From Local to Global Control

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***Abstract*— In this paper the authors study the problem of the existence of multiple local operating points in control systems. In particular, they consider a method of going from local to global control, i.e. given a number of local, linearized systems, from which global system do they come?, and, can global controllers be determined in this case?.**

I. INTRODUCTION

In many practical applications of the control of nonlinear systems, the dynamics are usually linearised about "operating points" (such as trim conditions in aircraft). These local models are then used to design local controllers and some gain scheduling procedure is used to switch between the controllers at different operating points. In fact, in many industrial plants or aerospace systems, only a small number of operating conditions are known and the global nonlinear dynamical system is not known. Gain scheduling methods are widely used and are object of many research papers, (see [1], [2], [3] and references within as example). In this paper, the problem of going from some known local models to a global one will be considered. Furthermore, the obtained global model can then be used instead of gain scheduling for control purposes.

The contains of this paper are as follows: *Section II* recalls the basics of nonlinear systems linearisation procedure, operating conditions and trackability properties. *Section III* introduces an algebraic method that allows the reconstruction of an unknown nonlinear system taking as starting points its linear representations at some given operating conditions. *Section IV* presents a global control method that can be used instead of the well-known of gain scheduling once the nonlinear system is known. *Section V* tackles the problem of going from the local models to the topology of the manifold on which the system is defined. *Section VI* contains a summary of the ideas here presented.

II. NONLINEAR SYSTEMS, OPERATING CONDITIONS AND TRACKABILITY

Consider the following nonlinear system:

$$\dot{x} = f(x,u) \tag{1}$$

defined on $\Re^n x \Re^m$. This is in fact, a kind of local model for systems on manifolds - this point will be addressed later.

Intuitively, by an *operating point* it is understood a pair $(x_d(t), u_d(t)) \in \Re$ consisting of an open loop control and a function $x_d(t)$ which satisfies equation (1) when the control $u_d(t)$ is applied, i.e.:

$$\dot{x}_d(t) = f(x_d(t), u_d(t)) \tag{2}$$

On the other hand, the desired function $x_d(t)$ is *trackable* if there exists a control $u_d(t)$ such that (2) holds.

If $x_d(t)$ is trackable for (1), the local variables around the operating point $y(t)$ and $v(t)$, are defined as:

$$\begin{aligned} y(t) &= x(t) - x_d(t) \\ v(t) &= u(t) - u_d(t) \end{aligned} \tag{3}$$

Then by applying Taylor′s theorem:

$$\begin{aligned} &\dot{y}(t) = \dot{x}(t) - \dot{x}_d(t) = f(x,u) - f(x_d(t), u_d(t)) \\ &\cong A(t)y(t) + B(t)v(t) \end{aligned} \tag{4}$$

where

$$A(t) = \frac{\partial f(x_d(t), u_d(t))}{\partial x}$$

and

$$B(t) = \frac{\partial f(x_d(t), u_d(t))}{\partial u}$$

for ′small′ $y(t)$ and $v(t)$.

If $(A(t), B(t))$ is a controllable pair, then the control of system (1) around the operating condition is achievable by using the control:

$$u(t) = u_d(t) + v(t) \tag{5}$$

In general, there could be a number i of these operating points:

$$\left(x_d^{(i)}, u_d^{(i)}\right) \in C\left([0,\infty]; \Re^{n+m}\right), \quad 1 \le i \le K \tag{6}$$

associated to a correspondent number i of $'$local models$'$ of the nonlinear system (1):

$$\dot{y}^{(i)}(t) = A^{(i)}(t)y^{(i)}(t) + B^{(i)}(t)v^{(i)}(t), \tag{7}$$

The problem to be considered now is the inverse one: How to go from a set of i local models (7) to the global one (1)?. For simplicity, it will be assumed that all the operating points are constant, so that the local models are linear, time-invariant systems of the form:

$$\dot{y}^{(i)} = A^{(i)}y^{(i)}(t) + B^{(i)}v^{(i)}(t), \quad 1 \le i \le K \tag{8}$$

Also, in this case, (2) becomes an algebraic condition of the form:

$$f(x_d^{(i)}, u_d^{(i)}) = 0 \tag{9}$$

if the values of $x_d^{(i)}$ are constant, $1 \le i \le K$.

