Quantitative Finance, Vol. 00, No. 00, Month 200x, 1–28

Lookback Option Pricing Using the Fourier Transform B-spline Method

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(11 November 2013)

We derive a new, efficient closed-form formula approximating the price of discrete lookback options, whose underlying asset price is driven by an exponential semimartingale process including (jump) diffusions, Lévy models, affine processes and other models. The derivation of our pricing formula is based on inverting the Fourier transform using B-spline approximation theory. We give an error bound for our formula and establish its fast rate of convergence to the true price. Our method provides lookback option prices across the quantum of strike prices with greater efficiency than for a single strike price under existing methods.

We provide an alternative proof to the Spitzer formula for the characteristic function of the maximum of a discretely observed stochastic process, which yields a numerically efficient algorithm based on convolutions. This is an important result which could have a wide range of applications where the Spitzer formula is utilized.

We illustrate the numerical efficiency of our algorithm by applying it in pricing fixed and floating discrete lookback options under Brownian motion, jump diffusion models, and the variance gamma process.

Keywords: Lookback option pricing, Fourier transform, B-spline interpolation, Spitzer formula, jump diffusion, variance gamma

1. Introduction

Lookback options are among the most popular path dependent exotic options traded over-the-counter or embedded in structured notes or insurance contracts. The lookback option provides a payoff at expiry that depends on the continuously or discretely monitored extremum of the underlying asset over the lifetime of the contract. Discrete lookback options commonly arise in life insurance applications, since they feature in variable annuity and equity indexed annuity contracts, which are popular in North America. Lin and Tan (2003) note that these protection features appeal to investors of savings products, but also are a source of significant volatility for

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Quantitative Finance
ISSN 1469-7688 print/ISSN 1469-7696 online
http://www.tandf.co.uk/journals
DOI: 10.1080/1469768YYxxxxxxxx

insurance companies that underwrite the product. Wong and Lam (2009) consider the valuation of dynamic fund protection in the context of equity indexed annuities. In particular, the valuation of the embedded options have not always been carried out using market consistent principles.

The theory for pricing continuously monitored lookback options is well developed, and a variety of methods exist under popular continuous-time asset price models. For example, Goldman *et al.* (1979b), Goldman *et al.* (1979a), and Conze and Viswanathan (1991) obtain closed-form prices under the Black-Scholes model assumptions. More recently, Boyle and Tian (1999) and Linetsky (2004) consider pricing continuous lookback options under the Constant Elasticity of Variance model of Beckers (1980). Kou and Wang (2004) and Cai and Kou (2011) consider pricing options under the Laplace transform (LT), where asset prices follow the double exponential jump diffusion model of Kou and Wang (2004) and the mixed exponential jump diffusion model. The latter was first considered by Lipton (2002a), and more recently by Cai and Kou (2011).

In contrast to the continuous case, the development of the theory of pricing discrete lookback options has been slower and relies largely on numerical methods. Heynen and Kat (1995) demonstrate that approximating discrete lookback options in terms of their continuous counterparts can lead to significant pricing errors, and Broadie *et al.* (1999) provide a detailed comparison of pricing continuous and discrete lookback options. Pricing of discrete lookback options has developed in several stages. Initially, pricing methods were based around (i) binomial methods, see, for example, Babbs (1992) and Hull and White (1993), (ii) partial differential equations (PDE), as shown by Wilmott *et al.* (1993) and Forsyth *et al.* (1999), and (iii) numerical integration in AitSahlia and Lai (1997) and Tse *et al.* (2001). Further extensions to the lattice based methods are proposed by Broadie *et al.* (1999) and Foufas and Larson (2004) that suggest a finite element based PDE methodology. However, with the exception of Tse *et al.* (2001), these methods focus on the Brownian motion model and are not easily extended to other models of asset return.

The next major breakthrough in the pricing of discrete lookback options was made by Öhgren (2001), who used the Spitzer identity to derive a recurrence formula for the evaluation of the characteristic function of the discretely monitored maximum asset price. The latter recurrence was used by Öhgren (2001) to price discrete lookback options at inception or at a monitoring point. This is an important result that Petrella and Kou (2004) and Borovkov and Novikov (2002) extend to a range of stochastic models, including the Black and Scholes (1973) model, and jump diffusion models, such as Merton (1976) and Kou (2002). Furthermore, Petrella and Kou (2004) extend the pricing method to value discrete lookback options at any point in time, not just at inception or at a monitoring point. It should be noted that related fluctuation identities were considered by Lipton (2002a) in the context of pricing path-dependent options. More recently, Broadie and Yamamoto (2005) propose a new pricing framework that utilizes the Fast Gauss Transform and demonstrate its use for pricing under the Merton (1976) jump diffusion model. Yamamoto (2005) applies this framework to pricing under the double exponential jump diffusion model of Kou and Wang (2004). Finally, two new pricing frameworks have been developed more recently. First, Atkinson and Fusai (2007) develop a method based on solving a Wiener-Hopf integral equation and show how this is closely related to the Spitzer formula. They provide a semi-closed-form solution for the option price under the Black-Scholes setting, and Green *et al.* (2010) propose that this methodology can be applied to other Lévy models. Second, Feng and Linetsky (2009) propose an approach based on the Hilbert transform and demonstrate that it provides efficient and accurate prices for a range of Lévy models, including the double exponential jump diffusion process.

In this paper we propose a new Fourier transform B-spline (FTBS) based methodology for pricing discrete lookback options under the class of general exponential semimartingale processes, for which the log-return process exhibits independent increments. The FTBS method is based on the development of a closed-form approximation to the pricing integral suggested by Lewis (2001) (see also Lewis (2000)), who applies the inverse Fourier transform to provide an option valuation formula as a contour integral in the complex plane. It should be noted that a similar approach to Lewis (2001) is described by Lipton (2002b) in the context of FX option pricing (see also Lipton (2001)). The Lewis-Lipton approach generalizes the method of Carr and Madan (1999) since their dampening factor corresponds to the path of the contour integral in the Fourier transform. Generalizations of the Lewis-Lipton framework have recently been considered by Dufresne *et al.* (2009) and by Eberlein *et al.* (2010). The latter authors have shown that the Lewis-Lipton option pricing framework is valid for the more general class of exponential semimartingale models of the underlying asset price dynamics. This class is rich and encompasses the majority of models utilized in finance for pricing derivatives. Examples include (i) the general families of (jump) diffusion processes such as the Black and Scholes (1973) model, the Merton (1976) model and the mixed exponential model of Lipton (2002a), and (ii) pure jump Lévy processes such as the CGMY processes of Carr *et al.* (2002), the generalized hyperbolic models (see Eberlein (2001)), and the general class of linear combinations of gamma (LG) processes, recently introduced by Kaishev (2013), which includes as special cases, the variance gamma (VG) process introduced by Madan *et al.* (1998), and the bilateral gamma process considered by Küchler and Tappe (2008). For properties of some of the particular examples of semimartingale models see Eberlein *et al.* (2008).

The FTBS method uses B-spline approximation theory to provide an accurate closed-form approximation to a Fourier transform representation of the discrete lookback option price. In common with Petrella and Kou (2004) and Borovkov and Novikov (2002), our method is based on the idea of applying the Spitzer formula to obtain the characteristic function of the discretely observed maximum asset price. A key advantage of the FTBS method is that it enables the valuation of discrete lookback options across a large number of strike prices in a single procedure with greater efficiency than for a single strike price under the methods of either Petrella and Kou (2004) or Borovkov and Novikov (2002). The FTBS method for pricing lookback options is an extension of the framework described in Haslip and Kaishev (2013) for pricing European options.

Several key innovations are introduced in this paper. First, we apply the Fourier transform of Lewis (2001) and Lipton (2002b) as extended by Eberlein *et al.* (2010) to obtain a semi-closed-form pricing formula for discrete lookback options in terms of the characteristic function of the maximum of the underlying log-return process (see Proposition 2), Second, we propose a new convolution algorithm (see Algorithm 1) for computing the characteristic function of the maximum, which is based on the Spitzer-recurrence expansion formula (see Theorem 4.1). The latter first appears in Wendel (1958) (see equation (1a) therein), but has since then remained unknown in the quantitative finance literature. In Theorem 4.1, we provide a new, simple proof of this formula (see Appendix B.1), and show that it gives rise to a new efficient, convolution based algorithm (see Algorithm 1) for the computation of the characteristic function of the maximum. We then apply it to evaluate the characteristic function of the maximum of the underlying log-return process over a large number of monitoring points. This provides a significant efficiency improvement over the Spitzer recurrence formula of Öhgren (2001). Third, we use B-spline interpolation theory to develop a strike-separable pricing formula that enables valuation of lookback options across a large number of strike prices in a single procedure with greater efficiency than for a single strike price under the methods of either Petrella and Kou (2004) or Borovkov and Novikov (2002). Finally, using residue calculus, we apply the FTBS method for efficient pricing of discrete lookback options under the VG process.

It is worth noting that, in Haslip and Kaishev (2013), the FTBS method has been thoroughly compared, based on the example of pricing European options, to the major existing option pricing methods, among which, the fast Fourier transform of Carr and Madan (1999), the fractional fast Fourier transform method of Chourdakis (2005), the Integration-Along-Cut method of Levendorskiĭ and Xie (2012), the Cosine method of Fang and Oosterlee (2008), and the Convolution method of Lord *et al.* (2008).

The paper is structured as follows. In Section 2, we first introduce the discrete lookback option, and then the Lewis-Lipton Fourier transform framework as extended by Eberlein *et al.* (2010), and use it to derive a discrete lookback option pricing formula (see Proposition 2.3). In Section 3 we develop our FTBS method for pricing discrete lookback options. In Section 3.2, we introduce a B-spline interpolation method by means of which we obtain our main result, the closed-form, Fourier transform B-spline discrete lookback option pricing formula given in Theorem 3.3. The latter theorem involves the characteristic function of the discretely observed asset price maximum. In order to efficiently evaluate this characteristic function, in Theorem 4.1 of Section 4.1, we derive an explicit expansion representation for it, based on the Spitzer recurrence formula (23), given by Öhgren (2001). We use the result of Theorem 4.1 which first appears in Wendel (1958) to develop an efficient convolution based algorithm (see Algorithm 1) for the evaluation of the characteristic function of the underlying log-return process. In Section 4.2, we provide a Fourier transform representation of the coefficients in the Spitzer-recurrence expansion representation formula given by Theorem 4.1. In Section 5 we illustrate how the FTBS method is applied to several popular models for the underlying asset price movement. Section 6 summarizes the numerical results and Section 7 provides conclusions and discussion. Proofs and auxiliary materials are given in Appendices A-B.

