Inevitability of Phase-locking in a Charge Pump Phase Lock Loop using Deductive Verification

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ABSTRACT

Phase-locking in a charge pump (CP) phase lock loop (PLL) is said to be inevitable if all possible states of the CP PLL eventually converge to the equilibrium, where the input and output phases are in lock and the node voltages vanish. We verify this property for a CP PLL using deductive verification. We split this complex property into two sub-properties defined in two disjoint subsets of the state space. We deductively verify the first property using multiple Lyapunov certificates for hybrid systems, and use the Escape certificate for the verification of the second property. Construction of deductive certificates involves positivity check of polynomial inequalities (which is an NP-Hard problem), so we use the sound but incomplete Sum of Squares (SOS) relaxation algorithm to provide a numerical solution.

Categories and Subject Descriptors

B.7.2 [**INTEGRATED CIRCUITS**]: Design Aids—*Verification and Simulation*

General Terms

Verification

Keywords

Deductive Verification; AMS Circuits; Lyapunov Certificate; Escape Certificate; SOS Programming

1. INTRODUCTION

Formal methods are in their infancy in Analog and Mixed Signal (AMS) circuits verification. Start up problems (Inevitability) have been very common in PLL circuits, i.e., for certain initial states of voltages on the nodes, the circuits do not converge to the desired behaviour. Furthermore, when perturbed by an external disturbance, designers need to know if the system will return to the desired behaviour.

Hybrid systems are well known modelling paradigm for a CP PLL [16], [2], [6]. Techniques for the verification of hybrid systems can be classified as, reach-set methods, abstraction based methods, and certificate based methods. Reach-set method has been used to prove the inevitability property. Hundreds of discrete transitions are required by the hybrid model of the CP PLL before it reaches the locking state. This results in a large number of continuous set computations followed by the guard conditions describing the switching laws, making the reachability computations prohibitively expensive.

In this paper, we use certificate based deductive verification of the inevitability of phase-locking in higher order CP PLL circuits. Due to its complexity, we adopt a two-pronged verification approach and divide the inevitability property in to the conjunction of two sub-properties. Verification of these two properties determine the truth value of the inevitability property in two disjoint subsets of the state space. The first property specifies, that in a compact set, all system trajectories eventually converge to the equilibrium locking state. The second property is specified, such that trajectories in the second subset, will eventually escape the set and reaches the set where the first property holds. The first property is verified by computing an attractive invariant region (ROA), utilizing the deductive Lyapunov stability theory for hybrid systems [5]. We construct multiple Lyapunov certificates for different modes of the CP PLL hybrid system. The maximized level curves of these Lyapunov certificates characterize the level sets whose union is the ROA. Similarly, we use Escape certificates to verify the second property, and establish that trajectories in the second set will eventually leave and reach the ROA. Our certificate (Lyapunov-Escape) based deductive approach involves checking positivity of polynomial inequalities, which is an NP-hard problem. Therefore, we utilize the sound but incomplete SOS relaxation to construct these certificates.

1.1 Related Work

A survey of the formal verification of AMS circuits can be found in [17]. In [6], the author verified the 'time to locking' property for a digitally extensive PLL. [16] verified 'global convergence' property for an all digital PLL. They divided the state space into linear and non-linear regions, and applied linear Lyapunov stability theory (using Quadratic Lyapunov Certificate) for linear and reachability analysis for non-linear regions respectively. Time-outs of the reachability tool has been reported by the author due to the large number of discrete transitions needed by the PLL hybrid automata. To avoid discrete jumps, [2] presented a continuization technique and verified the 'time to locking' property of a CP PLL.

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Inevitability of an equilibrium state is closely related to the global asymptotic stability of a dynamical system . Lyapunov theory is a well known approach for the verification of such properties [5]. Hybrid system models have different flavours that can be found in [8],[1]. Here we consider the framework outlined in [4]. In the last decade, SOS programming has been the major tool used in the algorithmic construction of Lyapunov certificates for continuous as well as hybrid systems [9], [11]. Barrier certificates has been used for safety verification of the hybrid systems in [11]. Deductive verification of continuous and hybrid systems have been demonstrated in [14], [13].

This paper is organized as follows: In Sec.II, we introduce the preliminaries of this paper. Sec.III illustrates verification of the inevitability of phase-locking in CP PLL. Experimental results are shown in Sec.IV. Sec.V concludes the paper.

