Asymptotic Results for Conditional Measures of Association of a Random Sum

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August 29, 2014

Abstract. Asymptotic results are obtained for several conditional measures of association. The chosen random variables are the first two order statistics and the total sum within a random sum. Many of the results have confirmed the "one-jump" property of the risk model. Non-trivial limits are obtained when the dependence among the first two order statistics is considered. Our results help in understanding the extreme behaviour of well-known reinsurance treaties that involve only few large claims. Interestingly, the Pearson product-moment correlation coefficient between the first two order statistics provides an alternative procedure to estimate the tail index of the underlying distribution.

Keywords and phrases: Extreme Value Theory; layer reinsurance; Long-Tailed Distribution; Gumbel Tail; Kendall's tau; Order statistics; Pearson Product-Moment Correlation Coefficient; Regular Variation; Spearman's rho; Tail index.

Mathematics Subject Classification: 60F99, 60G70.

1. Introduction

Let $X_1, \cdots, X_n$ be independent and identically distributed (iid) random variables (rv's) with distribution function (df) $F(\cdot)$, tail function $\bar{F} = 1 - F$ and infinite right-end point. *Extreme Value Theory (EVT)* assumes that there are constants $a_n > 0, b_n \in \Re$ such that

$$\lim_{n\to\infty} \Pr\left(a_n\Big(\max_{1\le i\le n} X_i - b_n\Big) \le x\right) = G(x), \qquad \text{for all } x.$$

Then, G is called an *Extreme Value Distribution* and F is said to belong to the *domain of attraction of* G. The Fisher-Tippett theorem states that if the limit distribution is non-degenerate then $G(x) = \exp\{-x^{-\alpha}\}$ for all $x, \alpha > 0$ or $G(x) = \exp\big\{ - e^{-x}\big\}$ for all $x \in \Re$, since F is assumed to have an infinite right bound. In the first case, F has the *regularly varying (RV)* property with tail index α, i.e.

$$\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-\alpha}, \qquad \text{for all } x > 0, \tag{1.1}$$

and we write $X \in RV_{-\alpha}$. In the second case, F has a *Gumbel* tail and it is well-known (see, for example, Embrechts *et al.*, 1997) that there exists a positive, measurable function $a(\cdot)$ such that

$$\lim_{t\to\infty} \frac{\bar{F}(t + xa(t))}{\bar{F}(t)} = e^{-x}, \qquad \text{for all } x \in \Re, \tag{1.2}$$

and we write $X \in \Lambda(a)$.

There are many characterizations of heavy-tailed distributions, but the largest one is the class $\mathcal{L}$ of *long-tailed* distributions. By definition, a df $F \in \mathcal{L}$ if

$$\lim_{t\to\infty} \frac{\Pr(X > x + t)}{\Pr(X > t)} = 1,$$

holds for all fixed $x \in \Re$. By the local uniformity of this convergence, it is not difficult to find out that there is some positive and increasing function l_t such that $l_t \to \infty$, $l_t = o(t)$ and $\overline{F}(t \pm l_t) \sim \overline{F}(t)$. A subclass of $\mathcal{L}$ is the set $\mathcal{S}$ of *subexponential* distributions. By definition, a df F with positive support belongs to $\mathcal{S}$ if

$$\lim_{x\to\infty} \frac{\Pr(X_1 + X_2 > x)}{\Pr(X > x)} = 2,$$

where X_1 and X_2 are iid copies of X, and we write $X \in \mathcal{S}$. A subclass of $\mathcal{S}$ is given by the set regularly varying distributions. The remaining well-known distributions, such as Log-Normal and Weibull, have a Gumbel tail. For more details of heavy-tailed distributions, we refer the reader to Bingham *et al.* (1987), Embrechts *et al.* (1997) and Foss *et al.* (2011).

Consider an iid sequence of random variables $X_1, X_2, \ldots, X_N$ with common distribution function F, where N is a non-negative integer valued random variables that is independent of X_1. Denote $X_N^{(1)} \geq X_N^{(2)} \geq \ldots$ the order sequence. Let $S = \sum_{i=1}^{N} X_i$ be the random sum, where by definition $S = 0$ if $N = 0$. A specific example is the classical risk model for which X_i's are the claim sizes within a finite horizon $[0, T]$ and $\{N(t), 0 \leq t \leq T\}$ is the claim arrival process (for details, see for example Embrechts *et al.* 1997).

The dependence between two or more random variables can be fully described by their copula whenever it exists (see Nelsen, 2006). Due to the data scarcity, fitting the dependence is often problematic, and more simple alternative methods would be more informative. For example, simple measures of association may be sufficient to estimate the quantity of interest, such as the tail dependence index, as we will find later in the considered setting. Recall that there are many measures of association that quantify the degree of dependence between two rv's, say (X, Y), and we will mainly focus on the three well-known ones in the literature (for more details, see Nelsen, 2006). *Kendall's tau*,

$$\tau := \Pr\big((X_1 - X_2)(Y_1 - Y_2) > 0\big) - \Pr\big((X_1 - X_2)(Y_1 - Y_2) < 0\big),$$

and *Spearman's rho rank correlation*,

$$\rho_R := 3\Big(P\big((X_1 - X_2)(Y_1 - Y_3) > 0\big) - P\big((X_1 - X_2)(Y_1 - Y_3) < 0\big)\Big),$$

are based on the concordance and discordance probabilities, where (X_i, Y_i), $i = 1, 2, 3$, are three iid copies from (X, Y). It is well-known that both measures of association are scale-invariant, and therefore robust, marginal-free whenever the marginal distributions are continuous. Besides these two, another measure of association is *Pearson product-moment correlation coefficient*,

$$\rho_L := \frac{cov(X, Y)}{\sqrt{Var(X)Var(Y)}},$$

which is the same as Spearman's rho if the marginals are uniform random variables. This third measure of association evaluates the linear correlation between two dependent rv's, and it has been criticized for its lack of robustness, but is still a well-accepted measure in the presence of linear dependence, which is our case since we are interested only in extreme events that happen to be strongly correlated in the tail.

There have been many papers in the last two decades that explored the tail behaviour of the order statistics within a random sum, such as Beirlant and Teugels (1992), Ladoucette and Teugels (2006), Asimit and Jones (2008), Jiang and Tang (2008), Hashorva and Li (2013), Li and Hashorva (2013). The asymptotic dependence, i.e. the joint tail behaviour, among various order statistics has been recently investigated in Hashorva (2007), Albrecher *et al.* (2014) and Peng (2014). A recent paper, namely, Asimit *et al.* (2014), studied the asymptotic behaviour of the conditional Kendall's tau from a statistical extremes perspective. The latter paper and current one lead to a conclusive assertion that conditional measures of association are useful in understanding individual and concomitant rare events. As a side note, this paper is able to show that a classical problem in

EVT of estimating the tail index (see for example, Lu and Peng, 2002, Li *et al.*, 2010, or any textbook in EVT) can be addressed via the conditional Pearson product-moment correlation coefficient as defined in this paper.