2. Discrete Lookback Options

In this section, we describe our proposed framework for pricing discrete lookback options. The risk-neutral dynamics of the asset price are modeled using an exponential semimartingale process given by $S(t) = S(0)e^{L_t}$, where $S(t)$ is the asset price at time $t \geq 0$, and L_t is a semimartingale process describing the log-return under probability measure $\mathbf{Q}$. We refer the reader to, for example, Jacod and Shiryaev (2003), for a primer on semimartingale processes. For brevity, we refer to discretely monitored lookback options simply as lookback options throughout this paper.

2.1. Notation and Background

Let T denote the expiry time of the lookback option and $[T_0, T]$ be the lookback period over which the asset price is observed at $q+1$ evenly spaced monitoring times, $t_j = T_0 + \frac{T-T_0}{q} j$, for $j = 0, \ldots, q$. The asset price at time t_j is denoted $S_j = S(t_j) = S(0)e^{L_{t_j}}$. Let j^* be such that $t_{j^*} \leq t < t_{j^*+1}$, where t is the current time ($T_0 \leq t \leq T$). We denote the maximum value of the asset price realized from T_0 to t over the discrete monitoring times, $t_j, j = 0, \ldots, j^*$, by

$$M = \max_{j=0,\ldots,j^*} S(t_j). \tag{1}$$

We denote by $X_k = \max_{j=1,\ldots,k} L_{j^*+j}$ the observed future maximum log-return process over the monitoring times $t_{j^*+1}, \ldots, t_{j^*+k}$ for $k = 1, \ldots, q - j^*$. To simplify the notation, denote by m the number of future monitoring time points, i.e. $m = q - j^*$. Several different types of lookback options exist, and without loss of generality in this paper we consider (i) the fixed-strike lookback call option, and (ii) the floating lookback put option. The pricing theory for the fixed-strike lookback put option and the floating lookback call option are similar and require only minor modification of the results presented in this paper.

For a contract at inception time T_0 that expires at future time T, the payoffs of the fixed call with strike price K and floating put lookback options are (i) $(\max_{i=0,\dots,q} S_i - K)^+$, and (ii) $\max_{i=0,\dots,q} S_i - S_T$, respectively. However, as highlighted by Petrella and Kou (2004), it is also important to be able to price a lookback contract at some current time t, after the time of inception, T_0, i.e., when $T_0 < t < T$. In this case, the payoff will be dependent on the observed maximum, M since inception. The following proposition explains how to price lookback options at any point in time by conditioning on the observed discrete extremum until the last monitoring point.

Proposition 2.1 Valuation of Lookback Options**:** *The price at the current time t ($T_0 \le t \le T$) of a fixed lookback call option, $FC(t,T)$, and floating lookback put option, $LP(t,T)$, that expires at future time $T > 0$ and were incepted at time T_0 in the past are given by*

$$FC(t,T) = \begin{cases} S_0 e^{-r(T-t)} E_{\mathbf{Q}}\left[\left(e^{X_m} - \frac{K}{S_0}\right)^+\right], & \text{if } M < K \\ S_0 e^{-r(T-t)} E_{\mathbf{Q}}\left[\left(e^{X_m} - \frac{M}{S_0}\right)^+\right] + (M-K)e^{-r(T-t)}, & \text{if } M \ge K \end{cases} \tag{2}$$

$$LP(t,T) = \begin{cases} S_0 e^{-r(T-t))} E_{\mathbf{Q}}\left[e^{X_m}\right] - S_0, & \text{if } M \le S_0 \\ S_0 e^{-r(T-t)} \left\{E_{\mathbf{Q}}\left[e^{X_m}\right] + E_{\mathbf{Q}}\left[\left(\frac{M}{S_0} - e^{X_m}\right)^+\right]\right\} - S_0, & \text{if } M > S_0, \end{cases} \tag{3}$$

respectively, where $M = \max_{j=0,\dots,j^*} S(t_j)$ and r is the risk-free rate of interest.

Proof: The valuation formula for $FC(t,T)$ and $LP(t,T)$ are derived in two stages: (i) identification of the payoff function by conditioning on the value of M relative to K and S_0, respectively, and (ii) application of the Fundamental Theorem of Asset Pricing in stage (i) to the payoff functions, see, for example, Delbaen and Schachermayer (1994). Thus, details are omitted. $\square$

Two useful and interesting simplifications to (2) and (3) occur in the case $M \ge K$, which we describe in the following lemma. Part (i) is provided in Petrella and Kou (2004), and part (ii) was first obtained by Öhgren (2001).

Lemma 2.2 Put-call Parity and Closed-form Price for Fixed Lookback Call Option**:**
(i) If $M \ge K$ then $FC(t,T) = LP(t,T) + S_0 - Ke^{-r(T-t)}$;
(ii) If $M \ge K$ and $M \le S_0$ then $FC(t,T) = S_0 e^{-r(T-t)}\left[\phi_{X_m}(-i) - \frac{K}{S_0}\right]$, where $\phi_{X_m}(z) = E_{\mathbf{Q}}(e^{izX_m})$ is the characteristic function of X_m.

Proof: (i) It is easy to see that if $M \ge K$ then $\max_{i=0,\dots,q} S_i \ge K$.
Therefore, $FC(t,T) = e^{-r(T-t)} E_{\mathbf{Q}}\left(\max_{i=0,\dots,q} S_i\right) - Ke^{-r(T-t)} = LP(t,T) + S_0 - Ke^{-r(T-t)}$.
(ii) This follows from (3) using (i) above and the condition $M \le S_0$. $\square$

2.2. The Fourier Transform

The FTBS method is built on the foundations of the Fourier transform method of option pricing. In this section, we introduce the Fourier transform pricing method and apply it to calculate the expectations in (2) and (3) of Proposition 2.1, and hence price lookback options. The Fourier transform was introduced to financial mathematics by Lewis (2001) who demonstrated its use for option pricing under a general exponential Lévy process. It should be noted that a similar approach is described by Lipton (2002b) in the context of FX option pricing (see also Lipton (2001)). This was later extended by Eberlein *et al.* (2010), to the more general setting of exponential semimartingale models and discontinuous payoff functions, see also Dufresne *et al.* (2009).

In order to evaluate the pricing formulae (2) and (3), we denote $C(T,R) = E_{\mathbf{Q}}\left[\left(e^{X_m} - R\right)^+\right]$, where $R > 0$ is the ratio of strike price or previous asset price maximum to the current asset price, that is, $R = \frac{K}{S_0}$ or $R = \frac{M}{S_0}$. The following proposition shows how to evaluate $C(T,R)$ using the Fourier transform.

Proposition 2.3 Fourier Pricing Formula**:** *If $M_{X_m}(v) = E_{\mathbf{Q}}\left(e^{vX_m}\right)$ exists for all $v \in (a,b)$, with $a < \frac{1}{2}$ and $b > 1$ then, using the Fourier transform, $C(T,R)$ is given by*

$$C(T,R) = \phi_{X_m}(-i) - \frac{\sqrt{R}}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,\left[\phi_{X_m}(-u - \tfrac{i}{2})R^{iu}\right]\frac{du}{u^2 + \frac{1}{4}}, \tag{4}$$

where $R > 0$, and ϕ_{X_m} is the characteristic function of the maximum log-return, $X_m = \max_{j=1,\ldots,m} L_{j^*+j}$, over all monitoring points from current time t ($T_0 \leq t \leq T$) to time T.

Proof: The proof closely follows e.g. Example 5.1 of Eberlein *et al.* (2010) and therefore, is omitted. □

Proposition 2.3 provides a semi-closed-form pricing formula that can be implemented independently of the choice of asset model and forms the basis of our pricing framework.

3. The Fourier Transform B-spline Pricing Framework

The standard approach in option pricing for the evaluation of (4) is to apply efficient numerical integration techniques, such as the Gauss-Kronrod quadrature method, to calculate the integral to the desired level of accuracy. For the type of characteristic functions encountered in, for example, pricing European options, this is a quick and efficient procedure. However, in the case of lookback options, the characteristic function ϕ_{X_m} of the asset log-return extremum is expensive to evaluate when the number of monitoring points m is large, as we describe in Section 4. Therefore, the standard application of quadrature methods results in slow calculation times since they require a large number of evaluations of the integrand to achieve good accuracy. In this section, we provide a full derivation of a general approach for evaluating the Fourier pricing formula (4) utilizing spline approximation theory that we refer to as the FTBS method.

3.1. Simplifying the Pricing Formula

In this section, we present an alternative representation of the pricing formula in (4) that decomposes the integrand into the product of (i) a trigonometric function dependent on R, the ratio of strike price or previous asset price maximum to the current asset price, and (ii) the real and imaginary parts of the semimartingale process characteristic function multiplied by the Fourier transform of the option payoff with strike 1, which is independent of the actual strike price.

Proposition 3.1 Strike-separable Pricing Formula**:** *An alternative form of* (4) *for evaluating $C(T,R)$, for $R > 0$, is given by*

$$C(T,R) = \phi_{X_m}(-i) - \frac{\sqrt{R}}{\pi}I(R) \tag{5}$$

where

$$I(R) = \int_0^1 \cos\left(\tfrac{1-t}{t}\log R\right) s_1(t)dt + \int_0^1 \sin\left(\tfrac{1-t}{t}\log R\right) s_2(t)dt \tag{6}$$

$$s_1(t) = \frac{\mathrm{Re}\left[\phi_{X_m}(-\frac{1-t}{t} - \frac{i}{2})\right]}{1 - 2t + \frac{5}{4}t^2} \quad \text{and} \quad s_2(t) = -\frac{\mathrm{Im}\left[\phi_{X_m}(-\frac{1-t}{t} - \frac{i}{2})\right]}{1 - 2t + \frac{5}{4}t^2}. \tag{7}$$

Proof: The proof follows by (i) decomposing the integrand in (4) into its real and imaginary parts, and (ii) changing variables $u = \frac{1-t}{t}$ to transform the limits of integration to the unit interval. □

Remark 1 : We make two important observations about strike-separable pricing formula in (5). First, integration is now performed over the unit interval. Specifically, we have applied the change of variables $u = \frac{1-t}{t}$ which transforms the upper limit of integration from infinity to zero, and the lower limits of integration from zero to one. Thus, integration is performed over the unit interval $[0, 1]$, which avoids making a truncation error at infinity. It should be noted that no truncation occurs at zero in the transformed integral, since as we will see in Section 3.3, we evaluate exactly the integrals in (6), by replacing functions $s_1(t)$ and $s_2(t)$ with their B-spline interpolants. This means that it is no longer necessary to carefully identify a truncation point for each asset price model by considering how quickly the characteristic function decays to zero.

