# Time-dependent massless Dirac fermions in graphene

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Abstract: Using the Lewis-Riesenfeld method of invariants we construct explicit analytical solutions for the massless Dirac equation in 2+1 dimensions describing quasi-particles in graphene. The Hamiltonian of the system considered contains some explicit time-dependence in addition to one resulting from being minimally coupled to a time-dependent vector potential. The eigenvalue equations for the two spinor components of the Lewis-Riesenfeld invariant are found to decouple into a pair of supersymmetric invariants in a similar fashion as the known decoupling for the time-independent Dirac Hamiltonians.

The two dimensional massless Dirac equation has recently attracted a lot of renewed attention because it describes quasi-particles in graphene [1, 2, 3], which is well known to possess a large amount of remarkable properties. Especially the Dirac equation in the presence of a magnetic field is of great interest as, unlike electrostatic potentials, such a configuration allows in principle to confine the Dirac fermions [4, 5, 6]. Many exact solutions have been provided for a variety of time-independent Hamiltonians and magnetic field configurations [7, 8, 9], including some for complex magnetic fields leading to pseudo/quasi-Hermitian interactions [10, 11]. While some solutions for the time-dependent Dirac equation in 1+1 dimensions have been constructed [12, 13, 14], little is known about the time-dependent setting with a magnetic field in 2+1 dimensions and no exact solutions have been reported. The aim of this manuscript is to commence filling that apparent gap. We shall demonstrate that the Lewis-Riesenfeld method of invariants [15] is a technique which can be employed successfully to solve this problem.

We consider here the time-dependent massless Dirac equation in two spacial dimensions in the form

$$H\mathbf{\Psi} = i\partial_t \mathbf{\Psi}, \qquad \text{with } H(x,y,t) = \boldsymbol{\sigma}\cdot\mathbf{p}, \tag{1}$$

and the two component wave function $\mathbf{\Psi} = (\psi_1, \psi_2)$. The effective Hamiltonian includes $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ comprised of the standard Pauli matrices and $\mathbf{p}$ being the two-dimensional momentum vector minimally coupled to a vector potential $A = (A_x, A_y)$ in the standard fashion $\mathbf{p} = (a(t)p_x + A_x, b(t)p_y + A_y)$. Besides the time-dependence entering through the vector potential, we allow here also for explicit time-dependent factors in front of the momenta, $a(t)$ and $b(t)$. One possibility to think of these factors is that they result from a time-dependent background, as discussed in [16] or alternatively as time-dependent velocities. The velocity of light, the reduced Planck constant and the charge are all set to one in our discussion, i.e., $c = \hbar = e = 1$.

We make now some specially simply choices for the vector potential by taking $A_x = 0$ and $A_y(x, t) = g(t)x$. In addition, we assume that the wave function separates with a free plane wave moving in the $y$-direction

$$\mathbf{\Psi}(x, y, t) = e^{i(ky-\omega t)}\mathbf{\Phi}(x, t), \tag{2}$$

with wave number $k$ and frequency $\omega$, such that we are left with the task to solve

$$\hat{H}\mathbf{\Phi} = i\partial_t\mathbf{\Phi}, \qquad \text{with } \hat{H}(x, t) = a(t)\sigma_x p_x + [kb(t) + g(t)x]\,\sigma_y - \omega\mathbb{I}, \tag{3}$$

for the two component wave function $\mathbf{\Phi} = (\phi_1, \phi_2)$. The first observation we make here is that the form of the explicit time-dependence does not allow for the standard decoupling of the systems into a pair of Hamiltonians related to each other by intertwining operators as common in the time-independent Dirac equation in analogy to standard supersymmetric quantum mechanics [17, 18].

