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Dirichlet Branes on Orbifolds

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Abstract

The D-brane spectrum of a class of orbifolds of toroidally compactified Type IIA and Type IIB string theory is analysed systematically. The corresponding K-theory groups are determined and complete agreement is found. The charge densities of the various branes are also calculated.

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1 Introduction

It is now generally appreciated that Dirichlet branes (D-branes) [1, 2, 3] play a central rôle in the non-perturbative description of string theory. D-branes are solitonic solutions of the underlying supergravity theory but they have also a description in terms of open strings; this allows for an essentially perturbative treatment of D-branes.

The D-branes that were first analysed were BPS states that break half the (spacetime) supersymmetry. It has now been realised however that, because of their description in terms of open strings, D-branes can be constructed and analysed in much more general situations. In fact, D-branes are essentially described by a boundary conformal field theory [4, 5, 6, 7], the consistency conditions of which are not related to spacetime supersymmetry [8, 9, 10]. In an independent development, D-branes that break supersymmetry have been constructed in terms of bound states of branes and anti-branes by Sen [11, 12, 13, 14, 15, 16]. This construction has been interpreted in terms of K-theory by Witten [17], and this has opened the way for a more mathematical treatment of D-branes [18, 19, 20].

The D-brane spectrum of a number of theories is understood in detail. These include the standard ten-dimensional Type IIA, IIB and I theory (see [21] for a review and [14, 15, 22] for more recent developments), as well as their non-supersymmetric cousins, Type 0A, 0B and 0 [8, 10, 23]. It is understood how the D-brane spectrum is modified upon compactification on tori, some supersymmetric orbifolds and near ALE singularities [24, 25, 26, 16, 27, 28, 29, 30]. There has also been progress in understanding the D-brane spectrum of Gepner models [31, 32] and WZW theories [33, 34, 35, 36, 37].

In this paper we analyse systematically the D-brane spectrum of certain orbifolds of toroidal compactifications of IIA/IIB superstring theory[1]. We describe in detail the boundary states that define the different D-branes, and show that the resulting spectrum is in agreement with the K-theory predictions which we determine independently. We also find the charge densities of the branes, and exhibit (for a number of examples) the difference between the K-theory group describing the D-brane charges and cohomology. Our analysis works equally well for supersymmetric theories as well as for theories without supersymmetry, thus emphasising once again that D-branes do not intrinsically depend on supersymmetry.

The paper is organised as follows. In section 2, we describe the perturbative and non-perturbative (D-brane) spectrum of the theories in question. Section 3 gives a brief account of the K-theory calculation in the uncompactified case. This is modified in section 4 to take into account the effect of the toroidal compactification. In section 5 we determine the charge densities, and section 6 contains some concluding remarks. We have included three appendices where some of the more technical details can be found.

[1] This class of compactifications contains supersymmetric orbifolds such as the ones analysed in [16, 27, 30], but also non-supersymmetric theories.

2 The Dirichlet brane spectrum

In the first subsection we describe briefly the spectrum of the various orbifold theories we shall be considering. In subsection 2.2 we then identify the consistent boundary states which form the building blocks for the Dirichlet branes.

2.1 Orbifold theories

Let us consider the orbifold of Type IIA or Type IIB that is generated by the non-trivial generator

$$g_1 = \mathcal{I}_n \qquad \text{or} \qquad g_2 = \mathcal{I}_n(-1)^{F_L} . \tag{2.1}$$

Here $\mathcal{I}_n$ describes the reflection of n coordinates, and $(-1)^{F_L}$ acts as ± 1 on left-moving spacetime bosons and fermions, respectively. The action of g_i describes a symmetry of Type II theories if n is even, and we shall therefore only consider this case. In the first instance we shall assume that the theory is ten-dimensional, but later on (in Sections 4 and 5) we shall also discuss how the analysis is modified if the n directions on which $\mathcal{I}_n$ acts are compactified on an n-torus. In this case, T-duality relates

$$\text{IIA on } T^n/\mathcal{I}_n \longleftrightarrow \text{IIB on } T^n/\mathcal{I}_n(-1)^{F_L} , \tag{2.2}$$

and similarly

$$\text{IIA on } T^n/\mathcal{I}_n(-1)^{F_L} \longleftrightarrow \text{IIB on } T^n/\mathcal{I}_n . \tag{2.3}$$

The orbifold group is $\mathbb{Z}_2$ if $n = 4 \bmod 4$, and in this case the theory is supersymmetric. For $n = 2 \bmod 4$, $g_1^2 = g_2^2 = (-1)^{F^s}$, where F^s is the spacetime fermion number, and the orbifold is actually $\mathbb{Z}_4$. In this case supersymmetry is broken since the orbifold theory does not contain any spacetime fermions. The orbifold of Type IIA/IIB by $(-1)^{F^s}$ is Type 0A/0B, and thus the $\mathbb{Z}_4$ orbifold can equivalently be described as the $\mathbb{Z}_2$ orbifold of Type 0A/0B by g_1 or g_2; this is the point of view we shall adopt.

For definiteness we shall phrase our results for the supersymmetric theories in the following, but we shall explain, where appropriate, the modifications that arise for $n = 2 \bmod 4$. In particular, we shall analyse carefully the construction of the boundary state components for all theories (see appendix B). Given the results of [8, 10, 23] it is then easy to determine the actual D-brane spectrum from this data. In fact the only modification that occurs is that the D-brane spectrum of the Type 0A/0B theories is doubled compared to that of Type IIA/IIB. Since the K-theory analysis is also doubled (as there are two different D9-branes in Type 0A and Type 0B), this is then also in agreement with the K-theory analysis.

Let us start by describing the bosonic (closed string) spectrum of these theories that is relevant for the description of the boundary states. In the untwisted sector there are the states in the

NS−NS and the R−R sectors that are invariant under the orbifold projection $\frac{1}{2}(1+g)$ (where $g=g_1$ or $g=g_2$), and the GSO-projection

$$\begin{array}{ll} \text{NS-NS} & \frac{1}{4}\left(1+(-1)^F\right)\left(1+(-1)^{\tilde{F}}\right) \\ \text{R−R} & \frac{1}{4}\left(1+(-1)^F\right)\left(1\mp(-1)^{\tilde{F}}\right) , \end{array} \tag{2.4}$$

where, in the second line, the − sign corresponds to Type IIA and the + sign to Type IIB. In the twisted sector, the moding for the excitations corresponding to the n directions along which $\mathcal{I}_n$ acts is half-integral for bosons and the R-sector world-sheet fermions, and integral for the NS-sector world-sheet fermions. Furthermore, the ground state energy vanishes in the twisted R-sector, and is

$$a_{\text{NS,T}} = \frac{(n-4)}{8} \tag{2.5}$$

in the twisted NS-sector. If the orbifold projection does not involve $(-1)^{F_L}$, *i.e.* if $g=g_1$, then the GSO-projection in the twisted sector is the same as (2.4). On the other hand, for $g=g_2$, the left-moving GSO-projection is opposite to (2.4) in all twisted sectors. Furthermore, the states in the twisted sector also have to be invariant under the orbifold symmetry.

The lowest lying states in the twisted R−R sector are always massless, and transform as a tensor product of two spinor representations of $SO(8-n)$ (where $SO(8-n)$ acts on the unreflected $8-n$ coordinates in light-cone gauge); this tensor product can be decomposed into antisymmetric tensor representations of $SO(8-n)$. The orbifold projection is very simple in this case since the twisted R-sector does not have any fermionic zero modes along the n directions of $\mathcal{I}_n$. All massless (GSO-invariant) states of the twisted R−R sector are therefore physical.

In the twisted NS-NS sector, the lowest lying states are massless for $n=4$, massive for $n>4$, and tachyonic for $n=2$. They transform as a tensor product of two spinor representations of $SO(n)$ (where $SO(n)$ acts on the n coordinates that are reflected by $\mathcal{I}_n$); from the point of view of the $8-n$ unreflected directions these states are scalars. On the ground states, the orbifold projection is proportional to the GSO-projection, and all GSO-invariant ground states are again physical.

We should mention at this stage that the definition of $\mathcal{I}_n$ in the untwisted R-R sector is in general ambiguous: on the ground states we can either define $\mathcal{I}_n$ to be

$$\mathcal{I}_n^{(1)} = \prod_{\mu=9-n}^{8}(\sqrt{2}\psi_0^\mu)\prod_{\mu=9-n}^{8}(\sqrt{2}\tilde{\psi}_0^\mu) \tag{2.6}$$

or

$$\mathcal{I}_n^{(2)} = \prod_{\mu=9-n}^{8}(2\psi_0^\mu\tilde{\psi}_0^\mu)\,. \tag{2.7}$$

These two expressions differ by the order of the fermionic zero modes; for $n = 4 \bmod (4)$, $\mathcal{I}_n^{(1)} = \mathcal{I}_n^{(2)}$ but for $n = 2 \bmod (4)$, $\mathcal{I}_n^{(1)} = -\mathcal{I}_n^{(2)}$. In this paper we shall use the convention that $\mathcal{I}_n$ refers to $\mathcal{I}_n^{(1)}$; for $n = 2 \bmod (4)$, we then have $\mathcal{I}_n(-1)^{F_L} = \mathcal{I}_n^{(2)}$.

For $n = 2 \bmod 4$, there is furthermore, at least *a priori*, an ambiguity in how to define the GSO-projection in each of the twisted sectors; this is due to the fact that in order to guarantee that $(-1)^F$ and $(-1)^{\tilde{F}}$ are of order two, one has to introduce non-trivial phases which are only determined up to signs. As we shall explain in appendix B, these ambiguities are uniquely fixed if we require that the theory has at least one 'fractional' D-brane. This convention leads to a D-brane spectrum that is consistent with the results obtained independently using K-theory.

2.2 The boundary state analysis

Let us first introduce a convenient notation to describe the allignement of the D-branes relative to the n preferred directions along which $\mathcal{I}_n$ acts. For definiteness, let us assume that $\mathcal{I}_n$ reflects the coordinates $x^{9-n}, \ldots, x^8$; we then say that a Dirichlet p-brane is of type (r, s) where $p = r + s$ if it has $r+1$ Neumann directions along $x^0, \ldots, x^{8-n}, x^9$, and s Neumann directions along $x^{9-n}, \ldots, x^8$. We shall always work in light-cone gauge (with light-cone directions x^0 and x^9), and therefore the actual D-brane states we shall analyse will have Dirichlet boundary conditions along the two light cone directions, and thus be D-instantons. As usual, we shall assume that we can perform an appropriate Wick-rotation to transform these states back to ordinary D-branes [38].