Second, we have separated the integrand into the product of $\cos\left(\frac{1-t}{t}\log R\right)$ or $\sin\left(\frac{1-t}{t}\log R\right)$, which are dependent on R, and functions $s_1(t)$ and $s_2(t)$, which are independent of R. This is an important feature of the FTBS method since it allows us to price lookback options at a range of different values of R, and hence strike prices, in an extremely efficient manner.

Pricing is one key aspect of the field of derivatives, another equally important aspect is hedging, which is crucial for effective risk management. We now consider the problem of computing the Greeks and demonstrate how (5) is easily extended to calculate the option price sensitivities. The following corollary provides option price sensitivities for the fixed lookback call option, and we note that this is easily extended to the floating lookback put option via the put-call parity relationship provided in Lemma 2.2.

Corollary 3.2 The Greeks**:** *The sensitivities of the fixed lookback call option price, $FC(T, K)$, to movements in the asset price are provided by $\Delta FC(T, K) = \frac{\partial FC(T,K)}{\partial S_0}$ and $\Gamma FC(T, K) = \frac{\partial^2 FC(T,K)}{\partial S_0^2}$. They are computed as*

$$\Delta FC(t, T) = e^{-r(T-t)}\left[\phi_{X_m}(-i) + \tfrac{\sqrt{R}}{\pi}\left(-\tfrac{1}{2}I(R) + \Delta I(R)\right)\right] \tag{8}$$

$$\Gamma FC(t, T) = \tfrac{e^{-r(T-t)}\sqrt{R}}{S_0\pi}\left[\tfrac{1}{4}I(R) - \Gamma I(R)\right], \tag{9}$$

where $R = \frac{K}{S_0}$ if $M < K$, and $R = \frac{M}{S_0}$ otherwise. Functions $\Delta I(R)$ and $\Gamma I(R)$ are related to the *derivatives of $I(R)$, defined in* (6)*. These are computed as*

$$\Delta I(R) = RI'(R) = \int_0^1 \cos\left(\tfrac{1-t}{t}\log R\right)\Delta s_1(t)dt + \int_0^1 \sin\left(\tfrac{1-t}{t}\log R\right)\Delta s_2(t)dt \tag{10}$$

$$\Gamma I(R) = R^2 I''(R) + \Delta I(R) = \int_0^1 \cos\left(\tfrac{1-t}{t}\log R\right)\Gamma s_1(t)dt + \int_0^1 \sin\left(\tfrac{1-t}{t}\log R\right)\Gamma s_2(t)dt, \tag{11}$$

where

$$\Delta s_1(t) = \frac{1-t}{t}s_2(t), \quad \text{and} \quad \Delta s_2(t) = -\frac{1-t}{t}s_1(t), \tag{12}$$

and

$$\Gamma s_1(t) = \frac{1-t}{t}\Delta s_2(t) = -\frac{(1-t)^2}{t^2}s_1(t),$$
$$\Gamma s_2(t) = -\frac{1-t}{t}\Delta s_1(t) = -\frac{(1-t)^2}{t^2}s_2(t).$$

Proof: See Haslip and Kaishev (2013) for a very similar proof in the case of European options. □

In the next section, we apply spline approximation theory to interpolate functions $s_i(t)$, $\Delta s_i(t)$, and $\Gamma s_i(t)$, for $i = 1, 2$.

3.2. Optimal Spline Interpolation

The standard approach in option pricing for the evaluation of (5) is to apply efficient numerical integration techniques, such as the Gauss-Kronrod quadrature method, to calculate the integral $I(R)$ to the desired level of accuracy. In this section, we introduce a general approach for evaluating the strike-separable pricing formula that utilizes spline approximation theory.

A spline function, $s(t)$ on $[a, b] \in \mathbf{R}$, of order n, degree $n-1$, is defined on the set of points called knots, $\{\eta_i\}_{i=1}^{2n+l}$ where $\eta_1 = \cdots = \eta_n = a$, $\eta_i < \eta_{i+1}$ for $i = n+1, \ldots, n+l$ and $\eta_{n+l+1} = \cdots = \eta_{2n+l} = b$, and l is some positive integer, as

$$s(t) = \sum_{i=1}^{p} c_i M_{i,n}(t), \tag{13}$$

where $p = l + n$, c_i are certain real coefficients and $M_{i,n}(t)$ are the B-splines, defined as the n-th order divided differences of the function $f(y) = n\left(\max\{(y-t), 0\}\right)^{n-1} = n(y-t)_+^{n-1}$, that is, $M_{i,n}(t) = M_{i,n}(t; \eta_i, \ldots, \eta_{i+n}) = [\eta_i, \ldots, \eta_{i+n}]f(y)$.

We recall that the n-th order ($n \geq 0$) divided difference of a function $f(t)$ is defined recurrently as

$$[\eta_i, \ldots, \eta_{i+n}]\, f(t) = \frac{[\eta_{i+1}, \ldots, \eta_{i+n}]\, f(t) - [\eta_i, \ldots, \eta_{i+n-1}]\, f(t)}{\eta_{i+n} - \eta_i}, \tag{14}$$

where $[\eta_i] f(t) = f(\eta_i)$ and the points $\{\eta_j\}_{j=i}^{i+n}$ are pairwise distinct. In the case when one or more points are repeated, as at the left and right end of the interval $[a, b]$, a derivative based formula provided in $(A1)$ of Appendix $A.1$, which appears in Ignatov and Kaishev (1989), can be used to calculate the divided difference.

For properties of splines and B-splines we refer to De Boor (2001). In order to prove our main result given by Theorem 3.3, we will need the following expression of a divided difference of a function $f(t)$, in terms of a B-spline, known as the Peano formula

$$[\eta_i, \ldots, \eta_{i+n}] f(x) = \int_{\mathbf{R}} M_{i,n}(t; \eta_i, \ldots, \eta_{i+n}) \frac{f^{(n)}(t)}{n!} dt, \tag{15}$$

where $f^{(n)}$ is the n-th derivative of f.

As a core step in developing the FTBS method we interpolate the functions $s_1(t)$ and $s_2(t)$ from the pricing formula (5), by appropriate quadratic ($n = 3$) splines, $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$, at some a priori selected interpolation sites, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_p)$, $\boldsymbol{\tau} \in [a, b] \equiv [0, 1]$ with $\tau_1 = 0$ and $\tau_p = 1$. We note that based on our numerical experience, quadratic splines provided very good results, their good approximation properties are also confirmed in the spline approximation literature (see e.g. Marsden (1974)). We briefly introduce this step on the example of $\tilde{s}_1(t)$, baring in mind that similar notation and reasoning applies for $\tilde{s}_2(t)$. We apply the optimal interpolation scheme of Gaffney and Powell (1976) independently obtained also by Micchelli *et al.* (1976) in order to determine the optimally located knots, $\{\tilde{\eta}_{1,i}\}_{i=n+1}^{n+l}$ of the spline interpolant $\tilde{s}_1(t)$. As noted by De Boor (2001) the optimal knot set $\{\tilde{\eta}_{1,i}\}_{i=n+1}^{n+l}$ is well approximated by

$$\eta_{1,n+i} = \frac{(\tau_{i+1} + \ldots + \tau_{i+n-1})}{n-1}, i = 1, \ldots, l, \tag{16}$$

We then substitute in the pricing formula (5), the optimal spline interpolants $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$ for $s_1(t)$ and $s_2(t)$ respectively, and apply the Peano representation (15) in order to express the integrals of B-splines as divided differences of appropriate functions. Details of this step are easily followed in the proof of Theorem 3.3. For further details of how this step is implemented, in particular, the selection of interpolation sites we refer to Haslip and Kaishev (2013), where the FTBS framework is successfully applied for pricing European options. We only note here that selection of $\boldsymbol{\tau}$ is robust with respect to the choice of the underlying semimartingale model. It is essential to point out that the FTBS approach leads to an extremely efficient lookback option pricing formula, given by Theorem 3.3. As shown in Proposition 3.4, the FTBS lookback option price quickly convergence to the true price. It is also important to note that the FTBS method allows the simultaneous evaluation of lookback options within a range of strike prices, not just for one strike. See Section 6 for details of the implementation. In the sequel we set $n = 3$ and utilize quadratic spline interpolation.

3.3. The Fourier Transform B-spline Pricing Formula

In this section, we utilize the optimal B-spline interpolation scheme presented above to develop the main result of this paper, the FTBS pricing formula. We first return to the strike-separable pricing formula (5) and define $\tilde{C}(T, R)$ to be the approximation to (5) provided by replacing $s_1(t)$ and $s_2(t)$ with their approximants $\tilde{s_1}(t)$ and $\tilde{s_2}(t)$ respectively. That is,

$$C(T, R) \approx \tilde{C}(T, R) = \phi_{X_m(-i)} - \frac{\sqrt{R}}{\pi} \tilde{I}(R), \tag{17}$$

where

$$\tilde{I}(R) = \int_0^1 \cos\left(\tfrac{1-t}{t}\log R\right)\tilde{s}_1(t)dt + \int_0^1 \sin\left(\tfrac{1-t}{t}\log R\right)\tilde{s}_2(t)dt.$$

To express (17) in a closed-form and eliminate integration, it suffices to (i) substitute $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$, which are splines of the form given by (13), (ii) take summation in front of integration, and (iii) apply the Peano formula in (15) to each integral in the linear combination. The latter integrals are of the form $\int_{\mathbf{R}} M_{i,n}(t;\eta_i,\ldots,\eta_{i+n})\psi(t)dt$, with $\psi(t) = \cos\left(\frac{1-t}{t}\log R\right)$ or $\psi(t) = \sin\left(\frac{1-t}{t}\log R\right)$, which are in the required form to apply the Peano representation, as described in Section 3.2. Therefore, one needs to find the function $\tilde{\psi}(t)$ whose n-th derivative coincides with $\psi(t)$. Following this approach, we give our main result stated by the following theorem.

