Verifying Inevitability of Phase-locking in a Charge Pump Phase Lock Loop using Sum of Squares Programming

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ABSTRACT

Phase-locking in a charge pump (CP) phase lock loop (PLL) is said to be inevitable if all possible states of the CP PLL eventually converge to the equilibrium where the input and output phases are in lock. We verify this property for a CP PLL using a mixed deductive and bounded verification methodology. This involves a positivity check of polynomial inequalities (which is an NP-Hard problem) so we use the sound but incomplete Sum of Squares (SOS) relaxation algorithm to provide a numerical solution.

1. INTRODUCTION

Formal methods are in their infancy in Analog and Mixed Signal (AMS) circuits verification. Start up problems have been very common in PLL circuits, i.e., for certain initial states of voltages, the circuits do not converge to the desired behaviour. In addition, while in phase-locking state and disturbed by an external input, it is important to know whether the PLL circuit retains its locking state. Verifying both these properties are closely related to the verification of inevitability property.

Hybrid systems are well known modelling paradigm for a CP PLL [16], [2], [6]. Techniques for the verification of hybrid systems can be classified as, reach-set methods, abstraction based methods, and certificate based methods. Reachability has been used to prove the above stated property. Hundreds of discrete transitions are required by the hybrid model of the CP PLL before it reaches the locking state. This results in large number of continuous set computations followed by the guard conditions describing the switching laws, making the reachability computation prohibitively expensive. In this paper, we use a mixture of certificate based deductive and bounded (reach-set) verification methodology. We verify the inevitability of the phase-locking in a CP PLL by adopting a two-pronged verification approach. Due to complexity of the property, we essentially divide the inevitability property in to the conjunction of two sub-properties. These two properties determine the truth value of the inevitability property in two disjoint subsets of the state space. The first property specifies, that in a compact set, all system trajectories eventually converge to the equilibrium locking state. The second property is specified, such that the set where the first property holds, is reachable from the second subset of the state space. The first property is verified by computing an attractive region (AR) utilizing the deductive Lyapunov stability theory for hybrid systems [5]. We construct multiple Lyapunov certificates for different modes of the CP PLL hybrid system. The maximized level curves of these Lyapunov certificates characterize the level sets whose union is the AR. We verify the second property utilizing bounded advection of level sets ([15]), and deductive Escape certificate method, showing the reachability of this AR from the states outside it. The deductive and bounded verification approaches involve checking positivity of polynomial inequalities, which is an NP-hard problem. We use the sound but incomplete SOS relaxation for the positivity verification of polynomial inequalities.

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1.1 Related Work

A survey of the formal verification of AMS circuits can be found in [17]. In [6], the author verified the 'time to locking' property for a digitally extensive PLL. [16] verified 'global convergence' property for an all digital PLL. They divided the state space into linear and non-linear regions, and applied linear Lyapunov stability theory (Using Quadratic Lyapunov Certificate) for linear and reachability analysis for non-linear regions respectively. Time-outs of the reachability tool has been reported by the author due to the large number of discrete transitions needed by the PLL hybrid automata. To avoid discrete jumps, [2] presented a continuization technique and verified the 'time to locking' property of a CP PLL.

Hybrid models have different flavours that can be found in [8],[1]. Here we consider the framework outlined in [4]. In the last decade, SOS programming has been the major tool used in the algorithmic construction of Lyapunov certificates for continuous, as well as hybrid systems [9], [11]. Deductive verification of continuous and hybrid systems have been demonstrated in [14], [13]. Recently an advection algorithm for polynomial level sets based on SOS programming has been presented in [15]. This advection algorithm has been used for reachable sets estimation for continuous dynamical systems. We extend this approach to bounded verification of the hybrid systems.

This paper is organized as follows: In Sec.II, we introduce the preliminaries of this paper. Sec.III illustrates verification of the inevitability of phase-locking in CP PLL. Experimental results are shown in Sec.IV. Sec.V concludes the paper.

2. PRELIMINARIES

2.1 Hybrid Systems Model

We use the hybrid system formalism described in [4]. We consider a hybrid system described by the tuple $(\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$. Here, $\{\mathcal{C} = \bigcup_{i \in I_C} C_i\} \subset \mathbb{R}^n$, and $\{\mathcal{D} = \bigcup_{i \in I_D} D_i\} \subset \mathbb{R}^n$ are the flow set and jump set for $i \in \mathbb{N}$, respectively. I_C and I_D are finite disjoint index sets and it is possible that $C_i \cap D_i \neq \emptyset$. The flow and jump maps are, respectively, $\mathcal{F} = \bigcup_{i \in I_C} F_i$, and $\mathcal{G} = \bigcup_{i \in I_C} G_i$, where each $F_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m+n}$, and, $G_i = \mathbb{R}^n \to \mathbb{R}^n$. These two mappings characterize the continuous and discrete evolution of the system, whereas C_i and D_i describe subsets of $\mathbb{R}^n$ where such evolution may occur. We represent a hybrid system $\mathcal{H}$ as

$$\mathcal{H} = \begin{cases} \dot{x} = F_i(x, u) \in \mathcal{F} & x \in \mathcal{C},\ u \in \mathcal{U} \\ x^+ = G_i(x) \in \mathcal{G} & x \in \mathcal{D} \end{cases} \tag{1}$$