In this paper, we seek to understand the asymptotic behaviour of the conditional version of the three measures of association. The conditioning event is chosen to restrict our analysis to joint extreme events between the random sum and/or the first two order statistics of the individual outcomes from the random sum. The next section summarises some definitions and preliminary results, while Sections 3, 4 and 5 provide our main results for the conditional Kendall's tau, Spearman's rho and Pearson product-moment correlation coefficient.

2. Definitions and Preliminaries Results

Throughout this paper, all limit relationships hold as $t \to \infty$. In addition, for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(\cdot) \sim cb(\cdot)$ for some positive constant c to mean strong equivalence, i.e., $\lim a(\cdot)/b(\cdot) = c$. Moreover, we say that $a(\cdot) = o\big(b(\cdot)\big)$ if $\lim a(\cdot)/b(\cdot) = 0$.

This paper deals with the model described in Section 1 for which the rv's of interest are S, $X_N^{(1)}$ and $X_N^{(2)}$. Sometimes, multiple realizations of the process will be needed to perform our calculations, and the three dimensional random vector of interest for the i^{th} realization will be denoted by $\big(S_i, X_{N,i}^{(1)}, X_{N,i}^{(2)}\big)$. In order to assess the strength of dependence between the extreme events arising from this process, the following two conditional Kendall's tau are investigated: for large values of t,

$$\begin{aligned}\tau^{+1}(t) &= \Pr\left((S_1 - S_2)\left(X_{N,1}^{(1)} - X_{N,2}^{(1)}\right) > 0 | X_{N,1}^{(1)}, X_{N,2}^{(1)} > t\right) \\ &\quad - \Pr\left((S_1 - S_2)\left(X_{N,1}^{(1)} - X_{N,2}^{(1)}\right) < 0 | X_{N,1}^{(1)}, X_{N,2}^{(1)} > t\right)\end{aligned}$$

and

$$\begin{aligned}\tau^{12}(t) &= \Pr\left(\left(X_{N,1}^{(1)} - X_{N,2}^{(1)}\right)\left(X_{N,1}^{(2)} - X_{N,2}^{(2)}\right) > 0 | X_{N,1}^{(2)}, X_{N,2}^{(2)} > t\right) \\ &\quad - \Pr\left(\left(X_{N,1}^{(1)} - X_{N,2}^{(1)}\right)\left(X_{N,1}^{(2)} - X_{N,2}^{(2)}\right) < 0 | X_{N,1}^{(2)}, X_{N,2}^{(2)} > t\right).\end{aligned}$$

Similarly, two conditional versions of Spearman's rho of interest are

$$\begin{aligned}\rho_R^{+1}(t) &= 3\Pr\left((S_1 - S_2)\left(X_{N,1}^{(1)} - X_{N,3}^{(1)}\right) > 0 | X_{N,1}^{(1)}, X_{N,2}^{(1)}, X_{N,3}^{(2)} > t\right) \\ &\quad -3\Pr\left((S_1 - S_2)\left(X_{N,1}^{(1)} - X_{N,3}^{(1)}\right) < 0 | X_{N,1}^{(1)}, X_{N,2}^{(1)}, X_{N,3}^{(1)} > t\right)\end{aligned}$$

and

$$\begin{aligned}\rho_R^{12}(t) &= 3\Pr\left(\left(X_{N,1}^{(1)}-X_{N,2}^{(1)}\right)\left(X_{N,1}^{(2)}-X_{N,3}^{(2)}\right)>0|X_{N,1}^{(2)},X_{N,2}^{(2)},X_{N,3}^{(2)}>t\right)\\ &\quad -3\Pr\left(\left(X_{N,1}^{(1)}-X_{N,2}^{(1)}\right)\left(X_{N,1}^{(2)}-X_{N,3}^{(2)}\right)<0|X_{N,1}^{(2)},X_{N,2}^{(2)},X_{N,3}^{(2)}>t\right).\end{aligned}$$

Finally, two conditional versions of Pearson product-moment correlation coefficient are

$$\rho_L^{+1}(t)=\frac{cov\left(S,X_N^{(1)}|X_N^{(1)}>t\right)}{\sqrt{Var\left(S|X_N^{(1)}>t\right)Var\left(X_N^{(1)}|X_N^{(1)}>t\right)}}$$

and

$$\rho_L^{12}(t)=\frac{cov\left(X_N^{(1)},X_N^{(2)}|X_N^{(2)}>t\right)}{\sqrt{Var\left(X_N^{(1)}|X_N^{(2)}>t\right)Var\left(X_N^{(2)}|X_N^{(2)}>t\right)}}.$$

Similar to Section 1, let $X_n^{(1)}\geq X_n^{(2)}\geq\cdots\geq X_n^{(n)}$ be the order statistics of a finite iid sample $X_1,\cdots,X_n$. It is worth mentioning that

$$\Pr\left(X_n^{(l)}>t\right)\sim\binom{n}{l}\bar{F}^l(t),\qquad 1\leq l\leq n, \tag{2.1}$$

which will be frequently used in our further derivations. It is not difficult to find that, for $1\leq l\leq n$,

$$\Pr\left(X_N^{(l)}>t\right)\sim E\binom{N}{l}\bar{F}^l(t),\qquad EN^l<\infty. \tag{2.2}$$

A classical result (see for example, Theorem 1.3.9 of Embrechts *et al.* 1997) that will often be used in our derivations is as follows:

Lemma 2.1. *If $F\in\mathcal{S}$ and $E(1+\epsilon)^N<\infty$ for some $\epsilon>0$, then* $\Pr(S>t)\sim EN\,\bar{F}(t)$.

Another important notion that is crucial for establishing our main results is *vague convergence*. Let $\{\mu_n,n\geq 1\}$ be a sequence of measures on a locally compact Hausdorff space $\mathbb{B}$ with countable base. Then μ_n converges vaguely to some measure μ, written as $\mu_n\xrightarrow{v}\mu$, if for every continuous function f with compact support we have

$$\lim_{n\to\infty}\int_{\mathbb{B}}f\,\mathrm{d}\mu_n=\int_{\mathbb{B}}f\,\mathrm{d}\mu.$$

A thorough background on vague convergence is given by Kallenberg (1983) and Resnick (1987).

3. Main Results: Kendall's tau

The main aim of this section is to find the limits for the Kendall's tau, i.e. $\tau_{+1}(t)$ and $\tau_{12}(t)$, as defined in Section 2. The first step is to establish some preliminary results, which are given in Proposition 3.1.