2. PRELIMINARIES

2.1 Hybrid Systems Model

We use the hybrid system formalism described in [4]. We consider a hybrid system described by the tuple $(\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$. Here, $\{\mathcal{C} = \bigcup_{i \in I_C} C_i\} \subset \mathbb{R}^n$, and $\{\mathcal{D} = \bigcup_{i \in I_D} D_i\} \subset \mathbb{R}^n$ are the flow set and jump set for $i \in \mathbb{N}$, respectively. I_C and I_D are finite disjoint index sets and it is possible that $C_i \cap D_i \neq \emptyset$. The flow and jump maps are, respectively, $\mathcal{F} = \bigcup_{i \in I_C} F_i$, and $\mathcal{G} = \bigcup_{i \in I_C} G_i$, where each $F_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m+n}$, and, $G_i = \mathbb{R}^n \to \mathbb{R}^n$. These two mappings characterize the continuous and discrete evolution of the system, whereas C_i and D_i describe subsets of $\mathbb{R}^n$ where such evolution may occur. We represent a hybrid system $\mathcal{H}$ as

$$\mathcal{H} = \begin{cases} \dot{x} = F_i(x, u) \in \mathcal{F} & x \in \mathcal{C},\ u \in \mathcal{U} \\ x^+ = G_i(x) \in \mathcal{G} & x \in \mathcal{D} \end{cases} \tag{1}$$

Here $u \in \mathcal{U} \subset \mathbb{R}^m$ is a vector of uncertain parameters. The state of the hybrid system consists of alternate flows in jumps through $\mathcal{C}$ and $\mathcal{D}$ according to F_i and G_i, respectively. This hybrid phenomena can be described by a notion of time called hybrid time.

Definition 1 (Hybrid Time Domain)**.**
A set $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if

$$\mathcal{T} = \bigcup_{j=0}^{j-1} ([t_j, t_{j+1}], j)$$

where $0 = t_0 \leq t_1 \leq t_2 \leq, ...,$ with the last interval possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$.

Definition 2 (Hybrid Arc)**.**
A mapping $x : \mathcal{T} \to \mathbb{R}^n$ is a hybrid arc if $\mathcal{T}$ is a hybrid time domain and for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is locally absolutely continuous on the interval $I_j = \{t : (t, j) \in \mathcal{T}\}$.

We denote by dom x, the domain of the hybrid arc which is the hybrid time domain. A hybrid arc x is a solution to the hybrid system $\mathcal{H}$, if $x(0, j) \in \mathcal{C} \cup \mathcal{D}$ and (i) for each $j \in \mathbb{N}$ such that I_j has a non-empty interior,

$$\begin{aligned} &\dot{x}(t, j) = F_i(x(t, j)),\ \forall t \in I_j, \\ &\qquad x(t, j) \in C_i,\ \forall i \in I_C,\ \forall t \in [min\ I_j, sup\ I_j) \end{aligned} \tag{2}$$

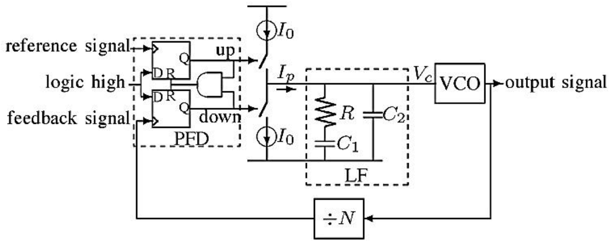

Figure 1: *Charge Pump (CP) Phase Lock Loop (PLL)*

Assumption 1.
The flow maps $F_i(x, u)$ and the jump maps $G_i(x)$ are polynomials.

Definition 3 (Equilibrium point)**.**
A point $x(t, j) \in \mathcal{C} \cup \mathcal{D}$ is called an equilibrium, if $\exists t$, $\exists j$, $F_j(x(t, j), u) = 0$.

Definition 4 (Inevitability of Equilibrium)**.**
The equilibrium point x_e is said to be inevitable, if $\forall x(0, j) \in \mathcal{C} \cup \mathcal{D}$ and bounded t, $x(t, j) \to x_e$.

2.2 CP PLL Model

A CP PLL circuit consists of a reference signal, a phase frequency detector (PFD), a charge pump (CP), a loop filter (LF), and a voltage controlled oscillator (VCO). In this paper we consider a single path higher order (Third and fourth) CP PLL shown in Fig. 1[1]. We use a behavioural model of the PLL, where we consider a linear model for VCO, a linear model for the third order LF, and a non-linear model for the PFD. We denote by ϕ_{ref}, and ϕ_{VCO}, the phases of the reference and VCO output feedback signals respectively. We model the CP PLL as a hybrid system such that the nonlinearities of the PFD is modelled as a piecewise continuous signal. Ignoring the cycle slip phenomena, the PFD output in the form of the charge pump current I_p, is given by the following piecewise linear inclusion:

$$I_p = \begin{cases} \in [I_p^U\, I_P^U] & \text{UP=1, Down=0, } 0 \leq \phi_{VCO} < 2\pi \leq \phi_{ref} \\ \in [I_p^D\, I_P^D] & \text{UP=0, Down=1, } 0 \leq \phi_{ref} < 2\pi \leq \phi_{VCO} \\ \in [0^R\ 0^R] & \text{UP=0, Down=0, } 0 \leq \phi_{VCO}, \phi_{ref} < 2\pi \end{cases} \tag{3}$$