Here $u \in \mathcal{U} \subset \mathbb{R}^m$ is a vector of uncertain parameters. The state of the hybrid system consists of alternate flows in jumps through $\mathcal{C}$ and $\mathcal{D}$ according to F_i and G_i, respectively. This hybrid phenomena can be described by a notion of time called hybrid time.

Definition 1 (Hybrid Time Domain).
A set $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if

$$\mathcal{T} = \bigcup_{j=0}^{j-1} ([t_j, t_{j+1}], j)$$

where $0 = t_0 \leq t_1 \leq t_2 \leq, ...,$ with the last interval possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$.

Definition 2 (Hybrid Arc).
A mapping $x : \mathcal{T} \to \mathbb{R}^n$ is a hybrid arc if $\mathcal{T}$ is a hybrid time domain and for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is locally absolutely continuous on the interval $I_j = \{t : (t, j) \in \mathcal{T}\}$.

Assumption 1.
The flow maps $F_i(x, u)$ and the jump maps $G_i(x)$ are polynomials.

Definition 3 (Equilibrium point).
A point $x(t, j) \in \mathcal{C} \cup \mathcal{D}$ is called an equilibrium, if $\exists t,\ \exists j,$ $F_j(x(t, j), u) = 0$.

Definition 4 (Inevitability of Equilibrium).
The equilibrium point x_e is said to be inevitable, if $\forall x(0, j) \in \mathcal{C} \cup \mathcal{D}$ and bounded t, $x(t, j) \to x_e$.

2.2 CP PLL Model

A CP PLL circuit consists of a reference signal, a phase frequency detector (PFD), a charge pump (CP), a loop filter (LF), and a voltage controlled oscillator (VCO). In this paper we consider a single path higher order (Third and fourth) CP PLL shown in Fig. 1[1]. We use a behavioural model of the PLL, where we consider a linear model for VCO, a linear model for the third order LF, and a non-linear model for the PFD. We denote by ϕ_{ref}, and ϕ_{VCO}, the phases of the reference and VCO output feedback signals respectively. We model the CP PLL as a hybrid system such that the non-linearities of the PFD is modelled as a piecewise continuous signal. Ignoring the cycle slip phenomena, the PFD output in the form of the charge pump current I_p, is given by the following piecewise linear inclusion:

[1]Third order shown here,fourth order has an additional resistor-capacitor in its LF

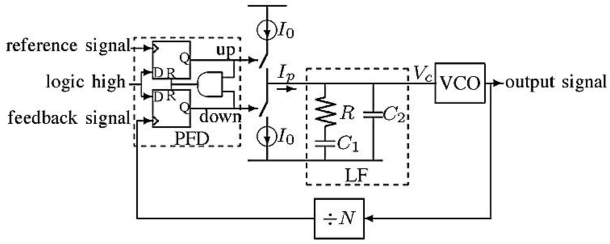

Figure 1: Charge Pump (CP) Phase Lock Loop (PLL)

$$I_p = \begin{cases} \in [I_p^U\, I_P^U] & \text{UP=1, Down=0, } 0 \leq \phi_{VCO} < 2\pi \leq \phi_{ref} \\ \in [I_p^D\, I_P^D] & \text{UP=0, Down=1, } 0 \leq \phi_{ref} < 2\pi \leq \phi_{VCO} \\ \in [0^R\ 0^R] & \text{UP=0, Down=0, } 0 \leq \phi_{VCO}, \phi_{ref} < 2\pi \end{cases} \tag{2}$$

We denote the three modes as mode1 (UP=0, Down=0), mode2 (UP=1, Down=0) and mode3 (UP=0, Down=1). The transition from one mode to another is based on the reference and feedback signals hitting the 2π threshold. Due to the cyclic behaviour of the PLL and to keep the analysis modulo 2π, we need to ensure the phases remain in the range $0 \leq \phi_{VCO}, \phi_{VCO} < 2\pi$ (Similarly ϕ_{ref}) after resetting the PFD. This is achieved by resetting the two phases such that ($\phi_{ref} := 0$, $\phi_{VCO} := \phi_{VCO} - 2\pi$), and ($\phi_{ref} := \phi_{ref} - 2\pi$, $\phi_{VCO} := 0$), while taking transitions from mode1 to mode2 and mode1 to mode3, respectively. Identity resets are used for transitions from mode2 to mode1 and mode3 to mode1.