We will attempt to solve equation (3) by using the Lewis-Riesenfeld method [15] originally designed to solve the time-dependent Schrödinger equation. The first step in this approach consists of solving the evolution equation

$$\frac{dI(t)}{dt} = \partial_t I(t) + \frac{1}{i}[I(t), \hat{H}(t)] = 0, \tag{4}$$

for the Hermitian time-dependent invariant $I(t)$. As usual in this context we take the invariant to be of the same order and form in the canonical variables as the Hamiltonian

$$I(t) = \alpha(t)p_x + \beta(t)x + \gamma(t), \tag{5}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are now unknown time-dependent matrices. Substituting our Ansatz (5) into the evolution equation (4) yields six constraining equations for the three coefficient matrices

$$[\alpha, \sigma_x] = 0, \qquad [\beta, \sigma_y] = 0, \qquad g\,[\alpha, \sigma_y] + a\,[\beta, \sigma_x] = 0, \tag{6}$$

$$-i\dot{\alpha} = kb\,[\alpha, \sigma_y] + a\,[\gamma, \sigma_x]\,, \tag{7}$$

$$-i\dot{\beta} = g\,[\gamma, \sigma_y]\,, \tag{8}$$

$$-i\dot{\gamma} = kb\,[\gamma, \sigma_y] + ia\sigma_x\beta - ig\alpha\sigma_y. \tag{9}$$

Expanding the matrices in the $su(2)$-basis, $\alpha(t) = \alpha_1(t)\mathbb{I} + \alpha_2(t)\sigma_x + \alpha_3(t)\sigma_y + \alpha_4(t)\sigma_z$ with $\alpha_i(t) \in \mathbb{R}$ for $i = 1, 2, 3, 4$ and similarly for $\beta(t)$, $\gamma(t)$, these equations are straightforward to solve. Starting with (6), the first two equations immediately imply that $\alpha_3 = \alpha_4 = 0$ and $\beta_2 = \beta_4 = 0$. The last equation in (6) then yields $\beta_3 = \alpha_2 g(t)/a(t)$. Proceeding in this way for (7)-(9), we find the following form for the time-dependent invariant

$$I(t) = (\alpha_1 p_x + \gamma_1)\,\mathbb{I} + \alpha_2 p_x \sigma_x + [\beta_3 x + \gamma_3(t)]\,\sigma_y, \tag{10}$$

with constants $\alpha_1, \gamma_1, \alpha_2, \beta_3$. The time-dependence of $I(t)$ is entirely contained in the function $\gamma_3(t)$ which is constrained by

$$\dot{\gamma}_3(t) = \alpha_1 g(t), \qquad \text{and} \qquad \gamma_3(t) = k\alpha_2 \frac{b(t)}{a(t)} = k\beta_3 \frac{b(t)}{g(t)}. \tag{11}$$

In addition we found that $a(t) = \mu g(t)$ with $\mu = \alpha_2/\beta_3$ being constant has to be satisfied. The equations (11) are most conveniently solved in terms of $b(t)$

$$\gamma_3(t) = \left(2k\alpha_1\beta_3 \int^t b(s)ds\right)^{1/2}, \qquad g(t) = \frac{k\beta_3}{2\alpha_1} b(s) \left(\int^t b(s)ds\right)^{-1/2}. \tag{12}$$

The next step in the Lewis Riesenfeld approach consists of solving the eigenvalue equation for the time-dependent invariant, i.e., we need to solve

$$I(t)\boldsymbol{\chi}(t) = \lambda \boldsymbol{\chi}(t), \tag{13}$$

for the time-dependent eigenfunction $\boldsymbol{\chi}(t) = (\chi_+(t), \chi_-(t))$ and time-independent eigenvalues $\lambda$. For this purpose we note at first that we can write (13) as

$$\begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{1}{\alpha_2}(\lambda - \alpha_1 - \gamma_1) \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}. \tag{14}$$

Thus we notice that unlike as the time-dependent Hamiltonian the invariant equation can be decoupled easily and acquires the form of a supersymmetric pair. Acting again with the off-diagonal invariant operator on (14) we obtain the two decoupled equations

$$I_\pm \chi_\pm = \left[p_x^2 + W^2 \pm W'\right] \chi_\pm = \frac{1}{\alpha_2^2}(\lambda - \alpha_1 - \gamma_1)^2 \chi_\pm, \tag{15}$$

for the two operators $I_\pm := L_\mp L_\pm$, where $W(x,t) = [\gamma_3(t) + \beta_3 x]/\alpha_2$ is the analogue to the superpotential in standard supersymmetric time-independent quantum mechanics. We observe that the potential is still time-dependent, but now a simple re-definition of our variables will move this dependence entirely into $\chi_\pm$. Defining the new time-dependent variable $\xi(t) = [\gamma_3(t) + \beta_3 x]/\alpha_2$ converts (15) into two eigenvalue equations for the time-independent quantum harmonic oscillator