The analysis we shall now describe is very similar to that performed in [8, 12, 9] (see also [4, 5]), and we shall therefore be rather sketchy. (Some details of the construction can however be found in appendix A.) A D-brane is described by a linear combination of physical boundary states that satisfies a certain compatibility condition. For each set of boundary conditions, there exists at most one non-trivial GSO and orbifold invariant boundary state in each sector (untwisted NS$-$NS and R$-$R , and twisted NS$-$NS and R$-$R), which is unique up to normalisation. The compatibility condition requires that the spectrum of states that is induced by the presence of a collection of D-branes defines open strings that have consistent interactions with the original closed string theory. These open strings can be determined, using world-sheet duality, from the corresponding closed string tree diagrams that describes the exchange of a closed string state between two D-branes.

One aspect of this consistency condition is the requirement that the string that begins and ends on the same D-brane must be a suitably projected open string. This implies that the actual D-brane state consists of non-trivial boundary states from different sectors. In fact, there exist three different possibilities: the boundary state describes either a *fractional* D-brane, *i.e.* it is a linear combination of non-trivial boundary states from all four sectors,

$$|D(r,s)\rangle = \frac{1}{2}\left(|D(r,s)\rangle_{\mathrm{NS-NS}} + \varepsilon_1 |D(r,s)\rangle_{\mathrm{R-R}} + \varepsilon_1\varepsilon_2 |D(r,s)\rangle_{\mathrm{NS-NS\,,T}} + \varepsilon_2 |D(r,s)\rangle_{\mathrm{R-R\,,T}}\right) , \tag{2.8}$$

a *bulk* D-brane, *i.e.* it is a linear combination involving only the untwisted NS−NS and R−R sectors

$$|D(r,s)\rangle_b = |D(r,s)\rangle_{\text{NS}-\text{NS}} + \varepsilon |D(r,s)\rangle_{\text{R}-\text{R}} \ , \tag{2.9}$$

or a *truncated* D-brane, *i.e.* it is a linear combination involving only the untwisted NS−NS and the twisted R−R sector,

$$|\hat{D}(r,s)\rangle = \frac{\mathcal{N}}{\sqrt{2}} \left(|D(r,s)\rangle_{\text{NS}-\text{NS}} + \varepsilon |D(r,s)\rangle_{\text{R}-\text{R}\,,\text{T}} \right) \ . \tag{2.10}$$

Here ε and $\varepsilon_{1,2}$ are ± 1 and describe the signs with respect to the different charges, the relative normalisations are determined by the consistency condition, and $\mathcal{N}$ is a normalisation constant that will be determined further below. The first two D-brane states (2.8) and (2.9) are conventional D-branes that are BPS provided that the orbifold preserves supersymmetry, whereas the state in (2.10) describes a non-BPS brane.

Not all of these D-brane states are independent: two fractional D-branes with $\varepsilon_1 = \varepsilon_1'$ and $\varepsilon_2 = -\varepsilon_2'$ can combine to form a bulk D-brane state (2.9); as it turns out, whenever a bulk D-brane exists, then so does the corresponding fractional brane, and we may thus restrict ourselves, without loss of generality, to considering only fractional and truncated D-brane states.

The condition that a string with both endpoints on the same truncated brane defines a consistent open string requires that $\mathcal{N}^2 \in \mathbb{N}$; the minimal value is therefore $\mathcal{N} = 1$. On the other hand, the compatibility condition also requires that a string which begins on any one of these branes and ends on any other one, must describe an open string. If for a given (r,s), a fractional D-brane state exists, then the open string from the fractional (r,s) to the truncated (r,s) D-brane leads to the partition function

$$\sqrt{2}\mathcal{N}(\text{NS} - \text{R})\frac{1}{2}\left(1 + g(-1)^F\right) \ . \tag{2.11}$$

This only defines an open string partition function if $\mathcal{N}$ is an integer multiple of $\sqrt{2}$, and the minimal value is then $\mathcal{N} = \sqrt{2}$. In this case, the mass and the charge of the truncated D-brane is precisely twice that of the fractional D-brane; thus the truncated D-brane can decay into two fractional D-branes with $\varepsilon_1 = -\varepsilon_1'$ and $\varepsilon_2 = \varepsilon_2'$, and does not describe an independent stable state.

In summary we therefore find that for a given (r,s) at most one of the above three D-brane states is fundamental, and that the other two (if they exist) can be obtained as bound states of the fundamental D-brane. This fundamental D-brane is either fractional or truncated. The fundamental D-brane is only stable provided that the open string that begins and ends on it does not have a tachyon. This is always the case for a fractional D-brane; in the case of a truncated D-brane the stability of the brane depends on the actual value of the compactification radii. In fact, if the theory is uncompactified, a truncated D-brane is unstable if and only if $s > 0$, and in the compactified case it is stable provided that the radii of the tangential circles are sufficiently small (and the radii of the circles transverse to the brane are sufficiently large) [16, 27, 30].

The D-brane spectrum can now be determined by analysing which of the different boundary components are GSO- and orbifold invariant. The detailed analysis is described in appendix B, and the final result is

(i)	**IIA by** $\mathcal{I}_n$:	Fractional (r, s) D-branes exist for r and s both even.
		Truncated (r, s) D-branes exist for r even and s odd.
(ii)	**IIB by** $\mathcal{I}_n$:	Fractional (r, s) D-branes exist for r odd and s even.
		Truncated (r, s) D-branes exist for both r and s odd.
(iii)	**IIA by** $\mathcal{I}_n(-1)^{F_L}$:	Fractional (r, s) D-branes exist for r and s both odd.
		Truncated (r, s) D-branes exist for r odd and s even.
(iv)	**IIB by** $\mathcal{I}_n(-1)^{F_L}$:	Fractional (r, s) D-branes exist for r even and s odd.
		Truncated (r, s) D-branes exist for both r and s even.

In the uncompactified theory, a fractional D-brane has two charges (ε_1 and ε_2), and a truncated D-brane has only a single charge; the corresponding K-theory groups should therefore either be $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z}$, depending on (r, s) as above. Strictly speaking, one should only trust this analysis for $s = 0$, since the truncated D-brane is otherwise unstable. However, as we shall see in the next section, the K-theory result agrees with the above even for $s \neq 0$.

The situation is somewhat clearer in the case where all n directions are compactified; this will be discussed in some detail in section 4.

3 K-theory analysis in the uncompactified theory

In this section we demonstrate that the above results can be reproduced in terms of K-theory. Here we shall only consider the uncompactified theory; the compactified case will be discussed in section 4. We shall discuss Type IIB in some detail in the first subsection, and Type IIA is analysed in section 3.2.

3.1 K-theory of Type IIB

The analysis of the D-brane spectrum in Type IIB orbifolds is relatively straightforward. If the $\mathbb{Z}_2$ orbifold leaves the D9-brane invariant (*i.e.* if it is of type $g = g_1$), the D-brane spectrum is described in terms of equivariant K-theory $K_{\mathbb{Z}_2}$ [39] of the space transverse to the worldvolume of the Dirichlet brane; on the other hand, if the orbifold maps the D9-brane to its anti-brane (*i.e.* if it is of type $g = g_2$), the relevant K-theory group is $K_{\pm}$ [17]. For a given (r, s) brane, the transverse space has dimension $9 - (r + s)$, of which $n - s$ directions are inverted under the action of $\mathcal{I}_n$. In order to distinguish between the directions on which $\mathcal{I}_n$ does or does not act, we denote the transverse space (as in [19]) by $\mathbb{R}^{n-s,9-n-r}$. The K-theory groups we want to determine are then

$$K_{\mathbb{Z}_2}(\mathbb{R}^{n-s,9-n-r}) \qquad \text{and} \qquad K_{\pm}(\mathbb{R}^{n-s,9-n-r})\,. \tag{3.1}$$

The D-brane configurations of interest are equivalent to the vacuum at transverse infinity; in terms of K-theory this means that the pairs of bundles (E,F) that define the elements of K-theory have the property that E is isomorphic to F near infinity [17]. The corresponding K-theory is usually called K-theory *with compact support*. By a theorem of Hopkins, $K_{\pm}(\mathbb{R}^{l,m})$ is given as

$$K_{\pm}(\mathbb{R}^{l,m}) = K^1_{\mathbb{Z}_2}(\mathbb{R}^{l+1,m}) = K_{\mathbb{Z}_2}(\mathbb{R}^{l+1,m+1})\,, \tag{3.2}$$

where the last equality follows from the compact support condition. The calculation of $K_{\pm}(\mathbb{R}^{l,m})$ therefore reduces to that of the equivariant K-theory groups. This has been calculated before by Gukov [19], and the result is

$$K_{\mathbb{Z}_2}(\mathbb{R}^{l,m}) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \mathbb{Z} & \text{if } m \text{ is even and } l \text{ is odd,} \\ R[\mathbb{Z}_2] = \mathbb{Z}\oplus\mathbb{Z} & \text{if } m \text{ and } l \text{ are even,} \end{cases} \tag{3.3}$$

where $R[G]$ is the representation ring of the group G. This is precisely in agreement with the results of the boundary analysis that we described in the previous section.

We shall now give a different derivation of (3.3) that is due to Segal [40]. Firstly, because of Bott periodicity, the answer depends only on the parity of l and m. If l is even we have

$$K_{\mathbb{Z}_2}(\mathbb{R}^{2\hat{l},m}) = K_{\mathbb{Z}_2}(\mathbb{R}^{0,m}) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ R[\mathbb{Z}_2] = \mathbb{Z}\oplus\mathbb{Z} & \text{if } m \text{ is even,} \end{cases} \tag{3.4}$$

where we have used the fact that $\mathbb{Z}_2$ acts freely on $\mathbb{R}^{0,m}$. In order to calculate $K_{\mathbb{Z}_2}(\mathbb{R}^{1,m})$ we observe that

$$K^*_{\mathbb{Z}_2}(X\times\mathbb{R}^{1,0}) = K^*_{\mathbb{Z}_2}(X\times D^1, X\times S^0)\,, \tag{3.5}$$

where X is an arbitrary manifold on which $\mathbb{Z}_2$ acts continuously, and $S^0\subset D^1\subset\mathbb{R}^{1,0}$ are the one-dimensional 'circle' (*i.e.* the two points ±1) and the one-dimensional 'disk' (*i.e.* the interval $[-1,1]$), respectively. The group $\mathbb{Z}_2$ acts on S^0 and D^1 by reflection. Because of (3.5), the long exact sequence can be written as

$$\begin{aligned} \cdots\rightarrow K^{-1}_{\mathbb{Z}_2}(X\times S^0) \;&\rightarrow\; K^0_{\mathbb{Z}_2}(X\times\mathbb{R}^{1,0}) \;\rightarrow\; K^0_{\mathbb{Z}_2}(X\times D^1) \\ &\rightarrow\; K^0_{\mathbb{Z}_2}(X\times S^0) \;\rightarrow\; K^1_{\mathbb{Z}_2}(X\times\mathbb{R}^{1,0}) \;\rightarrow\; K^1_{\mathbb{Z}_2}(X\times D^1) \;\rightarrow\cdots \end{aligned} \tag{3.6}$$