Our model consists of the state variables, ϕ_{VCO}, ϕ_{ref}, voltage v_1 across the capacitor C_1, and the voltage v_2 across the capacitor C_2 (Fourth order has an additional voltage variable across the third capacitor). Let f_{VCO}, and f_{ref}, represent the frequencies of the VCO output and the reference signal respectively. If K_p is the gain of the LF, then $f_{VCO} = K_p v_2 / 2\pi + f_O$, where f_O is the free running frequency of the VCO. Therefore, $\dot{\phi}_{VCO} = 2\pi f_{VCO}/N$, $\dot{\phi}_{ref} = 2\pi f_{ref}$. By Kirchhoff's current law and using the three modes of the PFD, we get the following hybrid system of the third order CP PLL

$$\mathcal{H} = \begin{cases} \dot{\begin{pmatrix} v_1 \\ v_2 \\ \phi_{ref} \\ \phi_{VCO} \end{pmatrix}} = A \begin{pmatrix} v_1 \\ v_2 \\ \phi_{ref} \\ \phi_{VCO} \end{pmatrix} + BI_p + c & x \in \mathcal{C}, \\ x^+ = G_i(x) & x \in \mathcal{D} \end{cases} \tag{3}$$

where, $A = \begin{pmatrix} -1/RC_1 & 1/RC_1 & 0 & 0 \\ 1/RC_2 & -1/RC_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & K_p/N & 0 & 0 \end{pmatrix}$,

$B = \begin{pmatrix} 0 \\ I_p/C_2 \\ 0 \\ 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 \\ 0 \\ 2\pi f_{ref} \\ 2\pi f_o/N \end{pmatrix}$, I_p as given by Eq. 2.

From the three modes and their invariants, we can easily find the sets $\mathcal{C}$, $\mathcal{D}$, and the jump maps $G_i(x)$. Here, we use the difference $\phi_{ref} - \phi_{VCO}$ as a state variable instead of ϕ_{ref}, and ϕ_{VCO}.

Remark 1.
This change of state variables transforms all jump maps G_i in to identity maps ($G_i(x) = x$), as the same constant 2π is subtracted from ϕ_{VCO} and ϕ_{ref}, leaving their difference $\phi_{ref} - \phi_{VCO}$ before and after the jumps unchanged.

2.3 Attractive Invariants in Hybrid Systems using Lyapunov Certificates

Contrary to safety properties where existence of an invariant set is sufficient for proving/dis-proving the property, for inevitability property, we use the concept of "attractive invariants". Attractive invariants are compact semi-algebraic sets where if the CP PLL hybrid system starts, will always remain there and will eventually converge to the equilibrium state. Stability and attractivity concepts for an equilibrium state of the continuous dynamical systems are discussed in [5], and have been extended to hybrid systems in [4]. The equilibrium point, $x_e = 0$, is called asymptotically stable (AS) if it is both stable and attractive. There are several versions of the stability theorems for hybrid systems based on the global Lyapunov function, and multiple Lyapunov certificates [12]. We use the following theorem of AS and define the attractive invariant,

Theorem 1.
Let, $\mathcal{I}_0 \subseteq I_C$ be the set of indices that contain the equilibrium. For a hybrid system $\mathcal{H}$ having an equilibrium point $x_e = 0$, if there exist Lyapunov certificates V_i such that,

1. $V_i(x) > 0,\ \forall i \in I_C,\ \forall x \in \mathcal{C} \setminus x_e$,
2. $V_i(0) = 0,\ \forall i \in \mathcal{I}_0$,
3. $\frac{\partial V_i}{\partial x}(x) F_i(x, u) < 0,\ \forall i \in I_C,\ \forall x \in \mathcal{C} \setminus x_e,\ F_i \in \mathcal{F}$, $u \in \mathcal{U}$,
4. $V_j(G_i(x)) - V_{j'}(x) < 0,\ \forall j\ \forall j' \in I_C,\ j \neq j'$, $\forall i \in I_D,\ \forall x \in \mathcal{D} \setminus x_e,\ G_i \in \mathcal{G}$,

then x_e is asymptotically stable. Furthermore, the set $\mathcal{AI} = \bigcup_i (V_i \leq c_{max}) \subset \mathcal{C} \cup \mathcal{D}$ is an "attractive invariant" set.

Proof. Similarly to [12]. □

We also use another important characteristic of the trajectories in a semi algebraic set and term it as the Escape property. This property ensures that trajectories in a compact set can not converge to an invariant set (Equilibrium,Limit Cycle) and will eventually leave that set.