Having obtained a global model from the i local ones, a global controller which will drive the system from one operating point to another has to be determined. This will involve an application of an iteration scheme [4] which replaces a nonlinear (not necessarily quadratic) optimal control problem which is linear (time-varying) and quadratic, which can be solved by classical methods.

Finally, the authors will consider the tracking global problem of determining (to some degree) the topology of the manifold on which a nonlinear system is defined from a knowledge of the local representatives assuming that the local systems are complete in the sense that their defining neighborhoods cover the manifold.

III. FROM LOCAL TO GLOBAL

Suppose the set of k unknown constant operating points:

$$\left(x_d^{(i)}, u_d^{(i)}\right), \quad 1 \le i \le k \tag{10}$$

at which there exist k linearisations of the form:

$$\dot{y}^{(i)}(t) = A^{(i)}y^{(i)}(t) + B^{(i)}v^{(i)}(t), \quad 1 \le i \le k \tag{11}$$

of some unknown nonlinear system on $\Re^n$.

If the unknown system is of the form (1)

$$\dot{x} = f(x,u) \tag{12}$$

then

$$\frac{\partial f(x_d^{(i)}, u_d^{(i)})}{\partial x} = A^{(i)}, \tag{13}$$

$$\frac{\partial f(x_d^{(i)}, u_d^{(i)})}{\partial u} = B^{(i)}, \tag{14}$$

for $1 \le i \le k$, and moreover, f must satisfy:

$$f(x_d^{(i)}, u_d^{(i)}) = 0, \quad 1 \le i \le k. \tag{15}$$

Consider the equations:

$$0 = f_p(x_d^{(l)}, u_d^{(l)}) \tag{16}$$

$$= \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} \cdots \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} a^p_{i_1\ldots i_n, j_1\ldots j_m} (x_d^{(l)})^i (u_d^{(l)})^j$$

with

$$1 \le l \le k, \quad 1 \le p \le n. \tag{17}$$

and

$$A_{pq}^{(l)} = \sum_{i_1=0}^{N_1} \cdots \sum_{i_q=1}^{N_q} \cdots \sum_{i_n=0}^{N_n} \cdots \sum_{j_1=0}^{M_1} \cdots \sum_{j_m=0}^{M_m} a^p_{i_1\ldots i_n, j_1\ldots j_m} i_q (x_d^{(l)})^{i-1_q} (u_d^{(l)})^j \tag{18}$$

$$B_{pq}^{(l)} = \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_u} \cdots \sum_{j_1=0}^{M_1} \cdots \sum_{j_q=1}^{M_q} \cdots \sum_{j_m=0}^{M_m} a^p_{i_1\ldots i_n, j_1\ldots j_m} j_q (x_d^{(l)})^{i} (u_d^{(l)})^{j-1_q} \tag{19}$$

where

$$i = (i_1, \cdots, i_n),$$

$$j = (j_1, \cdots, j_m)$$

and

$$1_q = (0, \cdots, 0, \underbrace{1}_{\text{q}}, 0, \cdots, 0)$$

in the $\prod_{i=1}^n (N_i+1) \prod_{j=1}^m (M_j+1)$ variables $a^p_{i,j}$, where f is assumed to be able to be approximated by a polynomial function. The number of equations is

$$(n + n^2 + nm)k \tag{20}$$

and they can be written in the form:

$$LV = W \tag{21}$$

where L is a linear operator, V is a vector of the unknown parameters $a^p_{i,j}$ and W contains the known local representations $A^{(i)}, B^{(i)}$. L maps $\Re^\alpha$ into $\Re^\beta$, where:

$$\alpha = \Pi(N_i+1)\Pi(M_j+1) \tag{22}$$

and

$$\beta = (n + n^2 + nm)k \tag{23}$$

The system (21) is:

- (i) Overdetermined if $\alpha < \beta$,
- (ii) Determined if $\alpha = \beta$ and L is invertible,
- (iii) Underdetermined if $\alpha > \beta$.