By homotopy equivalence, we have $K^*_{\mathbb{Z}_2}(X\times D^1)\cong K^*_{\mathbb{Z}_2}(X)$, and $K^*_{\mathbb{Z}_2}(X\times S^0)\cong K^*(X)$ since the $\mathbb{Z}_2$ action is free. Thus for $X=\mathbb{R}^{l,m}$ with l and m even, the exact sequence becomes

$$0\rightarrow K^0_{\mathbb{Z}_2}(\mathbb{R}^{l,m}\times\mathbb{R}^{1,0})\rightarrow R[\mathbb{Z}_2]\rightarrow\mathbb{Z}\rightarrow K^1_{\mathbb{Z}_2}(\mathbb{R}^{l,m}\times\mathbb{R}^{1,0})\rightarrow 0\,, \tag{3.7}$$

where we have used (3.4), $K^0(\mathbb{R}^{l,m}) = \mathbb{Z}$ and $K^1(\mathbb{R}^{l,m}) = 0$. The map from $R(\mathbb{Z}_2) = \mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$ is given by $(m,n) \mapsto m+n$ (which is onto), and it follows that

$$K^0_{\mathbb{Z}_2}(\mathbb{R}^{l,m} \times \mathbb{R}^{1,0}) = \mathbb{Z}\,, \tag{3.8}$$

$$K^1_{\mathbb{Z}_2}(\mathbb{R}^{l,m} \times \mathbb{R}^{1,0}) = 0\,. \tag{3.9}$$

Thus for $l = m = 0$ we obtain the desired result.

3.2 Orbifolds of Type IIA

The D-brane spectrum of Type IIA theory is given in terms of the K-theory groups $K^{-1}(X)$ [17, 18]. The elements of $K^{-1}(X)$ can be thought of as pairs (E, α), where E is a bundle on X, and α a bundle-automorphism [41]. In terms of string theory, the D-branes of Type IIA theory can be obtained from (unstable) D9-branes by tachyon condensation. This can then be interpreted directly in terms of the pairs (E, α), where E is the bundle on the D9-branes, and the automorphism α is related to the tachyon field T by

$$\alpha = -e^{\pi i T}\,. \tag{3.10}$$

This construction also applies to the orbifold theories in question, except that now the tachyon field has to be invariant under the orbifold action. In terms of K-theory, the relevant group is then the equivariant group $K^1_{\mathbb{Z}_2}(X)$ that consists of pairs (E, α), where α commutes with the $\mathbb{Z}_2$ action [40]. More precisely, the K-theory groups are

$$K^{-1}_{\mathbb{Z}_2}(\mathbb{R}^{n-s,9-n-r}) \qquad \text{and} \qquad K^{-1}_{\pm}(\mathbb{R}^{n-s,9-n-r})\,, \tag{3.11}$$

where again the first case corresponds to g_1, and the second to g_2 orbifolds. As before $\mathbb{R}^{n-s,9-n-r}$ denotes the transverse space to a (r,s)-brane. Because of the compact support condition, $K^{-1}_{\pm}$ can be evaluated using equation (3.2). Explicitly this gives

$$K^{-1}_{\mathbb{Z}_2}(\mathbb{R}^{n-s,9-n-r}) = K_{\mathbb{Z}_2}(\mathbb{R}^{n-s,10-n-r}) \qquad \text{and} \qquad K^{-1}_{\pm}(\mathbb{R}^{n-s,9-n-r}) = K_{\pm}(\mathbb{R}^{n-s,10-n-r})\,, \tag{3.12}$$

and thus the evaluation of K-groups relevant to the Type IIA orbifolds under consideration follows from equation (3.3), and is in precise agreement with the results of the boundary state analysis described in the previous section.

4 The compactified case

In this section we analyse the D-brane spectrum for the case when all n coordinates along which $\mathcal{I}_n$ acts are compactified on a torus. We shall first describe how the K-theory analysis is modified; we shall then explain how this matches precisely the results that can be obtained using the boundary state formalism.

4.1 Relative K-theory

The D-brane spectrum of toroidally compactified Type II string theories can be described in terms of relative K-theory [20], and the relevant K-theory groups are therefore

$$K^*_{\mathbb{Z}_2}(S^{9-n-r}\times T^n, T^n) \qquad \text{and} \qquad K^*_{\pm}(S^{9-n-r}\times T^n, T^n)\,. \tag{4.1}$$

Here the involution acts on T^n with 2^n fixed points and has trivial action on S^{9-n-r}, where S^{9-n-r} is the one-point compactification of the space $\mathbb{R}^{9-n-r}$ transverse to the brane in the uncompactified directions. The D-branes of interest are trivial at infinity, and the K-theory groups are therefore the relative K-theory groups that describe pairs of bundles (E,F) where E and F are isomorphic at $\{\infty\}\times T^n$. The K-theory groups only classify branes with fixed r; in terms of our previous discussion, these K-theory groups therefore correspond to products over different values of s.

Let us first compute the equivariant relative K-theory groups. Since T^n is a retract of $S^{9-n-r}\times T^n$ we have

$$K^*_{\mathbb{Z}_2}(S^{9-n-r}\times T^n, T^n)\oplus K^*_{\mathbb{Z}_2}(T^n) = K^*_{\mathbb{Z}_2}(S^{9-n-r}\times T^n)\,. \tag{4.2}$$

Here, we may take $K^*_{\mathbb{Z}_2}(T^n)$ and $K^*_{\mathbb{Z}_2}(S^{9-n-r}\times T^n)$ to denote the unreduced equivariant K-theory groups. The former have been computed [40]

$$K_{\mathbb{Z}_2}(T^n) \;=\; 3\cdot 2^{n-1}\,\mathbb{Z}\,, \tag{4.3}$$

$$K^1_{\mathbb{Z}_2}(T^n) \;=\; 0\,, \tag{4.4}$$

where we use the notation that $n\mathbb{Z}\equiv\mathbb{Z}^{\oplus n}$. Furthermore we have

$$K^*_{\mathbb{Z}_2}(X\times S^{2k}) \;=\; K^*_{\mathbb{Z}_2}(X)\oplus K^*_{\mathbb{Z}_2}(X)\,, \tag{4.5}$$

$$K^*_{\mathbb{Z}_2}(X\times S^{2k+1}) \;=\; K^*_{\mathbb{Z}_2}(X)\oplus K^{*-1}_{\mathbb{Z}_2}(X)\,. \tag{4.6}$$

Taking these identities together and using the fact that n is even we then find that

$$K_{\mathbb{Z}_2}(S^{9-n-r}\times T^n, T^n) \;=\; \begin{cases} 3\cdot 2^{n-1}\mathbb{Z} & r \text{ odd,}\\ 0 & r \text{ even,}\end{cases} \tag{4.7}$$

$$K^1_{\mathbb{Z}_2}(S^{9-n-r}\times T^n, T^n) \;=\; \begin{cases} 3\cdot 2^{n-1}\mathbb{Z} & r \text{ even,}\\ 0 & r \text{ odd.}\end{cases} \tag{4.8}$$

Next we consider $K_\pm$. As before we have

$$K^*_{\pm}(S^{9-n-r}\times T^n, T^n)\oplus \tilde{K}^*_{\pm}(T^n) = \tilde{K}^*_{\pm}(S^{9-n-r}\times T^n)\,, \tag{4.9}$$

which we have now written in terms of the reduced $\tilde{K}$-groups. Because of the theorem of Hopkins we have

$$\tilde{K}^m_{\pm}(T^n) \;=\; K^{m-1}_{\mathbb{Z}_2}(T^n\times \mathbb{R}^{1,0})\,, \tag{4.10}$$

$$\tilde{K}^m_{\pm}(S^{9-n-r}\times T^n) \;=\; K^{m-1}_{\mathbb{Z}_2}(S^{9-n-r}\times T^n\times \mathbb{R}^{1,0})\,. \tag{4.11}$$

Using (4.5) and (4.6) this implies that

$$K^m_\pm(S^{9-n-r}\times T^n, T^n) = \begin{cases} K^{m+1}_{\mathbb{Z}_2}(T^n\times \mathbb{R}^{1,0}) & r \text{ odd,}\\ K^m_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0}) & r \text{ even}\,. \end{cases} \tag{4.12}$$

To compute $K^*_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0})$ we consider next the long exact sequence (3.6) with $X=T^n$. Using (4.4) this becomes

$$0\to K^{-1}(T^n)\to K_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0})\to \tilde{K}_{\mathbb{Z}_2}(T^n)\to\tilde{K}(T^n)\to K^1_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0})\to 0\,, \tag{4.13}$$

where we have observed that for compact manifolds K-theory with compact support is the same as reduced K-theory. It is easy to see that (4.13) remains true if we replace the two reduced $\tilde{K}$-groups by their corresponding unreduced groups. We can then use (4.3) together with $K^*(T^n)=2^{n-1}\mathbb{Z}$ to rewrite this as

$$0\to 2^{n-1}\,\mathbb{Z}\xrightarrow{\alpha} K_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0})\xrightarrow{\beta} 3\cdot 2^{n-1}\,\mathbb{Z}\xrightarrow{\gamma} 2^{n-1}\,\mathbb{Z}\to K^1_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0})\to 0\,. \tag{4.14}$$

Since β is injective and the forgetting map γ is onto we then find that

$$K_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0}) = 3\cdot 2^{n-1}\,\mathbb{Z}\,, \tag{4.15}$$
$$K^1_{\mathbb{Z}_2}(T^n\times\mathbb{R}^{1,0}) = 0\,. \tag{4.16}$$

Finally, we thus have

$$K^{-1}_\pm(S^{9-n-r}\times T^n,T^n) = \begin{cases} 3\cdot 2^{n-1}\mathbb{Z} & r \text{ odd,}\\ 0 & r\text{ even,}\end{cases} \tag{4.17}$$
$$K_\pm(S^{9-n-r}\times T^n,T^n) = \begin{cases} 3\cdot 2^{n-1}\mathbb{Z} & r \text{ even,}\\ 0 & r\text{ odd.}\end{cases} \tag{4.18}$$

In agreement with T-duality (2.2), (2.3), these K-theory groups are the same as (4.7) and (4.8), respectively.

