Reduced Sensitivity Solutions to Global Linearisation of the Pole Assignment Map

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Abstract: The problem of pole assignment, by static output feedback controllers has been tackled as far as solvability conditions and the computation of solutions when they exist by a powerful method referred to as global linearisation. This is based on asymptotic linearisation (around a degenerate point) of the pole placement map. The essence of the present approach is to reduce the multilinear nature of the problem to the solution of a linear set of equations. The solution is given in closed form in terms of a one-parameter family of static feedback compensators, for which the closed-loop poles approach the required ones as $\epsilon \to 0$. The use of degenerate compensators makes the method numerically sensitive. This paper develops further the global linearisation framework by developing numerical techniques which make the method less sensitive to the use of degenerate solutions as the basis of the methodology. The proposed new computational framework for finding output feedback controllers improves considerably the sensitivity properties by using a predictor-corrector numerical method based on homotopy continuation. The modified method guarantees the maximum distance from the degenerate point. The current numerical method developed for the constant output feedback extends also to the case of dynamic output feedback.

Keywords: Linear systems; algebraic systems theory; degenerate compensators; global linearisation.

1. INTRODUCTION

The *Determinantal Assignment Problem* (DAP) belongs to the family of algebraic synthesis methods and has arisen as the abstract problem formulation of pole, zero assignment of linear systems (Karcanias and Giannakopoulos, 1984; Leventides and Karcanias, 1995) and thus unifies the study of the corresponding frequency assignment problems of multivariable systems, which have a determinantal character, under constant, dynamic centralised, or decentralised control structure. The DAP demonstrates the significance of exterior algebra and classical algebraic geometry for control problems. The importance of tools and techniques of algebraic geometry for control theory problems has been demonstrated by the work in (Brockett and Byrnes, 1981) etc. The construction of constant output feedback compensators that place the poles of a $p-$input, $m-$output, $n-$state MIMO system, to arbitrary chosen locations was always a challenging problem in Control theory. It is a highly nonlinear problem and multi-linear in the gain parameters and can be formulated as a problem of solving the algebraic equations $p(s) = det\{D(s) + \mathbf{K}N(s)\}$ (see Section 2 for the detailed problem formulation), which is of the general DAP type (Karcanias and Giannakopoulos, 1984). This equation has to be solved with respect to $\mathbf{K} \in \mathbb{R}^{p\times m}$, and contrary to the state feedback case, it is not always solvable. In fact, a necessary condition for its solvability has been given in (Wang, 1992) and states that $mp > n$, where the triple m, p, n denotes the number of outputs, inputs and states of the system respectively. Although $mp > n$ was proved to be a sufficient condition (Wang, 1992) for the generic solvability of the problem, the known constructive methods, such as dyadic or full rank output feedback, work only in the restricted case, i.e. when $m + p - 1 \geq n$ (Davison and Wang, 1975; Kimura, 1975). In (Leventides, 1993) a new constructive method was developed that treats the general case $mp > n$. This method is based on the asymptotic linearisation of the pole placement map (Leventides and Karcanias, 1995, 1998) by considering special sequences of feedback compensators, which in the limit, converge to a so-called degenerate compensator.

The advantage of this approach is that it asymptotically reduces the overall pole placement map to a linear one and thus reduces the overall solvability of the problem to a linear set of equations. When the differential of the related Pole Placement Map has full rank at the degenerate compensator, then the problem can be solved. It has been proved (Leventides, 2007) that this condition is satisfied generically when the number of controller parameters ex-

ceeds the number of independent equations and can lead to a numerical procedure for the construction of solutions. The solutions worked out within the given framework are given in a closed form and they are of the type of a one parameter family of multivariable compensators. It should be stressed that the approach is sufficient, but quite general, given that the condition $mp > n$ guarantees the success of the method for a generic system. On the other hand, a serious disadvantage of the method is that it is based on approaching a degenerate point, which is a point that the closed-loop system is not well defined; this has the effect that although the poles of the closed loop system are very close to the desired set, the sensitivity increases considerably (in fact tends to infinity as $\epsilon \to 0$).
The purpose of this paper is to improve the above linearisation method around a degenerate point, so that the desired set of poles can be approached whereas the feedback controller is far from the degenerate point and hence achieving pole placement which is less sensitive. An equivalent sufficient condition for arbitrary pole assignment is that the linearisation matrix (referred also as "Blow-up" matrix) should have full rank.
The paper is organized in the following way: Section 2 summarizes the theoretical background and results for the Global Linearisation method established in (Leventides and Karcanias, 1995) and formulates the constant DAP problem as Static Output Feedback (SOF) pole placement problem. In Section 3 the computational framework is introduced and a proposed numerical algorithm based on a Predictor-Corrector iterative method is presented which guarantees the maximum distance from the degenerate compensator. Finally, Section 4 contains the numerical example and the discussion of results.