Therefore, according to the above classification, there will be a unique solution in case (ii), a solution in case (i) if $W \in \text{Range}(L)$ and in case (iii) if $\text{Rank}(L) = \beta$.

IV. GLOBAL CONTROL

In the previous section, a way to find a global model from the local models around some operating conditions has been presented. Now, a $'$global$'$ controller which drives the system from one operating condition to another is found.

Suppose the existence of two distinct operating points $(x^{(1)}, u^{(1)})$ and $(x^{(2)}, u^{(2)})$, so that in the obtained global model of the form:

$$\dot{x} = f(x, u) \tag{24}$$

this will be,

$$f(x^{(i)}, u^{(i)}) = 0, \quad i = 1, 2. \tag{25}$$

Now the control objective is to drive the global nonlinear system (24) from $x^{(1)}$ to $x^{(2)}$, this is, to seek a desired trajectory $x_d(t)$ such that:

$$x_d(0) = x^{(1)}, \quad x_d(t_f) = x^{(2)}$$

Therefore, the question is: Does there exist a control $u_d(t)$ so that:

$$\dot{x}_d(t) = f(x_d(t), u_d(t)) \tag{26}$$

In order to answer this question, the notion of *projection field* should be introduced: This is defined as a section of the bundle of projection operators on $\Re^n$ of rank m. Thus, a projection field on $\Re^n$ associates a projection operator $P_x : \Re^n \to \Re^m$ to each point $x \in \Re^n$ such that the function $x \to P_x$ is smooth. The main result can be summarized as:

Theorem: Given a desired trajectory $x_d : \Re^+ \to \Re^n$, there is a control $u_d : \Re^+ \to \Re^m$ satisfying (26), if there exist a projection field $x \to P_x$ such that the function $g(x, u) = P_x f(x, u)$ satisfies:

$$\frac{\partial g(x_d(t), u_d(t))}{\partial u} \neq 0$$

for each $t \in \Re^+$ and $\dot{x}_d(t) \in RP_{x_d(t)}$ for all $t \in \Re^+$, where RP is the range of the projection P.

Proof:

Since $\dot{x}_d \in RP_{x_d}$, then:

$$P_{x_d(t)}\dot{x}_d = P_{x_d(t)} f(x_d(t), u_d(t)) = g(x_d(t), u_d(t))$$

Hence,

$$\dot{x}_d - g(x_d(t), u_d(t)) = 0$$

and since

$$\frac{\partial g(x_d(t), u_d(t))}{\partial u} \neq 0$$

the results follow from the implicit function theorem.$\square$

Suppose there exists a control $u_d(t)$ satisfying the above theorem, then from (24) and (26), it can be written that:

$$\begin{aligned} x - \dot{x}_d &= f(x, u) - f(x_d, u_d) \\ &= f(x - x_d + x_d, u - u_d + u_d) - f(x_d, u_d) \\ &= g(x - x_d, u - u_d, t) \end{aligned} \tag{27}$$

where $g(0, 0, t) = 0, \forall t$.

Hence, if $y = x - x_d$ and $v = u - u_d$, then the system (27) can be written as:

$$\dot{y} = g(y, v, t) \tag{28}$$

therefore, the regulator problem for v can be solved, and then write $u = v + u_d$.

To solve this problem, a well established technique of linear, time-varying approximations to the problem (see [4]) is applied: It is assumed that the system (28) can be written on the form:

$$\dot{y}(t) = A(y, t)y(t) + B(y, t)v(t), \tag{29}$$

In the case the system is not affine in the control, nonlinear control terms can be included in $A(y, t)$ and $B(y, t)$). Now, equation (29) is replaced by the following sequence of LTV systems:

$$\dot{y}^{[i]}(t) = A(y^{[i-1]}(t))y^{[i]}(t) + B(y^{[i-1]}(t))v^{[i]}(t) \tag{30}$$

and to each of the equations (30), the following quadratic cost functional is applied:

$$J = \frac{1}{2}x^{[i]T}(t_f)Fx^{[i]}(t_f) + \frac{1}{2}\int_0^{t_f}\left[x^{[i]T}(t)Qx^{[i]}(t) + u^{[i]T}(t)Ru^{[i]}(t)\right]dt \tag{31}$$

The problems (30) and (31) can then be solved by standard methods (see [5] or [7]).