$$\left(-\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{\mu^2}{2}\xi^2\right)\chi_\pm = \tilde{\lambda}^\pm \chi_\pm, \tag{16}$$

with $2\tilde{\lambda}_\pm = (\lambda - \alpha_1 - \gamma_1)^2/\beta_3^2 \mp \mu$. Demanding $\chi_\pm(\xi)$ to be a square integrable function $L^2(\mathbb{R}, d\xi)$ the solution is of course

$$\chi_{\pm,n}(\xi) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\mu}{\pi}\right)^{1/4} e^{-\mu/2\xi^2} H_n(\sqrt{\mu}\xi), \qquad \tilde{\lambda}_n^\pm = \mu\left(n + \frac{1}{2}\right), \tag{17}$$

with $H_n$ denoting the $n$-th Hermite polynomial. This means the two eigenvalues for the time-dependent spinor components of the invariant in (13) quantize to

$$\lambda_{n,s}^{\pm} = \alpha_1 + \gamma_1 + \beta_3 s\sqrt{\mu(2n+1\pm 1)}, \tag{18}$$

with $s = \pm 1$ being two possible signs of the square root. As expected these eigenvalues are indeed time-independent. Due to the supersymmetric structure we have the standard shift in the eigenvalues, that is $\lambda_{n+1}^{-} = \lambda_n^{+}$.

As argued by Lewis and Riesenfeld [15] the eigenfunction of the Hamiltonian and the invariant just differ by a phase

$$|\mathbf{\Phi}_n\rangle = e^{i\delta(t)}\, |\boldsymbol{\chi}_n\rangle \tag{19}$$

where the real function $\delta(t)$ in (19) must obey

$$\frac{d\delta(t)}{dt} = \langle \boldsymbol{\chi}_n|\, i\partial_t - \hat{H}(t)\, |\boldsymbol{\chi}_n\rangle\,. \tag{20}$$

The right hand side of (20) can be computed directly with our known eigenfunctions. We obtain

$$\begin{aligned}\langle \boldsymbol{\chi}_n|\, i\partial_t - \hat{H}(t)\, |\boldsymbol{\chi}_n\rangle &= \left\langle \chi_{+,n}\right| i\partial_t + \omega \left|\chi_{+,n}\right\rangle + \left\langle \chi_{+,n}\right| - ap_x + i(kb+gx) \left|\chi_{-,n}\right\rangle \qquad (21)\\ &\quad + \left\langle \chi_{-,n}\right| - ap_x - i(kb+gx) \left|\chi_{+,n}\right\rangle + \left\langle \chi_{-,n}\right| i\partial_t + \omega \left|\chi_{-,n}\right\rangle \\ &= 2\omega, \qquad (22)\end{aligned}$$

where we used the orthogonality of the eigenfunctions, $\left\langle \chi_{\pm,n}\right| \partial_t \left|\chi_{\pm,n}\right\rangle = 0$, $\left\langle \chi_{+,n}\right| p_x \left|\chi_{-,n}\right\rangle = -\left\langle \chi_{-,n}\right| p_x \left|\chi_{+,n}\right\rangle$ and $\left\langle \chi_{+,n}\right| x \left|\chi_{-,n}\right\rangle = \left\langle \chi_{-,n}\right| x \left|\chi_{+,n}\right\rangle$. The phase therefore simply becomes

$$\delta(t) = 2\omega t. \tag{23}$$

Thus assembling the results from equations (2), (17), (19) and (23) provides an exact solution to the Dirac equation (1).

We have demonstrated that the Lewis-Riesenfeld method can be applied to construct solutions to the 2+1 dimensional time-dependent Dirac equation. The time-dependence resulted from a background and a magnetic field. Exact solutions for this type of scenario have not been known previously. Clearly there are plenty of open problems and challenges left. For instance, just as in the time-independent scenario one would like to know exact solutions for more complicated vector field configurations, different background scenarios and possibly different assumptions about the motion in the $y$-direction. These tasks are left for future work, where this note can be taken as encouragement as it demonstrates the successful applications of a method to tackle these kind of problems.

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