We denote the three modes as mode1 (UP=0, Down=0), mode2 (UP=1, Down=0) and mode3 (UP=0, Down=1). The transition from one mode to another is based on the reference and feedback signals hitting the 2π threshold. Due to the cyclic behaviour of the PLL and to keep the analysis modulo 2π, we need to ensure the phases remain in the range $0 \leq \phi_{VCO}, \phi_{VCO} < 2\pi$ (Similarly ϕ_{ref}) after resetting the PFD. This is achieved by resetting the two phases such that ($\phi_{ref} := 0$, $\phi_{VCO} := \phi_{VCO} - 2\pi$), and ($\phi_{ref} := \phi_{ref} - 2\pi$, $\phi_{VCO} := 0$), while taking transitions from mode1 to mode2 and mode1 to mode3, respectively. Identity resets are used for transitions from mode2 to mode1 and mode3 to mode1.

Our model consists of the state variables, ϕ_{VCO}, ϕ_{ref}, voltage v_1 across the capacitor C_1, and the voltage v_2 across the capacitor C_2 (Fourth order has an additional voltage variable across the third capacitor). Let f_{VCO}, and f_{ref}, represent the frequencies of the VCO output and the refer-

[1]Third order shown here,fourth order has an additional resistor-capacitor in its LF

ence signal respectively. If K_p is the gain of the LF, then $f_{VCO} = K_p v_2/2\pi + f_O$, where f_O is the free running frequency of the VCO. Therefore, $\dot{\phi}_{VCO} = 2\pi f_{VCO}/N$, $\dot{\phi}_{ref} = 2\pi f_{ref}$. By Kirchhoff's current law and using the three modes of the PFD, we get the following hybrid system of the third order CP PLL

$$\mathcal{H} = \begin{cases} \dot{\begin{pmatrix} v_1 \\ v_2 \\ \phi_{ref} \\ \phi_{VCO} \end{pmatrix}} = A \begin{pmatrix} v_1 \\ v_2 \\ \phi_{ref} \\ \phi_{VCO} \end{pmatrix} + BI_p + c & x \in \mathcal{C}, \\ x^+ = G_i(x) & x \in \mathcal{D} \end{cases} \quad (4)$$

where, $A = \begin{pmatrix} -1/RC_1 & 1/RC_1 & 0 & 0 \\ 1/RC_2 & -1/RC_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & K_p/N & 0 & 0 \end{pmatrix}$,

$B = \begin{pmatrix} 0 \\ I_p/C_2 \\ 0 \\ 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 \\ 0 \\ 2\pi f_{ref} \\ 2\pi f_o/N \end{pmatrix}$, I_p as given by Eq. 3.

From the three modes and their invariants, we can easily find the sets $\mathcal{C}$, $\mathcal{D}$, and the jump maps $G_i(x)$. We notice, that the state variables ϕ_{ref} and ϕ_{VCO} do not settle to zero in their steady state. Instead the system has a limit cycle like behaviour in the (ϕ_{ref}, ϕ_{VCO}) plane. Here, we are interested in the stability of the equilibrium point (locking condition) where the difference of these two phases becomes zero. We therefore, use the difference $\phi_{ref} - \phi_{VCO}$ as a state variable instead of ϕ_{ref}, and ϕ_{VCO}.

Remark 1.
This change of state variables transforms all jump maps G_i in to identity maps $(G_i(x) = x)$, as the same constant 2π is subtracted from ϕ_{VCO} and ϕ_{ref}, leaving their difference $\phi_{ref} - \phi_{VCO}$ before and after the jumps unchanged.

2.3 Attractive Invariants in Hybrid Systems using Lyapunov Certificates

Contrary to the safety, where existence of an invariant set is sufficient for proving/dis-proving the property, we use the concept of "attractive invariants" to verify inevitability property. These are compact semi-algebraic sets inheriting the properties of invariance and attractivity; trajectories stay there indefinitely and eventually converge to the equilibrium state. Stability and attractivity concepts for an equilibrium state of the continuous dynamical systems are discussed in [5], and have been extended to hybrid systems in [4]. The equilibrium point, $x_e = 0$, is called asymptotically stable if it is both stable and attractive. There are several versions of the stability theorems for hybrid systems based on the global Lyapunov certificate, and multiple Lyapunov certificates [12]. We use the following theorem of asymptotic stability of the equilibrium and define the attractive invariant.

