Numerical Algebraic Geometry: A New Perspective on String and Gauge Theories

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Abstract

The interplay rich between algebraic geometry and string and gauge theories has recently been immensely aided by advances in computational algebra. However, these symbolic (Gröbner) methods are severely limited by algorithmic issues such as exponential space complexity and being highly sequential. In this paper, we introduce a novel paradigm of numerical algebraic geometry which in a plethora of situations overcomes these shortcomings. Its so-called 'embarrassing parallelizability' allows us to solve many problems and extract physical information which elude the symbolic methods. We describe the method and then use it to solve various problems arising from physics which could not be otherwise solved.

Contents

1 Introduction

Increasingly is the field of computational algebraic geometry becoming a significant tool to contemporary theoretical physics [1]. This is becoming ever more transparent especially in string theory and in gauge theory; indeed, the plethora of vacuum configurations in string and M-theory is clearly best suited for large-scale intensive computerization [2], so too can new hidden structures in gauge theory be uncovered by computing new geometric invariants (cf. e.g. [3–5]).

The rapid progress in algorithmic geometry as well as computer power has been a crucial ingredient to such advances. Key to much of the algorithms analyzing simultaneous multi-variate polynomial systems of high degree, to which the aforementioned physical situations can ultimately be reduced, is the Gröbner basis technique (for an excellent review in this context, cf. [6]). Roughly speaking, given the ideal generated by the vanishing set of the polynomials, the so-called Buchberger Algorithm (BA) or its refined variants can compute a new system of equations, called a Gröbner basis [7].

The BA reduces to Gaussian elimination in the case of linear equations. Similarly it is also a generalization of the Euclidean algorithm for the computation of the Greatest Common Divisors of a univariate polynomial. Recently, more efficient variants of the BA have been developed to obtain a Gröbner basis, e.g., F4 [8], F5 [9] and Involution Algorithms [10]. Symbolic computation packages such as Mathematica, Maple, Reduce, etc., have built-in commands to calculate a Gröbner basis. Moreover, Singular [11], COCOA [12] and Macaulay2 [13] are specialized packages for Gröbner basis and Computational Algebraic Geometry, available as freeware, and MAGMA [14] is also such a specialized package available commercially.

In [5, 15, 16], the power of Gröbner basis methods was harnessed for the sake of answering questions pertinent to string and particle phenomenology. A publicly available computational package, called Stringvacua, is designed to interface Mathematica with Singular and has built-in utilities for string phenomenology. By extracting important information such as the dimension of the ideal, the number of real roots in the system, etc., many issues such as stability or supersymmetry of the potential or the branches of moduli space of vacua, can be settled using only a regular desktop machine.

However, even with such reassuring techniques, there are a few problems with symbolic methods: the BA is known to suffer from *exponential complexity*, both the RAM required by the machine and the computation time increases exponentially with the number of variables, equations, degree and terms in each polynomial. Thus, often for even seemingly small sized systems, one may not be able to compute a Gröbner basis. Furthermore, BA is also usually less efficient for systems with irrational coefficients and one habitually has to resort to randomizing over the space of integer (or prime) coefficients and work over finite fields. Another drawback is the highly sequential nature of the BA and thus efficient

parallelizations currently do not exist.

When the information we are after is not sophisticated quantities such as cohomology of vector bundles, which, for example is central to understanding particle spectra and Yukawa couplings in heterotic compactifications, it is natural to wonder whether one could appeal to numerics. Indeed, in a typical situation when one is confronted with the minimization of a potential to see how many isolated solutions there are, what are their approximate values and whether supersymmetry is preserved, numerical recipes specifically tailored to polynomial systems would seem ideal.

The purpose of this paper is to explain precisely such a method, called the **Numerical Algebraic Geometry** (NAG) and to introduce it in our context. The method was first used in lattice field theory [17] where all the stationary points of a multivariate function called the lattice Landau gauge fixing functional [18–21] needed to be found. We will see how the shortcomings of the Gröbner basis methods may be overcome with this numerical method.

In general, the problems we encounter fall into two categories: (1) there is a finite number of isolated extrema and (2) there is a continuum of solutions. The first constitutes zero-dimensional ideals and arises, for example, when finding the number and numerical positions of minima of a given potential, be it from a heterotic string model or from a cosmological background. The second gives us an affine algebraic variety and involves queries from an algebraic-geometric perspective. These could come from analysis of the space of vacua of scalar fields in a supersymmetric gauge theory, or from complex geometries arising from AdS/CFT. The central method in NAG is the numerical polynomial homotopy continuation (NPHC): it can be shown that for a given system of polynomial equations to be solved, a homotopy between the system and a new system (which is easier to solve and share many features with the former system) can be constructed (see [22, 23] for a detailed description of this method in this context). Then, one tracks paths starting from each solution of the new system as one moves towards the original system along the homotopy, to finally obtain all the solutions of the original system. The NPHC method is used to find all the complex (which obviously include real) solutions of a system which is known to have only isolated solutions, i.e., the first category of the systems. The NPHC method has been immensely useful in various areas of theoretical physics including statistical mechanics, particle phenomenology, string phenomenology, lattice field theories, etc. [17, 18, 22–28]. For the second category of systems, the NPHC method can be extended to describe the solution space completely (i.e., dimension, degree, and number of irreducible components).

Indeed, one of the greatest computational hurdles to Gröbner techniques is the primary and irreducible decomposition of ideals which gives crucial data of the branches of the moduli space. We will address ample examples from both categories, demonstrating how many questions can be settled by NAG.

The organization of the paper is as follows. We begin in § 2 by describing

the NPHC method, first pedagogically for a univariate example before generalizing it to the multivariate polynomial ideal. We then apply the method to the two categories of circumstances. In § 3, we treat several systems from heterotic and M-theory models which the symbolic methods are already known to be prohibitively difficult. Next, in § 4, we discuss how to perform numerical irreducible decomposition to systems which are positive dimensional, and apply to four different situations arising from string and gauge theories. We conclude with prospects in § 6.

2 The Numerical Polynomial Homotopy Continuation Method

In this section, we will outline one of the most powerful tools in numerical algebraic geometry, the so-called *numerical polynomial homotopy continuation method*, or NPHC. Since this method is already introduced in particle physics and statistical mechanics areas in [17, 18, 22–25], we will only give a brief explanation as to the principles and algorithms involved as well as advertise some available softwares. We note that in these references only one kind of homotopy, called the Total Degree Homotopy, was used whereas in the present paper we introduce two new kinds of homotopies, the 2-Homogeneous Homotopy and the Polyhedral Homotopy, which in many ways are more powerful than the former.