Notation: Throughout the paper the following notation is adopted: If $\mathcal{F}$ is a field, or ring then $\mathcal{F}^{m\times n}$ denotes the set of $m \times n$ matrices over $\mathcal{F}$. If $\mathcal{V}$ is a vector space and $\{\underline{v}_{i_1}, \ldots, \underline{v}_{i_k}\}$ are vectors of $\mathcal{V}$ then $\underline{v}_{i1} \wedge \ldots \wedge \underline{v}_{ik} = \underline{v}_\omega \wedge$, $\omega = (i_1, \ldots, i_k)$ denotes their exterior product and $\wedge^r V$ the $r-$th exterior power of $\mathcal{V}$. If $H \in \mathcal{F}^{m\times n}$ and $r \le \min(m, n)$, then $C_r(H)$ is the $r-$th compound matrix of H.

2. GLOBAL LINEARISATION OF THE FREQUENCY ASSIGNMENT MAP

The method of Global Asymptotic Linearisation was first introduced in (Leventides (1993)) and further developed in (Leventides and Karcanias, 1995, 1998). The methodology was based on the remarkable property of the degenerate gains of a feedback configuration to "blow up" sequences of gains converging to them.

Problem Formulation: We consider linear systems described by the $m \times p$ transfer function matrix $G(s)$ with McMillan degree n and represented by the right coprime Matrix Fraction Description (MFD) $G(s) = N(s)D(s)^{-1}$.

For the Pole Assignment (PA) problem by Static Output Feedback (SOF) in the typical output feedback configuration the closed-loop TF matrix is given by

$$G_{cl}(s, \mathbf{K}) = [I_m + G(s)K]^{-1}G(s)$$

and the closed-loop characteristic polynomial is obtained by:

$$\begin{aligned} p(s,\mathbf{K}) &= f_n(k_{11},...,k_{mp})s^n + f_{n-1}(k_{11},...,k_{mp})s^{n-1} + \\ &\cdots + f_0(k_{11},...,k_{mp}) \\ &= \det\{D(s) + \mathbf{K}N(s)\} \end{aligned} \tag{1}$$

where $k_{11}, ..., k_{mp}$ indicate the entries of the output feedback matrix $\mathbf{K} \in \mathbb{R}^{p\times m}$. Hence, the SOF-PA problem involves the solution of (1) with respect to $\mathbf{K} \in \mathbb{R}^{p\times m}$, for an arbitrary given $p(s) \in \mathbb{R}[s]$, the so-called *prime or target polynomial.*

□

The *Frequency Assignment Map* associated with the problem is the map assigning $\mathbf{K}$ to the coefficient vector $\underline{p}$ of $p(s)$, i.e.

$$\mathrm{F} : \mathbb{R}^{p\times m} \to \mathbb{R}^{n+1} : \mathrm{F}(\mathbf{K}) = \underline{p}$$

For a system to have the arbitrary assignment property the map F has to be onto. A more relaxed condition for arbitrary pole assignment is that F is a dominant morphism. It has been shown (Leventides and Karcanias, 1995) that it is sufficient to find a degenerate compensator K_0 such that the the differential of F evaluated at K_0, $D\,\mathrm{F}_{K_0}$, has full rank. Also, for a generic proper system with $p-$inputs, $m-$outputs, $n-$states, represented by a transfer function $G(s) = N(s)D(s)^{-1}$ such that the condition $mp > n$ is satisfied, the map F is onto.

Degenerate gains were first introduced in (Brockett and Byrnes, 1981) in their generalized form as follows:

Definition 1. A generalized gain $rowspan[A, K]$ is degenerate if and only if it satisfies equation:

$$\det\{[A, K]M(s)\} = 0, \ \forall \mathrm{s} \in \mathbb{C} \tag{2}$$

□

Despite the fact the equation (2) is multilinear with respect to $[A, K]$, degenerate gains can be constructed easily from the null-spaces of certain matrices (Leventides and Karcanias, 1995). In the following, we denote by $\mathcal{M} = colsp_{\mathbb{R}[s]}\{M(s)\}$ the $\mathbb{R}[s]-$module generated by the columns of $M(s)$.