Theorem 1.
Let, $\mathcal{I}_0 \subseteq I_C$ be the set of indices that contain the equilibrium. For a hybrid system $\mathcal{H}$ having an equilibrium point $x_e = 0$, if there exist Lyapunov certificates V_i such that,

1. $V_i(x) > 0,\ \forall i \in I_C,\ \forall x \in \mathcal{C} \setminus x_e$,
2. $V_i(0) = 0,\ \forall i \in \mathcal{I}_0$,
3. $\frac{\partial V_i}{\partial x}(x) F_i(x,u) < 0,\ \forall i \in I_C,\ \forall x \in \mathcal{C} \setminus x_e,\ F_i \in \mathcal{F},$ $u \in \mathcal{U}$,
4. $V_j(G_i(x)) - V_{j'}(x) \le 0,\ \forall j\ \forall j' \in I_C,\ j \neq j',$ $\forall i \in I_D,\ \forall x \in \mathcal{D} \setminus x_e,\ G_i \in \mathcal{G}$,

then x_e is asymptotically stable. Furthermore, the set $\mathcal{AI} = \bigcup_i (V_i \le c^i_{max}) \subset \mathcal{C} \cup \mathcal{D}$ is an "attractive invariant" set.

Proof. Similarly to [12]. □

2.4 Escape of Trajectories from a Set using Escape Certificates

In this paper, we also use an another important characteristic of the trajectories in a semi-algebraic set and term it as the "Escape" property. This property ensures that trajectories in a compact set can not converge to an invariant set (Equilibrium,Limit Cycle), and will eventually leave that set.

Proposition 1.
For a compact set $\mathcal{X} \subset C_i$, if there is a differentiable Escape certificate, $E : \mathbb{R}^n \to \mathbb{R}$, and $\epsilon > 0$, such that

$$\frac{\partial E}{\partial x}(x) F_i(x,u) \le -\varepsilon,\ \forall x \in \mathcal{X},\ u \in \mathcal{U} \quad (5)$$

then $\forall x(t,i) \in \mathcal{X},\ x(t+T,i) \notin \mathcal{X},\ T > t$.

Proof. Assume that there exists $x_0 \in \mathcal{X}$ such that $x(t,i)$ starting at x_0 remain in $\mathcal{X}$ as $t \to \infty$. From Eq. 5, $E(x) = \int_0^\infty \frac{\partial E}{\partial x}(x) F_i(x,u) \le -\varepsilon$. As $t \to \infty,\ E(x) \to -\infty$. This contradicts the assumption as $E(x)$ should be bounded if $x(t,j)$ has to remain in the bounded set $\mathcal{X}$. Therefore, $x(t,j)$ has to eventually "Escape" the set $\mathcal{X}$ in finite time. □

Lemma 1.
If the hybrid arc $x(t,j)$ is bounded and belong to a set $\mathcal{X}$ for the hybrid time $(t,j) \ge 0$, then $x(t,j)$ approaches a compact invariant set as $(t,j) \to \infty$.

Proof. See [5]. □

2.5 SOS Programming

Our deductive verification approach involves checking the positivity of polynomials in semi-algebraic sets. To solve this NP-Hard problem, a sound relaxation method based on SOS programming has been presented in [10], [12]. A sufficient condition for a multivariate polynomial $p(x)$ to be non-negative everywhere, is that it can be decomposed as a sum of squares of polynomials. A polynomial $p(x)$ is a sum of squares, if there exist polynomials $p_1(x), ..., p_m(x)$ such that $p(x) = \sum_{i=1}^m p_i^2(x)$. We denote the set of polynomials in n variables with real coefficients by $\mathcal{P}_n$. A subset of this set is the set of SOS polynomials in n variables denoted by $\mathcal{S}_n$. For a differentiable scalar polynomial $q : \mathbb{R}^n \to \mathbb{R}$, we define the 0-sub-level-set of q as $\mathcal{Z}(q) = \{x \in \mathbb{R}^n \mid q(x) \le 0\}$. We present an important lemma to be used for polynomial level sets operations such as intersection, union, and set inclusion [15].

Lemma 2.
For polynomials $p1, p2 \in \mathcal{P}_n$, if there exist SOS polynomials $s0,\ s1\ \in\ \mathcal{S}_n$ such that

$$s0 - s1p1 + p2 = 0\ \forall x \in \mathbb{R}^n \quad (6)$$

Then $\mathcal{Z}(p1) \subset \mathcal{Z}(p2)$

Proof. See for example [15] and the references there in. □

3. VERIFICATION OF INEVITABILITY OF PHASE-LOCKING IN CP PLL

To verify inevitability of the CP PLL equilibrium, we introduce two compact sets $\mathcal{S}1$, and $\mathcal{S}2$, such that $\mathcal{S}1 \cap \mathcal{S}2 = \emptyset$, and $\mathcal{S}1 \cup \mathcal{S}2 = \mathcal{C} \cup \mathcal{D}$. We define two properties whose verification implies verification of the inevitability of the equilibrium.