4.2 Compact boundary states

Boundary states for D-branes that extend along compact (and inverted) directions have been analysed before [12, 30]. In essence the boundary states are described by the same formulae that we have given in section 2.2 and appendix A.1; there are however a few (minor) differences. Firstly, the branes can wind along the internal directions of the torus, and therefore carry appropriate winding numbers. Secondly, the orbifold has 2^n fixed points in the compactified case, and there are therefore 2^n different twisted sectors. For $s>0$ the D-branes extend along the internal directions, and the corresponding boundary states have a contribution from 2^s of these twisted sectors. (In fact, the 2^s twisted sectors correspond to the 2^s endpoints of the s-dimensional world-volume of the brane along the internal directions.) The structure of the boundary states, and in particular the normalisation of the different boundary components, is described in detail in appendix A.

In all known examples [42] (see also section 5), the only charges that are conserved in the various decay processes among D-branes are the R-R charges in the untwisted and twisted sectors. As we have just seen, there are 2^n different twisted R-R sectors; if for a given r, the corresponding twisted R-R boundary state is allowed, there are then 2^n different twisted R-R sector charges. As regards the charges in the untwisted R-R sector, these arise from the 10-dimensional R-R forms upon compactification on the different cycles of the torus. If the orbifold is of type g_1, the relevant cycles are even-dimensional, and we therefore get

$$N_1 = \sum_{\substack{l=0 \\ l \text{ even}}}^{n} \binom{n}{l} = 2^{n-1}\,, \tag{4.19}$$

whereas for g_2 we have

$$N_2 = \sum_{\substack{l=1 \\ l \text{ odd}}}^{n-1} \binom{n}{l} = 2^{n-1}\,. \tag{4.20}$$

It follows from the analysis in section 2 and appendix B that the condition on r for the twisted R-R sector boundary component to be consistent is the same as that for the untwisted R-R sector component. Thus, if r satisfies this condition, there are altogether

$$N = 2^n + 2^{n-1} = 3 \cdot 2^{n-1} \tag{4.21}$$

R-R charges that form a lattice of dimension $3 \cdot 2^{n-1}$ (and otherwise there are none). Since these are the only charges that are conserved in transitions between different D-branes, the actual D-brane charges form then a sublattice of this lattice.

In general this sublattice is not the whole lattice (see in particular appendix C for a concrete example), but it is of maximal rank; this follows from the fact that it contains yet another sublattice, namely the lattice of D-brane charges that is generated by the bulk and the truncated D-branes. (This is clearly of maximal rank since the bulk D-branes are only charged under the untwisted R-R sector, whereas the truncated D-branes are only charged under the different twisted R-R sectors.) It therefore follows that the lattice of D-brane charges is $3 \cdot 2^{n-1}\mathbb{Z}$ if r satisfies the appropriate condition, and zero otherwise. This is consistent with the result that follows from K-theory.

5 Charge densities

In this section we determine the charge densities of the different D-branes that we have described in this paper. We shall from now on always consider the compactified case; the descent relations apply equally to the Type 0A/0B case, but the overall normalisation is slightly different in that case.

5.1 Descent relations

As we have mentioned before, various D-branes can decay into one another. The basic phenomenon from which all others can be obtained is that of a truncated D-brane $\hat{\mathrm{D}}(r,s)$ decaying into two fractional $\mathrm{D}(r,s')$-branes with $s' = s+1$ or into two fractional $\mathrm{D}(r,s')$-branes with $s' = s-1$. Since the R-R sector charges are conserved by these processes, this implies certain relations between the charges of the different D-branes. In order to obtain these we introduce the following notation. Let us label the 2^n fixed points by $1,\ldots,2^n$. A (r,s) D-brane is charged under 2^s of these 2^n fixed points, and we include their labels as suffices, *i.e.* the (r,s) brane that is charged under the fixed points $n_1,\ldots,n_{2^s}$ is denoted by

$$(r,s)_{n_1,\ldots,n_{2^s}} \,. \tag{5.1}$$

If the charge with respect to one of the twisted R-R sectors is opposite, we place a bar over the corresponding label; similarly, if a fractional D-brane has opposite untwisted R-R sector charge, we place a bar over s in (5.1).

The basic decay processes can now be described as follows: a truncated $\hat{\mathrm{D}}(r,s)$ brane can decay into two fractional D-branes with $s' = s+1$ as

$$\hat{\mathrm{D}}(r,s)_{n_1,\ldots,n_{2^s}} \longrightarrow \mathrm{D}(r,s+1)_{n_1,\ldots,n_{2^s},m_1,\ldots,m_{2^s}} + \mathrm{D}(r,\overline{s+1})_{n_1,\ldots,n_{2^s},\overline{m_1},\ldots,\overline{m_{2^s}}} \,, \tag{5.2}$$

or it can decay into two fractional D-branes with $s' = s-1$ as

$$\hat{\mathrm{D}}(r,s)_{n_1,\ldots,n_{2^s}} \longrightarrow \mathrm{D}(r,s-1)_{n_1,\ldots,n_{2^{s-1}}} + \mathrm{D}(r,\overline{s-1})_{n_{2^{s-1}+1},\ldots,n_{2^s}} \,. \tag{5.3}$$

It follows from this observation that the twisted R-R sector charge of a fractional D-brane $\mathrm{D}(r,s+1)$ at each fixed point is *half* that of a truncated D-brane $\hat{\mathrm{D}}(r,s)$, and that the twisted R-R sector charge of a fractional D-brane $\mathrm{D}(r,s-1)$ at each fixed point is the *same* as that of a truncated D-brane $\hat{\mathrm{D}}(r,s)$. We can apply this argument repeatedly to express the charge of any D-brane in terms of that of the brane with $s=0$. There are two cases to consider: if the orbifold is of type g_1, the $(r,0)$ brane is fractional, and we have

$$\begin{array}{llcl} \mathrm{D}(r,2k) & \mu'_{(r,2k)} & = & 2^{-k}\mu'_{(r,0)} \,, \\ \hat{\mathrm{D}}(r,2k+1) & \mu'_{(r,2k+1)} & = & 2^{-k}\mu'_{(r,0)} \,, \end{array} \tag{5.4}$$

where $\mu'_{(r,s)}$ denotes the twisted R-R charge density of the a $\mathrm{D}(r,s)$-brane. Similarly, if the orbifold of type g_2, the $(r,0)$ brane is truncated, and we find instead

$$\begin{array}{llcl} \mathrm{D}(r,2k+1) & \mu'_{(r,2k+1)} & = & 2^{-k-1}\mu'_{(r,0)'} \,, \\ \hat{\mathrm{D}}(r,2k) & \mu'_{(r,2k)} & = & 2^{-k}\mu'_{(r,0)'} \,. \end{array} \tag{5.5}$$

These relations can also be obtained from the normalisation constants of the various branes that are determined in appendix A; see in particular Eq. (A.27), (A.29).

5.2 The overall normalisation

The above considerations only determine the twisted charges up to an overall factor. In order to find this normalisation constant we shall now compare the (open) string theory calculation with a field theory calculation [3]. We shall only consider the compactified case; because of the arguments of section 5.1 it is then sufficient to do the calculation for a brane with $s = 0$.

If the orbifold theory is of type g_1, the $(r, 0)$ brane is fractional, and its open string has the partition function

$$\int \frac{dt}{2t} \mathrm{Tr}_{NS-R} \left[\frac{1}{4}(1 + (-1)^F)(1 + \mathcal{I}_n) e^{-2tH_o} \right] . \tag{5.6}$$

The twisted R$-$R sector contribution comes from NS$(-1)^F \mathcal{I}_n$ and can be evaluated as

$$-\frac{1}{4} \frac{V_{r+1}}{(2\pi)^{r+1}} \int \frac{dt}{t} (4\pi\alpha' t)^{-(r+1)/2} e^{-t\frac{Y^2}{2\pi\alpha'}} \frac{f_4^{8-n}(\tilde{q}) f_3^n(\tilde{q})}{f_1^{8-n}(\tilde{q}) 2^{-n/2} f_2^n(\tilde{q})} , \tag{5.7}$$

where Y is the separation of the two branes, V_{r+1} denotes the (infinite) world-volume area of the $(r, 0)$-brane, and q, $\tilde{q}$ and the functions f_i are defined as in appendix A. In the field theory limit $(t \to 0)$ this gives

$$\begin{aligned} -\frac{1}{4} \frac{V_{r+1}}{(2\pi)^{r+1}} & \int \frac{dt}{t} (4\pi\alpha' t)^{-(r+1)/2} e^{-t\frac{Y^2}{2\pi\alpha'}} t^{(8-n)/2} (16 + O(e^{-\pi/t})) \\ & \simeq \quad -V_{r+1} \pi (4\pi^2 \alpha')^{3-r} G_{9-r}(Y) . \end{aligned} \tag{5.8}$$

where G_d is the scalar Green's function in d dimensions (*c.f.* Eq. (13.3.2) of [21]). This is to be compared with the field theory calculation, where the relevant terms in the effective action are

$$-\frac{1}{4\kappa_{10-n}^2} \int d^{10-n}x \sqrt{g} (H_t^{(r+2)})^2 + \mu'_{(r,0)} \int C_t^{(r+1)} . \tag{5.9}$$

The field theory amplitude is

$$-2 \left(\mu'_{(r,0)} \kappa_{10-n} \right)^2 G_{9-r}(Y) , \tag{5.10}$$

and comparison of the two results then gives

$$(\kappa_{10-n} \mu'_{(r,0)})^2 = \frac{1}{2} \pi (4\pi^2 \alpha')^{3-r} . \tag{5.11}$$

It follows from the arguments of section 5.1 that the general formula for the twisted R-R charge density $\mu'_{(r,s)}$ of a (r, s) brane on an $\mathcal{I}_n$ orbifold is then

$$(\kappa_{10-n} \mu'_{(r,s)})^2 = 2^{-(2k+1)} \pi (4\pi^2 \alpha')^{3-r} , \tag{5.12}$$

where $s = 2k$ or $s = 2k+1$ for fractional and truncated branes, respectively.

In the case of the g_2 orbifold the $(r,0)$ brane is truncated. The partition function of the relevant open string is then

$$\int \frac{dt}{2t} \mathrm{Tr}_{NS-R} \left[\frac{1}{2}(1 + (-1)^F \mathcal{I}_n) e^{-2tH_o} \right] . \tag{5.13}$$

The twisted R$-$R charge contribution comes again from the NS$(-1)^F \mathcal{I}_n$ sector and is exactly twice that in equation (5.7). On the other hand, the field theory action is as before, and we therefore find

$$(\kappa_{10-n} \mu'_{(r,0)'})^2 = \pi (4\pi^2 \alpha')^{3-r} . \tag{5.14}$$

Denoting by $\mu'_{(r,s)'}$ the twisted R$-$R charge density of an (r,s)-brane on a $(-1)^{F_L} \mathcal{I}_n$ orbifold we thus have

$$(\kappa_{10-n} \mu'_{(r,s)'})^2 = 2^{-2k} \pi (4\pi^2 \alpha')^{3-r} , \tag{5.15}$$

where now $s = 2k$ or $s = 2k-1$ for a truncated or fractional brane, respectively.