Theorem 3.3 Fourier Transform B-spline Pricing Formula **:** *Let* $\{\eta_{1,i}\}_{i=1}^{6+l_1}$, $\{\eta_{2,j}\}_{j=1}^{6+l_2}$ *and* $\{c_{1,i}\}_{i=1}^{p_1}$, $\{c_{2,j}\}_{j=1}^{p_2}$ *be the sets of knots and linear coefficients of the quadratic spline interpolants,* $\tilde{s}_1$ *and* $\tilde{s}_2$*, respectively, where* $p_1 = l_1 + 3$ *and* $p_2 = l_2 + 3$*. Additionally, let* $R > 0$*. The pricing formula* $\tilde{C}(T,R)$ *is provided in closed-form as*

$$\tilde{C}(T,R) = \phi_{X_m}(-i) - \frac{\sqrt{R}}{\pi}\tilde{I}(R), \tag{18}$$

where for $R \neq 1$,

$$\begin{aligned}\tilde{I}(R) = {} & 6\sum_{i=1}^{p_1} c_{1,i}[\eta_{1,i},\eta_{1,i+1},\eta_{1,i+2},\eta_{1,i+3}]f_1(t,\log R)\\ & + 6\sum_{i=1}^{p_2} c_{2,i}[\eta_{2,i},\eta_{2,i+1},\eta_{2,i+2},\eta_{2,i+3}]f_2(t,\log R),\end{aligned} \tag{19}$$

and f_1 *and* f_2 *are calculated as*

$$\begin{aligned}f_1(t,x) = \frac{1}{12}\Bigg[& t\left[-(x^2-2t^2)\cos\frac{x(t-1)}{t} - 5xt\sin\frac{x(t-1)}{t}\right]\\ & + x\,\mathrm{Ci}\left(\frac{x}{t}\right)\left[-6xt\cos x + (x^2-6t^2)\sin x\right]\\ & - x\,\mathrm{Si}\left(\frac{x}{t}\right)\left[6xt\sin x + (x^2-6t^2)\cos x\right]\Bigg]\end{aligned}$$

$$\begin{aligned}f_2(t,x) = \frac{1}{12}\Bigg[& t\left[-(x^2-2t^2)\cos\frac{x(t-1)}{t} - 5xt\sin\frac{x(t-1)}{t}\right]\\ & + x\,\mathrm{Ci}\left(\frac{x}{t}\right)\left[6xt\sin x + (x^2-6t^2)\cos x\right]\\ & + x\,\mathrm{Si}\left(\frac{x}{t}\right)\left[-6xt\cos x + (x^2-6t^2)\sin x\right]\Bigg].\end{aligned}$$

For $R = 1$, $\tilde{I}(R)$ simplifies to

$$\tilde{I}(R) = \sum_{i=1}^{p_1} c_{1,i} + \sum_{i=1}^{p_2} c_{2,i}. \tag{20}$$

Note that $\mathrm{Ci}(x)$ *and* $\mathrm{Si}(x)$ *are the trigonometric special functions defined as* $\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos y}{y} dy$ *and* $\mathrm{Si}(x) = \int_0^x \frac{\sin y}{y} dy$*, respectively.*

Proof: We begin with the first integral in (17).

$$\begin{aligned}
\int_0^1 \cos\left(\frac{1-t}{t}\log R\right) \tilde{s}_1(t)dt &= \int_0^1 \cos\left(\frac{1-t}{t}\log R\right) \sum_{i=1}^{p_1} c_{1,i} M_{i,3}(t)dt \\
&= \sum_{i=1}^{p_1} c_{1,i} \int_0^1 \cos\left(\frac{1-t}{t}\log R\right) M_{i,3}(t)dt \\
&= \sum_{i=1}^{p_1} c_{1,i} \left(3! \int_0^1 \frac{f_1^{(3)}(\frac{1-t}{t})}{3!} M_{i,3}(t)dt \right) = 6 \sum_{i=1}^{p_1} c_{1,i} [\eta_{1,i}, \cdots, \eta_{1,i+n}] f_1,
\end{aligned}$$

where f_1 is chosen to satisfy $f_1^{(3)}(\frac{1-t}{t}) = \cos(\frac{1-t}{t}\log R)$, and we have applied the Peano representation (15). We identify f_1 by repeated integration of $\cos(\frac{1-t}{t}\log R)$ with respect to t, e.g. using symbolic integration in tools such as Mathematica. The proof for the second integral is similar and is therefore omitted. □

The following proposition gives a bound for the absolute error of the FTBS lookback option price approximation. It shows that the FTBS method provides a fast convergence to the true price as the numbers, ν and l, of data sites, $\boldsymbol{\tau}$ and knots, $\{\tilde{\eta}_{j,i}\}_{i=3+1}^{3+l}$ increase, while the mesh sizes $|\tilde{\eta}_j| := \max_i (\tilde{\eta}_{j,i} - \tilde{\eta}_{j,i-1})$ go to zero.

Proposition 3.4 FTBS Lookback Option Price Error Bound**:** *The absolute error of the FTBS lookback option price, $\tilde{C}(T, R)$ is bounded by*

$$\left| C(T,R) - \tilde{C}(T,R) \right| \leq \frac{\sqrt{R}}{\pi} \left(C_1 \left\| s_1^{(3)}(t) \right\| + C_2 \left\| s_2^{(3)}(t) \right\| \right), \tag{21}$$

where C_j, $j = 1, 2$ are the constants obtained from the optimal spline interpolation, $\tilde{s}_j(t)$ of $s_j(t)$, $j = 1, 2$, following Gaffney and Powell (1976), and where $\|f\| = \max\{\,|f(t)|: 0 \leq t \leq 1\}$. The bound in (21) *converges to zero as the mesh sizes $|\tilde{\eta}_j| := \max_i (\tilde{\eta}_{j,i} - \tilde{\eta}_{j,i-1})$ go to zero, at a rate $O(|\tilde{\eta}|^3)$, where $\tilde{\eta} = \max(|\tilde{\eta}_1|, |\tilde{\eta}_2|)$, i.e.,*

$$\left| C(T,R) - \tilde{C}(T,R) \right| = O\left(|\tilde{\eta}|^3 \right) \tag{22}$$

Proof: Provided in Appendix *A*.2. □

Corollary 3.5 FTBS formulae for the Greeks**:**
Let $\{\eta_{1,i}\}_{i=1}^{6+l_1}$, $\{\eta_{2,j}\}_{j=1}^{6+l_2}$ *and* $\{c_{1,i}\}_{i=1}^{p_1}$, $\{c_{2,j}\}_{j=1}^{p_2}$ *be the sets of knots and linear coefficients of the quadratic spline interpolants,* $\tilde{s}_1(t)$ *and* $\tilde{s}_2(t)$*, respectively, where* $p_1 = l_1 + 3$ *and* $p_2 = l_2 + 3$.

Similarly, let the knots and linear coefficients of the quadratic spline interpolants $\widetilde{\Delta s}_1(t)$, $\widetilde{\Delta s}_2(t)$, *and* $\widetilde{\Gamma s}_1(t)$, $\widetilde{\Gamma s}_2(t)$ *be denoted by* $\{\eta^*_{1,i}\}_{i=1}^{6+l_1^*}$, $\{\eta^*_{2,j}\}_{j=1}^{6+l_2^*}$ *and* $\{c^*_{1,i}\}_{i=1}^{p_1^*}$, $\{c^*_{2,j}\}_{j=1}^{p_2^*}$*, where* $p_1^* = l_1^* + 3$ *and* $p_2^* = l_2^* + 3$ *for* $* \in \{\Delta, \Gamma\}$.

The closed-form approximations for the Greeks are obtained by replacing functions $I(R)$, $\Delta I(R)$, *and* $\Gamma I(R)$ *in Corollary* 3.2 *by their approximants* $\tilde{I}(R)$, $\widetilde{\Delta I}(R)$, *and* $\widetilde{\Gamma I}(R)$. $\tilde{I}(R)$ *is defined in Theorem* 3.3*, and* $\widetilde{\Delta I}(R)$ *and* $\widetilde{\Gamma I}(R)$ *are defined as*

$$\widetilde{\Delta I}(R) = 6\sum_{i=1}^{p_1^{\Delta}} c^{\Delta}_{1,i}[\eta^{\Delta}_{1,i}, \eta^{\Delta}_{1,i+1}, \eta^{\Delta}_{1,i+2}, \eta^{\Delta}_{1,i+3}] f_1(t, \log R)$$
$$+ 6\sum_{i=1}^{p_2^{\Delta}} c^{\Delta}_{2,i}[\eta^{\Delta}_{2,i}, \eta^{\Delta}_{2,i+1}, \eta^{\Delta}_{2,i+2}, \eta^{\Delta}_{2,i+3}] f_2(t, \log R)$$

$$\widetilde{\Gamma I}(R) = 6\sum_{i=1}^{p_1^{\Gamma}} c^{\Gamma}_{1,i}[\eta^{\Gamma}_{1,i}, \eta^{\Gamma}_{1,i+1}, t^{\Gamma}_{1,i+2}, t^{\Gamma}_{1,i+3}] f_1(t, \log R)$$
$$+ 6\sum_{i=1}^{p_2^{\Gamma}} c^{\Gamma}_{2,i}[\eta^{\Gamma}_{2,i}, \eta^{\Gamma}_{2,i+1}, \eta^{\Gamma}_{2,i+2}, \eta^{\Gamma}_{2,i+3}] f_2(t, \log R),$$

for $k \neq 0$*. If* $k = 0$*, this simplifies to* (20) *in Theorem* 3.3.