Property 1.
$\forall x(0,j) \in \mathcal{S}1,\ x(t,j) \to x_e$ for $t \to \infty$.

Property 2.
$\forall x(0,j) \in \mathcal{S}2 = (\mathcal{C} \cup \mathcal{D}) \setminus \mathcal{S}1,\ x(t,j) \in \mathcal{S}1$ for $t \to b \in \mathbb{R}_{>0}$.

If we denote the inevitability property by φ, Property.1 by $\varphi1$ and Property.2 by $\varphi2$, then $\varphi = \varphi1 \wedge \varphi2$. A hybrid arc x satisfies φ iff it satisfies $\varphi1$ in $\mathcal{S}1$ and $\varphi2$ in $\mathcal{S}2$, i.e., $\forall x \in \mathcal{C} \cup \mathcal{D},\ x \models \varphi \iff (x \models \varphi1\ \forall x \in \mathcal{S}1) \wedge (x \models \varphi2\ \forall x \in \mathcal{S}2)$. We verify property $\varphi1$ using Lyapunov certificates, and property $\varphi2$ using Escape certificates.

Theorem 2.
If there are feasible Lyapunov certificates (fulfilling Th. 1), $\{V_1,\ V_2,\ V_3\}$, then, $x \models \varphi1,\ \forall x(0,j) \in \mathcal{S}1 = \{(V_1 \leq \gamma1_{max}) \cup (V_2 \leq \gamma2_{max}) \cup (V_3 \leq \gamma3_{max})\}$, and $\mathcal{AI} = \mathcal{S}1$ is the attractive invariant set.

Proof. Follows directly from Th. 1, since the level sets defined by the level curves of the Lyapunov certificates represent attractive invariant sets with the negative Lie-derivative along the system trajectories. Therefore, eventually all system trajectories starting in these level sets converge to the equilibrium phase-locking state. □

Theorem 3.
If in a compact set $\mathcal{S}2$, such that $\mathcal{S}1 \cup \mathcal{S}2 = \mathcal{C} \cup \mathcal{D}$, where $\mathcal{S}1$ is an attractive invariant set, we have an Escape certificate $E = \cup_{i\in\{1,2,3\}} E_i(x)\ \forall x \in \mathcal{S}2$, then, $\forall x(t,j) \in \mathcal{S}2,\ x(t,j) \in \mathcal{S}1$ as $t + j \to \infty$.

Proof. Follows directly from Lemma. 1. The boundedness of $x(t,j)$ is guaranteed by the supply voltage and ground of the CP PLL circuit. Existence of an Escape certificate for $x(t,j) \in \mathcal{S}2$ (Prop. 1), guarantees that trajectories will eventually leave $\mathcal{S}2$, and being the only invariant set, they will eventually reach $\mathcal{S}1$. □

Following Th. 2, we verify $\varphi1$ using Alg. 1. The truth value of $\varphi1$ depends on the existence of the attractive invariant set $\mathcal{S}1$. The set $\mathcal{S}1$ is computed from the maximized level sets defined by the three candidate Lyapunov certificates V_1, V_2, V_3, Line (1-6). We compute these certificates using SOS programming and a mathematical technique called the S-procedure (to incorporate domain constraints [3]). We encode the verification of $\varphi1$ as two SOS programs. The first SOS program is given below:

(a) $V_i(x) - \epsilon - \sum_{k=1}^{n_{C_i}} s_1^{(ik)}(x) g_{ik}(x) \in \mathcal{S}_n,\ \forall x \neq 0,$
$i \in \{1,2,3\}, \forall k \in \{1,..,n_{C_i}\},\ s_1^{(ik)} \in \mathcal{S}_n,$

(b) $\left(-\frac{\partial V_i}{\partial x}(x) F_i(x,u) - \epsilon - \sum_{k=1}^{n_{C_i}} s_2^{(ik)}(x) g_{ik}(x) - \sum_{j=1}^{m} s_3^{(j)}(x) a_j(u) \right) \in \mathcal{S}_n,$
$\forall i \in \{1,2,3\}, \forall k \in \{1,..,n_{C_i}\},\ \forall j \in \{1,..,m\},\ s_2^{(ik)}, s_3^{(j)} \in \mathcal{S}_n,$