Theorem 2. For the system represented by $M(s) \in \mathbb{R}[s]^{(m+p)\times p}$, a $p-$dimensional space $\mathcal{D} = rowspan[A, K]$ corresponds to a degenerate gain, if and only if either of the following equivalent conditions holds true:

- **(i)** There exists a $((p+m)\times 1)$ polynomial vector $m(s) \in \mathcal{M}$ such that $[A, K]m(s) = 0, \forall s \in \mathbb{C}$.
- **(ii)** There exists a $((p+m)\times 1)$ polynomial vector $m(s) \in \mathcal{M}$ with coefficient matrix P such that the $rank\{P\} \leqslant m$.

□

Note that in the characterization of degenerate gains we consider all possible gains (bounded and unbounded) and we may classify them as:

- **(i)** a *finite degenerate* gain if $\mathcal{D} = rowspan[A, K]$ such that $det(A) \neq 0$;
- **(ii)** an *infinite degenerate* gain if $\mathcal{D} = rowspan[A, K]$ such that $det(A) = 0$;

Theorem (2) clearly, suggests that the parametrisation of the family of degenerate solutions, i.e. all degenerate gains, finite and infinite, is related to the properties of the module $\mathcal{M}$ (Karcanias et al., 2013) and in particular to the properties of minimal bases of $\mathcal{M}$ as these are defined by the corresponding minimal indices and the associated real invariant spaces (Karcanias, 1994, 1996). The results produced in (Karcanias et al., 2013) for the parametrisation of degenerate solutions will allow the selection of appropriate degenerate solutions shaping the properties of the Pole Assignment Map; how to choose the optimal degenerate point with the desired properties as far as spectrum assignment is currently being examined. For the generic properties of the pole assignment map and its relationship to system invariants see (Karcanias and Giannakopoulos, 1984; Leventides and Karcanias, 1993; Willems and Hesselink, 1978).

The importance of degenerate compensators is due to the following:

Lemma 3. If there exists a degenerate matrix $K \equiv K_0 \in \mathbb{R}^{p\times m}$ such that the differential of the Frequency Assignment Map is onto, then any polynomial of degree δ can be assigned via some static compensator.

□

Having constructed a degenerate gain is the starting point for our method and in order to achieve global linearisation, it is essential to consider sequences of generalized gains:

$$S(\epsilon) = [A, K] + \epsilon[A_1, K_1]$$

that converge to the degenerate gain $[A, K]$ as $\epsilon \to 0$. For the standard feedback configuration and using the gain matrix

$$(A + \epsilon A_1)^{-1}(K + \epsilon K_1)$$

the closed loop polynomial has the same roots as:

$$p_\epsilon(s) = \det\left\{ S(\epsilon) \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \right\} = \det\left\{S(\epsilon)M(s)\right\} \qquad (3)$$

where $p_\epsilon(s)$ tends to the prime polynomial $p(s)$ as $\epsilon \to 0$.

Remark 4. When $rowspan[A, K]$ is a degenerate gain, the prime polynomial $p(s)$ is not unique and depends on the direction $[A_1, K_1]$ and as the following theorems state (Leventides and Karcanias, 1995) the relation between them is linear.

Theorem 5. Let $rowspan[A, K]$ be a degenerate gain and $S(\epsilon)$ a sequence of gains converging to it. Then the corresponding sequence of closed-loop polynomial coefficient vectors $\left\langle \underline{p}_\epsilon \right\rangle$ converges as $\epsilon \to 0$ to a vector $\langle \underline{p} \rangle \in \mathrm{P}(\mathbb{R})^n$ which depends on $[A_1, K_1]$; furthermore the function τ which maps the direction $[A_1, K_1]$ to $\langle \underline{p} \rangle$ is linear.

