I_{\dot{a}}\right)\dot{Y}^a+\left(\gamma^I_{a\dot{a}}\eta\tilde{s}^I_b+\gamma^I_{a\dot{b}}\bar{\eta}s^I_{\dot{a}}\right)Y'^a\right]\\
&-\frac{\mu}{\sqrt{2}}(1-4\mu\alpha K)^{-1}\left[\left(\gamma^I_{a\dot{a}}\bar{D}_b\tilde{s}^I_b+\gamma^I_{a\dot{b}}D_bs^I_{\dot{a}}\right)(1-\Pi)^{ab}+2i\left(\gamma^I_{a\dot{a}}\eta\tilde{s}^I_b-\gamma^I_{a\dot{b}}\bar{\eta}s^I_{\dot{a}}\right)(1-\Pi)^a{}_bY_0^b\right]\\
&+\frac{i}{\sqrt{2}}\frac{\alpha'}{\alpha}\left[\mathcal{K}^I\mathcal{K}^J\gamma^J_{a\dot{a}}D^a\tilde{s}^I_b-\widetilde{\mathcal{K}}^I\widetilde{\mathcal{K}}^J\gamma^J_{a\dot{b}}\bar{D}^as^I_b\right]\Bigg)|V\rangle=0\,.
\end{aligned} \tag{C.7}$$

Next we remove the σ_r derivatives from Y by partial integration. This requires the identity [12][12]

$$\left(\gamma^I_{a\dot{a}}\eta\tilde{s}^I_b+\gamma^I_{a\dot{b}}\bar{\eta}s^I_{\dot{a}}\right)Y'^a=-\frac{2^{3/2}\alpha}{\alpha'}m'_{\dot{a}\dot{b}} \tag{C.8}$$

and

$$\sum_{r=1}^{3}\partial_{\sigma_r}|V\rangle=-\frac{i}{4}\frac{\alpha'}{\alpha}\Big((1-4\mu\alpha K)\big(\mathcal{K}^2-\widetilde{\mathcal{K}}^2\big)+4\big(Y\dot{Y}+i\mu(1-\Pi)YY_0\big)\Big)|V\rangle\,. \tag{C.9}$$

[12]There are some minor typos in appendices D and E of [12] which we correct for the interested reader to facilitate the comparison with this paper. The right-hand-side of equation (D.25) in [12] should be multiplied by $\frac{1}{2\alpha}$; on the r.h.s. of equations (D.26) and (D.27) it should be $\frac{1}{2^{3/2}}$ instead of $\frac{1}{\sqrt{2}}$; the r.h.s. of equation (E.8) should be multiplied by $\frac{1}{\alpha}$. This corrections modify equation (E.9) of [12] slightly, which is now in agreement with equations (4.33)–(4.35) of this paper. These typos do not affect the proof presented in appendix E of [12].

Using the above identities equation (C.7) becomes after integrating by parts

$$\int\prod_{r=1}^{3}d\sigma_r\,\Bigg((1-4\mu\alpha K)^{-1}\Big[\sqrt{2}\big(\gamma^I_{a\dot{a}}\eta\tilde{s}^I_{\dot{b}}-\gamma^I_{a\dot{b}}\bar{\eta}s^I_{\dot{a}}\big)-4im_{\dot{a}\dot{b}}Y_a\Big]\Big(\dot{Y}^a-\dot{Y}^a_0\Big)$$
$$-\frac{\mu}{\sqrt{2}}(1-4\mu\alpha K)^{-1}\Big(\gamma^I_{a\dot{a}}\bar{D}_b\tilde{s}^I_{\dot{b}}+\gamma^I_{a\dot{b}}D_bs^I_{\dot{a}}\Big)(1-\Pi)^{ab}$$
$$-i\mathcal{K}^I\mathcal{K}^J\Big[\delta^{IJ}m_{\dot{a}\dot{b}}-\frac{\alpha'}{\sqrt{2}\alpha}\gamma^J_{a\dot{a}}D^a\tilde{s}^I_{\dot{b}}\Big]+i\widetilde{\mathcal{K}}^I\widetilde{\mathcal{K}}^J\Big[\delta^{IJ}m_{\dot{a}\dot{b}}-\frac{\alpha'}{\sqrt{2}\alpha}\gamma^J_{a\dot{b}}\bar{D}^as^I_{\dot{a}}\Big]\Bigg)|V\rangle=0\,. \tag{C.10}$$

D The Functional Approach to the Prefactor

The bosonic constituents of the prefactor can be obtained via the operators [11, 12]

$$\begin{aligned}\partial X(\sigma)&=4\pi\frac{\sqrt{-\alpha}}{\alpha'}(\pi\alpha_1-\sigma)^{1/2}\big(\dot{x}_1(\sigma)+\dot{x}_1(-\sigma)\big)\,,\\ P(\sigma)&=-2\pi\sqrt{-\alpha}(\pi\alpha_1-\sigma)^{1/2}\big(p_1(\sigma)+p_1(-\sigma)\big)\,.\end{aligned} \tag{D.1}$$