By similar methods one can also determine the charge densities with respect to the untwisted R$-$R sector, and the tensions of the different branes.

6 Conclusions

In this paper we have analysed systematically the D-branes of certain orbifolds of (toroidal compactifications) of Type IIA/IIB string theory. We have determined the corresponding K-theory groups, and we have found complete agreement with the results obtained from a boundary state analysis. We have also calculated the relevant charge densities.

It would be interesting to determine the world-volume actions of the branes we have considered in this paper; work in this direction is in progress [43].

A Construction and normalisation of boundary states

In this appendix we determine the normalisation constants of the boundary states for the orbifold theories under consideration. We shall use the conventions of [12, 30]. Let us first consider the uncompactified case.

A.1 The uncompactified case

In each (bosonic) sector of the theory we can construct the boundary state

$$|B(r,s),k,\eta\rangle = \exp\left(\sum_{l>0}^{\infty} \left[\frac{1}{l} \alpha^{\mu}_{-l} S_{\mu\nu} \tilde{\alpha}^{\nu}_{-l} \right] + i\eta \sum_{m>0}^{\infty} \left[\psi^{\mu}_{-m} S_{\mu\nu} \tilde{\psi}^{\nu}_{-m} \right] \right) |B(r,s),k,\eta\rangle^{(0)} , \tag{A.1}$$

where, depending on the sector, l and m are integer or half-integer, and k denotes the momentum of the ground state. We shall always work in light-cone gauge with light-cone directions x^0 and x^9; thus μ and ν take the values $1, \ldots, 8$. We shall also drop the dependence on α' from now on.

The parameter $\eta = \pm$ describes the two different spin structures [4, 5], and the matrix S encodes the boundary conditions of the Dp-brane which we shall always take to be diagonal

$$S = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)\,, \tag{A.2}$$

where $p+1$ entries are equal to -1, $7-p$ entries are equal to $+1$, and $p = r+s$. If there are fermionic zero modes, the ground state in (A.1) satisfies an additional condition. (This will be relevant in appendix B; see for example (B.3) and (B.11).)

In order to obtain a localised D-brane, we have to take the Fourier transform of the above boundary state, where we integrate over the directions transverse to the brane,

$$|B(r,s), y, \eta\rangle = \int \left(\prod_{\mu \text{ transverse}} dk^\mu e^{ik^\mu y_\mu} \right) dk^0 e^{ik^0 y_0}\, dk^9 e^{ik^9 y_9}\, |B(r,s), k, \eta\rangle\,, \tag{A.3}$$

y denotes the location of the boundary state, and in the twisted sectors the momentum integral only involves transverse directions that are not inverted by the orbifold action. In the following we shall always consider (without loss of generality) the case $y=0$ in which case the boundary state is denoted by $|B(r,s), \eta\rangle$.

The invariance of the boundary state under the GSO-projection always requires that the physical boundary state is a linear combination of the two states corresponding to $\eta = \pm$. Using the conventions of appendix B, these linear combinations are of the form

$$\begin{aligned}
|B(r,s)\rangle_{\text{NS-NS}} &= \frac{1}{2}\Big(|B(r,s),+\rangle_{\text{NS-NS}} - |B(r,s),-\rangle_{\text{NS-NS}}\Big)\,, && \text{(A.4)}\\
|B(r,s)\rangle_{\text{R-R}} &= \frac{4i}{2}\Big(|B(r,s),+\rangle_{\text{R-R}} + |B(r,s),-\rangle_{\text{R-R}}\Big)\,, && \text{(A.5)}\\
|B(r,s)\rangle_{\text{NS-NS},\text{T}} &= \frac{2^{n/4}}{2}\Big(|B(r,s),+\rangle_{\text{NS-NS},\text{T}} + |B(r,s),-\rangle_{\text{NS-NS},\text{T}}\Big)\,, && \text{(A.6)}\\
|B(r,s)\rangle_{\text{R-R},\text{T}} &= \frac{2^{2-n/4}i}{2}\Big(|B(r,s),+\rangle_{\text{R-R},\text{T}} + |B(r,s),-\rangle_{\text{R-R},\text{T}}\Big)\,, && \text{(A.7)}
\end{aligned}$$

where, depending on the theory in question, these states are actually GSO-invariant provided that r and s satisfy suitable conditions. The normalisation constants have been introduced for later convenience.

In order to solve the open-closed consistency condition the actual D-brane state is a linear combination of physical boundary states from different sectors. There are two cases to consider, fractional and truncated D-branes. In the former case, the D-brane state can be written as

$$\begin{aligned}
|D(r,s)\rangle &= \mathcal{N}_{f,U}\left(|B(r,s)\rangle_{\text{NS-NS}} + \epsilon_1\, |B(r,s)\rangle_{\text{R-R}}\right)\\
&\quad + \epsilon_2\, \mathcal{N}_{f,T}\left(|B(r,s)\rangle_{\text{NS-NS},\text{T}} + \epsilon_1\, |B(r,s)\rangle_{\text{R-R},\text{T}}\right)\,,
\end{aligned} \tag{A.8}$$

where $\epsilon_i = \pm$ determines the sign of the charge with respect to the untwisted and twisted R$-$R sector charge. The closed string cylinder diagram is then of the form[2]

$$\begin{aligned}\mathcal{A} &= \int dl\, \langle B(r,s)|\, e^{-lH_c}\, |B(r,s)\rangle \\ &= \frac{1}{2}\mathcal{N}_{f,U}^2 \int dl\; l^{(p-9)/2} \left(\frac{f_3^8(q) - f_4^8(q) - f_2^8(q)}{f_1^8(q)}\right) \\ &\quad + \frac{1}{2}\mathcal{N}_{f,T}^2 \int dl\; l^{(r+n-9)/2} \left(\frac{f_3^{8-n}(q) f_2^n(q) - f_2^{8-n}(q) f_3^n(q) - \delta_{8,n} f_4^8(q)}{f_1^{8-n}(q) f_4^n(q)}\right), \qquad (A.9)\end{aligned}$$

where the functions f_i are defined as in [4], $q = e^{-2\pi l}$, and the closed string Hamiltonian is given by

$$H_c = \pi k^2 + 2\pi \sum_{\mu=1}^{8} \left[\sum_{l>0}^{\infty} (\alpha_{-l}^{\mu}\alpha_l^{\mu} + \tilde{\alpha}_{-l}^{\mu}\tilde{\alpha}_l^{\mu}) + \sum_{m>0}^{\infty} m(\psi_{-m}^{\mu}\psi_m^{\mu} + \tilde{\psi}_{-m}^{\mu}\tilde{\psi}_m^{\mu})\right] + 2\pi C_c\,. \qquad (A.10)$$

Here the constant C_c is -1 in the NS$-$NS sector, zero in the untwisted and twisted R$-$R sector, and $(n-4)/4$ in the twisted NS$-$NS sector. The corresponding open string amplitude is obtained by the modular transformation $t = 1/2l$, $\tilde{q} = e^{-\pi t}$,

$$\begin{aligned}\mathcal{A} &= 2^{(7-p)/2}\mathcal{N}_{f,U}^2 \int \frac{dt}{2t} t^{-(p+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_2^8(\tilde{q}) - f_4^8(\tilde{q})}{f_1^8(\tilde{q})}\right) \\ &\quad + 2^{(7-n-r)/2}\mathcal{N}_{f,T}^2 \int \frac{dt}{2t} t^{-(r+1)/2} \left(\frac{f_3^{8-n}(\tilde{q}) f_4^n(\tilde{q}) - f_4^{8-n}(\tilde{q}) f_3^n(\tilde{q}) - \delta_{8,n} f_2^8(\tilde{q})}{f_1^{8-n}(\tilde{q}) f_2^n(\tilde{q})}\right). \quad (A.11)\end{aligned}$$

This is to be compared with the open string one-loop diagram,

$$\begin{aligned}&\int \frac{dt}{2t} \mathrm{Tr}_{NS-R}\left(\frac{1+(-1)^F}{2}\frac{1+g}{2} e^{-2tH_o}\right) \\ &= \frac{V_{p+1}}{(2\pi)^{p+1}} 2^{-(p+5)/2} \int \frac{dt}{2t} t^{-(p+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_4^8(\tilde{q}) - f_2^8(\tilde{q})}{f_1^8(\tilde{q})}\right) \\ &\quad + \frac{V_{r+1}}{(2\pi)^{r+1}} 2^{(n-r-5)/2} \int \frac{dt}{2t} t^{-(r+1)/2} \left(\frac{f_3^{8-n}(\tilde{q}) f_4^n(\tilde{q}) - f_4^{8-n}(\tilde{q}) f_3^n(\tilde{q}) - \delta_{8,n} f_2^8(\tilde{q})}{f_1^{8-n}(\tilde{q}) f_2^n(\tilde{q})}\right),\end{aligned} \qquad (A.12)$$

[2] The minus sign in the term proportional to $\delta_{8,n}$ does not, at first sight, agree with the conventions of equation (A.7). However for $n = 8$ the twisted R$-$R sector does not have any zero modes, and the ground state is therefore unique; since the orbifold preserves supersymmetry [44], the sign of the GSO projection in this case is then determined by supersymmetry.

where g denotes the orbifold operator, V_{p+1} is the (infinite) $p+1$ dimensional volume of the brane, whilst V_{r+1} is the volume of the projection onto the directions unaffected by $\mathcal{I}_n$. The open string Hamiltonian is given by

$$H_o = \pi p^2 + \pi \sum_{\mu=1}^{8} \left[\sum_{l>0}^{\infty} \alpha^{\mu}_{-l} \alpha^{\mu}_{l} + \sum_{m>0}^{\infty} m \psi^{\mu}_{-m} \psi^{\mu}_{m} \right] + \pi C_o \,, \tag{A.13}$$

where, in the R sector, l and m run over the positive integers for NN and DD directions, and over positive half integers for ND directions. In the NS sector, the moding of the fermions (and therefore the values for m) are opposite to those in the R sector. C_o is zero in the R sector and is $\frac{4-t}{8}$ in the NS sector, where t is the number of ND directions. Comparison of equations (A.12) and (A.11) then gives

$$\mathcal{N}^2_{f,U} = \frac{V_{p+1}}{(2\pi)^{p+1}} \frac{1}{64} \,, \tag{A.14}$$

$$\mathcal{N}^2_{f,T} = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{2^n}{64} \,. \tag{A.15}$$