Proof: The proof follows the same logic as Theorem 3.3 and is therefore omitted. □

While the functions f_1 and f_2 in Theorem 3.3 may appear complicated, they can be evaluated efficiently. Their trigonometric integrals can be calculated using the Fortran library of Vandevender and Haskell (1982), which provides 17 significant figures of accuracy using series expansions with pre-computed coefficients. Calculation of the divided difference is simple using the standard recursive definition in (14) when the knots $[\eta_i, \eta_{i+1}, \eta_{i+2}, \eta_{i+3}]$ are distinct. In the case of repeated knots, as at the three leftmost and rightmost knots in the sets $\{\eta_{1,i}\}_{i=1}^{6+l_1}$ and $\{\eta_{2,j}\}_{j=1}^{6+l_2}$, the divided differences must be calculated using the derivative based definition in ($A1$) of Appendix A.1 (see also Haslip and Kaishev (2013).

4. The Characteristic Function of the Asset Price Maximum

In order to evaluate the lookback option price using the FTBS pricing formula in (18), we must first calculate the characteristic function of the asset price maximum. This can be achieved using the celebrated Spitzer formula, as adapted by Öhgren (2001). The Spitzer formula provides a recursive relationship between the characteristic function of the maximum of a process over m monitoring points and the characteristic function(s) over the previous $m-1$ monitoring points as

$$\phi_{X_0}(z) = 1, \quad \phi_{X_m}(z) = \frac{1}{m}\sum_{k=0}^{m-1} \phi_{X_k}(z) a_{m-k}(z), \quad \text{for } m = 1, 2, ... \tag{23}$$

where $a_{m-k}(z) = E_{\mathbf{Q}}\left(e^{izL^+_{j^*+m-k}}\right)$, and j^* is defined in (1). This recursive relationship allows us to evaluate the characteristic function of the maximum X_m in terms of the log-return process L_{j^*+j} at each observation point t_{j^*+j}, $j = 1, \ldots, m$. The calculation of $a_{m-k}(z)$, the characteristic function of the greater of the log-return process at time t_{j^*+m-k} and zero, is specific to the selected asset model. For asset models in which European option prices are available in closed-form, $a_{m-k}(z)$ may be evaluated directly, as shown by Petrella and Kou (2004) and Borovkov and Novikov (2002), and for others more advanced techniques may be required. We return to the problem of how to calculate $a_{m-k}(z)$, $k = 0, \ldots, m-1$ in Section 4.2.

4.1. Spitzer-recurrence Expansion Formula

A key difficulty with the Spitzer-recurrence formula in (23) is that it is not computationally efficient, since, as shown in the following theorem, it exhibits sub-exponential complexity. To address this, based on the Spitzer recurrence formula (23), we derive an expansion formula for ϕ_{X_m} that is significantly more effective for large m and provides reduced calculation times.

Theorem 4.1 Spitzer-recurrence Expansion Formula **:** *The characteristic function of the log-return of the asset price maximum X_m is given by*

$$\phi_{X_m}(z) = \mathrm{Coeff}_{t^m}\left(\sum_{j=0}^{m} \frac{1}{j!}\left(\sum_{k=1}^{m} \frac{a_k(z)}{k} t^k\right)^j\right), \tag{24}$$

where $\mathrm{Coeff}_{t^m}(.)$ *is the coefficient in front of t^m in (.), for $m = 1, 2, \ldots$.*

Proof: Provided in Appendix B.1. □

Remark 2 : Let us note that formula (24) can be obtained as a special case of the original Spitzer formula (see Spitzer 1956). This has also been noted by Wendel (1958), see formula (1a) therein, who has given an alternative proof. Here we give a significantly simpler proof of formula (24) (see Appendix B.1), based on the Spitzer-recurrence formula (23), due to Öhgren (2001). Formula (24) is much better suited to numerical implementation by following Algorithm 1 below.

Theorem 4.1 is an important result that enables the evaluation of $\phi_{X_m}(z)$ with much greater efficiency than is possible using the Spitzer recurrence formula (23). It is not difficult to see that $\mathrm{Coeff}_{t^m}\left(\sum_{j=0}^{m} \frac{1}{j!}\left(\sum_{k=1}^{m} \frac{a_k}{k} t^k\right)^j\right)$ can be obtained by convoluting the vector $(\frac{a_1}{1}, \ldots, \frac{a_m}{m})$ with itself j times for $j = 0, 1, \ldots, m$ and summing across the anti-diagonal. In Algorithm 1, we describe how to calculate ϕ_{X_m} using this result. Table 1 provides a comparison of the numerical complexity and shows that our method dominates the Spitzer formula for $m \geq 30$ and therefore provides greater efficiency in pricing discrete lookback options with a high number of monitoring points. For example, at $m = 200$, the Spitzer formula (23) requires approximately 9.25×10^{15} terms to calculate ϕ_{X_m} compared to 1.33×10^6 using formula (24).

Proposition 4.2 Complexity of Formulae (24) and (23)**:** *The following statements hold true: a) the Spitzer-recurrence expansion formula* (24) *is of complexity $O(m^3)$, and b) the Spitzer formula* (23) *is sub-exponential and requires computation of $p(m)$ individual terms involving a_k, where $p(m)$ is the partition function defined as the number of integer partitions of m.*

Proof: Provided in Appendix B.2 and B.3. □

Remark 3 : Note that the overall computational efficiency of FTBS depends not only on the asymptotic arithmetic complexity of Algorithm 1, but also on other characteristics of the method and the computational process implementing it. A key advantage of our approach, is that due to

Table 1. Number of Terms in Evaluating Spitzer Formula (24) and (23)

m	Algorithm **1** Implementing (24)	Spitzer Formula (23)	Ratio
30	4525	5604	8.1E-01
50	20875	204226	1.0E-01
100	166750	190569292	8.8E-04
150	562625	40853235313	1.4E-05
200	1333500	3.973E+12	3.4E-07
300	4500250	9.25308E+15	4.9E-10
500	20833750	2.30017E+21	9.1E-15

the strike-separable feature of the pricing formula (17), we are able to price options across multiple strikes in an extremely efficient manner. This is because the spline approximants, $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$, are independent of R and are computed only once with the first strike, and the divided differences in (19) can be pre-computed (see Haslip and Kaishev (2013)). Hence the efficiency of the FTBS method for pricing many options simultaneously across a quantum of strike prices is similar to that of pricing a single option.

Algorithm 1 Computation of $\phi_{X_m}(z)$ using the Spitzer-recurrence expansion formula (24)

```
Input: m ∈ N, z ∈ C
Output: φ_{X_m}(z)
  for i = 0, ..., m − 1 do
    P(0, i) ← a_i(z)/(i+1)
  end for
  φ ← P(0, m − 1)
  for i = 1, ..., m − 1 do
    for j = 0, ..., m − 1 − i do
      for k = 0, ..., j do
        P(i, j) ← P(i, j) + P(i − 1, k) ∗ P(0, j − k)
      end for
    end for
    φ ← φ + P(i, m−1−i)/(i+1)!
  end for
  return φ
```

4.2. Calculating $a_k(z)$

We now consider the problem of calculating $a_k(z) = E_{\mathbf{Q}}\left(e^{izL^+_{j^*+k}}\right)$, $k = 1, \ldots, m$, i.e. the characteristic function of the greater of the log-return process at time t_{j^*+k} and zero. For simple models of asset log-return, such as Brownian motion, this can be evaluated directly by integrating the probability distribution. For a general semimartingale process, the following theorem applies the Fourier transform to provide a general methodology for calculating a_k, $k = 1, \ldots, m$.

Proposition 4.3 Fourier Transform Representation of $a_k(z)$**:** *If $M_{L_{j^*+k}}(v) = E_{\mathbf{Q}}\left(e^{vL_{j^*+k}}\right)$ exists for all $v \in (a, b)$, where $a < 0$ and $b > -\operatorname{Im} z > \frac{1}{2}$, then the characteristic function $a_k(z)$ is given* *by*

$$a_k(z) = 1 - \frac{i}{2\pi} \int\limits_{i\alpha-\infty}^{i\alpha+\infty} \phi_{L_{j^*+k}}(-\xi) \frac{z}{\xi(\xi+z)} d\xi, \tag{25}$$

where $\alpha \geq \operatorname{Im} z > \frac{1}{2}$ *and* $k = 1, \ldots, m$.

Proof: The proof is based on applying the Cauchy residue theorem and the technical details are therefore omitted. □

To evaluate (25), note that it is possible to use either numerical methods, or analytic integration, as illustrated in (26) and (27) of Section 5.1. For example, residue calculus can be used to evaluate (25) in closed-form for some semimartingale processes, as illustrated in Section 5.2, Proposition 5.4 in the case of the VG process. In other cases, the FTBS method may be adapted to efficiently calculate the integral in (25) and hence evaluate $a_k(z)$ at all required z.

5. Pricing Under Selected Semimartingale Models

In this section, the FTBS method is applied to the problem of pricing discrete lookback options for some popular semimartingale processes. Before we proceed to consider the specifics of individual asset models, we provide a recap of the main algorithm proposed in this paper.

Algorithm 2 FTBS Pricing Framework

For a general semimartingale process of the log-return, L_t, the following steps provide the price of a discrete lookback option.

(i) Using Proposition 4.3, derive an analytic formula or an approximation to $a_k(z)$.
(ii) Apply the Spitzer-recurrence expansion formula in Theorem 4.1 to calculate the characteristic function $\phi_{X_m}(z)$ of the asset price maximum.
(iii) Fit optimal splines interpolants $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$ to approximate $s_1(t)$ and $s_2(t)$ in (7).
(iv) Calculate $\widetilde{C}(T, R)$ by using Algorithm *A1* of Appendix A.3 and use this in (2) or (3) to price the fixed-strike call or floating put lookback option.

It should be noted that the spline approximants, $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$, are independent of R, the ratio of strike price or previous asset price maximum to the current asset price. Since the divided differences can be calculated very quickly, the approximants' independence to R implies that the efficiency of the FTBS method for pricing many options simultaneously across a quantum of values of R is similar to that of pricing a single lookback option. This is analogous to the use of the FFT of Carr and Madan (1999) for evaluating the price of European options for a range of strike prices. Further, we note that the divided differences can be pre-computed for a particular asset price model, and recalled from memory as required. We refer to Haslip and Kaishev (2013) for further details on the pre-computation of divided differences.