Acting on the exponential part of the vertex these satisfy

$$\begin{aligned}\lim_{\sigma\to\pi\alpha_1}K(\sigma)|V\rangle&\equiv\lim_{\sigma\to\pi\alpha_1}\left(P(\sigma)+\frac{1}{4\pi}\partial X(\sigma)\right)|V\rangle=f(\mu)\mathcal{K}|V\rangle\,,\\ \lim_{\sigma\to\pi\alpha_1}\widetilde{K}(\sigma)|V\rangle&\equiv\lim_{\sigma\to\pi\alpha_1}\left(P(\sigma)-\frac{1}{4\pi}\partial X(\sigma)\right)|V\rangle=f(\mu)\widetilde{\mathcal{K}}|V\rangle\,.\end{aligned} \tag{D.2}$$

Here we defined

$$f(\mu)\equiv-2\frac{\sqrt{-\alpha}}{\alpha_1}\lim_{e\to0}\varepsilon^{1/2}\sum_{n=1}^{\infty}(-1)^n n\cos(n\varepsilon/\alpha_1)\bar{N}^1_n \tag{D.3}$$

which is equal to one for $\mu=0$ [11]. For the fermionic constituent of the prefactor one considers [11, 12]

$$Y(\sigma)=-2\pi\frac{\sqrt{-2\alpha}}{\sqrt{\alpha'}}(\pi\alpha_1-\sigma)^{1/2}\big(\lambda_1(\sigma)+\lambda_1(-\sigma)\big) \tag{D.4}$$

which satisfies [15]

$$\lim_{\sigma\to\pi\alpha_1}Y(\sigma)|V\rangle=f(\mu)(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\mathcal{Y}|V\rangle\,. \tag{D.5}$$

Comparing this identity with equation (4.22), we see that

$$\lim_{\sigma\to\pi\alpha_1}Y(\sigma)|V\rangle=Y|V\rangle\,,\qquad\Longleftrightarrow\qquad f(\mu)=(1-4\mu\alpha K)^{1/2}\,. \tag{D.6}$$

We conjecture that this is indeed the case. Unfortunately, it seems that there is no way of proving this conjecture, at least for large-μ, without knowing $\bar{N}^r$ for all μ. This is because in equation (D.3), the large-μ expansion does not commute with the infinite sum over n.

One can write the functional equivalent of equations (4.19), (4.20) and (4.21) as

$$H_3 = \lim_{\sigma\to\pi\alpha_1}\int d\mu_3\Big(\widetilde{K}^I(\sigma)K^J(\sigma) - \frac{\mu\alpha}{\alpha'}\delta^{IJ}\Big)v_{IJ}(Y(\sigma))\Phi(1)\Phi(2)\Phi(3)\,, \tag{D.7}$$

$$Q_{3\,\dot{a}} = \lim_{\sigma\to\pi\alpha_1}\int d\mu_3\widetilde{K}^I(\sigma)s^I_{\dot{a}}(Y(\sigma))\Phi(1)\Phi(2)\Phi(3)\,, \tag{D.8}$$

$$\widetilde{Q}_{3\,\dot{a}} = \lim_{\sigma\to\pi\alpha_1}\int d\mu_3 K^I(\sigma)\widetilde{s}^I_{\dot{a}}(Y(\sigma))\Phi(1)\Phi(2)\Phi(3)\,. \tag{D.9}$$

Here

$$d\mu_3 \equiv \left(\prod_{r=1}^{3} d\alpha_r D^8\lambda_r(\sigma)D^8p_r(\sigma)\right)\delta\big(\sum_s\alpha_s\big)\Delta^8\big[\sum_s\lambda_s(\sigma)\big]\Delta^8\big[\sum_s p_s(\sigma)\big] \tag{D.10}$$

is the functional expression leading to the exponential part of the vertex [11, 12].

Finally, let us mention the following subtlety. Using equation (2.20) it is easy to see that for example

$$\sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)}, \lim_{\sigma\to\pi\alpha_1}K(\sigma)]\,|V\rangle = \mu\gamma\Pi Y|V\rangle\,. \tag{D.11}$$

However, this is not equal to the commutator of $\sum_r\widetilde{Q}_{(r)}$ with $f(\mu)\mathcal{K}$. To see this, rewrite

$$\begin{aligned}\sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)}, f(\mu)\mathcal{K}]|V\rangle &= \sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)}, \lim_{\sigma\to\pi\alpha_1}K(\sigma)]\,|V\rangle \\ -\frac{\alpha'}{\alpha}\gamma Y(1-4\mu\alpha K)f^{-1}(\mu)[\lim_{\sigma\to\pi\alpha_1}K(\sigma), \widetilde{K}]|V\rangle &= \mu\gamma\Pi Y|V\rangle + \mu\gamma Y|V\rangle = \mu\gamma(1+\Pi)Y|V\rangle\,.\end{aligned} \tag{D.12}$$