Proposition 1.
For a compact set $\mathcal{X} \subset C_i$, if there is a differentiable Escape certificate, $E : \mathbb{R}^n \to \mathbb{R}$, and $\epsilon > 0$, such that

$$\frac{\partial E}{\partial x}(x) F_i(x, u) \leq -\varepsilon,\ \forall x \in \mathcal{X},\ u \in \mathcal{U} \tag{4}$$

then $\forall x(t, i) \in \mathcal{X},\ x(t + T, i) \notin \mathcal{X},\ T > t$.

Proof. Assume that there exists $x_0 \in \mathcal{X}$ such that $x(t, i)$ starting at x_0 remain in $\mathcal{X}$ as $t \to \infty$. From equation. 4, $E(x) = \int_0^\infty \frac{\partial E}{\partial x}(x) F_i(x) \leq -\varepsilon$. As $t \to \infty$, $E(x) \to -\infty$. This contradicts the assumption as $E(x)$ should be bounded if $x(t, j)$ has to be in the bounded set $\mathcal{X}$. Therefore, $x(t, j)$ has to eventually escape $\mathcal{X}$ in finite time. □

2.4 SOS Programming

Our hybrid deductive and bounded verification approach involves checking the positivity of polynomials in semi-algebraic sets. To solve this NP-Hard problem, a sound relaxation method based on SOS programming has been discussed in [10], [12]. A sufficient condition for a multivariate polynomial $p(x)$ to be non-negative everywhere, is that it can be decomposed as a sum of squares of polynomials. A polynomial $p(x)$ is a sum of squares, if there exist polynomials $p_1(x), ..., p_m(x)$ such that $p(x) = \sum_{i=1}^m p_i^2(x)$. We denote the set of polynomials in n variables with real coefficients by $\mathcal{P}_n$. A subset of this set is the set of SOS polynomials in n variables denoted by $\mathcal{S}_n$. For a polynomial $q : \mathbb{R}^n \to \mathbb{R}$, differentiable scalar function, we define the 0-sub-level-set of q as $\mathcal{Z}(q) = \{x \in \mathbb{R}^n \mid q(x) \leq 0\}$. We present an important lemma to be used for polynomial level sets operations such as intersection, union, and set inclusion [15].

Lemma 1.
For polynomials $p1, p2 \in \mathcal{P}_n$, if there exist SOS polynomials $s0,\ s1 \in \mathcal{S}_n$ such that

$$s0 - s1p1 + p2 = 0\ \forall x \in \mathbb{R}^n \tag{5}$$

Then $\mathcal{Z}(p1) \subset \mathcal{Z}(p2)$

Proof. See for example [15] and the references there in. □

2.5 Bounded Advection of Level Sets

Let we define a flow map $\psi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ for the hybrid system $\mathcal{H}$. A time t advection operator A_t is a map $A_t : Q(\mathbb{R}^n, \mathbb{R}) \to Q(\mathbb{R}^n, \mathbb{R})$, such that $U = A_t V$ for $U, V \in Q(\mathbb{R}^n, \mathbb{R})$, and $U(x) = V(\psi_{-t}(x))$ for all $x \in \mathcal{C} \cup \mathcal{D}$. This advection operator has an important property of linearity. For polynomial functions $U1, U2 \in Q(\mathbb{R}^n, \mathbb{R})$, if $U2 = A_t U1$, then $\mathcal{Z}(U2) = A_t(\mathcal{Z}(U1))$. For a detailed discussion on the advection operator see [15].

3. VERIFICATION METHODOLOGY

To verify inevitability of the CP PLL equilibrium, we introduce two compact sets $\mathcal{S}1$, and $\mathcal{S}2$, such that $\mathcal{S}1 \cap \mathcal{S}2 = \emptyset$, and $\mathcal{S}1 \cup \mathcal{S}2 = \mathcal{C} \cup \mathcal{D}$. We define two properties whose verification implies verification of the inevitability of the equilibrium.

Property 1.
$\forall x(0, j) \in \mathcal{S}1,\ x(t, j) \to x_e$ for $t \to \infty$.

Property 2.
$\forall x(0, j) \in \mathcal{S}2 = (\mathcal{C} \cup \mathcal{D}) \setminus \mathcal{S}1,\ x(t, j) \in \mathcal{S}1$ for $t \to b \in \mathbb{R}_{>0}$.

If we denote the inevitability property by φ, Property.1 by $\varphi1$ and Property.2 by $\varphi2$, then $\varphi = \varphi1 \wedge \varphi2$. A hybrid arc x satisfies φ iff it satisfies $\varphi1$ in $\mathcal{S}1$ and $\varphi2$ in $\mathcal{S}2$ i.e., $\forall x \in \mathcal{C} \cup \mathcal{D},\ x \models \varphi \iff (x \models \varphi1\ \forall x \in \mathcal{S}1) \wedge (x \models \varphi2\ \forall x \in \mathcal{S}2)$. We verify property $\varphi1$ by the deductive Lyapunov theory, and use a combination of bounded advection and Escape certificate for the verification of property $\varphi2$.