Algorithm 1 Verification of Property $\varphi1$

INPUT: : Hybrid System Model of CP PLL
OUTPUT: : $\varphi1$ Verified/No-answer, $\mathcal{S}1$

1: $\mathcal{S}1 \leftarrow \emptyset$
2: **for** $n \leftarrow 1$ **to** $n \leftarrow 3$ **do**
3: $\quad V_n \leftarrow Parametrize(V_n)$; Setting degree of V_n Polynomials
4: **end for**
5: **if** $V_n, \forall n \in \{1,2,3\}$, are feasible (fulfilling Th. 1) **then**
6: $\quad V_{multiple} \leftarrow \{V_{multiple}, V_n\},\ \forall n \in \{1,2,3\}$
7: $\quad \mathcal{S}1 \leftarrow (V_{multiple}(1) \leq \gamma1_{max}) \cup (V_{multiple}(2) \leq \gamma2_{max}) \cup (V_{multiple}(3) \leq \gamma3_{max})$
8: $\quad x \models \varphi1,\ \forall x \in \mathcal{S}1$
9: $\quad$ **break**
10: **else**
11: $\quad V_n \leftarrow Infeasible$
12: $\quad$ Increase degree of Lyapunov Certificates
13: **end if**
14: **if** degree = maximum possible value & $\mathcal{S}1 = \emptyset$ **then**
15: $\quad$ No Answer about $\varphi1$
16: **end if**
17: **return** $\mathcal{S}1$

(c) $\left(V_j(x) - V_{j'}(G_i(x)) - s_4^{(i0)}(x) h_{i0}(x) - \sum_{k=1}^{m_{D_i}} s_5^{(ik)}(x) h_{ik}(x) \right) \in \mathcal{S}_n,\ \forall j \forall j' \in \{1,2,3\},$
$j \neq j',\ \forall i \in \{1,2,3,4\}, \forall k \in \{1,..,n_{D_i}\},$
$s_4^{(i0)} \geq 0, s_5^{(ik)} \in \mathcal{S}_n, \epsilon > 0$

Here $V_i(x),\ V_j(x),\ V_{j'}(x),\ s_1^{(ik)},\ s_2^{(ik)},\ s_3^{(j)},\ s_4^{(i0)},\ s_5^{(ik)}$, are polynomials of degree d.

SOS constraints (a) and (b) enforce positive definiteness on the Lyapunov certificates, and negative semi-definiteness on their Lie-derivatives respectively. Furthermore, these constraints have to be satisfied in their respective domains C_i's, where, $C_i = \{x \in \mathbb{R}^n\ : g_{ik} \geq 0,\ \text{for}\ k \in \{1,..,n_{C_i}\},\ i \in \{1,2,3\}\}$. Constraint (b) also ensures parameters u to belong to the set, $\{a(u) \geq 0,\ \text{for}\ j \in \{1,..,m\}\}$. Constraint in (c) ensures that Lyapunov certificates $V_j(x)$ decrease along the discrete jumps in the sets, $D_i = \{x \in \mathbb{R}^n\ : h_{ik} \geq 0,\ h_{i0} = 0,\ \text{for}\ k \in \{1,..,n_{D_i}\},\ i \in \{1,2,3,4\}\}$, through the mappings $G_i(x)$'s. SOS polynomials $s_1^{(ik)},\ s_2^{(ik)},\ s_3^{(j)},\ s_4^{(i0)},\ s_5^{(ik)}$ are used to enforce domain constraints through the S-procedure. A feasible solution of the above SOS program results in Lyapunov certificates V_i. If this SOS program is infeasible, then either the program is repeated for an increased degree d of the polynomials, or we conclude that the truth value of the property $\varphi1$ can not be established Line (11-16).

The second SOS program for maximizing the level curves for every $V_i \leq (\gamma_i \in \mathbb{R}_{>0})$ is (Line-7),

$$\begin{aligned} &\text{maximize} && \gamma_i \\ &\text{subject to} && s5 + \sum_{k=1}^{n_{C_i}} s6^{ik}(-g_{ik}) - (V_i - \gamma_i) + \epsilon = 0, \\ &&& s5, s6^{ik} \in \mathcal{S}_n,\ i \in \{1,2,3\},\ k \in \{1,...,n_{C_i}\}. \end{aligned}$$

This algorithm maximizes the level curves of the Lyapunov certificates V_i such that $\mathcal{Z}(V_i - \gamma_i) \subset \mathcal{Z}(-g_{ik})$, for $k \in \{1,..,n_{C_i}\}$ (Lemma. 2). The set $\mathcal{S}1 = \bigcup_{i=1}^{3} (V_i \leq (\gamma_i)_{max})$. The non-emptiness of the set $\mathcal{S}1$ shows that $x \models \varphi1,\ \forall x \in \mathcal{S}1$.