Here we used equation (4.18) and

$$[\lim_{\sigma\to\pi\alpha_1}K(\sigma), \widetilde{K}]|V\rangle = -\frac{\mu\alpha}{\alpha'}(1-4\mu\alpha K)^{-1}f(\mu)|V\rangle\,. \tag{D.13}$$

Equation (D.12) is equivalent to equation (4.14) of section 4. It is this appearance of the matrix $1+\Pi$ as opposed to just Π, that is responsible for the term proportional to $\mu\delta^{IJ}$ in the cubic interaction vertex.

E Large-μ Expansions

In this appendix we collect some results on the large-μ expansion. We need the explicit expressions for the matrices $A^{(r)}$ and the vector B. For $m, n > 0$ they are

$$\begin{aligned} A^{(1)}_{mn} &= (-1)^n \frac{2\sqrt{mn}\beta}{\pi} \frac{\sin m\pi\beta}{m^2\beta^2 - n^2}\,, \\ A^{(2)}_{mn} &= \frac{2\sqrt{mn}(\beta+1)}{\pi} \frac{\sin m\pi\beta}{m^2(\beta+1)^2 - n^2}\,, \\ A^{(3)}_{mn} &= \delta_{mn}\,, \\ B_m &= -\frac{2}{\pi}\frac{\alpha_3}{\alpha_1\alpha_2} m^{-3/2} \sin m\pi\beta\,. \end{aligned} \tag{E.1}$$

Writing

$$\Gamma^{-1} = \frac{1}{2} C C^{-1}_{(3)} + R \tag{E.2}$$

it was shown in [36] that

$$R = a_R \frac{\pi}{(\mu\alpha_3)^4} \left(\frac{\alpha_1\alpha_2}{\alpha_3}\right)^2 C^3 B B^T C^3 + \cdots \tag{E.3}$$

with a_R an undetermined number. The next term in the expansion would be of order μ^{-6}. It follows that

$$\bar{N}^3 = \frac{1}{\mu\alpha_3}\left[-\frac{1}{2}C^{1/2} + \frac{1}{(\mu\alpha_3)^2}\left(\frac{1}{4} - a_R\right) C^{5/2} + \ldots\right] B\,. \tag{E.4}$$

Furthermore the large-μ expansion of $1 - 4\mu\alpha K$ is [36]

$$1 - 4\mu\alpha K = \frac{a_K}{\pi\mu\alpha_3} \frac{1}{\beta(\beta+1)} + \cdots\,. \tag{E.5}$$

The next term in the expansion would be of order μ^{-3}. Here, a_K is undetermined as well, however it was shown in [36] that

$$a_R a_K = \frac{1}{64}\,. \tag{E.6}$$

From the definition of $\bar{N}^{rs}_{mn}$ it is easy to see that

$$\bar{N}^{r3}_{mn} \sim -\frac{2}{\pi}(-1)^{(m+1)r} \left(-\frac{\alpha_r}{\alpha_3}\right)^{3/2} \frac{n \sin n\pi\beta}{n^2 \left(\frac{\alpha_r}{\alpha_3}\right)^2 - m^2} \tag{E.7}$$

at leading order. Here and below $r,\ s \in \{1, 2\}$. Using the factorization theorem equation (3.15), the above formulae are sufficient to determine the leading large-μ behavior of all the other Neumann matrices as well. We find

$$\bar{N}^r_n \sim (-1)^{(n+1)r} \frac{a_K}{\pi} \frac{1}{\mu^2 \alpha_1 \alpha_2 \alpha_r} \tag{E.8}$$

and therefore

$$\bar{N}^{rs}_{mn} \sim (-1)^{(m+1)r+(n+1)s} \frac{2a_K}{\pi\mu\sqrt{\alpha_r\alpha_s}}\,, \qquad \bar{N}^{33}_{mn} \sim \frac{2a_K}{\pi(\mu\alpha_3)^3}\big(m\sin m\pi\beta\big)\big(n\sin n\pi\beta\big)\,. \tag{E.9}$$

References

[1] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, Phys. Lett. B **134** (1984) 194.

[2] C. M. Hull, *Exact pp wave solutions of 11-dimensional supergravity*, Phys. Lett. B **139** (1984) 39.

[3] J. Figueroa-O'Farrill and G. Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, JHEP **08** (2001) 036 [arXiv:hep-th/0105308].

[4] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, JHEP **01** (2002) 047 [arXiv:hep-th/0110242].