□

The matrix representation of the linear map τ can be deduced from the next theorem (Leventides and Karcanias, 1995):

Theorem 6. Let $\mathcal{D} = rowspan[A, K]$ be a degenerate point of a system defined by the composite coprime MFD representation $M(s)$; then the prime polynomial of the given system with respect to $\mathcal{D}$ and the direction $[A_1, K_1] = [b_{ij}]$ can be written as:

$$p(s) = \sum \left(b_{ij} \times p_{ij}(s)\right) \qquad (4)$$

where $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, p + m$ and $p_{ij}(s)$ is the determinant of the $p \times p$ polynomial matrix $D_{ij}(s)$ having the same rows as the matrix $[AD(s) + KN(s)]$ apart from the $i-$th, which is replaced by the $j-$th row of $M(s)$.

□

The prime polynomial, in terms of its coefficient vector $\underline{p}$ can be written in a linear matrix form as:

$$\underline{p} = \mathbf{L}\underline{k} \qquad (5)$$

where $\underline{k}$ is the vector formed by all the columns of the direction $[A_1, K_1]$ and $\mathbf{L}$ denotes the linearisation matrix, i.e. the matrix representation of the linear map, that is the coefficient matrix of the polynomial vector $\left[p_{11}(s), p_{12}(s), \ldots, p_{p(p+m)}(s)\right]$ as described in Theorem 6.

The theoretical background for the Global Linearisation method has been established in (Leventides and Karcanias, 1995) for the constant output feedback problem, where a theoretical procedure for the construction of (approximate) solutions is given; whereas in (Leventides and Karcanias, 1998) the method is extended to cover the dynamic output feedback problem as well.

3. COMPUTATIONAL SCHEME: THE PREDICTOR-CORRECTOR METHOD

The Global Linearisation method for the SOF pole assignment problem is based on the results of Theorem 5. The linearisation method, as a constructive method can provide solutions which allows considerably large number of states in the open loop system compared with the existing ones and with feedback compensators which in general are of low order. The disadvantage is that it has inherent certain limitations which stems from the fact that the method is based on a point of singularity of the feedback configuration, that is the degenerate compensator. Solutions close to the degenerate point, have infinite sensitivity and they result to an explosion of the norm of the sensitivity function $[I + K(\epsilon)G(s)]^{-1}$ and hence small perturbations in the parameters may result to very big perturbations in the set of closed-loop poles. Thus, such solutions, have only a theoretical significance. Using, however, this degenerate compensator (which can be found easily) and assuming that the resulting linearisation matrix is of full rank, the following proposed numerical scheme can be used iteratively to provide solutions in closed form far from the degenerate compensator and thus with improved sensitivity.

In the following, let $\underline{k} \equiv [k_{ij}] = [k_{11}, k_{12}, \ldots, k_{mp}, \ldots, k_\sigma]^T$ with $\sigma = \begin{pmatrix} m+p \\ p \end{pmatrix} - 1$ be all the elements k_{ij} of the augmented output feedback matrix $\tilde{\mathbf{K}} \in \mathbb{R}^{p(m+p)}$, stacked in one vector, whose elements are defined as inhomogeneous coordinates of the Grassmann space, $Grass(p, m+p)$ and are constrained in Quadratic Plucker Relations (QPRs); let also $\underline{p} = [1, a_1, a_2, \ldots, a_n]^T \in \mathbb{R}^{n+1}$ be the vector contains all the coefficients of the target polynomial $p(s)$ we want to assign, i.e.

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_n.$$

Let also define the differential of the Pole Placement Map F as the $(n+1)\times p(m+p)$ matrix, symbolized as $DF_{\underline{k}}$, which is the Jacobian $\partial F_i/\partial k_j$, evaluated at a given solution $\underline{k}$.

Based on the above setting, the problem under investigation can be formulated as the integration of a high-order differential equation which is defined as

$$\mathrm{DF}_k \cdot \dot{\underline{k}} = \underline{p}\ ,\ k(0) = \mathbf{K_0} : \text{degenerate point} \qquad (6)$$

and therefore we can use numerical integration methods, or homotopy continuation methods, in order to provide adequate linearised solutions in a closed form. The following numerical scheme proposed here guarantees the maximum distance from the degenerate point by maximizing the angle between the degenerate compensator and the final one.

3.1 Numerical Scheme

Solution of (6) can be achieved by using a *Predictor-Corrector* iterative scheme. The numerical procedure requires as input data: the given MIMO (p, m, n)-system described by the composite MFD $M(s) \in \mathbb{R}^{(m+p)\times p}$; the real coefficient vector $\underline{p} \in \mathbb{R}^{n+1}$ of the closed loop polynomial to be assigned and the degenerate compensator $\mathbf{K_0}$ which fulfils the pole placement equations at limit and can be constructed easily as described in Section 2. The maximum number of iterations, the step size Δt and the degrees of proximity (or tolerances) has to be given initially as well.