The analysis for the case of the truncated D-branes is similar. In this case the D-brane boundary state is given by

$$|\hat{D}(r,s)\rangle = \mathcal{N}_{t,U} \, |B(r,s)\rangle_{\mathrm{NS-NS}} + \epsilon \mathcal{N}_{f,T} \, |B(r,s)\rangle_{\mathrm{R-R\ ,T}} \,, \tag{A.16}$$

where $\epsilon = \pm$ determines the sign of the twisted R$-$R sector charge. The closed string tree diagram now only produces some of the terms of (A.9), and the corresponding open string amplitude is

$$\begin{aligned} &\int \frac{dt}{2t} \mathrm{Tr}_{NS-R} \left(\frac{1+g(-1)^F}{2} e^{-2tH_o} \right) \\ &\quad = \frac{V_{p+1}}{(2\pi)^{p+1}} 2^{-(p+3)/2} \int \frac{dt}{2t} t^{-(p+1)/2} \left(\frac{f_3^8(\tilde{q}) - f_2^8(\tilde{q})}{f_1^8(\tilde{q})} \right) \\ &\qquad - \frac{V_{r+1}}{(2\pi)^{r+1}} 2^{(n-r-3)/2} \int \frac{dt}{2t} t^{-(r+1)/2} \left(\frac{f_4^{8-n}(\tilde{q}) f_3^n(\tilde{q}) - \delta_{8,n} f_2^8(\tilde{q})}{f_1^{8-n}(\tilde{q}) f_2^n(\tilde{q})} \right) . \end{aligned} \tag{A.17}$$

(A.18)

Comparison with the corresponding closed string calculation then gives

$$\mathcal{N}^2_{t,U} = \frac{V_{p+1}}{(2\pi)^{p+1}} \frac{1}{32} \,, \tag{A.19}$$

$$\mathcal{N}^2_{t,T} = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{2^n}{32} \,. \tag{A.20}$$

A.2 The compactified case

The construction in the compactified case is essentially the same as in the above uncompactified case; however there are the following differences.
1. In the localised boundary state (A.3) the integral over compact transverse directions is replaced by a sum

$$\int dk^\nu e^{ik^\nu y_\nu} \longrightarrow \sum_{m^\nu \in \mathbb{Z}} e^{im^\nu y_\nu / R_\nu} , \tag{A.21}$$

where R_ν is the radius of the compact x^ν direction.
2. In the two untwisted sectors, the ground state is in addition characterised by a winding number w_ν for each compact direction that is tangential to the world-volume of the brane. The localised bound state (A.3) then also contains a sum over these winding states

$$\sum_{w_\mu} e^{i\theta^\mu w_\mu} , \tag{A.22}$$

where θ^μ is a Wilson line; as required by orbifold invariance, $\theta^\mu \in \{0, \pi\}$.
3. For general s, the contribution in the two twisted sectors consists of a sum of terms that are associated to 2^s of the 2^n different twisted sectors that define the endpoints of the world-volume of the brane in the internal space. For convenience we may assume that one of the 2^s fixed points is always the origin.
4. The open and closed string Hamiltonians, H_o and H_c, each acquire an extra term $1/4\pi(\sum_\mu w_\mu^2)$.

Let us now construct in more detail the boundary state for a fractional D(r, s) brane. This is of the form

$$\begin{aligned} |D(r,s)\rangle \;&=\; \mathcal{N}_{f,U} \Big(|B(r,s)\rangle_{\rm NS-NS} \; + \epsilon_1 \, |B(r,s)\rangle_{\rm R-R} \Big) \\ &\quad + \epsilon_2 \, \mathcal{N}_{f,T} \sum_{\alpha=1}^{2^s} e^{i\theta_\alpha} \Big(|B(r,s)\rangle_{\rm NS-NS\,,T_\alpha} + \epsilon_1 \, |B(r,s)\rangle_{\rm R-R\,,T_\alpha} \Big) , \end{aligned} \tag{A.23}$$

where α labels the different fixed points between which the brane stretches (where we choose the convention that T_1 is the twisted sector at the origin), and θ_α is the Wilson line that is associated to the difference of the fixed point α and the origin. (Thus if α has coordinates $(n_i \pi R_i)$, $i = 9-n, \ldots, 8$, $\theta_\alpha = n_i \theta^i$.) The closed string tree diagram is now

$$\begin{aligned} \mathcal{A}_c \;&=\; \int dl \, \langle B(r,s)| \, e^{-lH_c} \, |B(r,s)\rangle \\ &=\; \frac{1}{2} \mathcal{N}_{f,U}^2 \int dl \; l^{(r+n-9)/2} \left(\frac{f_3^8(q) - f_2^8(q) - f_4^8(q)}{f_1^8(q)} \right) \\ &\qquad \times \prod_{i=1}^{s} \sum_{w_{j_i} \in \mathbb{Z}} e^{-l\pi R_{j_i}^2 w_{j_i}^2} \prod_{i=1}^{n-s} \sum_{n_{k_i} \in \mathbb{Z}} e^{-l\pi (n_{k_i}/R_{k_i})^2} \end{aligned}$$

$$+\frac{2^s}{2}\mathcal{N}_{f,T}^2\int dl\, l^{(r+n-9)/2}\left(\frac{f_3^{8-n}(q)f_2^n(q)-f_2^{8-n}(q)f_3^n(q)-\delta_{8,n}f_4^8(q)}{f_1^{8-n}(q)f_4^n(q)}\right), \quad (A.24)$$

where R_{j_i}, $i=1,\ldots,s$ are the radii of the circles that are tangential to the world-volume of the brane, and R_{k_i}, $i=1,\ldots,n-s$ are the radii of the directions transverse to the brane. Upon the substitution $t=1/2l$, using the Poisson resummation formula (see for example [12, 30]), this amplitude becomes

$$\begin{aligned}\mathcal{A}_c &= \mathcal{N}_{f,U}^2\frac{\prod_{i=1}^{n-s}R_{k_i}}{\prod_{i=1}^{s}R_{j_i}}2^{(7-r)/2}\int\frac{dt}{2t}t^{-(r+1)/2}\left(\frac{f_3^8(\tilde q)-f_2^8(\tilde q)-f_4^8(\tilde q)}{f_1^8(\tilde q)}\right)\\ &\quad\times\prod_{i=1}^{s}\sum_{n_{j_i}\in\mathbb{Z}}e^{-2t\pi n_{j_i}^2/R_{j_i}^2}\prod_{i=1}^{n-s}\sum_{w_{k_i}\in\mathbb{Z}}e^{-2t\pi w_{k_i}^2R_{k_i}^2}\\ &\quad+2^{(7-n-r)/2}2^s\mathcal{N}_{f,T}^2\int\frac{dt}{2t}t^{-(r+1)/2}\left(\frac{f_3^{8-n}(\tilde q)f_4^n(\tilde q)-f_4^{8-n}(\tilde q)f_3^n(\tilde q)-\delta_{8,n}f_2^8(\tilde q)}{f_1^{8-n}(\tilde q)f_2^n(\tilde q)}\right).\end{aligned}$$

This is to be compared with the open string amplitude

$$\begin{aligned}\int\frac{dt}{2t}\mathrm{Tr}_{NS-R}\left(\frac{1+(-1)^F}{2}\frac{1+g}{2}e^{-2tH_o}\right) &= \frac{V_{r+1}}{4(2\pi)^{r+1}}2^{-(r+1)/2}\int\frac{dt}{2t}t^{-(r+1)/2}\frac{f_3^8(\tilde q)-f_4^8(\tilde q)-f_2^8(\tilde q)}{f_1^8(\tilde q)}\\ &\quad\times\prod_{i=1}^{s}\sum_{n_{j_i}\in\mathbb{Z}}e^{-2t\pi n_{j_i}^2/R_{j_i}^2}\prod_{i=1}^{n-s}\sum_{w_{k_i}\in\mathbb{Z}}e^{-2t\pi w_{k_i}^2R_{k_i}^2}\\ &\quad+\frac{V_{r+1}}{(2\pi)^{r+1}}2^{(n-r-5)/2}\int\frac{dt}{2t}t^{-(r+1)/2}\\ &\quad\times\frac{f_3^{8-n}(\tilde q)f_4^n(\tilde q)-f_4^{8-n}(\tilde q)f_3^n(\tilde q)-\delta_{8,n}f_2^8(\tilde q)}{f_1^{8-n}(\tilde q)f_2^n(\tilde q)}.\end{aligned} \quad (A.25)$$

By comparison this then fixes the normalisation constants as

$$\mathcal{N}_{f,U}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}}\frac{1}{64}\frac{\prod_{i=1}^{s}R_{j_i}}{\prod_{i=1}^{n-s}R_{k_i}}, \quad (A.26)$$

$$\mathcal{N}_{f,T}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}}\frac{2^{n-s}}{64}. \quad (A.27)$$

The analysis for the truncated D-branes is almost identical; the boundary state is the truncation of (A.23) to the untwisted NS−NS and the twisted R−R sector; this then only depends on one parameter $\epsilon=\epsilon_1\epsilon_2$ as well as the Wilson lines θ_α. The open string amplitude contains also only the corresponding terms. Furthermore, since the projection operator is now $\frac{1}{2}(1+g(-1)^F)$

each of the terms that appears is twice as large as in the fractional case. This implies that the relevant normalisation constants are given as

$$\mathcal{N}_{t,U}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{1}{32} \frac{\prod_{i=1}^{s} R_{j_i}}{\prod_{i=1}^{n-s} R_{k_i}}, \tag{A.28}$$

$$\mathcal{N}_{t,T}^2 = \frac{V_{r+1}}{(2\pi)^{r+1}} \frac{2^{n-s}}{32}. \tag{A.29}$$

We should mention that the normalisation constants of the twisted sector boundary states (A.27) and (A.29) in the compactified theory differ from the corresponding normalisation constants in the uncompactified theory (A.15) and (A.20) by a factor of 2^s. This seems contradictory since the twisted charge that is carried by a D-brane should not depend on whether the directions transverse to the orbifold plane are compact or not. Presumably this means that we can only trust the boundary state calculation if all directions tangential to the brane are compact; this is anyway necessary for the normalisation constants in the untwisted sector to make sense (since they are proportional to the volume of the brane).[3]

B Consistency conditions of boundary states

In this appendix we analyse which boundary states are GSO- and orbifold invariant. Let us first consider the condition that comes from GSO-invariance. It is well known that the untwisted NS−NS component is invariant under the GSO-projection for all (r, s). In the untwisted R−R sector, the GSO-projection acts as in the theory before orbifolding, and therefore the boundary state is only GSO-invariant if $r + s$ is even in the case of a Type IIA orbifold, and odd in the case of Type IIB [8].