Algorithm 2 Verification of Property $\varphi 2$

INPUT: : Hybrid System Model of CP PLL, Set $\mathcal{S}2 = \cup_{i \in 1,2,3} \mathcal{S}2_i$
OUTPUT: : $\varphi 2$ Verified/No-answer
1: **for** $i \leftarrow 1$ **to** $i \leftarrow 3$ **do**
2: $\quad E_i \leftarrow Parametrize(E_i)$; Setting degree of E_i Polynomials
3: **end for**
4: **if** $E_i, \forall i \in \{1,2,3\}$, are feasible (fulfilling Prop. 1) **then**
5: $\quad E_{multiple} \leftarrow \{E_{multiple}, E_i\},\ \forall i \in \{1,2,3\}$
6: $\quad x \models \varphi 2,\ \forall x \in \mathcal{S}2 = \cup_{i \in 1,2,3} \mathcal{S}2_i$
7: **else**
8: $\quad E_i \leftarrow Infeasible$
9: $\quad$ Increase degree of Escape Certificates
10: **end if**
11: **if** degree = maximum possible value & E_i are Infeasible **then**
12: $\quad$ No Answer about $\varphi 2$
13: **end if**
14: **return** Truth value of $\varphi 2$

Similarly, following Th. 3, we verify property $\varphi 2$ utilizing Alg. 2. We search for three Escape certificates (Prop. 1) in three disjoint sets $\mathcal{S}2_i,\ i \in \{1,2,3\}$, such that $\mathcal{S}2 = \cup_{i \in 1,2,3} \mathcal{S}2_i$. After parametrizing the three Escape certificates, we establish the feasibility of these Escape certificates by the following SOS program,

$$-\frac{\partial E_i}{\partial x}(x) F_i(x,u) - \sum_{k=1}^{n_{C_i}} s_1^{(ik)}(x) g2_{ik}(x) - .. \\ \sum_{j=1}^{m} s_2^{(j)}(x) a_j(u) + \varepsilon \in \mathcal{S}_n \\ s_1^{(ik)}, s_2^{(j)} \in \mathcal{S}_n,\ \varepsilon > 0$$

This SOS program ensures that the Lie-derivative of E_i is strictly negative in the set, $\mathcal{S}2_i = \{x \in \mathbb{R}^n \ : g2_{ik} \geq 0,\ \text{for } k \in \{1,..,n_{C_i}\},\ i \in \{1,2,3\}\}$. The second constraint in this SOS program is such that the parameters u belong to the set, $\{a(u) \geq 0,\ \text{for } j \in \{1,..,m\}\}$. Here ε is a small positive real number. Feasibility of this SOS program indicates existence of the Escape certificates for each mode of the CP PLL hybrid system, and consequently the property $\varphi 2$ is verified Line(4-6). Alternatively, in case of infeasible solution of the SOS program, we increase the degree of the Escape certificates and repeat the process Line(8-10). If the property $\varphi 2$ is still not verified, we conclude inconclusiveness about the truth value of $\varphi 2$ (respectively φ) Line 12.

4. EXPERIMENTAL EVALUATION

We used YALMIP [7] solver within MATLAB for the verification of the inevitability property (respectively subproperties) on a 2.6 GHZ Intel Core i5 machine with 4 GB of memory. The CP PLL parameters are listed in Table. 1, with all phases normalized by 2π. We computed degree-6 multiple Lyapunov certificates for the third order, and degree-4 multiple Lyapunov certificates for the fourth order CP PLL. Their attractive invariant sets as projected onto different planes are shown in Fig. 2, and Fig. 3 respectively. We constructed three Escape certificates for each mode of the the third order and fourth order CP PLL hybrid models. For both benchmarks, we computed degree 2 Escape certificates for mode2 and mode3. For mode1, we computed degree 12 and degree 10 Escape certificates for third and fourth order CP PLL respectively. We chose, $\varepsilon = 1e-4$, for all Escape certificates. We noticed that decreasing the value of ε resulted in a higher degree Escape certificate for both benchmarks. However, this is at the cost of higher computation time. We therefore opted for the value $1e-4$. A simulation trace along with the derivative of the Escape certificate patched up from the three Escape certificates of each benchmark are depicted in Fig. 4 and Fig. 5 respectively. Note that due to space constraints, we have shown projections on only two planes for each benchmark. Simulation traces show that the derivative of the Escape certificate is negative along the trajectories. Computation time of different steps of our verification methodology is given in Table. 2. Though the maximum degree of certificates (Lyapunov,Escape) for the fourth order is less than that of the third order, however, the dimensionality factor is dominant as far as the computation time is concerned.