Proposition 3.1. *Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be to two iid samples with common continuous df F. In addition, $X_m^{(1)} \geq X_m^{(2)} \geq \ldots \geq X_m^{(m)}$ and $Y_n^{(1)} \geq Y_n^{(2)} \geq \ldots \geq Y_n^{(n)}$ be the corresponding order statistics. Moreover, let $S_k = \sum_{i=1}^{k} X_i$, $1 \leq k \leq m$, and $T_l = \sum_{i=1}^{l} Y_i$, $1 \leq l \leq n$, be the partial sums.*

i) *If $F \in \mathcal{L}$, then it holds for every integers $m, n \geq 1$ that*

$$\lim_{t\to\infty} \Pr\left(S_m > T_n, X_m^{(1)} > Y_n^{(1)} | X_m^{(1)}, Y_n^{(1)} > t\right) = \frac{1}{2}\ ;$$

ii) *It holds for every integers $m, n \geq 2$ that*

$$\lim_{t\to\infty} \Pr\left(X_m^{(1)} > Y_n^{(1)}, X_m^{(2)} > Y_n^{(2)} | X_m^{(2)}, Y_n^{(2)} > t\right) = \frac{1}{3}\ .$$

Proof. i) Note first that

$$\begin{aligned} \Pr\left(S_m > T_n, X_m^{(1)} > Y_n^{(1)} | X_m^{(1)}, Y_n^{(1)} > t\right) &= \frac{\Pr\left(S_m > T_n, X_m^{(1)} > Y_n^{(1)} > t\right)}{\Pr\left(X_m^{(1)}, Y_n^{(1)} > t\right)} \\ &\sim \frac{A}{mn\overline{F}(t)^2}. \end{aligned} \tag{3.1}$$

Now,

$$\begin{aligned} A &= mn \int_{x>y>t} \Pr\left(S_m > T_n, X_m^{(2)} \leq x, X_m \in dx, Y_n^{(2)} \leq y, Y_n \in dy\right) \\ &= mn \int_{x>y>t} \Pr\left(S_{m-1} + x > S_{n-1} + y, X_m^{(2)} \leq x, Y_n^{(2)} \leq y\right) dF(x) dF(y)\ . \end{aligned}$$

Keeping in mind that $\Pr\left(X_m^{(2)} \leq x, Y_n^{(2)} \leq y\right) \to 1$ holds uniformly on $\{(x,y) : x > y > t\}$ and $\Pr\left(A_1\right) - \Pr\left(\bar{A}_2\right) \leq \Pr\left(A_1 \cap A_2\right) \leq \Pr\left(A_1\right)$ is true for any sets A_1 and A_2, it follows that

$$\begin{aligned} A &= mn \int_{x>y>t} \Big(\Pr\left(S_{m-1} + x > T_{n-1} + y\right) + o(1)\Big) dF(x) dF(y) \\ &= mn \int_{x>y>t} \Pr\left(T_{n-1} - S_{m-1} < x - y\right) dF(x) dF(y) + o\big(\bar{F}^2(t)\big) \\ &= mn \left(\int_{x-y>l_t, y>t} + \int_{0<x-y\leq l_t, y>t} \right) \Pr\left(T_{n-1} - S_{m-1} < x - y\right) dF(x) dF(y) + o\big(\bar{F}^2(t)\big) \\ &= mn\big(I_1(t) + I_2(t)\big) + o\big(\bar{F}^2(t)\big). \end{aligned}$$

Since $\Pr\big(T_{n-1}-S_{m-1} < x-y\big) \to 1$ holds uniformly, then we have

$$I_1(t) \sim \int_{x-y>l_t,y>t} dF(x)dF(y) = \int_{y>t} \bar{F}\big(y+l_t\big)dF(y) \sim \int_{y>t} \bar{F}(y)dF(y) = \frac{1}{2}\bar{F}^2(t).$$

Clearly,

$$I_2(t) \le \int_{0<x-y\le l_t,y>t} dF(x)dF(y) \le \int_t^{\infty} \bar{F}(y)dF(y) = o\big(\bar{F}^2(t)\big).$$

The last two relations imply that $A \sim \frac{1}{2}mn\bar{F}^2(t)$, which together with (3.1) one may conclude this part.

ii) Simple derivations help in finding

$$\begin{aligned}\Pr\big(X_m^{(1)} > Y_n^{(1)}, X_m^{(2)} > Y_n^{(2)} | X_m^{(2)}, Y_n^{(2)} > t\big) &= \frac{\Pr\big(X_m^{(1)} > Y_n^{(1)}, X_m^{(2)} > Y_n^{(2)} > t\big)}{\Pr\big(X_m^{(2)}, Y_n^{(2)} > t\big)} \\ &\sim \frac{B}{\frac{m(m-1)n(n-1)}{4}\bar{F}^4(t)}\end{aligned}$$

and

$$\begin{aligned} B &= \int_{x>y>t}\int_{u>v,u>x,v>y} \Pr\big(X_m^{(1)} \in du, X_m^{(2)} \in dx\big)\Pr\big(Y_n^{(1)} \in dv, Y_n^{(2)} \in dy\big) \\ &= mn(m-1)(n-1)\int_{x>y>t}\left(\int_{u>v>y,u>x} dF(u)dF(v)\right)F^{m-2}(x)F^{n-2}(y)dF(x)dF(y) \\ &\sim mn(m-1)(n-1)\int_{x>y>t}\left(\int_{u>v>y,u>x} dF(u)dF(v)\right)dF(x)dF(y) \\ &= mn(m-1)(n-1)\int_{x>y>t}\left(\frac{1-\bar{F}^2(x)}{2} - \big(1-\bar{F}(x)\big)\bar{F}(y)\right)dF(x)dF(y) \\ &= mn(m-1)(n-1)\bar{F}^4(t), \end{aligned}$$

which justify part ii) in full. $\square$

We are now ready to provide the main results of this section, stated as Theorem 3.1.

Theorem 3.1. *Assume that N is not degenerate at 0 and $EN < \infty$. In addition, F is a continuous function.*

i) *If $F \in \mathcal{L}$ then $\tau^{+1}(t) \to 1$.*

ii) *If $\Pr(N \ge 2) > 0$ then $\tau^{12}(t) \to 1/3$.*

Proof. i) Simple calculations show that

$$\tau^{+1}(t) = 4\frac{\Pr\big(S_1 > S_2, X_{N,1}^{(1)} > X_{N,2}^{(1)} > t\big)}{\Pr\big(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t\big)} - 1.$$

For an arbitrarily fixed $C > 0$, observe that

$$\frac{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t, N_1 > C \text{ or } N_2 > C\right)}{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t\right)}$$

$$\leq \frac{E\left[\Pr\left(X_{N,1}^{(1)} > t | N_1\right) \Pr\left(X_{N,2}^{(1)} > t | N_2\right) 1_{\left\{N_1 \text{ or } N_2 \leq C\right\}}\right]}{E\left[\Pr\left(X_1, Y_1 > t\right) 1_{\left\{N_1, N_2 \geq 1, N_1 \text{ or } N_2 \leq C\right\}}\right]}$$

$$\leq \frac{E\left[N_1 N_2 1_{\left\{N_1 \text{ or } N_2 \leq C\right\}}\right]}{E\left[1_{\left\{N_1, N_2 \geq 1, N_1 \text{ or } N_2 \leq C\right\}}\right]}$$

$$= \frac{E\left[N_1 N_2 1_{\left\{N_1 \text{ or } N_2 \leq C\right\}}\right]}{E\left[1_{\left\{N_1, N_2 \geq 1, N_1 \text{ or } N_2 \leq C\right\}}\right]}.$$