5.1. Brownian Motion and the Merton Jump Diffusion Model

We illustrate Step 1 of the FTBS pricing framework under the Black-Scholes assumptions.

Proposition 5.1 Calculation of $a_k(z)$ for Brownian motion**:**
If $L_t = \sigma W(t) + \mu t \sim N(\mu t, \sigma^2 t)$, *where* $W(t)$ *is standard Brownian motion and* $\mu = r - \frac{1}{2}\sigma^2$, $\sigma > 0$ *then*

$$a_k(z) = \Phi\left(-\frac{\mu\sqrt{t_{j^*+k}}}{\sigma}\right) + \frac{1}{2} e^{z t_{j^*+k}(i\mu - \frac{1}{2}z\sigma^2)} erfc\left(-\frac{1}{\sqrt{t_{j^*+k}}}\left(\frac{\mu}{\sigma} + iz\sigma\right)\right), \tag{26}$$

where erfc$(.)$ *is the complex complimentary error function and* $k = 1, \ldots, m$.

Proof: The proof is based on directly integrating with respect to the p.d.f. of the Normal distribution. □

In the following proposition, we show how to apply this approach to the Merton jump diffusion model, as described by Petrella and Kou (2004).

Proposition 5.2 Calculation of a_k for the Merton Jump Diffusion Model**:**
If $L_t = \sigma W(t) + \mu t + \sum_{i=1}^{N_t} Y_i$, *where* $\mu = r - \frac{1}{2}\sigma^2 - \lambda\zeta$, $\zeta = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$, $N_t \sim Po(\lambda)$, *and* $Y_i \sim N(\mu_J, \sigma_J^2)$ *with* $\sigma, \sigma_J, \lambda > 0$ *then*

$$a_k(z) = \sum_{j=0}^{\infty}\left\{\Phi\left(-\frac{\mu_j}{\sigma_j}\right) + \frac{1}{2} e^{z(i\mu_j - \frac{1}{2}z\sigma_j^2)} erfc\left(-\frac{1}{\sqrt{2}}\left(\frac{\mu_j}{\sigma_j} + iz\sigma_j\right)\right)\right\} e^{-\lambda t_k}\frac{(\lambda t_{j^*+k})^j}{j!}, \tag{27}$$

where $\mu_j = \mu t_{j^*+k} + j\mu_J$, $\sigma_j = \sqrt{j\sigma_J^2 + \sigma^2 t_{j^*+k}}$, *and erfc*$(.)$ *is the complex complimentary error function.*

Proof: This follows by conditioning on the number of jumps occurring in $[t_{j^*+1}, t_{j^*+k}]$ under the Poisson distribution. The full derivation is provided in Petrella and Kou (2004). □

5.2. The Variance Gamma Process

The VG process can be interpreted as Brownian motion with time changed by an independent Gamma Lévy process. The subordinated time process is often viewed as "business time", reflecting the random speedups and slowdowns in real-time economic and business activity. We refer the reader to Madan *et al.* (1998) for a detailed introduction to the VG process, and Kaishev and Dimitrova (2009) for a more recent account on the VG related literature. The risk-neutral dynamics of the log-return process are given by $L_t = rt + \mu\gamma_t + \sigma W(\gamma_t) - \omega t$, where $W(t)$ is standard Brownian motion, $\sigma > 0$, and $\gamma_t \sim \Gamma\left(\frac{t}{\nu}, \nu\right)$ with $\nu > 0$. The compensator $\omega = -\frac{1}{\nu}\ln\left(1 - \mu\nu - \frac{1}{2}\sigma^2\nu\right)$ is chosen to ensure that $E_{\mathbf{Q}}(S(t)) = S(0)e^{rt}$, and this imposes the requirement that $(\mu + \frac{1}{2}\sigma^2)\nu < 1$ on the parameters of the VG process. To calculate $a_k(z)$, we first need to compute the characteristic function $\phi_{L_t}(z)$.

Proposition 5.3: *Let* L_t *follow a VG process so that* $L_t = rt + \mu\gamma_t + \sigma W(\gamma_t) - \omega t$, *where* $\sigma > 0, \nu > 0$, *and* $(\mu + \frac{1}{2}\sigma^2)\nu < 1$, *under the risk-neutral probability measure. The characteristic function of* L_t *is given by*

$$\phi_{L_t}(\xi) = e^{i\xi(r-\omega)t}\left(\frac{1}{1 - i\mu\nu\xi + \frac{1}{2}\sigma^2\nu\xi^2}\right)^{\frac{t}{\nu}}, \ \xi \in \mathbf{C}. \tag{28}$$

Proof: The proof follows from a result by Madan *et al.* (1998) for the characteristic function of the real-world log-return process, appropriately extending the domain of the characteristic function to the complex plain. □

To calculate $a_k(z)$ for the VG process, we utilize Proposition 4.3 and apply residue calculus to evaluate the contour integral.

Proposition 5.4: *Let L_t follow a VG process, where $r > \omega$ or, equivalently, $(\mu+\frac{1}{2}\sigma^2)\nu < 1-e^{-r\nu}$ is satisfied. If $s = \frac{t_{j^*+k}}{\nu}$ for $k = 1, \ldots, m$ is integer valued then*

$$a_k(z) = -2\pi i \left[R_1(z) + R_2(z)\right], \tag{29}$$

where

$$R_1(z) = \frac{i}{2\pi}\left(1 - i\mu\nu z + \frac{1}{2}\sigma^2\nu z^2\right)^{-s} e^{iz(r-\omega)s\nu} \tag{30}$$

$$\begin{aligned} R_2(z) = {} & \frac{iz}{2\pi(s-1)!}\left(\tfrac{1}{2}\sigma^2\nu\right)^{-s} e^{i\xi_2 s(r-w)\nu} \sum_{j_1=0}^{s-1} \frac{(s+j_1-1)!}{j_1!}\left(\xi_2 - xi_3\right)^{-s-j_1} \\ & \sum_{j_2=0}^{s-1-j_1} \frac{(-1)^{s-1-j_2}}{j_2!}\left[is(r-w)v\right]^{j_2} \sum_{j_3=0}^{s-1-j_1-j_2} \xi_2^{-1-j_3}\left(\xi_2+z\right)^{j_1+j_2+j_3-s}, \end{aligned} \tag{31}$$

where $\xi_1 = -z$ and $\xi_2 = -\frac{i}{\sigma^2\nu}\left(\mu\nu + \sqrt{\mu^2\nu^2 + 2\sigma^2\nu}\right)$. If s is not integer valued for some k, then we have $s = s_I + w$, where $s_I = \lfloor s \rfloor$ is the integer part of s and $w = s - s_I$, $0 < w < 1$. Then, we estimate $a_k(z)$ for a given s as $\hat{a}_k(z) = -2\pi i\left[R_1(z) + \hat{R}_2(z)\right]$, where $\hat{R}_2(z) = (1 - w)R_2(z)|_{s=\tilde{s}_I} + wR_2(z)|_{s=\tilde{s}_I+1}$.

Proof: The proof utilizes the Cauchy residue theorem to evaluate the integral along the straight line $\xi = \alpha i$ as the sum of residues lying below this line and details are therefore omitted. □

To avoid computational difficulties with large numbers, we write (31) in terms of logarithms. Additionally, in our implementation we identify a condition on the third summation index j_3 to avoid calculating terms that are smaller than a specified level to improve efficiency. We note that the requirement for $s \in \mathbf{Z}$, where $\mathbf{Z}$ is the set of integers, imposes the restriction that the number of monitoring points, m, should not exceed the time period of the contract (measured in the time units of the VG process calibration). For example, if the calibration is carried out in annual time units, then we are limited to annual monitoring points. However, if half-yearly monitoring points are required, this is achieved by calibrating the VG process to semi-annual time units.

6. Numerical Results

In this section, we demonstrate the FTBS method through several numerical examples. First, using the Laplace transform (LT) based results of Petrella and Kou (2004) as reference prices, we compute the prices and sensitivities of discrete floating lookback put options under the Brownian motion and the Merton jump diffusion models of log-returns. Second, using the Dirichlet bridge sampling method of Kaishev and Dimitrova (2009), we generate reference prices for discrete fixed lookback call options under the VG process and compare them to the FTBS results. To quantify the level of uncertainty around the simulated option price, we repeat the simulation 20 times for each maturity and strike price with ten million simulations at each pricing point. Our unbiased estimator of the option price is the sample mean, and we calculate the standard error of the simulated price to provide an estimate of the simulation error.

In our implementation, we fit optimal splines to both $s_1(t)$ and $s_2(t)$ in (7) using 200 data points in $[0, 1]$. We allocate the data sites in uniform bands: 60% to $[0, 0.2)$, 20% to $[0.2, 0.6)$ and the remainder to $[0.6, 1]$ (see Haslip and Kaishev (2013) for a further details on this choice of data site allocation). The numerical implementation is in C++, and all computations have been performed using an Intel Core I7-3610QM processor. The complex-valued error function is calculated, as in Petrella and Kou (2004), via the Faddeeva function using an algorithm by Poppe and Wijers (1990), which ensures an accuracy of 13 digits in almost the entire complex plane.

In Table 2 and 3, we provide a comparison of the results under the FTBS method to the LT method of Petrella and Kou (2004), under Brownian motion and the Merton jump diffusion process respectively. We note that Petrella and Kou (2004) use an Intel Pentium 1.8Ghz processor and do not report the software implementation. To illustrate the efficiency of the FTBS method, instead of computing the price of two options for observed asset price maximum $M = 110$ and $M = 120$, we price 41 options for $M = 101$ to $M = 140$ in steps of one. As described in Section 3.1, the strike-separable pricing formula is extremely efficient for pricing lookback options at a range of different strike prices or observed asset price maximum. Our calculations utilize pre-computation of the required divided differences at the 41 specified asset price maximum levels, as described in Haslip and Kaishev (2013). It should be noted that the divided difference calculation is independent of the choice of the exponential semimartingale process. Therefore, for a fixed set of data sites and corresponding optimal knots, we can pre-compute the divided differences for different values of R and do not need to calculate this each time an option is priced for a different value of R. Furthermore, in our implementation, we use same data sites and knots for the Greeks, and hence the pre-computed divided differences are applied for both pricing and computing the option price sensitivities.