Before we start applying the Predictor-Corrector iterative scheme and proceed to the computation of solutions, a degenerate point which satisfies the necessary conditions for pole assignment must be computed. This is a point of singularity with infinite sensitivity where the closed-loop characteristic polynomial is not well defined, however, we use it as a starting point to calculate a series of solutions with lower sensitivity far from the degenerate point by tracing a holomorphic curve in $Grass(p, m+p)$.

Secondly, the initial step-size Δt must be chosen in a systematic way in accordance with the desired predictor tolerance (*etol1*). For the particular degenerate point (which fulfils the necessary condition for PA) and the target closed loop polynomial a parametric function is computed as:

$$f(\varepsilon) = coef\{(\mathbf{K_0} + \varepsilon B)\cdot M(s)\}$$

where B is the direction via we approach the particular degenerate point. Next, the graph of

$$g(\varepsilon) = \left\| f(\varepsilon) - \varepsilon \cdot \underline{p} \right\|$$

is produced, which is the one shows the distance from the curve of the target polynomial $\underline{p}$ as ε varies.
For the desired predictor tolerance (*etol1*) we would like to have we choose the corresponding Δt from the graph $g(\varepsilon)$. The corrector tolerance (*etol2*) should be smaller than (*etol1*) and in particular we fix it as $etol2 = \gamma \cdot etol1$, where $\gamma \simeq 0.02 - 0.07$.

Following that, the Pole Assignment Map F and the differential of F at the degenerate compensator, DF_{K_0}, needs to be calculated. If DF_{K_0} has full rank then we may proceed to compute the series of solutions $\mathbf{K}_{i+1}$ for $i = 0, 1, 2, ...$ using the recursive scheme as described below.

The Predictor-Corrector method consists of repeatedly performing predictor and corrector steps. In our implementation we utilize the well known Euler method as Predictor, as shown below:

$$\underline{\mathbf{K}}_{i+1} = \underline{\mathbf{K}}_i + \Delta t \cdot \left[DF_{\underline{\mathbf{K}}_i}\right]^{\dagger} \cdot \underline{p} \qquad (7)$$

whereas for Corrector steps we will make use of a Newton-type iterative scheme such as:

$$\underline{\mathbf{K}}_{j+1} = \underline{\mathbf{K}}_j - [DF_{\underline{\mathbf{K}}_j}]^{\dagger}(F(\underline{\mathbf{K}}_j) - t\cdot \underline{p}) \qquad (8)$$

The basic steps of the algorithm in pseudo-code are given in *Algorithm 1*.

Algorithm 1 Predictor-Corrector Iterative Scheme

Input: $M(s)$, $p(s)$, K_0, $etol1$, $etol2$, Δt and $maxiter$
Output: The Output feedback matrix $K \in \mathbb{R}^{p\times m}$

```
Compute the PPM: F
Compute the differential of the PPM: D(F) ≡ DF
K_i ← K_0
t ← 0
for i = 0 to maxiter do
    Evaluate the differential of the PPM at K_i, denoted
    as DF_{K_i}
    repeat
        K_{i+1} = K_i + [DF_{K_i}]^† * Δt * p
        F(K_{i+1})
        dist = |F(K_{i+1}) − t * p|
        Δt = γ * Δt, where γ < 1 // Step-size adaptation
    until dist ≤ etol1
    t ← t + (1/γ) × Δt
    K_cor ← K_{i+1}
    while dist > etol2 do
        Compute the DF_{K_cor}
        Corrector steps:
        K_cor = K_cor − [DF_{K_cor}]^† * (F(K_cor) − t * p)
        F(K_cor) // Calculate the PPM at K_cor
        dist = |F(K_cor) − t * p|
    end while
    K_{i+1} ← K_cor
end for
```

Using as initial point of the method the degenerate compensator $\mathbf{K} = \mathbf{K}_0$ and starting the iterations from $\epsilon = 0$ and gradually increase it ($\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots$) by using the predefined step size Δt we construct a series of static compensators $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, ...$ etc.
As we have seen before, the bigger the ϵ_i the further away from the degenerate compensator the solution $\mathbf{K}_i$ is and hence with a less sensitivity. Iterations will be continued until the stopping criterion is satisfied, which may be one of the following: (a) maximum iterations; (b) solution reaches a specified degree of proximity.