Parameters	Third Order	Fourth Order
C_1	$[1.98\ 2.2]e-12F$	$[31\ 29]e-12F$
C_2	$[6.1\ 6.4]e-12F$	$[3.2\ 3.4]e-12F$
C_3		$[1.8\ 2.2]e-12F$
R	$[7.8\ 8.2]e3\Omega$	$[48\ 52]e3\Omega$
$R2$		$[7\ 9]e3\Omega$
f_{ref}	27MHZ	5MHZ
f_O	27e3MHZ	5MHZ
I_p	[495 505]e-6A	[395 405]e-6A
K_p	[198 202]	[495 502]

Table 1: *PLL Parameters used in the Experimentation*

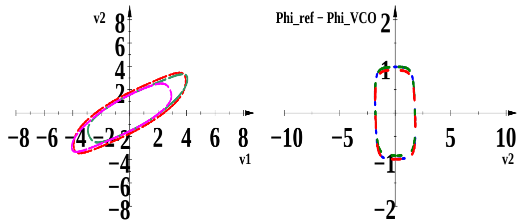

Figure 2: *3-Order $\mathcal{AI}$ Projected onto $(v1, v2)$, and$(v2, (\phi_{ref} - \phi_{VCO})$*

Results show the effectiveness of our approach to the verification of the inevitability property of a complex real circuit. We have proved the inevitability property avoiding hundred of discrete transitions as well as the complex continuization as in [2]. Computation time is comparable to [2], and infact is less by an order of atleast three considering their approach using gridding of the state space for a third order PLL only. Our Lyapunov and Escape certificate based deductive methods, though needs user input in the formalization of the problem, are applicable to infinite domain (oppose to bounded) and avoid approximating (under or over) solutions of the differential equations. Furthermore, SOS based relaxation, in addition to solve the NP-hard problem of positivity check, offers an easy way of incorporating parameter variations as well.

5. CONCLUSION

We have presented a scalable deductive verification methodology for the inevitability verification of phase-locking in higher order CP PLL. We benefited from the Lyapunov sta-

Verification Step	3-Order Time(Sec)	4-Order Time(Sec)
Attractive Invariants	1381.7(Degree 6)	10021(Degree 4)
Max.Level Curves	15.5	12
Escape Certificates	100	900

Table 2: *Computation Time of the Inevitability Verification*

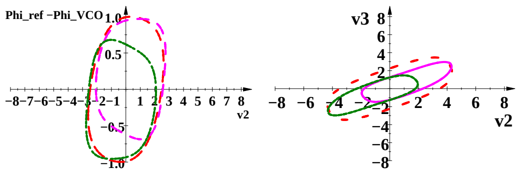

Figure 3: *4-Order $\mathcal{AI}$ Projected onto* $(v2, v3)$, *and* $(v2, (\phi_{ref} - \phi_{VCO})$

bility of hybrid systems and Non-linear continuous systems theory, and came up with some interesting local properties. Successful verification of these local properties resulted in the global inevitability property verification of a CP PLL. Involving an NP-Hard problem of checking the positivity of polynomials inequalities, we used the sound SOS relaxation algorithm for the verification of these properties. Experimental results show the effectiveness of our approach avoiding expensive discretization and reach set computations.

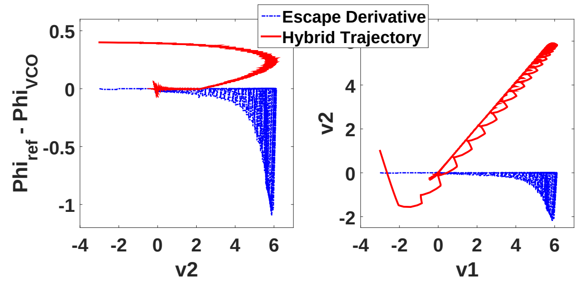

Figure 4: *3-Order Derivative of Escape Certificates, Trajectory Trace Projected onto* $(v1, v2)$, *and* $(v2, (\phi_{ref} - \phi_{VCO}))$

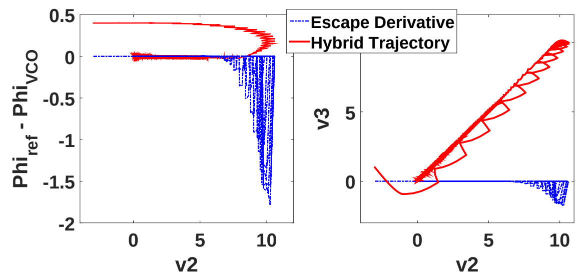

Figure 5: *4-Order Derivative of Escape Certificates, Trajectory Trace Projected onto* $(v2, v3)$, *and* $(v2, (\phi_{ref} - \phi_{VCO}))$

6. REFERENCES

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