The reported CPU time for the FTBS method is the time taken to calculate all 41 option prices and sensitivities and is compared to the time taken for a single option price and sensitivities under the LT method. We note that our reported CPU times exclude the time taken to pre-compute the required divided differences, but this is a very quick process which took only 60ms in total for the 41 individual options. The FTBS method provides results in very close agreement to the LT method using 200 knots in the optimal interpolation scheme. The FTBS method is faster than the LT method for all choices of the number of monitoring points m, and the CPU time of the FTBS is faster by a factor of approximately $5 - 15$ times. The substantial improvement in efficiency for larger m can be explained by the use of the Spitzer-recurrence expansion formula (24) in computing the characteristic function of the asset price maximum. For example, at $m = 80$, for the Merton jump diffusion model, the FTBS method takes 256ms to price 41 options, whereas the LT method takes $4,070$ms to compute two option prices.

In Table 4, we provide prices for the discrete fixed lookback call option under the VG process for strike prices $K = 105$, $K = 110$, and $K = 115$ for contracts with annual monitoring points and contract duration, T, ranging from 5 to 75 years. Again, the CPU times provided correspond to the total time for computing the price of 40 individual options with strike prices ranging from $K = 101$ to $K = 140$ in steps of one. The results show close agreement with the Monte Carlo simulation, and differences are well within one standard deviation of the Monte Carlo results.

We note that options with strike price or previous asset price maximum less than or equal to 100 have been intentionally excluded from the numerical study. This is because the price of in-the-money or at-the-money options can be determined exactly using the closed-form solution in Lemma 2.2.

Table 2. Pricing Results for Brownian Motion Discrete Floating Lookback Put Option

M	n	Price		Delta		Gamma		Time (ms)	
		FTBS	LT	FTBS	LT	FTBS	LT	FTBS	LT
110	5	13.300	13.300	-0.3567	-0.3568	0.0288	0.0287	2	20
120	5	18.837	18.837	-0.5924	-0.5924	0.0244	0.0244		10
110	10	14.123	14.123	-0.3032	-0.3034	0.0310	0.0309	4	30
120	10	19.323	19.323	-0.5547	-0.5547	0.0260	0.0260		20
110	20	14.806	14.806	-0.2632	-0.2633	0.0320	0.0319	12	60
120	20	19.743	19.743	-0.5238	-0.5237	0.0272	0.0273		70
110	40	15.345	15.345	-0.2332	-0.2333	0.0324	0.0324	40	190
120	40	20.083	20.083	-0.4998	-0.4999	0.0281	0.0281		190
110	80	15.754	15.754	-0.2111	-0.2112	0.0327	0.0327	145	640
120	80	20.346	20.346	-0.4819	-0.4819	0.0287	0.0287		640
110	160	16.059	16.059	-0.1951	-0.1952	0.0329	0.0329	663	2,340
120	160	20.544	20.544	-0.4687	-0.4687	0.0291	0.0291		2,350

Brownian motion parameters: $S_0 = 100, \sigma = 0.3, r = 0.1, T = 0.5$.
The times (FTBS) are for computing the prices and sensitivities of 41 options, whereas (LT) are the times quoted by Petrella and Kou (2004) are for computing the prices and sensitivities of a single option.

Table 3. Pricing Results for Merton Jump Diffusion Discrete Floating Lookback Put Option

M	n	Price		Delta		Gamma		Time (ms)	
		FTBS	LT	FTBS	LT	FTBS	LT	FTBS	LT
110	5	12.684	12.683	-0.3919	-0.3919	0.0313	0.0312	10	80
120	5	18.528	18.528	-0.6320	-0.6320	0.0233	0.0233		90
110	10	13.312	13.311	-0.3488	-0.3488	0.0332	0.0331	18	180
120	10	18.886	18.886	-0.6031	-0.6031	0.0249	0.0249		170
110	20	13.812	13.812	-0.3163	-0.3163	0.0342	0.0341	41	350
120	20	19.180	19.180	-0.5805	-0.5805	0.0261	0.0261		350
110	40	14.193	14.193	-0.2925	-0.2924	0.0349	0.0348	98	810
120	40	19.408	19.408	-0.5634	-0.5634	0.0269	0.0269		820
110	80	14.476	14.476	-0.2752	-0.2752	0.0353	0.0353	256	2,000
120	80	19.579	19.580	-0.5508	-0.5508	0.0275	0.0275		2,070
110	160	14.682	14.681	-0.2629	-0.2629	0.0356	0.0356	887	5,550
120	160	19.705	19.706	-0.5417	-0.5417	0.0279	0.0279		5,690

Merton parameters: $S_0 = 100, \sigma = 0.21213, \lambda = 2.23881, \mu_J = -0.01, \sigma_J = 0.14142, r = 0.1, T = 0.5$.
The times (FTBS) are for computing the prices and sensitivities of 41 options, whereas (LT) are the times quoted by Petrella and Kou (2004) are for computing the prices and sensitivities of a single option.

7. Conclusion

In this paper, we have presented an entirely new approach to pricing discrete lookback options that utilizes B-spline interpolation theory to provide an efficient closed-form solution. The FTBS framework works across the family of exponential semimartingale processes and other models

Table 4. Variance Gamma Option Pricing Results

T,m	K	MC Price	MC S.E.	FTBS Price	Time (sec)
5	105	39.079	0.003	39.080	0.03
	110	36.110	0.003	36.110	
	115	33.303	0.003	33.303	
10	105	64.127	0.008	64.126	0.20
	110	61.626	0.009	61.626	
	115	59.203	0.007	59.205	
20	105	94.634	0.014	94.639	1.16
	110	93.134	0.015	93.096	
	115	91.635	0.014	91.580	
50	105	125.940	0.635	125.797	15.44
	110	125.520	0.354	125.484	
	115	125.350	0.653	125.173	
75	105	132.082	2.822	131.086	56.83
	110	132.791	4.286	131.006	
	115	131.611	2.754	130.925	

$S_0 = 100, \mu = -0.2859, \sigma = 0.1927, \nu = 0.25, r = 0.0548.$

whose characteristic function satisfies certain regularity constraints. Our novel use of the Peano representation of a divided difference makes the FTBS method extremely efficient. It enables us to evaluate the pricing integral in closed-form once the integrands have been replaced by the B-spline approximants.

Numerical examples have demonstrated the accuracy and efficiency of the FTBS method. Our method also has an important advantage over existing methods for pricing discrete lookback options in that, once the B-spline interpolant has been fitted to the strike-separable function, we are able to price lookback options at a range of different strike prices with virtually no additional computational cost.

Finally, we have given an alternative derivation of the expansion representation of the Spitzer formula due to Wendel (1958), which we call the Spitzer-recurrence expansion formula. This provides a substantial efficiency gain over the Spitzer formula. This is one of the key reasons why our numerical results for the FTBS method have shown such a significant improvement over the Laplace Transform method of Petrella and Kou (2004) for large numbers of monitoring points m. Algorithm 1 which implements the Spitzer-recurrence expansion formula is of fundamental importance since it has many potential applications in wider fields in which the Spitzer formula is employed.

Acknowledgements

We would like to thank the two anonymous referees for their valuable comments and suggestions and in particular for pointing to us the papers by Lipton (2002b,a) and Wendel (1958) which helped to significantly improve the revised version of our paper. We are also grateful to Dimitrina Dimitrova and Zvetan Ignatov for valuable comments and suggestions on earlier versions of the paper.

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Appendix A: Fourier Transform B-spline Method

A.1. The Divided Difference at Repeated Knots (see Ignatov and Kaishev (1989))

$$[\underbrace{\eta_0, \cdots, \eta_0}_{v_0}, \cdots, \underbrace{\eta_m, \cdots, \eta_m}_{v_m}]f = \sum_{i=0}^{m} \frac{D^{v_i-1} g_i(\eta_i)}{(v_i-1)!}, \tag{A1}$$

where

$$g_i(t) = \frac{f(t)}{\prod_{\substack{j=0 \\ j\neq i}}^{l} (t-\eta_j)^{v_j}},$$

and $D^{v_i-1} g_i(t_i)$ is the (v_i-1)-th derivative of $g_i(t)$ evaluated at $t = \eta_i$.

A.2. Proof of Proposition 3.4 (FTBS Lookback Option Price Error Bound)

From Theorem 1 (formula for the price $C(T,R)$) we have

$$\begin{aligned}
&\left|C(T,R) - \tilde{C}(T,R)\right| = \\
&= \left|\frac{-\sqrt{R}}{\pi}\left(\int_0^1 \cos\left(\frac{1-t}{t}\log R\right)(s_1(t) - \tilde{s}_1(t))\,dt + \int_0^1 \sin\left(\frac{1-t}{t}\log R\right)(s_2(t) - \tilde{s}_2(t))\,dt\right)\right| \\
&\leq \frac{\sqrt{R}}{\pi}\left(\int_0^1 \left|\cos\left(\frac{1-t}{t}\log R\right)(s_1(t) - \tilde{s}_1(t))\right| dt + \int_0^1 \left|\sin\left(\frac{1-t}{t}\log R\right)(s_2(t) - \tilde{s}_2(t))\right| dt\right) \\
&\leq \frac{\sqrt{R}}{\pi}\left(\int_0^1 |s_1(t) - \tilde{s}_1(t)|\,dt + \int_0^1 |s_2(t) - \tilde{s}_2(t)|\,dt\right) \\
&\leq \frac{\sqrt{R}}{\pi}\left(\underset{t\in[0,1]}{max}|s_1(t) - \tilde{s}_1(t)| + \underset{t\in[0,1]}{max}|s_2(t) - \tilde{s}_2(t)|\right) \\
&\leq \frac{\sqrt{R}}{\pi}\left(\left\|s_1^{(3)}(t)\right\| C_1 + \left\|s_2^{(3)}(t)\right\| C_2\right)
\end{aligned}$$

which is the asserted bound. In the last inequality we have used the optimal spline interpolation bounds $\underset{t\in[0,1]}{max}|(s_i(t) - \tilde{s}_i(t))| = \|s_i(t) - \tilde{s}_i(t)\| \leq C_i \left\|s_i^{(3)}(t)\right\|$, $i = 1, 2$ obtained by Gaffney and Powell (1976). The rate $O\left(|\tilde{\eta}|^3\right)$ in (22) now follows applying to (21) a result by De Boor (2001) (Theorem 22, page 154). This completes the proof of Proposition 3.4. □

A.3. Algorithm for Calculating $\tilde{C}(T, R)$

Algorithm *A1* Computation of Lookback option prices using the FTBS method (using identical sets of knots for interpolation of functions $s_1(t)$ and $s_2(t)$)

Input:

- Number of data sites p.
- Array specifying data sites $\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_p\}$ with $\tau_1 = 0$, $\tau_p = 1$, and $\tau_i \in (0,1)$ for $j = 2, \ldots, p-1$, allocated uniformly, according to the empirically justified rule: 60% in $[0, 0.2)$, 20% in $[0.2, 0.6)$ and 20% in $[0.6, 1)$.
- The constants r, q, T, R.