Theorem 2.
If there are feasible Lyapunov certificates (fulfilling Th. 1), $\{V_1,\ V_2,\ V_3\}$, then, $x \models \varphi 1,\ \forall x(0,j) \in \mathcal{S}1 = \{(V_1 \leq \gamma 1_{max}) \cup (V_2 \leq \gamma 2_{max}) \cup (V_3 \leq \gamma 3_{max})\}$, and $\mathcal{AI} = \mathcal{S}1$ is the attractive invariant set.

Proof. Follows directly from Th. 1, since the level sets defined by the level curves of the Lyapunov certificates represent attractive invariant sets with the negative Lie-derivative along the system trajectories. Therefore, eventually all system trajectories starting in these level sets converge to the equilibrium point. □

Following Th. 2, we encode the verification of $\varphi 1$ as two SOS programs. The truth value of $\varphi 1$ depends on the existence of the attractive invariant set $\mathcal{S}1$. The set $\mathcal{S}1$ is computed from the maximized level sets defined by the three candidate Lyapunov certificates V_1, V_2, V_3. We compute these certificates using a sound mathematical technique called the S-procedure [3]. The first SOS program is given below:

(a) $V_i(x) - \sum_{k=1}^{n_{C_i}} s_1^{(ik)}(x) g_{ik}(x) \in \mathcal{S}_n,\ \forall x \neq 0,$
$i \in \{1,2,3\}, \forall k \in \{1,..,n_{C_i}\},\ s_1^{(ik)} \in \mathcal{S}_n,$

(b) $$\left(-\frac{\partial V_i}{\partial x}(x) F_i(x,u) - \sum_{k=1}^{n_{C_i}} s_2^{(ik)}(x) g_{ik}(x) - \sum_{j=1}^{m} s_3^j(x) a_j(u) \right) \in \mathcal{S}_n,$$
$\forall i \in \{1,2,3\}, \forall k \in \{1,..,n_{C_i}\},\ \forall j \in \{1,..,m\},$
$(s_2^{(ik)},\ s_3^j) \in \mathcal{S}_n,$

(c) $$\left(V_j(x) - V_{j'}(G_i(x)) - s_4^{(i0)}(x) h_{i0}(x) - \sum_{k=1}^{m_{D_i}} s_5^{(ik)}(x) h_{ik}(x) \right) \in \mathcal{S}_n,\ \forall j \forall j' \in \{1,2,3\},$$
$j \neq j',\ \forall i \in \{1,2,3,4\}, \forall k \in \{1,..,n_{D_i}\},$
$s_4^{(i0)} \geq 0,\ s_5^{(ik)} \in \mathcal{S}_n$

Here $V_i(x),\ V_j(x),\ V_{j'}(x),\ s_1^{(ik)},\ s_2^{(ik)},\ s_3^{(j)},\ s_4^{(i0)},\ s_5^{(ik)}$, are polynomials of degree d.

SOS constraints (a) and (b) enforce positive definiteness on the Lyapunov certificates, and negative semi-definiteness on their Lie-derivatives respectively. Furthermore, these constraints have to be satisfied in their respective domains C_i's, where, $C_i = \{x \in \mathbb{R}^n\ : g_{ik} \geq 0,\ \text{for}\ k \in \{1,..,n_{C_i}\},\ i \in \{1,2,3\}\}$. Constraint (b) also ensures parameters u to belong to the set, $\{a(u) \geq 0,\ \text{for}\ j \in \{1,..,m\}\}$. Constraint in (c) ensures that Lyapunov certificates $V_j(x)$ decrease along the discrete jumps in the sets, $D_i = \{x \in \mathbb{R}^n\ : h_{ik} \geq 0,\ h_{i0} = 0,\ \text{for}\ k \in \{1,..,n_{D_i}\},\ i \in \{1,2,3,4\}\}$, through the mappings $G_i(x)$'s. SOS polynomials $s_1^{(ik)},\ s_2^{(ik)},\ s_3^{(j)},\ s_4^{(i0)},\ s_5^{(ik)}$ are used to enforce domain constraints through the S-procedure. A feasible solution of the above SOS program results in Lyapunov certificates V_i. If this SOS program is infeasible, then either the program is repeated for an increased degree d of the polynomials, or we conclude that the truth value of the property $\varphi 1$ can not be established.