In the twisted NS−NS sector there exist fermionic zero modes along the n directions that are inverted by $\mathcal{I}_n$, and therefore the condition only affects s. Let us introduce

$$\psi_{\pm}^{\mu} = \frac{1}{\sqrt{2}} \left(\psi_0^{\mu} \pm i\tilde{\psi}_0^{\mu} \right), \tag{B.1}$$

where μ takes the n values $\mu = 9-n, \ldots, 8$ and $\{\psi_0^{\mu}, \psi_0^{\nu}\} = \{\tilde{\psi}_0^{\mu}, \tilde{\psi}_0^{\nu}\} = \delta^{\mu\nu}$; these modes satisfy then the Clifford algebra

$$\{\psi_{\pm}^{\mu}, \psi_{\pm}^{\nu}\} = 0, \qquad \{\psi_{+}^{\mu}, \psi_{-}^{\nu}\} = \delta^{\mu,\nu}. \tag{B.2}$$

The GSO-invariant boundary state is a linear combination of the two states $|B(r,s),\eta\rangle_{\mathrm{NS-NS\,,T}}$ with $\eta = \pm$. On the ground states $|B(r,s),\eta\rangle^0$, the fermionic zero modes satisfy

$$\begin{aligned} &\psi_{\eta}^{\nu}|B(r,s),\eta\rangle_{\mathrm{NS-NS\,,T}}^0 = 0 \quad \text{for the Neumann directions } \nu = 9-n, \ldots, 8-n+s, \\ &\psi_{-\eta}^{\nu}|B(r,s),\eta\rangle_{\mathrm{NS-NS\,,T}}^0 = 0 \quad \text{for the Dirichlet directions } \nu = 9-n+s, \ldots, 8, \end{aligned} \tag{B.3}$$

[3] We thank Ashoke Sen for a discussion on this point.

where we have suppressed the dependence of the ground state on k that is irrelevant for the present discussion. We may choose the relative normalisation between the ground states corresponding to $\eta = \pm$ to be defined by

$$|B(r,s),+\rangle^0_{\mathrm{NS-NS},\mathrm{T}} = a \prod_{\nu=9-n+s}^{8} \psi^\nu_- \prod_{\nu=9-n}^{8-n+s} \psi^\nu_+ |B(r,s),-\rangle^0_{\mathrm{NS-NS},\mathrm{T}} \,. \tag{B.4}$$

It then follows that

$$|B(r,s),-\rangle^0_{\mathrm{NS-NS},\mathrm{T}} = b \prod_{\nu=9-n+s}^{8} \psi^\nu_+ \prod_{\nu=9-n}^{8-n+s} \psi^\nu_- |B(r,s),+\rangle^0_{\mathrm{NS-NS},\mathrm{T}} \,, \tag{B.5}$$

where

$$b = \frac{(-1)^{n(n-1)/2}}{a} \,. \tag{B.6}$$

The expression $(-1)^{n(n-1)/2}$ is $+1$ for $n = 0 \bmod(4)$, and equals -1 for $n = 2 \bmod(4)$. On the ground states the two GSO-projections $(-1)^F$ and $(-1)^{\tilde{F}}$ take the form

$$\begin{aligned} (-1)^F &= c \prod_{\mu=9-n}^{8} (\sqrt{2}\psi^\mu_0) = c \prod_{\mu=9-n}^{8} (\psi^\mu_+ + \psi^\mu_-) \,, \\ (-1)^{\tilde{F}} &= d \prod_{\mu=9-n}^{8} (\sqrt{2}\tilde{\psi}^\mu_0) = d i^n \prod_{\mu=9-n}^{8} (\psi^\mu_+ - \psi^\mu_-) \,. \end{aligned} \tag{B.7}$$

Since both $(-1)^F$ and $(-1)^{\tilde{F}}$ have to be of order 2, c and d satisfy

$$c^2 = d^2 = (-1)^{n(n-1)/2} \,. \tag{B.8}$$

Applying (B.7) to the boundary states we then find

$$\begin{aligned} (-1)^F |B(r,s),-\rangle_{\mathrm{NS-NS},\mathrm{T}} &= \tfrac{c}{a} |B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}} \,, \\ (-1)^F |B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}} &= \tfrac{c}{b} |B(r,s),-\rangle_{\mathrm{NS-NS},\mathrm{T}} \,, \\ (-1)^{\tilde{F}} |B(r,s),-\rangle_{\mathrm{NS-NS},\mathrm{T}} &= i^n \tfrac{d}{a} (-1)^{n-s} |B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}} \,, \\ (-1)^{\tilde{F}} |B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}} &= i^n \tfrac{d}{b} (-1)^{s} |B(r,s),-\rangle_{\mathrm{NS-NS},\mathrm{T}} \,. \end{aligned} \tag{B.9}$$

We may choose for convenience

$$a = b = c = (-1)^{n(n-1)/4} \,, \qquad i^n d = \kappa c \,,$$

where $\kappa = \pm 1$; this is then consistent with (B.6) and (B.8).

If the orbifold does not involve $(-1)^{F_L}$, *i.e.* if it is a g_1 orbifold, then the left- and right GSO-projections have to be the same, and thus a GSO-invariant combination only exists if $\kappa(-1)^s = +1$. In the case of a g_2 orbifold the situation is precisely opposite, *i.e.* a GSO-invariant boundary state exists provided that $\kappa(-1)^s = -1$. In either case, the GSO-invariant boundary state is then, up to normalisation, given by

$$|B(r,s)\rangle_{\mathrm{NS-NS},\mathrm{T}} = |B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}} + |B(r,s),-\rangle_{\mathrm{NS-NS},\mathrm{T}} . \tag{B.10}$$

The analysis in the twisted R−R sector is very similar; in this case the fermionic zero modes occur for $\mu = 1,\dots,8-n$, and the condition therefore only involves r. We can similarly introduce modes $\psi^\mu_\pm$ that satisfy a Clifford algebra, and the boundary states $|B(r,s),\eta\rangle_{\mathrm{R-R},\mathrm{T}}$ then satisfy

$$\begin{array}{ll} \psi^\nu_\eta |B(r,s),\eta\rangle^0_{\mathrm{R-R},\mathrm{T}} = 0 & \text{for the Neumann directions } \nu = 1,\dots,r+1, \\ \psi^\nu_{-\eta} |B(r,s),\eta\rangle^0_{\mathrm{R-R},\mathrm{T}} = 0 & \text{for the Dirichlet directions } \nu = r+2,\dots,8-n, \end{array} \tag{B.11}$$

where the suffix 0 again denotes the ground state. We can choose the relative normalisation of the ground states as

$$|B(r,s),+\rangle^0_{\mathrm{R-R},\mathrm{T}} = \hat{a} \prod_{\nu=1}^{r+1} \psi^\nu_- \prod_{\nu=r+2}^{8-n} \psi^\nu_+ |B(r,s),-\rangle^0_{\mathrm{R-R},\mathrm{T}} , \tag{B.12}$$

and

$$|B(r,s),-\rangle^0_{\mathrm{R-R},\mathrm{T}} = \hat{b} \prod_{\nu=1}^{r+1} \psi^\nu_+ \prod_{\nu=r+2}^{8-n} \psi^\nu_- |B(r,s),+\rangle^0_{\mathrm{R-R},\mathrm{T}} , \tag{B.13}$$

where

$$\hat{a}\hat{b} = (-1)^{(8-n)(7-n)/2} . \tag{B.14}$$

On the ground states the two GSO-projections $(-1)^F$ and $(-1)^{\tilde{F}}$ take the form

$$\begin{array}{ll} (-1)^F & = \hat{c}\prod_{\mu=1}^{8-n}(\sqrt{2}\psi^\mu_0) = \hat{c}\prod_{\mu=1}^{8-n}(\psi^\mu_+ + \psi^\mu_-) , \\ (-1)^{\tilde{F}} & = \hat{d}\prod_{\mu=1}^{8-n}(\sqrt{2}\tilde{\psi}^\mu_0) = \hat{d} i^n \prod_{\mu=1}^{8-n}(\psi^\mu_+ - \psi^\mu_-) , \end{array} \tag{B.15}$$

and since $(-1)^F$ and $(-1)^{\tilde{F}}$ have to be of order 2,

$$\hat{c}^2 = \hat{d}^2 = (-1)^{(8-n)(7-n)/2} . \tag{B.16}$$

We thus find

$$\begin{array}{rcl} (-1)^F|B(r,s),-\rangle_{\mathrm{R-R},\mathrm{T}} & = & \frac{\hat{c}}{\hat{a}}|B(r,s),+\rangle_{\mathrm{R-R},\mathrm{T}} , \\ (-1)^F|B(r,s),+\rangle_{\mathrm{R-R},\mathrm{T}} & = & \frac{\hat{c}}{\hat{b}}|B(r,s),-\rangle_{\mathrm{R-R},\mathrm{T}} , \\ (-1)^{\tilde{F}}|B(r,s),-\rangle_{\mathrm{R-R},\mathrm{T}} & = & i^n\frac{\hat{d}}{\hat{a}}(-1)^{7-n-r}|B(r,s),+\rangle_{\mathrm{R-R},\mathrm{T}} , \\ (-1)^{\tilde{F}}|B(r,s),+\rangle_{\mathrm{R-R},\mathrm{T}} & = & i^n\frac{\hat{d}}{\hat{b}}(-1)^{r+1}|B(r,s),-\rangle_{\mathrm{R-R},\mathrm{T}} . \end{array} \tag{B.17}$$

As before we may choose for convenience

$$\hat{a} = \hat{b} = \hat{c} = (-1)^{(8-n)(7-n)/4} , \qquad i^n\hat{d} = \hat{\kappa}\hat{c} ,$$

where $\hat{\kappa}$ is again ± 1. In the case of a g_1 orbifold of Type IIB or a g_2 orbifold of Type IIA, the left- and right-GSO-projections are the same, and therefore a GSO-invariant boundary state exists provided that $\hat{\kappa}(-1)^{r+1} = +1$; similarly a GSO-invariant boundary state exists in the other two cases provided that $\hat{\kappa}(-1)^{r+1} = -1$; in either case, the boundary state for the twisted R−R sector is then, up to normalisation, given by

$$|B(r,s)\rangle_{\mathrm{R-R\,,T}} = |B(r,s),+\rangle_{\mathrm{R-R\,,T}} + |B(r,s),-\rangle_{\mathrm{R-R\,,T}}. \tag{B.18}$$