Remark 7. Note that since the matrix $[DF_K]$ is not a square matrix, in order to compute the solutions of (6)-(8) we need to find the generalized inverse (or pseudoinverse) denoted here by $[DF_K]^{\dagger}$. For that we use the Moore-Penrose pseudoinverse given by $A^{\dagger} = A^T(AA^T)^{-1}$.

For measures of sensitivity we consider the following:

(a) The norm of the Differential (or Jacobian) $\left\| D(F)_{K(\epsilon)} \right\|$ of the Pole Placement Map F evaluated at the final compensator we found.

(b) The angle θ° between the degenerate point $\mathbf{K}_0$ and the final (solution) compensator $\mathbf{K}(\epsilon)$ obtained by

$$\cos\theta = \frac{tr\{K_0 \cdot K(\epsilon)\}}{\|K_0\| \cdot \|K(\epsilon)\}\|} \tag{9}$$

As a measure of accuracy, the norm $\left\| \Delta_{\underline{p}} \right\|$ of the difference of the closed loop polynomial $p_\epsilon(s)$ and the desired prime polynomial $p(s)$ is used.
The following Example illustrates the improved method as described above.

4. EXAMPLE

Consider the proper multivariable system with $p = 3$ inputs, $m = 4$ outputs and $n = 11$ states defined by the composite MFD:

$$M(s) = \begin{bmatrix} s^4 & 0 & 0 \\ 1 & s^4 & s-3 \\ s^3+1 & s-1 & s^3-s^2+1 \\ \hline s^2+3 & s^2+1 & 2s^2-1 \\ s^2+s+1 & s+1 & s+1 \\ s^3-2 & s^3+2s-1 & 2s^2+3s \\ 1 & -1 & s^2+s+1 \end{bmatrix} = \begin{bmatrix} \frac{D(s)}{N(s)} \end{bmatrix} \tag{10}$$

A degenerate point for this system is defined by $\mathcal{D} = rowspan[A, K]$ and calculated as:

$$\mathbf{K}_0 = \begin{bmatrix} 0 & 1 & 0 & -4 & -9 & 0 & 8 \\ 0 & -1 & 0 & -2 & -5 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let for simplicity the desired closed-loop characteristic polynomial be set by $p(s) = (s+1)^{11}$ with a real coefficient vector

$$\underline{p}^T = [1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1]$$

Having constructed the degenerate compensator (i.e. the starting point) the initial step size Δt has to be selected. For $etol1 = 0.5$(predictor tolerance), by using the produced graph $g(\epsilon)$, the corresponding step size should be $\Delta t = 0.55$ and $etol2 = 0.01$ (corrector tolerance).

Using the method as described in Algorithm 1 for a maximum number of 1000 iterations we calculate a set of 1000 different solutions (static output feedback compensators). The results we get are summarized next.
By observing the plot of the distance $\|\Delta_p\|$ in Figure (1) of the closed loop polynomial from the prime polynomial to be assigned we are able to select solutions for various ϵ where the distance $\|\Delta_p\|$ is minimum. For instance, when $\epsilon = 1000$ the distance $\|\Delta_p\| \leqslant 1.093191 \times 10^{-12}$.

As a result we get the augmented output feedback matrix, selected for $\epsilon = 1000$, $[I_p, \mathbf{K}_f]$, where the output feedback compensator K_f which places the poles at the exact desired locations is given by:

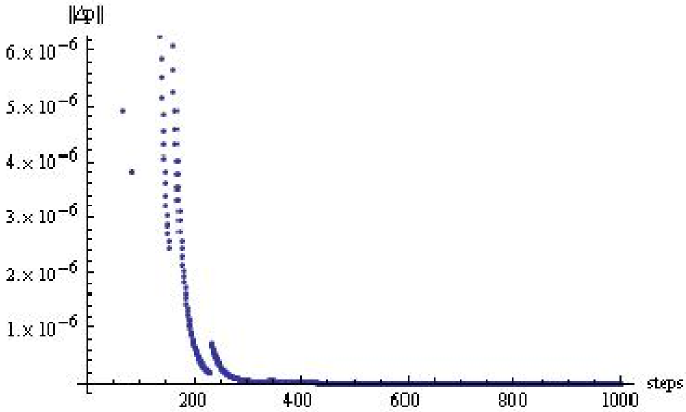

Fig. 1. Distance of closed loop polynomial and the prime polynomial to be assigned