We know from the Fundamental Theorem of Algebra that for a single variable polynomial equation of degree $k \in \mathbb{Z}_{>0}$, say $f(x) = \sum_{i=0}^{k} a_i x^i$, with coefficients a_i and the variable x both defined over $\mathbb{C}$, the number of solutions is exactly k if $a_k \neq 0$, counting multiplicities. To solve a univariate equation, many numerical methods such as the *companion matrix* trick for low degree polynomials, and divide-and-conquer techniques for high degree polynomials, are available. Here, we introduce the NPHC method, which can then be extended to the multivariate case in a straightforward manner [29, 30].

Briefly, in the NPHC method, first one writes down the equation or system of equations to be solved in a more general parametric form, then solves this system at a point in parameter space where its solutions can be easily found, before finally tracking these solutions from this point in parameter space to the point in parameter space corresponding to the original system/problem. This approach can be applied even to non-algebraic equations which exhibit a continuous dependence of the solutions on the parameters. However, there exist many difficulties, which do not arise in the polynomial setting, in making this method a primary candidate method to solve a set of non-algebraic equations.

Let us consider the univariate equation $z^2 - 5 = 0$, pretending that we do not know its solutions, i.e., $z = \pm\sqrt{5}$. We thus begin by defining the more general

parametric family

$$H(z,t) = (1-t)(z^2-5) + \gamma t(z^2-1) = 0, \tag{2.1}$$

where $t \in [0,1]$ is a parameter and γ is a complex number. For $t=1$, we have $z^2-1=0$ and at $t=0$ we recover our original problem. The problem of getting all solutions of the original problem now reduces to tracking solutions of $H(z,t)=0$ from $t=1$ where we know the solutions, i.e., $z=\pm 1$, to $t=0$. The choice of z^2-1 in Eq. (2.1), called the *start system*, is made for the following reasons: this system has the same number of solutions as the original problem and is easy to solve. For multivariate systems, a clever choice of a start system is essential in reducing the computation as we shall see.

We briefly mention the numerical methods used in path-tracking from $t=1$ to $t=0$. One of the ways to track the paths is to solve a differential equation that is satisfied along all solution paths, say $z_i^*(t)$ for the i^{th} solution path,

$$\frac{dH(z_i^*(t),t)}{dt} = \frac{\partial H(z_i^*(t),t)}{\partial z}\frac{dz_i^*(t)}{dt} + \frac{\partial H(z_i^*(t),t)}{\partial t} = 0. \tag{2.2}$$

This equation is called the **Davidenko differential equation**. We can solve this initial value problem numerically – again, pretending that an exact solution is not known – with the initial conditions as $z_1^*(0)=1$ and $z_2^*(0)=-1$. (Since we also know that $H(z_i^*(t),t)=0$, predictor-corrector methods, such as Euler's predictor and Newton's corrector, are used in practice to solve this initial value problem.) We shall not discuss the actual path tracking algorithms further, but it is important to mention that in these algorithms are designed to handle almost all apparent difficulties such as tracking singular solutions, multiple roots, solutions at infinity, etc.

We now return to the complex number γ, where we will consider $\gamma = e^{i\theta}$ with $\theta \in [0, 2\pi)$ chosen generically. It is known that for all but finitely values of $\theta \in [0,2\pi)$, the paths are well-behaved for $t \in (0,1]$, i.e., for the whole path except the end-point [29]. This makes sure that there is no singularity or bifurcation along the paths. This is a remarkable technique, called the **γ-trick** in the literature, and constitutes the reason why *NPHC is guaranteed to find all isolated solutions.*

There are several sophisticated numerical packages well-equipped with path trackers such as Bertini [31], PHCpack [32], PHoM [33], HOMPACK [34] and HOM4PS2 [35,36]. They all are available as freewares from the respective research groups. In the above example, PHCpack with its default settings yields the solutions $z = -2.23606797749979 + i\,0.00000000000000$ and $z = 2.23606797749979 + i\,0.00000000000000$. Thus, it gives the expected two solutions of the system with a very high numerical accuracy.

2.1 Multivariate Polynomial Homotopy Continuation

We will now generalize the NPHC method explained above to the multivariate case, say $P(x) = 0$, where $P(x) = (p_1(x), \ldots, p_m(x))$ and $x = (x_1, \ldots, x_m)$,

that is *known to have isolated solutions* (i.e., a 0-dimensional ideal). Of course, in many cases where one does not know the dimensionality of the solutions to start with, one has to check if the given system is an 0-dimensional first. In the following sections, where we explain the extension of the NPHC method to the positive dimensional systems, we will describe a concrete way of finding if the system possesses only isolated solutions. One quick and dirty way of seeing that a square system (i.e., number of equations and number of variables are the same) has only isolated nonsingular solutions is to add the determinant of the Jacobian matrix to the system and verifying that the combined system has no solutions. In the 0-dimensional case, the system has maximal rank at each solution. Having said this, in many physical applications, we know *a priori* that the given system has only isolated solutions.

We can construct a homotopy, or a set of problems, similar to the aforementioned one-dimensional case, as

$$H(x,t) = (1-t)P(x) + \gamma t Q(x), \tag{2.3}$$

where $Q(x)$ is a system of polynomial equations, $Q(x) = (q_1(x), \ldots, q_m(x))$ with the following properties:

1. The solutions of $Q(x) = H(x,1) = 0$ are known or can be easily obtained. $Q(x)$ is called the *start system* and the solutions are called the *start solutions*.

2. The number of solutions of $Q(x) = H(x,1) = 0$ is equal to an estimated number (or an upper bound) of the solutions for $P(x) = 0$.

3. The solution set of $H(x,t) = 0$ for $0 < t \leq 1$ consists of a finite number of smooth paths, called homotopy paths, each parameterized by $t \in (0,1]$.

4. Every isolated solution of $H(x,0) = P(x) = 0$ can be reached by some path originating at a solution of $H(x,1) = Q(x) = 0$.

We can then track all the paths corresponding to each solution of $Q(x) = 0$ from $t = 1$ to $t = 0$ and reach $H(x,0) = P(x) = 0$. By implementing an efficient path tracker algorithm, we can get all the isolated solutions of a system of multivariate polynomials just as in the univariate case. There are several upper bounds on the number of solutions of the system $P(x) = 0$, which yield alternatives to constructing $Q(x)$ and the homotopy $H(x,t)$ for the multivariate case. The ones that we will consider are the (1) Total Degree Homotopy, (2) 2-Homogeneous Homotopy, and (3) Polyhedral Homotopy. We introduce the Total Degree Homotopy below and discuss the 2-Homogeneous and Polyhedral Homotopies in Appendices.