Now, since the right-hand side from above tends to 0 as $C \to \infty$, then for an arbitrarily fixed $0 < \varepsilon < 1$, we can find some large C such that

$$\frac{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t, N_1 \text{ or } N_2 > C\right)}{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t\right)} \leq \varepsilon.$$

is true for all $t > 0$, and in turn we have

$$\frac{\Pr\left(S_1 > S_2, X_{N,1}^{(1)} > X_{N,2}^{(1)} > t\right)}{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t\right)}$$

$$\geq \frac{\Pr\left(S_1 > S_2, X_{N,1}^{(1)} > X_{N,2}^{(1)} > t, N_1, N_2 \geq 1, N_1 \text{ or } N_2 \leq C\right)}{\frac{1}{1-\varepsilon} \Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t, N_1, N_2 \geq 1, N_1 \text{ or } N_2 \leq C\right)}$$

$$\to \frac{1}{2}(1 - \varepsilon)$$

and that

$$\frac{\Pr\left(S_1 > S_2, X_{N,1}^{(1)} > X_{N,2}^{(1)} > t\right)}{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t\right)}$$

$$\leq \frac{\Pr\left(S_1 > S_2, X_{N,1}^{(1)} > X_{N,2}^{(1)} > t, N_1, N_2 \geq 1, N_1 \text{ or } N_2 \leq C\right)}{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)} > t, N_1, N_2 \geq 1, N_1 \text{ or } N_2 \leq C\right)} + \varepsilon$$

$$\to \frac{1}{2} + \varepsilon,$$

where the last steps are due to Proposition 3.1. Therefore, by taking $\varepsilon \downarrow 0$, the claim from part i) can be retrieved.

ii) The proof follows the same steps as in part ii) with the additional note

$$\tau^{12}(t) = 4 \Pr \left(X_{N,1}^{(1)} > X_{N,2}^{(1)}, X_{N,1}^{(2)} > X_{N,2}^{(2)} | X_{N,1}^{(2)}, X_{N,2}^{(2)} > t \right) - 3,$$

which completes the proof. □

4. Main Results: Spearman's rho

The current section provides the mirror results of Section 3 for the Spearman's rho measure of association, i.e. $\rho_{+1}(t)$ and $\rho_{12}(t)$, as defined in Section 2. As before, we first need to show some useful results, where the samples have deterministic sizes.

Proposition 4.1. *Let $X_1, X_2, \ldots, X_m$, $Y_1, Y_2, \ldots, Y_n$ and $Z_1, Z_2, \ldots, Z_n$ be three iid samples with common continuous df F. Moreover, denote $X_m^{(1)} \geq X_m^{(2)} \geq \ldots \geq X_m^{(m)}$, $Y_n^{(1)} \geq Y_n^{(2)} \geq \ldots \geq Y_n^{(n)}$ and $Z_r^{(1)} \geq Z_r^{(2)} \geq \ldots \geq Z_r^{(r)}$ the corresponding order statistics. Moreover, let $S_k = \sum_{i=1}^{k} X_i$, $1 \leq k \leq m$ and $T_l = \sum_{i=1}^{l} Y_i$, $1 \leq l \leq n$ be the partial sums.*

i) *If $F \in \mathcal{L}$, then it holds for every integers $m, n, k \geq 1$ that*

$$\lim_{t\to\infty} \Pr \left(S_m > T_n, X_m^{(1)} > Z_r^{(1)} > t | X_m^{(1)}, Y_n^{(1)}, Z_r^{(1)} > t \right) = \frac{1}{3}.$$

ii) *It holds for every $m, n, k \geq 2$ that*

$$\lim_{t\to\infty} \Pr \left(X_m^{(1)} > Y_n^{(1)}, X_m^{(2)} > Z_r^{(2)} | X_m^{(2)}, Y_n^{(2)}, Z_r^{(2)} > t \right) = \frac{13}{45}.$$

Proof. i) It is useful to first note that

$$\begin{aligned} &\Pr \left(S_m > T_n, X_m^{(1)} > Z_r^{(1)} > t | X_m^{(1)}, Y_n^{(1)}, Z_r^{(1)} > t \right) \\ &= \frac{\Pr \left(S_m > T_n, X_m^{(1)} > Z_r^{(1)}, X_m^{(1)}, Y_n^{(1)}, Z_r^{(1)} > t \right)}{\Pr \left(X_m^{(1)}, Y_n^{(1)}, Z_r^{(1)} > t \right)} \\ &\sim \frac{D}{mnk\bar{F}(t)^3} \end{aligned}$$

and

$$
\begin{aligned}
D &= mnk \int_{y,z>t,x>z} \Pr\big(S_{m-1}+x > T_{n-1}+y, X_m^{(2)} \le x, Y_n^{(2)} \le y, Z_r^{(2)} \le z\big) \\
&\qquad\qquad\qquad\qquad \times \Pr\left(X_m \in dx, Y_n \in dy, Z_r \in dz\right) \\
&\sim mnk \int_{y,z>t,x>z} \Pr\big(T_{n-1} - S_{m-1} < x-y\big) dF(x)dF(y)dF(z) \\
&= mnk \Big(\int_{y,z>t,,x>z,x-y>l_t} + \int_{y,z>t,x>z,x-y\le -l_t} + \int_{y,z>t,x>z,-l_t<x-y\le l_t} \Big) \\
&\qquad\qquad\qquad\qquad \Pr\big(T_{n-1} - S_{m-1} < x-y\big) dF(x)dF(y)dF(z) \\
&= mnk\big(J_1(t)+J_2(t)+J_3(t)\big).
\end{aligned}
$$

Since $\Pr\big(T_{n-1} - S_{m-1} < x-y\big)$ tends to 1 uniformly, one may find that

$$
J_1(t) \sim \int_{y,z>t,x>z,x-y>l_t} dF(x)dF(y)dF(z) = \int_{y,z>t} \bar{F}\Big(\min\big(z, y+l_t,\big)\Big) dF(y)dF(z).
$$

The fact that $\bar{F}\big(\min(z,y)+l_t\big) \le \bar{F}\Big(\min\big(z, y+l_t\big)\Big) \le \bar{F}\big(\min(z,y)\big)$ implies that $\bar{F}\Big(\min\big(z, y+l_t\big)\Big) \sim \bar{F}\big(\min(z,y)\big)$ holds uniformly on $y, z > t$, and in turn we get

$$
J_1(t) \sim \int_{y,z>t} \bar{F}\big(\min(z,y)\big) dF(y)dF(z) = 2\int_{y>z>t} \bar{F}(y)dF(y)dF(z) = \frac{1}{3}\bar{F}^3(t).
$$

Clearly,

$$
J_2(t) \le \Pr\big(T_{n-1} - S_{m-1} < -l_t\big) \int_{y,z>t,x>z} dF(x)dF(y)dF(z) = o\big(\bar{F}^3(t)\big)
$$

and

$$
J_3(t) \le \int_{y,z>t,x>z,-l_t<x-y\le l_t} dF(x)dF(y)dF(z) = o\big(\bar{F}^3(t)\big),
$$

and by putting all the results together one may fully justify part i).

ii) The proof is similar to the proof of Proposition 3.1ii), and therefore it is left to the reader. □

We can now provide the main results of this section, stated as Theorem 4.1.