[5] R. R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. B **625** (2002) 70 [arXiv:hep-th/0112044].

[6] R. R. Metsaev and A. A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, Phys. Rev. D **65** (2002) 126004 [arXiv:hep-th/0202109].

[7] S. Mandelstam, *Interacting string picture of dual resonance models*, Nucl. Phys. B **64** (1973) 205; *ibid.*, *Dual - resonance models*, Phys. Rept. **13** (1974) 259.

[8] E. Cremmer and L. Gervais, *Combining and splitting relativistic strings*, Nucl. Phys. B **76** (1974) 209; *ibid.*, *Infinite component field theory of interacting relativistic strings and dual theory*, Nucl. Phys. B **90** (1975) 410.

[9] M. Kaku and K. Kikkawa, *The field theory of relativistic strings, pt. 1: Trees*, Phys. Rev. D **10** (1974) 1110; *ibid.*, *The field theory of relativistic strings. pt. 2: Loops and pomerons*, Phys. Rev. D **10** (1974) 1823.

[10] S. Mandelstam, *Interacting string picture of the Neveu-Schwarz-Ramond model*, Nucl. Phys. B **69** (1974) 77.

[11] M. Green and J. Schwarz, *Superstring interactions*, Nucl. Phys. B **218** (1983) 43.

[12] M. Green, J. Schwarz and L. Brink, *Superfield theory of type (II) superstrings*, Nucl. Phys. B **219** (1983) 437.

[13] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background*, hep-th/0204146.

[14] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background II*, hep-th/0206073.

[15] A. Pankiewicz, *More comments on superstring interactions in the pp-wave background*, JHEP **09** (2002) 056 [arXiv:hep-th/0208209].

[16] R. Penrose, *Any space-time has a plane-wave as a limit* in 'Differential Geometry and Relativity', M. Cahen and M. Flato eds., Reidel, Dordrecht, Netherlands 1976, pp.271-275.

[17] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

[18] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B **428** (1998) 105 [arXiv:hep-th/9802109].

[19] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150].

[20] D. Berenstein, J. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang Mills*, JHEP **04** (2002) 013 [arXiv:hep-th/0202021].

[21] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and PP-wave strings*, Nucl. Phys. B**643** (2002) 31 [arXiv:hep-th/0205033].

[22] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, *PP-wave string interactions from perturbative Yang-Mills theory*, JHEP **07** (2002) 017 [arXiv:hep-th/0205089].

[23] D. J. Gross, A. Mikhailov and R. Roiban, *Operators with large R charge in $N = 4$ Yang-Mills theory*, Annals Phys. **301** (2002) 31 [arXiv:hep-th/0205066].

[24] A. Santambrogio and D. Zanon, *Exact anomalous dimensions of $N = 4$ Yang-Mills operators with large R charge*, hep-th/0206079.

[25] G. Arutyunov and E. Sokatchev, *Conformal fields in the pp-wave limit*, JHEP **08** (2002) 014 [arXiv:hep-th/0205270].

[26] H. Verlinde, *Bits, matrices and 1/N*, arXiv:hep-th/0206059.

[27] J. G. Zhou, *pp-wave string interactions from string bit model*, hep-th/0208232.

[28] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, *BMN correlators and operator mixing in $N = 4$ super Yang-Mills theory*, hep-th/0208178.

[29] N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, *Operator mixing and the BMN correspondence*, hep-th/0209002.

[30] H. Verlinde and D. Vaman, *Bit Strings from $\mathcal{N} = 4$ gauge theory*, hep-th/0209215.

[31] R. A. Janik, *BMN operators and string field theory*, hep-th/0209263.

[32] J. Pearson, M. Spradlin, D. Vaman, H. Verlinde and A. Volovich, *Tracing the string: BMN correspondence at finite J^2/N*, hep-th/0210102.

[33] J. Gomis, S. Moriyama and J. Park, *SYM description of SFT Hamiltonian in a pp-wave background*, hep-th/0210153.

[34] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, JHEP **06** (2002) 025 [arXiv:hep-th/0204054].

[35] O. Bergman, M. R. Gaberdiel and M. B. Green, *D-brane interactions in type IIB plane-wave background*, hep-th/0205183.

[36] J. Schwarz, *Comments on superstring interactions in a plane-wave background*, JHEP **09** (2002) 058 [arXiv:hep-th/0208179].

[37] Y. Kiem, Y. Kim, S. Lee and J. Park, *PP-wave/Yang-Mills correspondence: An explicit check*, Nucl. Phys. B**642** (2002) 389 [arXiv:hep-th/0205279].

[38] P. Lee, S. Moriyama and J. Park, *A note on cubic interactions in pp-wave light cone string field theory*, Phys. Rev. D **67**, 086001 (2003) [arXiv:hep-th/0209011].