2.1.1 Total Degree Homotopy

The Total Degree Homotopy arises from the *Classical Bézout Theorem*, which asserts that a system of m polynomial equations in m variables has at most $\prod_{i=1}^{m} d_i$ isolated solution in $\mathbb{C}^m$, where d_i is the degree of the ith polynomial. This bound, called the Classical Bézout Bound (CBB), is sharp for generic values (i.e., roughly speaking, non-zero random values) of the coefficients. The *genericity* is well-defined and the interested reader is referred to [29] for details. The homotopy constructed using the CBB is called the *Total Degree Homotopy*. The start system $Q(x) = 0$ can be taken for example as

$$Q(x) = \begin{pmatrix} x_1^{d_1} - 1 \\ x_2^{d_2} - 1 \\ \vdots \\ x_m^{d_m} - 1 \end{pmatrix} = 0, \tag{2.4}$$

where d_i is the degree of the i^{th} polynomial of the original system $P(x) = 0$. Eq. (2.4) is easy to solve and guarantees that the total number of start solutions is $\prod_{i=1}^{m} d_i$, all of which are nonsingular. The Total Degree Homotopy is a very effective and popular homotopy which is used in actual path trackers.

The advantages of the Total Degree Homotopy are (1) the CBB is easy to compute, and (2) the start system based on the CBB can be solved quickly. The drawback of it is that the CBB does not take the sparsity of the system into account: systems arising in practice have far fewer solutions than the CBB, so a large portion of the computational effort is wasted.

The 2-Homogeneous Homotopy is constructed by first writing $\mathbb{C}^m = \mathbb{C}^k \times \mathbb{C}^{k-m}$ for some $0 < k < m$, which is accomplished by partitioning the original variables into two groups. This has the advantage of incorporating some of the structure of the given polynomial system $P(x)$ into the start system $Q(x)$. The corresponding bound, called the 2-Homogeneous Bézout Bound (2HomBB), is often tighter than the CBB when the polynomial system $P(x)$ has a naturally arising partition of the variables, which occurs in the examples below. Given a partition, the 2HomBB is easy to compute and the start system can be solved quickly via linear algebra.

The Polyhedral Homotopy uses the monomial structure of the given polynomial system $P(x)$ based on the Bernstein-Khovanskii-Kushnirenko (BKK) Theorem [37–39] to yield the BKK bound. Essentially, this upper bound on the number of complex solutions is obtained by computing the mixed volume of the convex hull of the Newton polytope (which is based on the exponents of the monomials appearing) of each equation. Since we must introduce some jargon to fully describe the 2-Homogeneous and Polyhedral Homotopies, we do it in Appendix A and B, respectively. We note that, as with the CBB, the 2HomBB and BKK bound are also generically sharp with respect to the family of polynomial systems under consideration.

Since the BKK count utilizes sparsity of the system, the BKK bound is tighter than the CBB and 2HomBB resulting in fewer homotopy paths to track. The apparent drawback is that to compute the BKK bound and solve the start system itself, the Polyhedral Homotopy requires additional computational effort. As we will see, the difference between the CBB and BKK bound suppress this drawback of the Polyhedral Homotopy by saving a large portion of the computational effort over the Total Degree Homotopy. The relationship between the 2HomBB and BKK count vary based on the system.

2.1.2 Parallelizability of the NPHC Method

While going through the algorithm for the NPHC method, one could have noticed that to track paths from $t = 1$ to $t = 0$, i.e., from the start system to the original system, each start solution does not need any knowledge about any other path. In other words, each solution path can be tracked completely independently of all others. This feature makes the method *e*mbarrassingly parallelizable. This is sharply different than the BA which is known to be a sequential algorithm, i.e., each step requires the knowledge of the previous step. Thus, the NPHC method is destined to be much more efficient to solve bigger systems which are way beyond the reach of the Gröbner basis techniques.

2.1.3 Numerical Solutions of Multivariate System

While tracking all the paths using either Total Degree, 2-Homogeneous, or Polyhedral Homotopies, only those paths leading to solutions of the original system will converge at $t = 0$, that is, some of the paths will diverge. The values of the variables at $t = 0$, i.e., $\vec{x}(t = 0)$, after the path-tracking are the solutions of the original system. Obviously, they will be in the numerical form. It is useful to define what we mean by a solution here. For the multivariate case, a solution is a set of numerical values of the variables which is within a given tolerance $\triangle_{\text{sol}}$ (we will take $\sim 10^{-10}$ in our ensuing calculations) of an actual solution. Since the variables are allowed to take complex values, all the solutions come with real and imaginary parts. A solution is a real solution if the imaginary part of each of the variables is less than or equal to a given tolerance, $\triangle_{\mathbb{R}}$ ($\sim 10^{-7}$ is a robust tolerance for the equations we will be dealing with in the next section, below which the number of real solutions does not change). All these solutions can be further refined to within an *arbitrary precision* (up to the memory and computational limits placed on by the machine).

The obvious question at this stage would be if the number of real solutions depends on $\triangle_{\mathbb{R}}$. To resolve this issue, we use a recently developed algorithm called "alphaCertified" which is based on the so-called **Smale's α-theory**. This algorithm certifies which nonsingular solutions of the given polynomial system are real using either exact rational arithmetic and arbitrary precision floating

point arithmetic [40]. This is a remarkable step, because using alphaCertified, we can prove that a solution classified as a real solution is actually a real solution independent of $\triangle_{\mathbb{R}}$, and hence these numerical approximations are as good as the *exact solutions*.

3 Isolated Solutions and Phenomenology

We will now apply the principles introduced in the preceding section on NPHC to questions which arise in theoretical physics, especially in string phenomenology and supersymmetric gauge theory. Our discussions will roughly fall into two categories to which we alluded in the discussions above: (1) polynomials systems which have any isolated solutions; these are clearly of importance when dealing with issues such as finding and stabilizing vacua given some effective potential, and (2) systems which can have solution components; these are crucial to understanding various geometrical properties of gauge theories such as the moduli space of vacua or Calabi-Yau spaces in the AdS/CFT context. We will first discuss (1) and then turn to (2) in the next section, illustrating in each case with ample examples, particularly those which defy more standard techniques.