Theorem 4.1. *Assume that N is not degenerate at 0 and $EN < \infty$. In addition, F is a continuous function.*

i) *If $F \in \mathcal{L}$ then $\rho^{+1}(t) \to 1$.*
ii) *If $\Pr(N \ge 2) > 0$ then $\rho^{12}(t) \to 7/15$.*

Proof. The proofs are similar to the ones given in Theorem 3.1, and therefore we will only justify part i). Note that

$$\rho_R^{+1}(t) = 12 \frac{\Pr\left(S_1 > S_2, X_{N,1}^{(1)} > X_{N,2}^{(1)}, X_{N,1}^{(1)}, X_{N,2}^{(1)}, X_{N,3}^{(1)} > t\right)}{\Pr\left(X_{N,1}^{(1)}, X_{N,2}^{(1)}, X_{N,3}^{(1)} > t\right)} - 3,$$

which together with Proposition 4.1i) clarify our claim. The proof is now complete. □

5. Main Results: Pearson correlation

The main results are now developed for the Pearson's measures of association. The proofs are different than in the previous two sections, and the two cases, $F \in RV_\alpha$ and $F \in \Gamma(a)$, will be treated separately. We first need to find conditional higher moments, which are found in Lemma 5.1.

Lemma 5.1. *i) If $X \in RV_{-\alpha}$ is a positive rv, then $E(X^k|X > t) \sim t^k \frac{\alpha}{\alpha-k}$ holds for all $0 < k < \alpha$. Consequently, $Var(X|X > t) \sim t^2 \frac{\alpha}{(\alpha-2)(\alpha-1)^2}$ is also true for $\alpha > 2$.*

ii) If $X \in \Gamma(a)$ is a positive rv, then $E(X^k|X > t) = t^k + kt^{k-1}a(t)\big(1 + o(1)\big)$ and $Var(X|X > t) = a^2(t)\big(1 + o(1)\big)$ are true, where k is a positive integer.

Proof. Note first that $\bar{F}(t) = o\big(t^k\big)$. Thus, integration by parts and an obvious change of variables lead to

$$\begin{aligned} E(X^k|X > t) &= t^k + k\int_t^\infty x^{k-1}\frac{\bar{F}(x)}{\bar{F}(t)}\,dx \\ &= t^k + kt^k\int_1^\infty y^{k-1}\frac{\bar{F}(ty)}{\bar{F}(t)}\,dy \\ &\sim t^k + kt^k\int_1^\infty y^{k-\alpha-1}\,dy \\ &= t^k\frac{\alpha}{\alpha-k}, \end{aligned}$$

where the second last implication is verified by the Dominated Convergence Theorem justified via the Potter's bound

$$\bar{F}(ty)/\bar{F}(t) \le (1+\varepsilon)y^{-\alpha+\varepsilon}, \text{ for arbitrary } 0 < \varepsilon < \alpha, \text{ large } t \text{ and all } y > 1$$

(see Theorem 1.5.6(iii) of Bingham *et al.*, 1987). The conditional variance result is a straightforward implication of the above result.

ii) Note first that $\bar{F}(t) = o\big(t^k\big)$, which is a consequence of the representation for Von Mises functions (see p.40, Resnick, 1987). Thus, integration by parts and a change of variables, $x = t + \xi a(t)$, lead to

$$\begin{aligned}
E(X^k|X>t) &= t^k + k\int_t^\infty x^{k-1}\frac{\bar F(x)}{\bar F(t)}\,dx \\
&= t^k + kt^{k-1}a(t)\int_0^\infty\left(1+\xi\frac{a(t)}{t}\right)^{k-1}\frac{\bar F(t+\xi a(t))}{\bar F(t)}\,d\xi \\
&= t^k + kt^{k-1}a(t)\sum_{l=0}^{k-1}\left(\frac{a(t)}{t}\right)^{k-l-1}\int_0^\infty \xi^{k-l-1}\frac{\bar F(t+\xi a(t))}{\bar F(t)}\,d\xi \\
&= t^k + kt^{k-1}a(t)\sum_{l=0}^{k-1}\left(\frac{a(t)}{t}\right)^{k-l-1}\big((k-l-1)!+o(1)\big) \\
&= t^k + kt^{k-1}a(t) + o\big(t^{k-1}a(t)\big),
\end{aligned}$$

where the second last implication is due to the uniform convergence in equation (1.2), Lemma 3.4 from Tang and Yang (2012) and the Dominated Convergence Theorem, while the very last implication is a consequence of $a(t) = o(t)$.

Now, the rate of convergence for the conditional variance cannot be obtained from the above results, but similar derivations to the one displayed in the last equation show that

$$\begin{aligned}
Var(X|X>t) &= E(X^2|X>t) - \big(E(X|X>t)\big)^2 \\
&= t^2 + 2ta(t)\int_0^\infty\frac{\bar F(t+\xi a(t))}{\bar F(t)}\,d\xi + 2a^2(t)\int_0^\infty \xi\frac{\bar F(t+\xi a(t))}{\bar F(t)}\,d\xi \\
&\quad - \left(t + a(t)\int_0^\infty\frac{\bar F(t+\xi a(t))}{\bar F(t)}\,d\xi\right)^2 \\
&= 2a^2(t)\int_0^\infty \xi\frac{\bar F(t+\xi a(t))}{\bar F(t)}\,d\xi - a^2(t)\left(\int_0^\infty\frac{\bar F(t+\xi a(t))}{\bar F(t)}\,d\xi\right)^2 \\
&= 2a^2(t)\big(1+o(1)\big) - a^2(t)\big(1+o(1)\big)^2 \\
&= a^2(t)\big(1+o(1)\big),
\end{aligned}$$

where the second last implication is due to the uniform convergence in equation (1.2), Lemma 3.4 from Tang and Yang (2012) and the Dominated Convergence Theorem. The proof is now complete. $\square$

Further, some crucial convergence results are developed in Proposition 5.1.

Proposition 5.1. *Assume that X is a positive rv and N is non-degenerate at* 0.

(a) *If $X \in RV_{-\alpha}$ with $\alpha > 0$ and $EN < \infty$, then for all positive (ξ_1, ξ_2) we have*

$$\lim_{t\to\infty}\frac{\Pr\left(S > t\xi_1, X_N^{(1)} > t\xi_2\right)}{\bar F(t)} = EN\left(\max\left(\xi_1,\xi_2\right)\right)^{-\alpha};$$