The second SOS program for maximizing the level curves for every $V_i \leq \gamma_i \in \mathbb{R}_{>0}$, is,

$$\begin{aligned} &\text{maximize} \quad \gamma_i \\ &\text{subject to} \quad s5 + \sum_{k=1}^{n_{C_i}} s6^{ik}(-g_{ik}) - (V_i - \gamma_i) + \epsilon = 0, \\ &\qquad (s5, s6^{ik}) \in \mathcal{S}_n,\ i \in \{1,2,3\},\ k \in \{1,...,n_{C_i}\}. \end{aligned}$$

This algorithm maximizes the level curves of the Lyapunov certificates V_i such that $\mathcal{Z}(V_i - \gamma_i) \subset \mathcal{Z}(-g_{ik})$, for $k \in \{1,..,n_{C_i}\}$ (Lemma. 1). The set $\mathcal{S}1 = \bigcup_{i=1}^{3}(V_i \leq (\gamma_i)_{max})$. The non-emptiness of the set $\mathcal{S}1$ shows that $x \models \varphi 1,\ \forall x \in \mathcal{S}1$.

To verify property $\varphi 2$, we extend the advection of level sets [15] to hybrid systems, and show that for all $x(0,j) \in \mathcal{S}2,\ x(t,j) \to \mathcal{S}1$ for some bounded t. This is given in Alg. 1 where Line-5 is the set advection implemented as a SOS program. Let the sets, $\mathcal{S}2 = \mathcal{Z}(P_{Initial})$, and $\mathcal{S}2_{next} = \mathcal{Z}(P_{next})$, where, $(P_{Initial}, P_{next}) : \mathbb{R}^n \to \mathbb{R}$ are polynomial functions. The SOS algorithm for bounded advection of level sets is given below,

$$\begin{aligned} &\text{minimize } \eta \\ &\text{s.t.} \quad P_{next}(0) < 0, \\ &\nabla P_{next}.(v_1,\ v_2,\ \phi_{ref} - \phi_{VCO})^T > 0, \\ &s1_i - s2_i P_{Initial} + A_{-h} P_{next} + \eta\ + \sum_{k=1}^{m_{C_i}} s3_{ik} g_{ik} = 0, \\ &s4_i + s5_i(P_{Initial} - \mu) - A_{-h} P_{next} + \eta\ + \sum_{k=1}^{m_{C_i}} s6_{ik} g_{ik} = 0, \\ &s7_i - s8_i(P_{Initial} - \mu) + \sum_{k=1}^{m_{C_i}} s9_{ik} g_{ik}\ + \nabla^2 P_{next} \frac{h^2}{2} - \eta = 0, \\ &s10_i - s11_i(P_{Initial} - \mu) + \sum_{k=1}^{m_{C_i}} s12_{ik} g_{ik} \nabla^2 P_{next} \frac{h^2}{2} - \eta = 0, \\ &(s1_i,\ s2_i,\ s3_{ik},\ s4_i,\ s5_i,\ s6_{ik},\ s7_i,\ s8_i,\ s9_{ik},\ s10_i,\ s11_i, \\ &s12_{ik}) \in \mathcal{S}_n. \end{aligned} \tag{6}$$

Here P_{next} is of degree $d_r,\ \mu > 0,\ \eta > 0,\ h > 0,$ $u \in [L\ U]$, and $s1_i,\ s2_i,\ s3_{ik},\ s4_i,\ s5_i,\ s6_{ik},\ s7_{ik},\ s8_i,\ s9_{ik},\ s10_i,\ s11_i,\ s12_{ik}$, are polynomials of degree d.

The first two constraints of this SOS program ensure the advected level sets are closed and connected (See [15] and the references there in). The next two constraints search for a polynomial P_{next}, such that when the set $\mathcal{Z}(P_{next})$ is backward advected by the first order Taylor advection map A_{-h}, we obtain a set such that,

$\mathcal{Z}(P_{Initial}) \subset \mathcal{Z}(A_{-h}P_{next} + \eta) \subset \psi_{-h}(\mathcal{Z}(P_{next})) \subset \mathcal{Z}(A_{-h}P_{next} - \eta) \subset \mathcal{Z}(P_{Initial} - \mu).$

Here μ is used as a precision parameter determining, how closely we want the set $\mathcal{Z}(P_{Initial})$ to be approximated by the set $\mathcal{Z}(A_{-h}P_{next} + \|\eta\|)$. The last two constraints enforce the truncation error of the first order Taylor approximation such that, $\|\nabla^2 P_{next} \frac{h^2}{2}\| \leq \eta$, in the set $\mathcal{Z}(P_{Initial} - \mu)$. Note that we solve the above optimization SOS program by using bisection on η. To be conservative, and use an over-approximation to the set $\psi_h(\mathcal{Z}(P_{Initial}))$, the set membership in Line-6 of the Alg. 1, is encoded as a SOS program utilizing Lemma. 1 for the sets $\mathcal{Z}(P_{next} - \eta)$ and $\mathcal{S}1$.