EXAMPLE

Let (ξ_{10}, ν_{10}), $(\xi_{20}, \nu_{20}) \in \Re^2$ be two points which satisfy:

$$\nu_{10} = \xi_{20} - 2\xi_{10} + \xi_{10}^3$$

$$\nu_{20} = -\xi_{10} + \xi_{20}^3,$$

and suppose the two local systems:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 - \xi_{i0}^2 & -1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot u, \quad i = 1, 2. \tag{32}$$

These systems are local versions of:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 - \xi_{10}^2 & -1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot u \tag{33}$$

and if:

$$x_{1d}(t) = \xi_{10}(1 - t) + \xi_{20}(t)$$

$$x_{2d}(t) = \dot{x}_{1d} - x_{1d} + x_{1d}^3$$

Then the control:

$$u_d = \dot{x}_{2d} + x_{1d} = -\dot{x}_{1d} + 3x_{1d}^2 \cdot \dot{x}_{1d} - x_{1d}$$

will derive the system between the two points.

The system (28) becomes:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & ((y_1+x_{1d})^2+2(y_1+x_{1d})x_{1d}+x_{1d}^2) \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot v \tag{34}$$

and this generates a closed-loop control so that the following expression of $u(t)$,

$$u = v + u_d, \tag{35}$$

will regulate the system between the two given operating values.

V. GLOBAL SYSTEMS ON MANIFOLDS

In this section, a more general problem is considered; that of going from local models to the topology of the ambient manifold on which the system is defined.

Hence, consider a compact n-dimensional differentiable manifold M and let $X(u)$ be a parameterized vector field on M, where $u \in \Re^m$. Let (ϕ_i, U_i), $1 \leq i \leq K$ be a finite covering of M by local parameters, where each $\phi_i : U_i \Re^n$ is a homeomorphism.

In each coordinate neighborhood, the local form of the dynamical system corresponding to the vector field $X(u)\|_{u_i}$ can be written as:

$$\dot{x}_i = f^{(i)}(x^{(i)}, u^{(i)}). \tag{36}$$

It will be assumed that, in each neighborhood U_i there is an operating point $x_d^{(i)} \in \phi_(i)(U_i)$. This, as before, means that there is a (constant) open loop control $u_d^{(i)}$ such that

$$f^{(i)}(x_d^{(i)}, u_d^{(i)}) = 0 \tag{37}$$

Then the new local coordinates are defined,

$$y^{(i)} = x^{(i)} - x_d^{(i)} \tag{38}$$

and the new control

$$v^{(i)} = u^{(i)} - u_d^{(i)} y \tag{39}$$

so the $'$local model$'$ is of the form:

$$y^{(i)} = g^{(i)}(y^{(i)}, v_d^{(i)}) \tag{40}$$

where

$$g^{(i)}(y^{(i)}, v_d^{(i)}) = f^{(i)}(y^{(i)} + x_d^{(i)}, v^{(i)} + u_d^{(i)})$$

Note that $g(0,0) = 0$.

The question is now: Knowing the local models (40) and the operating points $(x_d^{(i)}, u_d^{(i)})$, what can be said about the topology of M?.

It is assumed that complete information is known, in the sense that the coordinate patches on which the local systems are defined cover the (unknown) ambient manifold. So the problem is, how do they fit together?. Taking zero controls from (40):

$$\dot{y}^{(i)} = g^{(i)}(y^{(i)}, 0), \quad 1 \leq i \leq K$$

each of these systems is defined on some region V_i, say of $\Re^n$.