(b) *If $X \in RV_{-\alpha}$ with $\alpha > 0$, $EN^2 < \infty$ and* $\Pr(N \geq 2) > 0$, *then for all positive (ξ_1, ξ_2) is true*

$$\lim_{t\to\infty} \frac{\Pr\left(X_N^{(1)} > t\xi_1, X_N^{(2)} > t\xi_2\right)}{\bar{F}^2(t)} = E\binom{N}{2} \begin{cases} 2\xi_1^{-\alpha}\xi_2^{-\alpha} - \xi_1^{-2\alpha}, & \textit{if} \quad 0 < \xi_2 < \xi_1, \\ \xi_2^{-2\alpha}, & \textit{if} \quad 0 < \xi_1 \leq \xi_2; \end{cases}$$

c) *If $F \in \mathcal{S} \bigcap \Gamma(a)$ and $E(N) < \infty$, then for all positive (ξ_1, ξ_2) we have*

$$\lim_{t\to\infty} \frac{\Pr\left(S > t + \xi_1 a(t), X_N^{(1)} > t + \xi_2 a(t)\right)}{\bar{F}(t)} = \exp\left\{ -max(\xi_1, \xi_2)\right\}.$$

d) *If $F \in \Gamma(a)$, $EN^2 < \infty$ and* $\Pr(N \geq 2) > 0$, *then for all positive (ξ_1, ξ_2) is true*

$$\frac{\Pr\left(X_N^{(1)} > t + \xi_1 a(t), X_N^{(2)} > t + \xi_2 a(t)\right)}{\bar{F}^2(t)} \sim E\binom{N}{2} \begin{cases} 2e^{-\xi_1-\xi_2} - e^{-2\xi_1}, & \textit{if} \quad 0 \leq \xi_2 < \xi_1, \\ e^{-2\xi_2}, & \textit{if} \quad 0 \leq \xi_1 \leq \xi_2. \end{cases}$$

Proof. a) The case in which $0 \leq \xi_1 \leq \xi_2$ is trivial due to $S \geq X_N^{(1)}$ and Lemma 2.1. Now, consider the other case in which $0 \leq \xi_2 < \xi_1$. Clearly, for every $C > 0$

$$\frac{\Pr\left((X_1/t, \dots, X_n/t) \in \cdot\right)}{\bar{F}(t)} \xrightarrow{v} \mu(\cdot) \tag{5.1}$$

holds on $[-C, \infty]^n \setminus \left\{[-C, 0)^n \bigcup \{\underline{0}\}\right\}$, where the measure μ puts its entire mass on its axes due to the fact that X's are positive iid rv's. That is, for all $1 \leq j \leq n$

$$\mu\big((x_1, \infty] \times \cdots \times (x_n, \infty]\big) = x_j^{-\alpha}, \text{ if } -C \leq x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n \leq 0 \text{ and } 0 < x_j,$$

and null if there are $1 \leq i \neq j \leq n$ such that $x_i > 0$ and $x_j > 0$. Denote

$$D_n := \left\{ \sum_{i=1}^n x_i > \xi_1, \max_{1\leq i\leq n} x_i > \xi_2 \right\}.$$

Obviously, $\mu(\partial D_n) = 0$, and therefore relation (5.1) implies that

$$\lim_{t\to\infty} \frac{\Pr\left(S_n > t\xi_1, X_n^{(1)} > t\xi_2\right)}{\bar{F}(t)} = \mu(D_n) = n\xi_1^{-\alpha}.$$

Thus, the latter and the fact that

$$\frac{\Pr\left(S > t\xi_1, X_N^{(1)} > t\xi_2\right)}{\bar{F}(t)} \leq \sum_{n=1}^{\infty} \Pr(N = n) \frac{\Pr\left(X_n^{(1)} > t\right)}{\bar{F}(t)} \leq EN < \infty.$$

allow us to apply the Dominated Convergence Theorem, which concludes part (a).

b) Whenever $0 \leq \xi_1 \leq \xi_2$, our claim is trivial due to (2.2), and therefore we further assume that $0 < \xi_2 < \xi_1$. Now, for every positive t, there exists a constant $C > 0$ that does not depend upon t such that

$$\begin{aligned}\frac{\Pr\left(X_N^{(1)} > t\xi_1, X_N^{(2)} > t\xi_2\right)}{\bar{F}^2(t)} &= \sum_{n=2}^{\infty} \Pr(N=n)\, \frac{\Pr\left(X_n^{(1)} > t\xi_1, X_n^{(2)} > t\xi_2\right)}{\bar{F}^2(t)} \\ &\leq C \sum_{n=2}^{\infty} \Pr(N=n)\, n^2\, \frac{\bar{F}^2(t\xi_2)}{\bar{F}^2(t)},\end{aligned}$$

which is finite due to the Potter's bound (see Theorem 1.5.6(iii) of Bingham *et al.*, 1987). Thus, we may apply the Dominated Convergence Theorem by noting that

$$\begin{aligned}\frac{\Pr(X_n^{(1)} > t\xi_1, X_n^{(2)} > t\xi_2)}{\bar{F}^2(t)} &\sim \binom{n}{2} \frac{\Pr(X_1 > t\xi_1, X_2 > t\xi_2) + \Pr(t\xi_2 < X_1 \leq t\xi_1, X_2 > t\xi_1)}{\bar{F}^2(t)} \\ &\sim \binom{n}{2} \left(2\xi_1^{-\alpha}\xi_2^{-\alpha} - \xi_1^{-2\alpha}\right),\end{aligned}$$

which is a consequence of (1.1).

c) As before, we only need to justify the case in which $0 < \xi_2 < \xi_1$, since the other scenario can be simply recovered by using equation (1.2). Now, for any positive integer n we have

$$\frac{\Pr\left(X_n^{(1)} > t + \xi_1 a(t)\right)}{\bar{F}(t)} \leq \frac{\Pr\left(S_n > t + \xi_1 a(t), X_n^{(1)} > t + \xi_2 a(t)\right)}{\bar{F}(t)} \leq \frac{\Pr\left(S_n > t + \xi_1 a(t)\right)}{\bar{F}(t)}.$$

Since both bounds are equal to $ne^{-\xi_1}\left(1 + o(1)\right)$ due to (2.1) and the fact that $S_n \in \Gamma(a)$, we may conclude that

$$\frac{\Pr\left(S_n > t + \xi_1 a(t), X_n^{(1)} > t + \xi_2 a(t)\right)}{\bar{F}(t)} \sim ne^{-\xi_1}.$$

The latter and the fact that

$$\frac{\Pr\left(S > t + \xi_1 a(t), X_N^{(1)} > t + \xi_2 a(t)\right)}{\bar{F}(t)} \leq \sum_{n=1}^{\infty} \Pr(N=n) \frac{\Pr\left(X_n^{(1)} > t\right)}{\bar{F}(t)} \leq EN < \infty,$$

is true for all t allow us to apply the Dominated Convergence Theorem in order to replicate our claim.

d) A combination of some steps used in b) and c) lead to our statement. □

The main results of this section are now ready and are stated as Theorems 5.1 and 5.2. It is interesting to note that Theorem 5.1b) provides a novel estimator for the tail index α, which is the topical problem in statistical extremes (for further details, see Embrechts *et al.*, 1997). Moreover, taking $\alpha \to \infty$ in Theorem 5.1b), one may recover the final result from 5.2b) Theorem 5.1b), which is not surprising, but nevertheless, both derivations are needed.