Algorithm 1 Verification of Property $\varphi 2$

INPUT: : Hybrid System Model of CP PLL, Sets $\mathcal{S}1$, $\mathcal{S}2$
OUTPUT: : $\varphi 2$ Verified in Bounded Time/No-answer

1: $\mathcal{S}2_{next} \leftarrow \emptyset$
2: $\mathcal{S}2_{advect} \leftarrow \emptyset$
3: $\mathcal{S}2_{advect} \leftarrow \mathcal{S}2$
4: **for** $j \leftarrow 1$ **to** $j \leftarrow m$ **do**
5: $\quad \mathcal{S}2_{next} \leftarrow Advect(\mathcal{S}2_{advect})$
6: $\quad$ **if** $\mathcal{S}2_{next} \not\subset \mathcal{S}1$ **then**
7: $\quad\quad \mathcal{S}2_{advect} \leftarrow \mathcal{S}2_{next}$
8: $\quad$ **else**
9: $\quad\quad x \models \varphi 2, \ \forall x \in \mathcal{S}2$
10: $\quad\quad$ **break**
11: $\quad$ **end if**
12: **end for**
13: Try a large value of m
14: **if** $\mathcal{S}2_{next} \not\subset \mathcal{S}1$ **then**
15: $\quad$ For $\mathcal{S}2_{next} \setminus (\mathcal{S}2'_{next} = \mathcal{S}1 \cap \mathcal{S}2_{next})$ find the Escape Certificate E.
16: $\quad$ **if** E exists **then**
17: $\quad\quad x \models \varphi 2, \ \forall x \in \mathcal{S}2$
18: $\quad\quad$ **break**
19: $\quad$ **else**
20: $\quad\quad$ No Answer about $\varphi 2$
21: $\quad$ **end if**
22: **end if**

Remark 2.
As for the transformed CP PLL hybrid system (Remark.1) we have identity jump maps, there is therefore no need of constraints on the level sets due to discrete jumps.

After each iteration of the advection of level sets, if the set inclusion $\mathcal{S}2_{next} \subset \mathcal{S}1$ is true, then property $\varphi 2$ is verified. Alternatively, the algorithm keeps on advecting the set $\mathcal{S}2_{next}$ for a user defined bounded number of iterations (Line 7-13). If the property $\varphi 2$ is still not verified (This can happen when the advection of the level sets is unsymmetrical and a subset of the set $\mathcal{S}2$ is not immersed in $\mathcal{S}1$), we compute the Escape certificate E for the set, $\mathcal{S}2_{next} \setminus \mathcal{S}2'_{next}$, showing that trajectories in this set will eventually leave and reach $\mathcal{S}1$ (Line 14-18, as they can not reach $\mathcal{S}2_{next-1}$). This either results in the verification of the property $\varphi 2$ (respectively φ) in the set $\mathcal{S}2$ or we conclude inconclusiveness about the truth value of $\varphi 2$ (respectively φ). Line 15 is implemented by the following SOS program,

$$-\frac{\partial E_i}{\partial x}(x)F_i(x,u) - s1(x)g2(x) + s2(x)g2'(x) + \varepsilon \in \mathcal{S}_n$$
$$(s1, s2) \in \mathcal{S}_n$$

where, $\mathcal{S}2_{next} := g2(x) \geq 0$, and $\mathcal{S}2'_{next} := g2'(x) \geq 0$.

4. EXPERIMENTAL EVALUATION

We used YALMIP [7] solvers within MATLAB for the verification of the inevitability property (respectively sub-properties) on a 2.6 GHZ Intel Core i5 machine with 4 GB of memory. The CP PLL parameters are listed in Table. 1, with all phases normalized by 2π. We computed degree-6 multiple Lyapunov certificates for the third order, and degree-4 multiple Lyapunov certificates for the fourth order CP PLL. Their attractive invariant sets as projected onto different planes are shown in Fig. 2, and Fig. 3. Results of their bounded advection are shown in Fig. 4 and Fig. 5 respectively. Note that due to space constraint, we have shown projections on only two planes for each benchmark. The outer set plotted in solid is the initial set inside which we aim to prove the inevitability of the phase-locking in the CP PLL. The advected level curves are shown in dotted. We used the time step h=1e-3 seconds, and $\mu = 1e-4$ in the computation of advected sets. It can be observed, for the third order level sets eventually symmetrically immersed in the central attractive set after bounded iterations. For the fourth order CP PLL, the advection of level sets is unsymmetrical as the progress in one direction is more abrupt than another. We have therefore the level sets immersed in the attractive invariant from one direction, but advection is inconclusive for a subset in the other direction shown by the pink shaded area in Fig. 5. For the inconclusive subset, we searched a degree-4 Escape certificate for two modes (In one mode2 bounded advection proved convergence to attractive invariant set) showing convergence of the trajectories to the attractive invariant. Computation time of different steps of our verification methodology are given in Table. 2.