Much of current research in string phenomenology is focused on developing methods to find and analyze vacua of four dimensional effective theories for supergravity descended from flux compactifications. Stated in explicit terms, one is interested in finding all the vacua (usually, isolated stationary points) of the scalar potential V of such a theory. In particular, given a Kähler potential K, and a superpotential W, for uncharged moduli fields, the scalar potential can be obtained as

$$V = e^K [K^{A\bar{B}}\ D_A W\ D_{\bar{B}} \bar{W} - 3|W|^2] \ , \tag{3.5}$$

where D_A is the Kähler derivative $\partial_A + \partial_A K$ and $K^{A\bar{B}}$ is the inverse of $K_{A\bar{B}} = \partial_A \partial_{\bar{B}} K$. Once the vacua are found, one can then classify them by either using the eigenvalues of the Hessian matrix of V or by introducing further constraints such as $W = 0$.

Finding all the stationary points of a given potential V amounts to solving the stationary equations, i.e. solving the system of equations consisting of the first derivatives of V with respect to all the fields equated to zero. The stationary equations for V arising in the string phenomenological models are usually non-linear. In the perturbative limit, W usually has a polynomial form. This is an important observation since we can then use algebraic geometric concepts and methods to extract information about V.

For the systems known to have only isolated solutions, we are in the situation of 0-dimensional ideals, a Gröbner basis using a lexicographic ordering of the monomials always has at least one univariate equation and the subsequent equations consist of an increasing number of variables, i.e., it is in a *triangular*

form. The solutions of a Gröbner basis is always the same as the original system, but the former is easier to solve due to its triangular form as the univariate equation can be solved either analytically or numerically using straightforward methods. Then, by back-substituting the solutions in the subsequent equations and continually solving them, we can find all the solutions of the original system.

3.1 Some Toy Examples

One of the main features of the "Stringvacua" package [5, 15, 16] is to address problems with a 0-dimensional ideal using Gröbner basis techniques. Let us first take some of the toy examples discussed therein and see how our numerical techniques can extract some of the requisite information without resorting to the often expensive Gröbner algorithms.

Sys1: A Single-Modulus Example: We begin with a single-modulus toy example. Let the Kähler potential K and superpotential W be given as

$$K = -3\log(T+\bar{T}) \ , \quad W = a + bT^8 \ . \tag{3.6}$$

Note that the field T comes along with its complex conjugate. So even though they can be treated as different variables by merely relabeling them, they are not actually independent variables. To avoid this problem, we can write them in terms of real and imaginary parts, i.e., $T = t + i\,\tau$ where t and τ are real. The potential is

$$V = \frac{1}{3t}(4b(5b(t^2+\tau^2)^7 - 3a(t^6 - 21t^4\tau^2 + 35t^2\tau^4 - 7\tau^6))) \tag{3.7}$$

which has 2 variables. To find the stationary points of V, we need to compute the zero locus of the partial derivatives of V with respect to variables t and τ:

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{1}{3t^2}(4b(5b(13t^2-\tau^2)(t^2+\tau^2)^6 - 3a(5t^6 - 63t^4\tau^2 + 35t^2\tau^4 + 7\tau^6))) = 0, \\
\frac{\partial V}{\partial \tau} &= \frac{1}{3t}(56b\tau(5b(t^2+\tau^2)^6 + a(9t^4 - 30t^2\tau^2 + 9\tau^4))) = 0 \ .
\end{aligned} \tag{3.8}$$

For general values of parameters a and b it was already known [6, 15] that the system becomes prohibitively difficult to analyze using symbolic methods. For now, we take $a = b = 1$.

We also note that the stationary equations in this example involve denominators. Since we are not interested in the solutions for which the denominators are zero, we clear them out by multiplying them with the numerators appropriately. In these equations, all the denominators are multiples of t. The condition that none of the denominators is zero can be imposed algebraically by adding a constraint equation as $1 - z\,t = 0$ with z being an additional variable. Thus

there are now 3 equations in 3 variables. One could impose this by the method of saturation [15].

Stringvacua can easily solve this system: this system has 6 real solutions. Note that this computation uses the Gröbner basis technique implemented in Singular.

We now turn to solve the above system using Bertini and HOM4PS2. The CBB of this system is 364, while the 2HomBB, using $\{t, \tau\} \times \{z\}$, and the BKK bound are both 182. In the end, there are 56 finite nonsingular complex solutions, of which six are real. Thus, we have produced the results known from symbolic methods by using the NPHC method. Note that the choice of a and b was crucial here. For a generic choice for a and b, the computation using symbolic methods via Stringvacua becomes extremely difficult already for this small system, whereas the NPHC method takes the same time to solve it as before. Since we have all the real solutions, we can then compute the Hessian of V at the real solutions and separate out the physically interesting vacua.

Sys2: Two-Moduli Model: Consider the Kähler potential and superpotential

$$\begin{aligned} K &= -3\log(T+\bar{T}) - \log(S+\bar{S}), \\ W &= aS + bST + cT^2, \end{aligned} \tag{3.9}$$

with two fields $T = t + i\tau$ and $S = s + i\sigma$. Again, a, b and c are the parameters and for convenience we chose them to be $1, -1, 1$, respectively. The potential is

$$\begin{aligned} V &= \frac{1}{48st^3}(-5t^4 + 3(s^2+\sigma^2) - 2t^2\tau^2 + 3\tau^4 - (s^2+\sigma^2)(5t^2 - 3\tau^2) - 6(5st^2 - \\ &\quad t(s^2+\sigma^2) + 2t\sigma\tau - s\tau^2) + 2(13st^3 + t^2\sigma\tau + 9st\tau^2 - 3\sigma\tau^3)) \end{aligned} \tag{3.10}$$

and the stationary equations are

$$\begin{aligned} 0 &= \frac{1}{48st^4}(-5t^4 - 9(s^2+\sigma^2) + 2t^2\tau^2 - 9\tau^4 + (s^2+\sigma^2)(5t^2 - 9\tau^2) \\ &\quad +6(5st^2 - 2t(s^2+\sigma^2) + 4t\sigma\tau - 3s\tau^2) - 2\tau(t^2\sigma + 18st\tau - 9\sigma\tau^2)), \\ 0 &= \frac{1}{48s^2t^3}(5t^4 + 3(s^2-\sigma^2) + 2t^2\tau^2 - 3\tau^4 + 6t(s^2 - \sigma^2 + 2\sigma\tau) \\ &\quad -2\sigma\tau(t^2 - 3\tau^2) - (s^2-\sigma^2)(5t^2 - 3\tau^2)), \\ 0 &= \frac{1}{24st^3}(-6t\sigma + t^2\sigma + 6s\tau + 18st\tau + 3(s^2+\sigma^2)\tau - 9\sigma\tau^2 \\ &\quad +2\tau(-t^2 + 3\tau^2)), \\ 0 &= \frac{1}{24st^3}(3\sigma - 5t^2\sigma + t^2\tau + 3\sigma\tau^2 - 3\tau^3 - 6t(-\sigma+\tau)) \ . \end{aligned} \tag{3.11}$$

With the denominator equation $1 - zst = 0$, there are 5 equations in 5 variables. The system possesses only isolated solutions and both the Stringvacua package

and the NPHC method can solve this system. The CBB is 432, the 2HomBB, using $\{t, \tau, s, \sigma\} \times \{z\}$ is 144, and the BKK root count is 100. There are 6 real solutions.