$$\mathbf{K}_f = \begin{bmatrix} -2.772 & -4.169 & -0.133 & 6.697 \\ 49.885 & 76.095 & 1.195 & -117.031 \\ -29.299 & -44.938 & -1.474 & 72.484 \end{bmatrix} \tag{11}$$

with the corresponding closed-loop characteristic polynomial as:

$$p(s) = s^{11} + 11s^{10} + 55s^9 + 165s^8 + 330s^7 + 462s^6 + \\ + 462s^5 + 330s^4 + 165s^3 + 55s^2 + 11s + 1 \tag{12}$$

One may verifies if compute the $\det\{D(s) + \mathbf{K}_f N(s)\}$. It is worth noting that the norm of the differential at the final compensator is calculated

$$\left\| DF_{K_f} \right\| = 316.7$$

whereas the norm of the feedback compensator in (11) is

$$\|K_f\| = 70.189$$

and the angle (in degrees) between the degenerate compensator $\mathbf{K}_0$ and $\mathbf{K}_f$ as defined in (9) is

$$\theta = 46.8768$$

and guarantees the maximum distance from the degenerate point and hence the lower sensitivity solution. The variation of angle θ for all the iterations is indicated in Figure (2).
The advantages of the Global Linearisation framework in contrast with the conventional methods (eg. state-space or LMI methods) are:

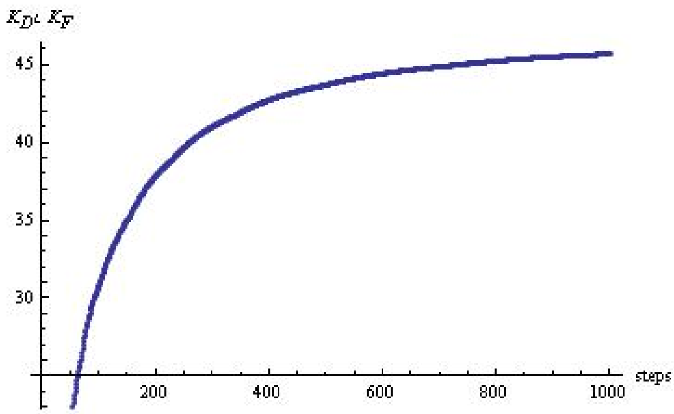

Fig. 2. Angle θ between degenerate and final compensators

a) Provides linearised solutions (in linear matrix form) which gives rise to apply numerical computational procedures;
b) Provides the geometric insight by investigating the Grassmann invariant condition as proposed by (Karcanias and Giannakopoulos, 1984)
c) May be applied to a wide coverage of MIMO systems for pole assignment due to the condition $mp > n$ in contrast with the $m + p > n + 1$ in state-space conventional methods.

More precisely in the improved method presented here has been developed an effective size adaptation, in order to be able to handle and approximate successfully special points on the curve (such as singular or turning points). Current work involves the study of the convergence properties of the numerical method and the implementation of additional objectives in order to achieve optimization goals while achieving pole placement. Possible optimization goals might be:

(i) The feedback matrix should have the minimum norm;
(ii) Minimum sensitivity objective;
(iii) Maximizing the controllability measure of the resulting closed-loop system.

5. CONCLUSION

An improvement of the global linearisation framework has been introduced that reduces the overall sensitivity of the methodology and its inherent dependence on degenerate compensators by using a Predictor-Corrector numerical method based on Homotopy continuation. The modified method guarantees the maximum distance from the degenerate point. The proposed new computational framework for finding output feedback controllers exploits an effective adaptive scheme of the step size Δt, in order to be able to trace successfully complicated manifolds (curves) and improves considerably the sensitivity properties of the scheme. The algorithm can be readily extended to cover the dynamic output feedback pole assignment problem as well. Alternative numerical techniques which make the method less sensitive to the use of degenerate solutions are currently under investigation. The convergence properties of the predictor-corrector method, as presented here, needs to be examined as well in the near future. Furthermore, the selection of the degenerate point around which global linearisation is achieved is a possible factor that affects the overall performance and convergence properties. Hence, the optimal selection of degenerate points is still an open issue which need to be further investigated in a systematic way.

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