Results show the effectiveness of our approach to the verification of the inevitability property of a complex real circuit. We have proved the inevitability property avoiding hundred of discrete transitions as well as the complex continuization as in [2]. Computation time is comparable to [2], and infact is less by an order of atleast considering their approach using gridding of the state space for a third order PLL only. Though user input is needed in the formalization of the problem, our Lyapunov and Escape certificate based deductive methods are applicable to infinite domain (oppose to bounded) and avoid approximating (under or over) solutions of the differential equations. Furthermore, our bounded advection of level sets has the advantage of dealing with larger sets in a single iteration as compare to the existing bounded model checking approaches.

Parameters	Third Order	Fourth Order
C_1	$[1.98\ 2.2]e-12F$	$[31\ 29]e-12F$
C_2	$[6.1\ 6.4]e-12F$	$[3.2\ 3.4]e-12F$
C_3		$[1.8\ 2.2]e-12F$
R	$[7.8\ 8.2]e3\Omega$	$[48\ 52]e3\Omega$
$R2$		$[7\ 9]e3\Omega$
f_{ref}	27MHZ	5MHZ
f_O	27e3MHZ	5MHZ
I_p	[495 505]e-6A	[395 405]e-6A
K_p	[198 202]	[495 502]

Table 1: PLL Parameters used in the Experimentation

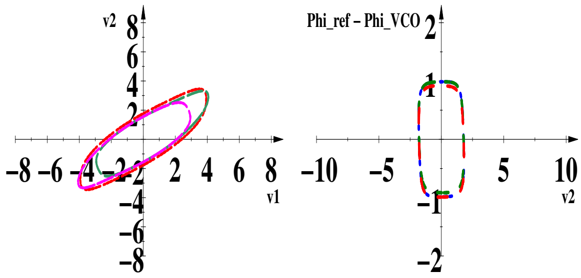

Figure 2: 3-Order $\mathcal{AI}$ Projected onto $(v1, v2)$, and $(v2, (\phi_{ref} - \phi_{VCO})$

Verification Step	3-Order Time(Sec)	4-Order Time(Sec)
Attractive Invariant	1381.7(Degree 6)	10021(Degree 4)
Max.Level Curves	15.5	12
Advection	106.8487 (14 iterations)	140.678 (7 Iterations)
Checking Set Inclusion	13	10.2
Escape Certificate		18 (2 Certificates)

Table 2: Computation Time of the Inevitability Verification

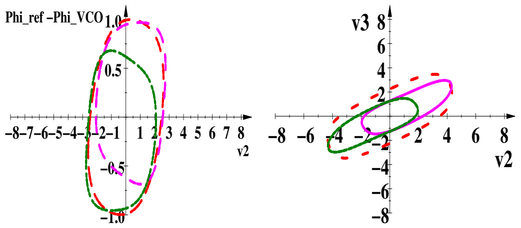

Figure 3: 4-Order $\mathcal{AI}$ Projected onto $(v2, v3)$, and $(v2, (\phi_{ref} - \phi_{VCO})$

5. CONCLUSION

We have presented a scalable verification methodology benefiting from both deductive and bounded verification approaches. We tailored these approaches to verify the complex inevitability property for a practical AMS CP PLL circuit of higher order. As the problem is known to be NP-hard, we used the sufficient SOS relaxation algorithm for the verification of these properties. Experimental results show the effectiveness of our methodology avoiding expensive discretization and reach set computations.

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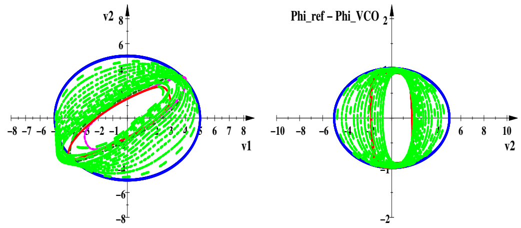

Figure 4: 3-Order Advection Projected onto $(v1, v2)$, and $(v2, (\phi_{ref} - \phi_{VCO}))$

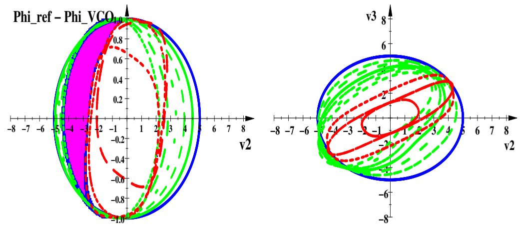

Figure 5: 4-Order Advection Projected onto $(v2, v3)$, and $(v2, (\phi_{ref} - \phi_{VCO}))$

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