Again, for a generic choice of the values for a, b and c, it becomes extremely difficult for Stringvacua to solve the corresponding system unlike the NPHC method.

Sys3-1: Two Moduli Example : We now take a four dimensional $\mathcal{N} = 1$ supergravity theory with Kähler potential and superpotential

$$\begin{aligned} K &= -3\log(T_1 + \bar{T}_1) - 3\log(T_2 + \bar{T}_2), \\ W &= -T_1^2 - T_1 T_2 - T_2^2 + 10(T_1 + T_2) - 100. \end{aligned} \tag{3.12}$$

Taking $T_j = t_j + i\tau_j$, for $j = 1, 2$, the potential is

$$\begin{aligned} V = \; & \frac{1}{192 t_1^3 t_2^3}(t_1^4 + t_1^3(20 - 14t_2) + 20t_2^3 + t_2^4 - 60t_2(100 + 2\tau_1\tau_2 + \tau_2^2) \\ & + t_1^2(-500 + 280t_2 - 37t_2^2 + 10\tau_1^2 + 10\tau_1\tau_2 + 7\tau_2^2) \\ & + t_2^2(7\tau_1^2 + 10\tau_1\tau_2 + 10(-50 + \tau_2^2)) - 2t_1(-140t_2^2 + 7t_2^3 \\ & + 30(100 + \tau_1^2 + 2\tau_1\tau_2) + 3t_2(200 + \tau_1^2 + 4\tau_1\tau_2 + \tau_2^2)) \\ & + 9(10000 + \tau_1^4 + 2\tau_1^3\tau_2 - 100\tau_2^2 + 2\tau_1\tau_2^3 + \tau_2^4 + \tau_1^2(-100 + 3\tau_2^2))) \end{aligned} \tag{3.13}$$

with the stationary points dictated by

$$\begin{aligned} \frac{\partial V}{\partial \tau_1} = \; & \tfrac{1}{96 t_1^3 t_2^3}(-900\tau_1 - 60t_1\tau_1 + 10t_1^2\tau_1 - 6t_1t_2\tau_1 + 7t_2^2\tau_1 + 18\tau_1^3 - 60t_1\tau_2 \\ & + 5t_1^2\tau_2 - 60t_2\tau_2 - 12t_1t_2\tau_2 + 5t_2^2\tau_2 + 27\tau_1^2\tau_2 + 27\tau_1\tau_2^2 + 9\tau_2^3), \\ \frac{\partial V}{\partial \tau_2} = \; & \tfrac{1}{96 t_1^3 t_2^3}(-60t_1\tau_1 + 5t_1^2\tau_1 - 60t_2\tau_1 - 12t_1t_2\tau_1 + 5t_2^2\tau_1 + 9\tau_1^3 \\ & - 900\tau_2 + 7t_1^2\tau_2 - 60t_2\tau_2 - 6t_1t_2\tau_2 + 10t_2^2\tau_2 + 27\tau_1^2\tau_2 + 27\tau_1\tau_2^2 + 18\tau_2^3), \\ \frac{\partial V}{\partial t_1} = \; & \tfrac{1}{192 t_1^4 t_2^3}(-270000 + 12000t_1 + 500t_1^2 + t_1^4 + 18000t_2 + 2400t_1t_2 - 280t_1^2t_2 \\ & + 1500t_2^2 - 560t_1t_2^2 + 37t_1^2t_2^2 - 60t_2^3 + 28t_1t_2^3 - 3t_2^4 \\ & + 2700\tau_1^2 + 120t_1\tau_1^2 - 10t_1^2\tau_1^2 + 12t_1t_2\tau_1^2 - 21t_2^2\tau_1^2 - 27\tau_1^4 \\ & + 240t_1\tau_1\tau_2 - 10t_1^2\tau_1\tau_2 + 360t_2\tau_1\tau_2 + 48t_1t_2\tau_1\tau_2 - 30t_2^2\tau_1\tau_2 \\ & - 54\tau_1^3\tau_2 + 2700\tau_2^2 - 7t_1^2\tau_2^2 + 180t_2\tau_2^2 + 12t_1t_2\tau_2^2 - 30t_2^2\tau_2^2 \\ & - 81\tau_1^2\tau_2^2 - 54\tau_1\tau_2^3 - 27\tau_2^4), \\ \frac{\partial V}{\partial t_2} = \; & \tfrac{1}{192 t_1^3 t_2^4}(-270000 + 18000t_1 + 1500t_1^2 - 60t_1^3 - 3t_1^4 \\ & + 12000t_2 + 2400t_1t_2 - 560t_1^2t_2 + 28t_1^3t_2 + 500t_2^2 - 280t_1t_2^2 + 37t_1^2t_2^2 + t_2^4 \\ & + 2700\tau_1^2 + 180t_1\tau_1^2 - 30t_1^2\tau_1^2 + 12t_1t_2\tau_1^2 - 7t_2^2\tau_1^2 - 27\tau_1^4 \\ & + 360t_1\tau_1\tau_2 - 30t_1^2\tau_1\tau_2 + 240t_2\tau_1\tau_2 + 48t_1t_2\tau_1\tau_2 \\ & - 10t_2^2\tau_1\tau_2 - 54\tau_1^3\tau_2 + 2700\tau_2^2 - 21t_1^2\tau_2^2 + 120t_2\tau_2^2 \\ & + 12t_1t_2\tau_2^2 - 10t_2^2\tau_2^2 - 81\tau_1^2\tau_2^2 - 54\tau_1\tau_2^3 - 27\tau_2^4). \end{aligned} \tag{3.14}$$

We can easily clear out the denominators from each of the equations and add a constraint equation $1 - z\, t_1\, t_2 = 0$ representing the fact that none of the

denominators is zero. This combined system contains only isolated solutions, but solving this system using the symbolic Gröbner basis methods is a prohibitively difficult task [15].