In summary the following boundary states are thus GSO-invariant

$$\begin{array}{lll} |B(r,s)\rangle_{\mathrm{NS-NS}} & \text{for all } (r,s), & \\ |B(r,s)\rangle_{\mathrm{R-R}} & \text{if } r+s \text{ is} & \begin{cases} \text{even:} & \text{IIA-orbifold,} \\ \text{odd:} & \text{IIB-orbifold,} \end{cases} \\ |B(r,s)\rangle_{\mathrm{NS-NS\,,T}} & \text{if } \kappa(-1)^s \text{ is} & \begin{cases} +1: & g_1\text{-orbifold,} \\ -1: & g_2\text{-orbifold,} \end{cases} \\ |B(r,s)\rangle_{\mathrm{R-R\,,T}} & \text{if } \hat{\kappa}(-1)^r \text{ is} & \begin{cases} +1: & g_1\text{-orbifold of IIA or } g_2\text{-orbifold of IIB,} \\ -1: & g_2\text{-orbifold of IIA or } g_1\text{-orbifold of IIB.} \end{cases} \end{array} \tag{B.19}$$

It is reasonable to assume that every theory possesses at least one fractional brane; if we make this assumption, it follows that κ and $\hat{\kappa}$ must be the same. For example, for the g_1 orbifold of IIA, a GSO-invariant boundary state exists in all sectors provided that $(-1)^{r+s}$, $\kappa(-1)^s$ and $\hat{\kappa}(-1)^r$ are all $+1$; this requires that $\kappa\hat{\kappa} = +1$. The other cases are similar. Actually, the value of κ is completely determined by this assumption once we consider the conditions that come from the requirement that the boundary states must also be invariant under the orbifold projection. Again, the boundary state in the untwisted NS−NS sector is invariant under both g_1 or g_2, but a non-trivial condition arises in the untwisted R−R sector. Indeed, since there are fermionic zero modes, $\mathcal{I}_n$ acts on the corresponding ground states (that are analogously defined to (B.3) and (B.11)) as

$$\begin{aligned} \mathcal{I}_n|B(r,s),+\rangle^0_{\mathrm{R-R}} &= \prod_{\mu=9-n}^{8}(\sqrt{2}\psi^\mu_0)\prod_{\mu=9-n}^{8}(\sqrt{2}\tilde{\psi}^\mu_0)|B(r,s),+\rangle^0_{\mathrm{R-R}} && \text{(B.20)} \\ &= i^n(-1)^{n(n-1)/2}(-1)^s|B(r,s),+\rangle^0_{\mathrm{R-R}}\ , && \text{(B.21)} \end{aligned}$$

and similarly

$$\mathcal{I}_n|B(r,s),-\rangle^0_{\mathrm{R-R}} = i^n(-1)^{n(n-1)/2}(-1)^{n-s}|B(r,s),-\rangle^0_{\mathrm{R-R}}\ . \tag{B.22}$$

In the case of the g_1 orbifold, the boundary state is invariant under the orbifold projection provided that s is even; in the case of a g_2 orbifold the condition is that s is odd. If this is to be consistent with what we found above, we have to choose $\kappa = \hat{\kappa} = +1$. Incidentally, this is the convention for which

$$(-1)^F(-1)^{\tilde{F}} = \mathcal{I}_n\,, \tag{B.23}$$

on the ground states of the twisted NS−NS sector.

The condition in the twisted sectors is more difficult to analyse since the definition of $\mathcal{I}_n$ in the twisted sector is *a priori* ambiguous. The correct prescription seems to be that the physical states in both twisted sectors have to have eigenvalue +1 (−1) with respect to the standard orbifold projection if the orbifold is of type $g = g_1$ ($g = g_2$).[4] This does not give any further restrictions for the states in the twisted R−R sector (since the action of $\mathcal{I}_n$ on the ground states does not involve any fermionic zero modes), and in the twisted NS−NS sector, we find

$$\begin{aligned} \mathcal{I}_n|B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}} &= \prod_{\mu=9-n}^{8}(\sqrt{2}\psi_0^\mu)\prod_{\mu=9-n}^{8}(\sqrt{2}\tilde{\psi}_0^\mu)|B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}} && \text{(B.24)} \\ &= i^n(-1)^{n(n-1)/2}(-1)^{n-s}|B(r,s),+\rangle_{\mathrm{NS-NS},\mathrm{T}}\,, && \text{(B.25)} \end{aligned}$$

and similarly

$$\mathcal{I}_n|B(r,s),-\rangle_{\mathrm{NS-NS},\mathrm{T}} = i^n(-1)^{n(n-1)/2}(-1)^{s}|B(r,s),-\rangle_{\mathrm{NS-NS},\mathrm{T}}\,. \qquad \text{(B.26)}$$

Thus s has to be even if $g = g_1$, and odd if $g = g_2$; this reproduces precisely the above constraints. Taking everything together, the allowed boundary states are then given as in the main part of the paper.

C K-theory versus Cohomology

As was pointed out in Section 4, the lattice of D-brane charges that is described by K-theory is a sublattice of maximal rank of the (suitably normalised) cohomology lattice. This is something one may expect on general grounds: K-theory and cohomology are equivalent over the rational numbers (see for example [45]), and over the integers, they can therefore only differ by a finite torsion group. In this appendix we shall illustrate this difference between K-theory and cohomology by working out the simplest case, the $\mathcal{I}_2(-1)^{F_L}$ orbifold, in detail. We shall also give the corresponding results for $\mathcal{I}_2$ and for the two theories with $n = 4$.

For $n = 2$ mod 4, the orbifold in question is a $\mathbb{Z}_2$ orbifold of Type 0A/0B, and therefore the the D-brane spectrum (*i.e.* the K-theory group), as well as the R-R spectrum (*i.e.* the relevant cohomology group) is doubled; for simplicity we shall only consider one copy, *i.e.* we shall pretend that the theory is really a $\mathbb{Z}_2$ orbifold of Type IIA/IIB. The relevant K-theory group for the case of the $\mathcal{I}_2(-1)^{F_L}$ orbifold is then $\mathbb{Z}^{\oplus 6}$ if r is even (odd) in Type B (A), and it is trivial otherwise. The non-trivial elements arise from fractional branes that exist for $s = 1$, and truncated branes that exist for $s = 0$ and $s = 2$. For fixed r, the cohomology charges can be described by a six-component vector, whose first two entries correspond to the charge with respect to the untwisted

[4] This is well known in the case of the orbifold of Type IIB by $\mathcal{I}_4(-1)^{F_L}$ [9] where it follows from considerations of supersymmetry.

R−R form $C^{(r+2)}_{\mu_1\ldots\mu_{r+1}7}$ and $C^{(r+2)}_{\mu_1\ldots\mu_{r+1}8}$. The remaining four entries describe the charge with respect to the four twisted R-R forms. We choose the normalisation so that the minimal charges are all integer; then a $\hat{\mathrm{D}}(r,0)$-brane has charges

$$(0,0;\pm 2,0,0,0)\,, \tag{C.1}$$

and the fractional $\mathrm{D}(r,1)$-branes are described by

$$(\pm 1,0;\pm 1,\pm 1,0,0), \quad (\pm 1,0;0,0,\pm 1,\pm 1)\,, \quad (0,\pm 1;\pm 1,0,\pm 1,0), \quad (0,\pm 1;0,\pm 1,0,\pm 1)\,, \tag{C.2}$$

where all possible sign combinations are allowed. Finally the $\hat{\mathrm{D}}(r,2)$-branes have the charges

$$(0,0;\pm 1,\pm 1,\pm 1,\pm 1)\,, \tag{C.3}$$

where the number of $+$-signs is even. Let us denote by Λ_K the lattice that is generated by these D-branes, and by Λ_H the cohomology lattice, *i.e.* the lattice generated by the basis vectors

$$e_i = (0,\ldots,0,1,0,\ldots,0)\,, \qquad i=1,\ldots,6\,, \tag{C.4}$$

where the 1 is placed in the i-th position. Clearly Λ_K is a sublattice of Λ_H, and they differ by the finite (torsion) group Λ_H/Λ_K.

The group Λ_H/Λ_K is generated by the elements e_3, e_4 and e_5 since

$$\begin{aligned} e_1 &= (1,0;-1,-1,0,0)+e_3+e_4\,, && \text{(C.5)}\\ e_2 &= (0,1;-1,0,-1,0)+e_3+e_5\,, && \text{(C.6)}\\ e_6 &= (0,0;1,1,1,1)-e_3-e_4-e_5\,. && \text{(C.7)} \end{aligned}$$

Each of these three elements is of order two since $2e_i$ (for $i=3,4,5$) corresponds to a truncated $\hat{\mathrm{D}}$(r,0)-brane and is hence in Λ_K. Finally, none of the combinations

$$e_3+e_4, \quad e_3+e_5, \quad e_4+e_5, \quad e_3+e_4+e_5 \tag{C.8}$$

are elements of Λ_K, and thus

$$\Lambda_H/\Lambda_K = \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\,. \tag{C.9}$$

In the case of the $\mathcal{I}_2$ orbifold, the six relevant R−R charges are the untwisted R−R form $C^{(r+1)}_{\mu_1\ldots\mu_{r+1}}$ and $C^{(r+3)}_{\mu_1\ldots\mu_{r+1}78}$, as well as the four twisted R−R forms. Choosing again the normalisation of the charges so that every D-brane has integer components, the fractional $\mathrm{D}(r,0)$-brane is described by

$$(\pm 1,0;\pm 2,0,0,0)\,, \tag{C.10}$$

where all sign choices are allowed, and the 2 can be placed in any of the four last entries. The $\hat{\mathrm{D}}(r,1)$-branes have charges

$$(0,0;\pm 2,\pm 2,0,0), \quad (0,0;0,0,\pm 2,\pm 2), \quad (0,0;\pm 2,0,\pm 2,0), \quad (0,0;0,\pm 2,0,\pm 2)\,, \tag{C.11}$$

whilst the fractional D$(r,2)$-branes are described by

$$(0,\pm 1;\pm 1,\pm 1,\pm 1,\pm 1)\,. \tag{C.12}$$

Here any combination of signs with an even number of plus signs in the last four entries is allowed. In this case the group Λ_H/Λ_K is generated by $e_3, e_4-e_3, e_5-e_3, e_6-e_3$, where e_3 is of order four while each of the other three elements is of order two in Λ_H/Λ_K. There are no further relations, and the group is therefore

$$\Lambda_H/\Lambda_K = \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\,. \tag{C.13}$$

We have also determined the torsion groups for the two orbifolds with $n=4$[5]. For the $\mathcal{I}_4$ orbifold the torsion group is

$$\Lambda_H/\Lambda_K = \mathbb{Z}_8\times(\mathbb{Z}_4)^5\times(\mathbb{Z}_2)^{10}\,, \tag{C.14}$$

while in the case of $\mathcal{I}_4(-1)^{F_L}$ the answer is

$$\Lambda_H/\Lambda_K = (\mathbb{Z}_4)^5\times(\mathbb{Z}_2)^{10}\,. \tag{C.15}$$

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