Theorem 5.1. *Assume that $X \in RV_{-\alpha}$ with $\alpha > 0$ is a positive rv and N is non-degenerate at* 0.

a) *If $E(N) < \infty$, then*
   i) $E\big(S^k | X_N^{(1)} > t\big) \sim t^k \frac{\alpha}{\alpha-k}$ *for all $\alpha > k$;*
   ii) $E\big(SX_N^{(1)} > t | X_N^{(1)} > t\big) \sim t^2 \frac{\alpha}{\alpha-2}$ *whenever $\alpha > 2$.*
   Consequently, $\lim_{t\to\infty} \rho_L^{+1}(t) = 1$ *whenever $\alpha > 2$.*

b) *If $EN^2 < \infty$ and $\Pr(N \geq 2) > 0$, then*
   i) $E\Big(\big(X_N^{(1)}\big)^k | X_N^{(2)} > t\Big) \sim t^k \frac{2\alpha^2}{(2\alpha-k)(\alpha-k)}$ *for all $\alpha > k$;*
   ii) $E\big(X_N^{(1)} X_N^{(2)} | X_N^{(2)} > t\big) \sim t^2 \frac{\alpha^2}{(\alpha-1)^2}$ *whenever $\alpha > 1$.*
   Consequently, $\lim\limits_{t\to\infty} \rho_L^{12}(t) = \sqrt{\dfrac{\alpha(\alpha-2)}{(5\alpha-1)(\alpha-1)}}$ *whenever $\alpha > 2$.*

Proof. The results from parts a) and b) can be derived in the same manner, and thus, we will focus only on part a).

a)i) It is not difficult to find the claim from this part by applying the result from Proposition 5.1 a), Potter's bound (see Theorem 1.5.6(iii) of Bingham *et al.*, 1987) and the Dominated Convergence Theorem.

a)ii) Note first that $\limsup\limits_{x\to\infty} x \Pr\big(SX_N^{(1)} > x | X_N^{(1)} > t\big) = 0$ for all $t > 0$ since $\alpha > 2$, which implies that $SX_N^{(1)}$ has a finite mean. Thus, integration by parts and an obvious change of variables lead to

$$\begin{aligned} E\big(SX_N^{(1)} | X_N^{(1)} > t\big) &= \int_{t^2}^{\infty} x \Pr\big(SX_N^{(1)} \in dx | X_N^{(1)} > t\big) \\ &= t^2 \Pr\big(SX_N^{(1)} > t^2 | X_N^{(1)} > t\big) + \int_{t^2}^{\infty} \Pr\big(SX_N^{(1)} > x | X_N^{(1)} > t\big)\, dx \\ &= t^2 + t^2 \int_1^{\infty} \Pr\big(SX_N^{(1)} > t^2 \xi | X_N^{(1)} > t\big)\, d\xi. \end{aligned} \tag{5.2}$$

Note that Proposition 5.1a) implies the following weak convergence

$$\Pr\Big(\big(S/t, X_N^{(1)}/t\big) \in \cdot | X_N^{(1)} > t\Big) \xrightarrow{w} \mu_+(\cdot)$$

holds on $[1,\infty]^2$, where the probability measure $\mu_+\big((x,\infty] \times (y,\infty]\big) = \big(\max(x,y)\big)^{-\alpha}$. For every fixed $\xi > 0$, let $D_\xi := \{xy > \xi\}$. Since $\mu_+(\partial D_\xi) = 0$, the latter shows that

$$\lim_{t\to\infty} \Pr\big(SX_N^{(1)} > t^2 \xi | X_N^{(1)} > t\big) = \mu_+\big(D_\xi\big) = \xi^{-\alpha/2}.$$

Thus, the Dominated Convergence Theorem can be used in equation (5.2) as a result of the Potter's bound (see Theorem 1.5.6(iii) of Bingham *et al.*, 1987) and the fact that

$$\Pr\left(SX_N^{(1)} > t^2\xi | X_N^{(1)} > t\right) \le \frac{\Pr(S > t\sqrt{\xi})}{\Pr(S > t)},$$

which explain our claim. □

Theorem 5.2. *Assume that $X \in \Gamma(a)$ is a positive rv and N is non-degenerate at 0. If*

a) $E(1+\epsilon)^N < \infty$ *for some* $\epsilon > 0$ *and* $X \in \mathcal{S}$, *then* $E\big(S_T | X_T^{(1)} > t\big) - t \sim a(t)$ *and*

$$Var\big(S_T | X_T^{(1)} > t\big) = a^2(t)\big(1 + o(1)\big) = cov\big(S_T, X_T^{(1)} | X_T^{(1)} > t\big).$$

Consequently, $\rho_L^{+1}(t) \sim 1$

b) $EN^2 < \infty$ *and* $\Pr(N \ge 2) > 0$, *then* $E\big(X_T^{(1)} | X_T^{(2)} > t\big) = t + a(t)\big(3/2 + o(1)\big)$,

$$Var\big(X_T^{(1)} | X_T^{(2)} > t\big) \sim 5a^2(t)/4 \quad \text{and} \quad cov\big(X_T^{(1)}, X_T^{(2)} | X_T^{(2)} > t\big) \sim a^2(t)/4.$$

Consequently, $\rho_L^{12}(t) \sim 1/\sqrt{5}$.

Proof. a) A direct implication of Lemma 2.1 and equation (2.2) lead to

$$\Pr\left(S > t + \xi a(t) | X_N^{(1)} > t\right) = e^{-\xi}\big(1 + o(1)\big), \quad \text{for all } \xi \ge 0,$$

and together with Lemma 3.4 from Tang and Yang (2012), one may justify the use of the Dominated Convergence Theorem, and we get

$$E\big(S | X_N^{(1)} > t\big) = t + a(t)\int_0^\infty \Pr\left(S > t + \xi a(t) | X_N^{(1)} > t\right)\, d\xi = t + a(t)\big(1 + o(1)\big),$$

where the first equality can be justified by following the same steps shown in the proof of Lemma 5.1 ii). The proof of the conditional variance may be found in a similar manner as the equivalent result from Lemma 5.1 ii), and thus, its proof is omitted.

We now derive the rate of convergence for the conditional covariance. Clearly,

$$\begin{aligned} &cov\big(S, X_N^{(1)} | X_N^{(1)} > t\big) \\ &= E\Big(\big(S - t\big)\big(X_N^{(1)} - t\big) | X_N^{(1)} > t\Big) - E\Big(\big(S - t\big) | X_N^{(1)} > t\Big) E\Big(\big(X_N^{(1)} - t\big) | X_N^{(1)} > t\Big), \end{aligned}$$

which together with the first result of this part, relation (2.2) and Lemma 5.1 ii), one may conclude the desired result, as long as the following is true

$$E\Big(\big(S - t\big)\big(X_N^{(1)} - t\big) | X_N^{(1)} > t\Big) = 2a^2(t)\big(1 + o(1)\big). \tag{5.3}$$

Note that Proposition 5.1c) implies the following weak convergence

$$\Pr\left(\left(\frac{S - t}{a(t)}, \frac{X_N^{(1)} - t}{a(t)}\right) \in \cdot | X_N^{(1)} > t\right) \xrightarrow{w} \mu_+(\cdot)$$

holds on $[0,\infty]^2$, where the probability measure $\mu_+\big((x,\infty]\times(y,\infty]\big)=\exp\big\{-\max(x,y)\big\}$. Let us denote $D_\xi:=\{xy>\xi\}$ for every fixed $\xi\geq 0$. Clearly, $\mu_+(\partial D_\xi)=0$, and therefore the above convergence yields that

$$\Pr\left(\frac{(S-t)(X_N^{(1)}-t)}{a^2(t)}>\xi|X_T^{(1)}>t\right)\sim\mu_+\big(D_\xi\big)=\mu_+\big(x=y>\sqrt{\xi}\big)=\exp\big\{-\sqrt{\xi}\big\}.$$