However, using the NPHC method, we can solve this system in less than a minute on a standard desktop machine. The CBB of this system is 432, the 2HomBB, using $\{t_1, \tau_1, t_2, \tau_2\} \times \{z\}$, and the BKK root count are 144. In the end, there are 70 solutions, of which 6 are real.

Sys3-2: Furthermore, the equations for F-terms are

$$\begin{aligned} 0 &= 300 - 10t_1 - t_1^2 - 30t_2 + t_1t_2 + 3t_2^2 - 3\tau_1^2 - 3\tau_1\tau_2 - 3\tau_2^2, \\ 0 &= 300 - 30t_1 + 3t_1^2 - 10t_2 + t_1t_2 - t_2^2 - 3\tau_1^2 - 3\tau_1\tau_2 - 3\tau_2^2, \\ 0 &= -30\tau_1 + 2t_1\tau_1 + 3t_2\tau_1 - 30\tau_2 + t_1\tau_2 + 6t_2\tau_2, \\ 0 &= -30\tau_1 + 6t_1\tau_1 + t_2\tau_1 - 30\tau_2 + 3t_1\tau_2 + 2t_2\tau_2 \end{aligned}$$

The CBB and the BKK root count both are the same in this case, namely 16. Due to the lack of a good partition of the variables, the 2HomBB is actually larger, namely 24 for $\{t_1, t_2\} \times \{\tau_1, \tau_2\}$, than the CBB for example. These equations have only 12 solutions, of which 4 are real.

3.2 Examples from Heterotic and M-Theory Effective Potentials

Having warmed up with some toy examples and seen that the NPHC method can extract information which is too difficult for Gröbner basis methods or for ordinary numerical methods untuned for polynomial systems, let us see some "real" scenarios from phenomenology.

Sys4: A Heterotic Example: Let us begin with a Kähler potential and superpotential taken from a heterotic compactification [41, 42]:

$$\begin{aligned} K &= -3\ln(T+\bar{T}) - 3\ln(Z+\bar{Z}), \\ W &= i(\xi + ieT) + (\epsilon + ipT)Z + \frac{i}{2}(\mu + iqT)Z^2 + \frac{1}{6}(\rho + irT)Z^3 \ , \end{aligned} \tag{3.15}$$

where $\xi, r, \epsilon, q, \mu, p, \rho, e$ are parameters which satisfy the constraint $\xi r - \epsilon q + \mu p - \rho e = 0$. We write $T = t + i\tau$ and $Z = z + i\zeta$, and take $\xi = -13, r = 0, \epsilon = -4, q = 2, \mu = 2, p = 1, \rho = 5, e = -7$ to ensure that the solution space is zero-dimensional.

The critical points satisfy

$$
\begin{aligned}
0 &= -\tfrac{1}{6912t^4z^3}(164268+10152z^2-4032tz^3+2268z^4\\
&\quad -252t^2z^4+360tz^5+675z^6+101088\xi-11232tz\xi+25380z^2\xi\\
&\quad +1620z^4\xi+40824\xi^2-1728tz\xi^2+9720z^2\xi^2-216t^2z^2\xi^2\\
&\quad +720tz^3\xi^2+2025z^4\xi^2+28836\xi^3+3240z^2\xi^3+7452\xi^4\\
&\quad +36t^2\xi^4+360tz\xi^4+2025z^2\xi^4+1620\xi^5+675\xi^6-8424z^2T\\
&\quad -216z^4T-4320z^2\xi T-1620z^4\xi T-25272\xi^2T-2160z^2\xi^2T\\
&\quad -7776\xi^3T-3240z^2\xi^3T-1944\xi^4T-1620\xi^5T+108z^4T^2\\
&\quad +1080z^2\xi^2T^2+972\xi^4T^2+1764(t^2+27T^2)\\
&\quad -36(t^2(7z^2-\xi^2)-3(z^2+9\xi^2)T^2)+84(-6t^2\xi\\
&\quad -6tz(-24+25z^2-12\xi-15\xi^2)-27T(78+24\xi+6\xi^2+5\xi^3\\
&\quad +z^2(2+5\xi)+6\xi T)+6(t^2(11z^2+\xi^2)+9(z^2+3\xi^2)T^2))\\
&\quad -36(2tz(-78+6\xi^2+10\xi^3+z^2(14+10\xi))-3(5z^4\\
&\quad +z^2(8+10\xi+20\xi^2)+3\xi(78+24\xi+6\xi^2+5\xi^3))T\\
&\quad +2(t^2(-3z^2\xi+\xi^3)+3\xi(5z^2+9\xi^2)T^2))),\\
0 &= -\tfrac{1}{2304t^3z^4}(54756+1128z^2-252z^4+84t^2z^4\\
&\quad -120tz^5-225z^6+33696\xi-3744tz\xi+2820z^2\xi-180z^4\xi\\
&\quad +13608\xi^2-576tz\xi^2+1080z^2\xi^2-72t^2z^2\xi^2\\
&\quad -225z^4\xi^2+9612\xi^3+360z^2\xi^3+2484\xi^4+36t^2\xi^4\\
&\quad +120tz\xi^4+225z^2\xi^4+540\xi^5+225\xi^6-936z^2T+24z^4T\\
&\quad -480z^2\xi T+180z^4\xi T-8424\xi^2T-240z^2\xi^2T-2592\xi^3T\\
&\quad -360z^2\xi^3T-648\xi^4T-540\xi^5T-12z^4T^2\\
&\quad +120z^2\xi^2T^2+324\xi^4T^2+1764(t^2+9T^2)\\
&\quad +12(t^2(-7z^2+3\xi^2)+(z^2+27\xi^2)T^2)\\
&\quad +12(-2tz(-78+6\xi^2+10\xi^3)-(5z^4\\
&\quad +z^2(-8-10\xi-20\xi^2)-9\xi(78+24\xi+6\xi^2+5\xi^3))T\\
&\quad +2\xi(3t^2(z^2-\xi^2)-(5z^2+27\xi^2)T^2))\\
&\quad +84(-6t^2\xi+6tz(8+\xi(4+5\xi))+2(t^2(11z^2+3\xi^2)\\
&\quad +3(z^2+9\xi^2)T^2)-3T(72\xi+z^2(2+5\xi)\\
&\quad +3(78+6\xi^2+5\xi^3+6\xi T)))),\\
0 &= \tfrac{1}{384t^3z^3}(-3276-240z^2-4z^4-1008\xi-290z^2\xi-30z^4\xi-720\xi^2-40z^2\xi^2-354\xi^3\\
&\quad -60z^2\xi^3-36\xi^4-30\xi^5+1764T+168z^2T+4z^4T\\
&\quad +504\xi^2T+40z^2\xi^2T+36\xi^4T+4(z^2+9\xi^2)T\\
&\quad +2(5z^4+z^2(8+\xi(10+20\xi-20T))+3\xi(24\xi+5\xi^3\\
&\quad +3\xi^2(2-4T)-6(-13+14T)))),\\
0 &= \tfrac{1}{1152t^3z^3}(5616-936tz+1410z^2\\
&\quad +90z^4+4536\xi-288tz\xi+1080z^2\xi-72t^2z^2\xi\\
&\quad +120tz^3\xi+225z^4\xi+4806\xi^2+540z^2\xi^2\\
&\quad +1656\xi^3+24t^2\xi^3+120tz\xi^3+450z^2\xi^3\\
&\quad +450\xi^4+225\xi^5-240z^2T-90z^4T-2808\xi T-240z^2\xi T\\
&\quad -1296\xi^2T-540z^2\xi^2T-432\xi^3T-450\xi^4T\\
&\quad +120z^2\xi T^2+216\xi^3T^2+12\xi(t^2+9T^2)\\
&\quad -42(-12tz-4t^2\xi-30tz\xi+72T+15z^2T+36\xi T+45\xi^2T\\
&\quad -36\xi T^2+2(t^2+9T^2))+2(-3(2tz(5z^2+6\xi+15\xi^2)\\
&\quad +(-144\xi-z^2(10+40\xi)-3(78+18\xi^2+20\xi^3))T)\\
&\quad +6(3t^2(z^2-\xi^2)-(5z^2+27\xi^2)T^2))).
\end{aligned}
\tag{3.16}
$$