Now, it is said that two local systems

$$\dot{y}_{(i)} = g^{(i)}(y^{(i)}, 0), \quad \dot{y}_{(j)} = g^{(j)}(y^{(j)}, 0) \tag{41}$$

are *compatible* if there exist a diffeomorphism $\phi_{ij} : \hat{V}_{(i)} \hat{V}_{(j)}$ from some nonempty subset $\hat{V}_{(i)}$ of V_i onto some subset $\hat{V}_{(j)}$ of V_j such that the system is a topologically conjugate on $\hat{V}_j$, i.e. ϕ_{ij} maps trajectories of systems (i) on those of system (j). Note that, if these two systems are compatible, then

$$y^{(i)}(t) = \phi_{ij}(y^{(i)}(t))$$

and so

$$\dot{y}^{(j)} = \frac{\partial \phi_{ij}}{\partial y^{(i)}} \dot{y}^{(i)}$$

i.e.,

$$g^{(j)}(y^{(i)}, 0) = \frac{\partial \phi_{ij}}{\partial y^{(i)}} g^{(i)}(y^{(i)}, 0).$$

Hence the matrices $\frac{\partial \phi_{ij}}{\partial y^{(i)}} g^{(i)}$ from the transmition matrices for this tangent bundle of a manifold. The transition matrices are denoted by

$$\gamma_{(ij)} = \frac{\partial \phi_{ij}}{\partial y^{(i)}}(y^{(i)})$$

Note that they obey the standard cocycle conditions:

$\gamma_{ij} - \gamma_{jk} = \gamma_{ik}$ on $U_i \cap U_j \cap U_k$,
$\gamma_{ii} = I$,
$\gamma_{ij} \cdot \gamma_{ji} = I$ on $U_i \cap U_j$.

For example, for a sphere S^2, regarded as $S^2 = \mathbb{C} \cup \{\infty\}$:

$$g_{\infty 0} : U_0 \cap U_\infty \to GL(I, \mathbb{C})$$

where $g_{\infty 0}(z) = \frac{1}{z^n}$ for each integer n and U_0, U_∞ are neighborhoods of 0 and ∞ respectively. This defines a complex line bundle on S^2, usually denoted by H^n.

A *connection* on the vector bundle defined by transition functions γ_{ij} is a collection of differential operators $d + w_i$ defined on U_i such that $w_i = \gamma_{ij} d\gamma_{ij}^{-1} + \gamma_{ij} w_j \gamma_{ij}^{-1}$ on $U_i \cap U_j$ where d is the exterior derivative. These glue together to give a global map:

$$d_A : \Omega^0(E) \to \Omega^1(E) \tag{42}$$

for a bundle E.

The *curvature* of d_A is (d_A^2) and is represented locally by a matrix K_i of two-forms for which

$$K_i = \gamma_{ij} K_j \gamma_{ij}^{-1} \tag{43}$$

The K^{th} characteristic class of the bundle is defined by

$$\tau_k(E) = [\tau_k(A)] \in H^{2k}(M; \Re) \tag{44}$$

where $\tau(A)$ =trace$\left[(\frac{i}{2\pi}K_j)^k\right]$. The first Chem class $c_1(E) = \tau_1(E)$.

Then, $< c_1(\tau M), [M] >= \frac{1}{2\pi}\int_M KdA = X(M)$ by the Gauss-Bonnet theorem, so the Chem class states if the system is trivial or not.

EXAMPLE

The systems:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot u \tag{45}$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot u \tag{46}$$

Generate a system in S^2, since the transition functions look like $g_{\infty 0}(z) = \frac{1}{z}$ in local coordinates. It can be seen that $< c, (\tau M), [M] >= 2 = X(M)$

VI. Conclusions

In this paper the authors have considered the problem of piecing together a set of given local systems to form a global one on some manifold. If the information is incomplete (as in most practical cases) interpolation methods can be used and if it is complete, then topological manifold theory will be used to describe the ambient manifold. Using an approximation method for the obtained nonlinear system, global controllers can be now designed in order to drive the system from one local operating point to another.

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