The latter and Lemma 3.4 from Tang and Yang (2012) help in using the Dominated Convergence Theorem, which in turn implies our claim from (5.3) as follows

$$\begin{aligned}
E\Big(\big(S-t\big)\big(X_N^{(1)}-t\big)|X_N^{(1)}>t\Big) &= a^2(t)\int_0^\infty \Pr\left(\frac{S-t}{a(t)}\frac{X_N^{(1)}-t}{a(t)}>\xi|X_N^{(1)}>t\right)d\xi\\
&\sim a^2(t)\int_0^\infty \exp\Big\{-\sqrt{\xi}\Big\}\ d\xi\\
&= 2a^2(t),
\end{aligned}$$

which justifies in full the last result of part (a).

b) Proposition 5.1d) suggests that the following weak convergence

$$\Pr\left(\left(\frac{X_N^{(1)}-t}{a(t)},\frac{X_N^{(2)}-t}{a(t)}\right)\in\cdot|X_N^{(2)}>t\right)\xrightarrow{w}\mu_1(\cdot) \tag{5.4}$$

holds on $[0,\infty]^2$, where the limiting probability measure is given by

$$\mu_1\big((\xi_1,\infty]\times(\xi_2,\infty]\big)=\begin{cases}2\exp\{-\xi_1-\xi_2\}-\exp\{-2\xi_1\}, & \text{if}\quad 0\leq\xi_2<\xi_1,\\ \exp\{-2\xi_2\}, & \text{if}\quad 0\leq\xi_1\leq\xi_2.\end{cases}$$

Therefore, $\Pr\big(X_N^{(1)}>t+\xi a(t)|X_N^{(2)}>t\big)\sim 2e^{-\xi}-e^{-2\xi}$ for all non-negative ξ, and due Lemma 3.4 from Tang and Yang (2012), one may justify the use of the Dominated Convergence Theorem as follows:

$$\begin{aligned}
E\big(X_N^{(1)}|X_N^{(2)}>t\big) &= t+a(t)\int_0^\infty \Pr\left(\frac{X_N^{(1)}-t}{a(t)}>\xi|X_N^{(2)}>t\right)d\xi\\
&= t+a(t)\left(\int_0^\infty\big(2e^{-\xi}-e^{-2\xi}\big)\ d\xi+o(1)\right)\\
&= t+a(t)\big(3/2+o(1)\big).
\end{aligned}$$

We now justify the conditional variance result. As before, one may show that

$$
\begin{aligned}
& Var\big(X_N^{(1)}|X_N^{(2)}>t\big)\\
= \;& 2a^2(t)\int_0^\infty \xi \Pr\left(\frac{X_N^{(1)}-t}{a(t)}>\xi|X_N^{(2)}>t\right)d\xi\\
& -a^2(t)\left(\int_0^\infty \Pr\left(\frac{X_N^{(1)}-t}{a(t)}>\xi|X_N^{(2)}>t\right)d\xi\right)^2\\
= \;& 2a^2(t)\left(\int_0^\infty \xi\big(2e^{-\xi}-e^{-2\xi}\big)\,d\xi+o(1)\right)-a^2(t)\left(\int_0^\infty \big(2e^{-\xi}-e^{-2\xi}\big)\,d\xi+o(1)\right)^2\\
= \;& 2a^2(t)\big(7/4+o(1)\big)-a^2(t)\big(3/2+o(1)\big)^2\\
= \;& a^2(t)\big(5/4+o(1)\big),
\end{aligned}
$$

where once again Lemma 3.4 from Tang and Yang (2012) and the Dominated Convergence Theorem are the key ingredients for coming up with the needed result.

Finally, it only remains to show the conditional covariance result. Clearly,

$$
\begin{aligned}
cov\big(X_N^{(1)},X_N^{(2)}|X_N^{(2)}>t\big) &= a^2(t)cov\left(\frac{X_N^{(1)}-t}{a(t)},\frac{X_N^{(2)}-t}{a(t)}|X_N^{(2)}>t\right)\\
&= a^2(t)E\left(\frac{\big(X_N^{(1)}-t\big)\big(X_N^{(2)}-t\big)}{a^2(t)}|X_N^{(2)}>t\right)\\
&\quad -a^2(t)E\left(\frac{X_N^{(1)}-t}{a(t)}|X_N^{(2)}>t\right)E\left(\frac{X_N^{(2)}-t}{a(t)}|X_N^{(2)}>t\right).
\end{aligned}
\tag{5.5}
$$

Note that $X_N^{(2)}\in\Gamma(2a)$ due to equation (2.2), and therefore Lemma 5.1 ii) yields that $E\left(\frac{X_N^{(2)}-t}{2a(t)}|X_N^{(2)}>t\right)=1+o(1)$. Now, for every fixed ξ, denote $E_\xi:=\{xy>\xi\}$. The absolutely continuous property of μ_1 implies that $\mu_1(\partial E_\xi)=0$, and in turn, the weak convergence from (5.4) justifies

$$
\begin{aligned}
\Pr\left(\frac{\big(X_N^{(1)}-t\big)\big(X_N^{(2)}-t\big)}{a^2(t)}>\xi|X_N^{(2)}>t\right) &\sim \mu_1\big(E_\xi\big)\\
&= 2\int_0^{\sqrt{\xi}} e^{x+\xi/x}\,dx+2\int_{\sqrt{\xi}}^\infty e^{-2x}\,dx\\
&= 2\int_0^{2\sqrt{\xi}} e^{-y}\left(1-\frac{y}{\sqrt{y^2-4\xi}}\right)\,dy+e^{-2\sqrt{\xi}}\\
&= 2\sqrt{\xi}K\big(1,2\sqrt{\xi}\big),
\end{aligned}
$$

where a change of variables, $y = x + \xi/x$ or equivalently $x = y - \sqrt{y^2 - 4\xi}$, is made in the third implication. Also, $K(\cdot)$ represents the modified Bessel function of the second kind. As before, Lemma 3.4 from Tang and Yang (2012) explains the use of the Dominated Convergence Theorem

$$\begin{aligned} E\left(\frac{\left(X_N^{(1)}-t\right)\left(X_N^{(2)}-t\right)}{a^2(t)}|X_N^{(2)}>t\right) &= \int_0^\infty \Pr\left(\frac{\left(X_N^{(1)}-t\right)\left(X_N^{(2)}-t\right)}{a^2(t)}>\xi|X_N^{(2)}>t\right)d\xi \\ &= 2\int_0^\infty \sqrt{\xi}K\left(1,2\sqrt{\xi}\right)\,d\xi + o(1) \\ &= 1+o(1). \end{aligned}$$

The latter and relation (5.5) imply $cov\left(X_N^{(1)}, X_N^{(2)}|X_N^{(2)} > t\right) = a^2(t)(1/4 + o(1))$, which completes the proof. $\square$

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