Output: $\tilde{C}(T, R)$

(i) Compute knots $\{\eta_i\}_{i=1}^{p+3}$, using the approximation in (16) for the interior knots. Therefore, $\eta_1 = \eta_2 = \eta_3 = 0$, $\eta_{3+j} = \frac{\tau_{i+j}+\tau_{i+j}}{2}$ for $j = 1, \ldots, p-3$, and $\eta_{p+1} = \eta_{p+2} = \eta_{p+3} = 1$. Note that in our computational implementation, the same knots will be utilized to interpolate functions $s_1(t)$ and $s_2(t)$, that is we take $p = p_1 = p_2$ and $\eta_i = \eta_{1,i} = \eta_{2,i}$, for $i = 1, \ldots, p+3$.

(ii) Compute divided differences $[\eta_i, \eta_{i+1}, \eta_{i+2}, \eta_{i+3}]f_1(t, \log R)$, and $[\eta_i, \eta_{i+1}, \eta_{i+2}, \eta_{i+3}]f_2(t, \log R)$, for $i = 1, \ldots, p$.

(iii) Compute quadratic spline approximants, $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$ to $s_1(t)$ and $s_2(t)$ using function SPLINT of De Boor (2001). This routine will return the linear coefficients $\{c_{1,i}\}_{i=1}^p$ and $\{c_{2,i}\}_{i=1}^p$ (see Section 3.2).

(iv) Compute option prices using Theorem 3.2. That is, first compute $\tilde{I}(R)$, using formula (19), and then $\tilde{C}(T, R)$ using formula (18) (see Section 3.3).

Appendix B: Spitzer Formula

B.1. Proof of Theorem 4.1 *(Spitzer-recurrence Expansion Formula)*

We start with the Spitzer-recurrence formula (23). For brevity, we omit the dependence of $\phi_{X_m}(z)$ on the argument z and write ϕ_m instead of ϕ_{X_m}. We have

$$\phi_m = \frac{1}{m} \sum_{k=0}^{m-1} \phi_k a_{m-k}, \quad m = 1, 2 \ldots . \tag{B1}$$

We can rewrite $(B1)$ as

$$m\phi_m = \sum_{k=0}^{m} \phi_k a_{m-k} - \phi_m a_0, \text{ where } a_0 \equiv 1.$$

The latter equality can further be written as

$$(m+1)\phi_m = \sum_{k=0}^{m} \phi_k a_{m-k}. \tag{B2}$$

Let us now introduce the function

$$U(u) = \sum_{m=0}^{\infty} u^m \phi_m, \text{ with } U(0) \equiv 1. \tag{B3}$$

Noting that $U'(u) = \frac{dU}{du} = \sum_{m=0}^{\infty} m u^{m-1} \phi_m$, we have that

$$U + uU' = \sum_{m=0}^{\infty} (m+1) u^m \phi_m. \tag{B4}$$

Now, multiplying both sides of $(B2)$ by u^m and summing over m from 0 to ∞, we have

$$\sum_{m=0}^{\infty} (m+1) u^m \phi_m = \sum_{m=0}^{\infty} u^m \sum_{k=0}^{m} \phi_k a_{m-k} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \phi_k u^k a_{m-k} u^{m-k} = UA \tag{B5}$$

where $A = \sum_{j=0}^{\infty} a_j u^j$, and $a_0 \equiv 1$. It follows from $(B4)$ and $(B5)$ that $U + uU' = UA$, which can be written as

$$\frac{U'(u)}{U(u)} = \frac{A(u)-1}{u} \implies d\log(U) = \frac{A-1}{u} du \implies \int_0^t d\log(U) = \log(U(t)) = \int_0^t \frac{A-1}{u} du$$

$$\implies U(t) = e^{\int_0^t \frac{A(u)-1}{u} du}. \tag{B6}$$

Now, consider the integral

$$\int_0^t \frac{A(u)-1}{u} du = \int_0^t \frac{\sum_{j=0}^{\infty} a_j u^j - 1}{u} du = \int_0^t \sum_{k=1}^{\infty} a_k u^{k-1} du$$

$$= \sum_{k=1}^{\infty} a_k \int_0^t u^{k-1} du = \sum_{k=1}^{\infty} a_k \frac{t^k}{k}, \tag{B7}$$

where in the last equality we have assumed that a_k are bounded, that is, $|a_k| < B$. Now, from $(B3)$, $(B6)$, and $(B7)$ we have

$$U(t) = \sum_{m=0}^{\infty} t^m \phi_m = \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{k} t^k\right) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\sum_{k=1}^{\infty} \frac{a_k}{k} t^k\right)^j.$$

Hence we can see that ϕ_m is given by the coefficient of the power t^m for $m = 1, 2 \dots$ in the above expansion. This completes the proof of the theorem. $\square$

B.2. Proof of Proposition **4.2** *(Efficiency of Spitzer-recurrence Expansion Formula* **(24)***)*

It is easy to see that the total number of loops in Algorithm **1** is given by

$$m + \sum_{i=1}^{m-1} \sum_{j=0}^{m-1-i} \sum_{k=0}^{j} 1 = \frac{1}{6} m(m^2+5),$$

and hence the algorithm has complexity $O(m^3)$ $\square$.

B.3. Proof of Proposition **4.2** *(Efficiency of Spitzer Formula* **(23)***)*

First, we note that, if the Spitzer formula for ϕ_m is expanded computationally as a recursive function, then there is a total of $m!$ terms to evaluate (to see this expand the formula for $\phi_1, \phi_2, \ldots$). This basic implementation is shown in Algorithm *B2* and is a standard way to proceed given its definition.

Algorithm *B2* Computation of $\phi_{X_m}(z)$ using the Spitzer recurrence formula

Input: $m \in \mathbf{N}, z \in \mathbf{C}$
Output: $\phi_{X_m}(z)$
  if $m = 0$ **then**
    return 1
  else
    $\phi \leftarrow 0$
    for $k = 0, \ldots, m-1$ **do**
      $\phi \leftarrow \phi + \phi_{X_k}(z) * a_{m-k}(z)$
    end for
    return $\frac{\phi}{m}$
  end if

However, Algorithm *B2* would not be an efficient implementation, since the coefficients and expressions in the expansion can be simplified and pre-computed in terms of $a_k(z)$. The following lemma describes a more efficient computational implementation and shows that the number of terms can be reduced to $p(m)$, where p is the number of integral partitions of m.

Lemma B.1 General Form of Spitzer Recurrence Formula**:** *When fully expanded, the recursive form of the Spitzer formula* (23) *has the following general form*

$$\phi_{X_m}(z) = \frac{1}{m} \sum_{(i_1,\ldots,i_m)\in\Omega} C(i_1,...,i_m) a_1^{i_1}(z) \ldots a_m^{i_m}(z), \tag{B8}$$

where $\Omega = \left\{ (i_1,...,i_m) : \sum_{j=1}^{m} i_j j = m \text{ and } i_j \in \mathcal{Z}^+, j = 1,...,m \right\}$ *and* $C(i_1,...,i_m)$ *are real valued coefficients that must be determined using numerical methods. The cardinality of* Ω *is given by* $p(m)$*, the number of integer partitions of the number* m*.*

Proof: We show this holds by induction. Define $D = \{m \in \mathbf{N} :$ equation $(B8)$ holds for all $l \leq m, \quad l \in \mathbf{N}\}$. First note that $\phi_1 = \phi_0 a_1 = a_1^1$ and $m = 1 \equiv 1 \times 1$. Hence, $1 \in D$. Now assume

$m \in D, l \in \mathbf{N}$ and observe that

$$\phi_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} \phi_k a_{m+1-k} = \frac{1}{m+1} \left(\sum_{k=0}^{m-1} \phi_k a_{m+1-k} + \phi_m a_1 \right).$$

We consider each product on the right hand side of the above. We have

$$\begin{aligned}
\phi_0 a^1_{m+1} &= a^1_{m+1} \text{ and } (m+1) \times 1 = m+1 \\
\phi_1 a^1_m &= \phi_0 a^1_1 a^1_m = a^1_1 a^1_m \text{ and } 1 \times 1 + m \times 1 = m+1 \\
&\cdots \\
\phi_k a^1_{m+1-k} &= \frac{1}{k} \sum_{(i_1,\ldots,i_k)\in\Omega} C(i_1,...,i_k) a_1^{i_1}(z)...a_k^{i_k}(z) a^1_{m+1-k} \text{ and } \sum_{j=1}^{k} j i_j + (m+1-k) \times 1 = m+1 \\
&\cdots \\
\phi_{m-1} a^1_{m+1-(m-1)} &= \phi_{m-1} a^1_2 = \sum_{(i_1,\ldots,i_{m-1})\in\Omega} C(i_1,...,i_{m-1}) a_1^{i_1}(z)...a_{m-1}^{i_{m-1}}(z) a^1_2 \text{ and } \sum_{j=1}^{m-1} j i_j + 2 \times 1 = m+1.
\end{aligned}$$

Hence, we have $(m+1) \in D$, and by induction $D = \mathbf{N}$. Finally, by definition of integer partitions, we see that $\Omega = p(m)$. Proposition 4.2 is a direct consequence of Lemma B.1. □