We can clear out the denominators from all the equations and add the additional constraint equation $1 - y\,z\,t = 0$ with additional variable y, to get a combined system which only possess of isolated solutions. The system is prohibitively difficult to solve using the Gröbner basis techniques. However, the NPHC method solves it in less than a minute. The CBB is 2700, the 2HomBB using $\{t, \tau, z, \zeta\} \times \{y\}$ is 900, and the BKK root count is 340. There are a total of 62 solutions, of which only 6 are real.

Sys5: A Model from M-Theory: Here, we take an example of M theory compactified on the coset $\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ [43]. The coset has SU(3) structure. The corresponding Kähler and superpotential are

$$\begin{aligned} K &= -4\log(-i(U-\bar{U})) - \log(-i(T_1-\bar{T}_1)(T_2-\bar{T}_2)(T_3-\bar{T}_3)), \\ W &= \frac{1}{\sqrt{8}}(4U(T_1+T_2+T_3) + 2T_2T_3 - T_1T_3 - T_1T_2 + 200). \end{aligned}$$

Using $T_j = \tau_j - it_j$, for $j = 1, 2, 3$, and $U = y - ix$, the potential is

$$\begin{aligned} V &= \frac{1}{256t_1t_2t_3x^4}(40000 + t_3^2\tau_1^2 - 400\tau_1\tau_2 - 4t_3^2\tau_1\tau_2 + 4t_3^2\tau_2^2 + \tau_1^2\tau_2^2 - 400\tau_1\tau_3 \\ &\quad +800\tau_2\tau_3 + 2\tau_1^2\tau_2\tau_3 - 4\tau_1\tau_2^2\tau_3 + \tau_1^2\tau_3^2 - 4\tau_1\tau_2\tau_3^2 + 4\tau_2^2\tau_3^2 - 24t_2t_3x^2 \\ &\quad +4t_3^2x^2 - 24t_1(t_2+t_3)x^2 + 4\tau_1^2x^2 + 8\tau_1\tau_2x^2 + 4\tau_2^2x^2 + 8\tau_1\tau_3x^2 + 8\tau_2\tau_3x^2 \\ &\quad +4\tau_3^2x^2 + 1600\tau_1y - 8t_3^2\tau_1y + 1600\tau_2y + 16t_3^2\tau_2y - 8\tau_1^2\tau_2y - 8\tau_1\tau_2^2y \\ &\quad +1600\tau_3y - 8\tau_1^2\tau_3y + 16\tau_2^2\tau_3y - 8\tau_1\tau_3^2y + 16\tau_2\tau_3^2y + 16t_3^2y^2 + 16\tau_1^2y^2 \\ &\quad +32\tau_1\tau_2y^2 + 16\tau_2^2y^2 + 32\tau_1\tau_3y^2 + 32\tau_2\tau_3y^2 + 16\tau_3^2y^2 \\ &\quad +t_1^2(t_2^2 + t_3^2 + \tau_2^2 + 2\tau_2\tau_3 + \tau_3^2 + 4x^2 - 8\tau_2y - 8\tau_3y + 16y^2) \\ &\quad +t_2^2(4t_3^2 + \tau_1^2 - 4\tau_1(\tau_3+2y) + 4(\tau_3^2 + x^2 + 4\tau_3y + 4y^2)). \end{aligned}$$

There are 9 equations to be solved: 8 equations corresponding to the derivatives with respect to 8 variables plus one constraint equation $1 - z(t_1t_2t_3x) = 0$ with z being an additional variable. The CBB is 103,680, the 2HomBB, using $\{t_j, \tau_j, x, y\} \times \{z\}$ is 20,736, and the BKK root count is 18,624. The NPHC method yields 516 solutions, of which only 12 are real. The computation based on the BKK root count takes around 10 minutes using HOM4PS2.

Sys6: A $SU(2)\times SU(2)/\mathbb{Z}_2\times\mathbb{Z}_2$ Model: Compactifying type IIA supergravity on $SU(2) \times SU(2)/\mathbb{Z}_2 \times \mathbb{Z}_2 = S^3 \times S^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ and restricting to the modes that are left-invariant under the action of $SU(2) \times SU(2)$ one obtains an $\mathcal{N} = 1$ supergravity in 4D, which was studied in [44–46]. There are no D-terms so that the scalar potential V is determined by the Kähler potential K and the

superpotential W which are

$$\begin{aligned} K = &- \log((t_1+\bar{t}_1)(t_2+\bar{t}_2)(t_3+\bar{t}_3)) \\ &- \log(16(N_1+\bar{N}_1)(N_2+\bar{N}_2)(N_3+\bar{N}_3)(N_4+\bar{N}_4)) \end{aligned}$$

and

$$\begin{aligned} W = &-if_0 t_1 t_2 t_3 + t_1 t_2 f_3^{(2)} + t_1 t_3 f_2^{(2)} + t_2 t_3 f_1^{(2)} + (-i\,h + t_1 + t_2 + t_3)N_1 \\ &+ (-i\,h + t_1 - t_2 - t_3)N_2 + (i\,h + t_1 - t_2 + t_3)N_3 + (-i\,h - t_1 - t_2 + t_3)N_4. \end{aligned}$$

Here t_i are the Kähler moduli ($J - iB = t_i Y^{(2-)i}$), N_K are the complex structure moduli ($e^{-\phi} Im(\Omega) + iC_3 = N_K Y^{(3+)K}$), f_0 is the mass parameter (of massive type IIA i.e. the flux F_0), $f_i^{(2)}$ is the F_2-flux ($F_2 = f_i^{(2)} Y^{(2-)i}$) and h is the H-flux (the F_4-flux is not turned on and also included $O6$-planes). The Y are the elements of the cohomology of $M = SU(2) \times SU(2)/\mathbb{Z}_2 \times \mathbb{Z}_2$. The superscript $(i-)$ means that they are elements of $H^i(M, R)$ and the superscript $+/-$ means that they are even/odd under the orientifold projection. i runs over the number of odd 2-forms in cohomology. The fluxes and fields have fixed transformations under the orientifold projection and have to be expanded either even or odd cohomology in order to survive the orientifold projection. By rescaling the fields and the overall scale of V we can set $f_0, f_1^{(2)}, f_2^{(2)}, f_3^{(2)}$ to unity. Furthermore, the scalar potential simplifies substantially if we analytically solve for the four C_3 axions (i.e. $Im(N_K)$). This leaves us with a model having 10 real fields and one parameter h.

As was shown in [45, 46], the scalar potential V for this model circumvents all known no-go theorems against dS solutions and actually allows for dS critical points that are however unstable. The results in [45, 46] where obtained by numerically minimizing $\epsilon = \frac{K^{A\bar{B}}(\partial_A V)\partial_{\bar{B}} V}{V^2}$ using Mathematica. This method however does not give all possible solutions. Using HOM4PS2, we have now solved $\partial_A V = 0$ and found all critical points for any given value of h. There are 10 variables and 10 equations, plus an additional constraint equation with additional variable to make sure that the denominator is non-zero, yielding a total of 11 equations. The CBB for this system is 279,936,000, the 2HomBB, using the original variables in one group and the additional variable in the other, is 34,992,000, but the BKK bound is only 574,080. For a fixed value of $h(= -5)$ we have found that there are 60 real solutions to these equations. By stability analysis of the solutions, we conclude that only one (the previously known) unstable dS critical point exists in this case. Furthermore, we also find (previously unknown) that up to permutation of the fields there are 4 additional AdS solutions. Thus, we have now completely classified the stationary points for this model using the numerical homotopy continuation method.

Model	# Eqns	CBB	2HomBB	BKK	Real Sols	Timing	GB Technique
Sys1	3	364	182	182	6	< 20s	Yes
Sys2	5	432	144	100	6	< 10s	Yes
Sys3-1	5	432	144	144	6	< 30s	No
Sys3-2	4	16	24	16	4	< 10s	Yes
Sys4	5	2700	900	340	6	< 10s	No
Sys5	9	103680	20736	18624	12	~ 590s	No
Sys6	11	279936000	34992000	574080	60	11 hrs	No

Table 1: The short names of the various models used in the main text are written in the first column. The subsequent columns are the number of equations, CBB bound, 2HomBB bound, BKK bound, number of real solutions, and the time taken by the NPHC method (either Bertini or HOM4PS2). The final column lists whether or not the Gröbner Basis technique (via Stringvacua) could solve the corresponding system.

4 Numerical Algebraic Geometry

We have presented, in the above, extensive examples wherein the power of numerical algebraic geometry may be harnessed in various phenomenological contexts, especially in the identification of isolated extrema of potentials coming from heterotic and M-theory scenarios. One might naturally wonder as to situations where the solution space of algebraic systems has positive dimensionality. Such cases arise naturally as a central subject in the study of supersymmetric gauge theories.

Indeed, the vacuum moduli space, parameterized by the scalar components of multiplets, in generic gauge theories with supersymmetry is a continuous manifold, or, more strictly speaking, an affine algebraic variety. This fact is particularly pronounced in string theory when the gauge theory manifests as the world-volume theory on a brane and the vacuum moduli space is some space of special holonomy such as Calabi-Yau or G_2 manifolds: this is the AdS/CFT correspondence.

The geometrical engineering of world-volume gauge theories given an affine (Calabi-Yau) geometry has been a vast subject over the last decade (cf. e.g. [47] for an introduction) and conversely, the geometric analysis of the vacuum of a given supersymmetric gauge theory (whether it comes from string theory or not) could uncover hidden phenomenological symmetries [3, 48, 49]. The algebraic geometry of the vacuum could often be extremely complicated since the coordinates parameterizing it are the gauge invariant operators, subject to superpotential constraints. Though in AdS/CFT, the geometries are affine Calabi-Yau threefolds by construction, reverse engineering [50] can lead to unlimited possibilities of Kähler manifolds. In super-symmetric QCD with gauge group $SU(N_c)$ with N_f fundamental flavours, for example, the vacuum is an affine variety of dimension

as high as $2N_cN_f - (N_c^2 - 1)$ when $N_f \geq N_c$.

Furthermore, the vacuum moduli space could be composed of (possibly intersecting) unions of components - or *branches* - at various dimensions all the way from 0 to that of the top (or coherent) component. This phenomenon is especially marked in the investigation of the so-called "master space", which is a join of mesonic and baryonic branches [51, 52].

The importance of studying the algebraic geometric structure of the vacuum, as can be seen from the above-mentioned, should not be undermined by the technical difficulties which we are inevitably lead to encounter. The usual method of attack is to recast the algebraic equations describing the vacuum as a polynomial ideal, transform into standard Gröbner basis, and then perform primary decomposition to extract the irreducible branches. As is by now well recognized, this is a very expensive computation, given especially the double-exponential running time of computing Gröbner bases (cf. [6]). In this section, we will study examples where if one wishes to know simply some crucial but preliminary information such as the number, dimension and degree of the components, one could bypass the prohibitive step of primary and irreducible decomposition and turn to the virtues of numerical algebraic geometry.

For positive dimensional varieties, since there are infinitely many solutions, one needs a proper representation of the solutions. The important first question to be asked is how we need to represent the solutions. For a 0-dimensional variety, the solutions are just a finite set of points, so they can be represented by complex numbers. For positive dimensional varieties the situation is more involved. The solutions in this case form, for example, curves, surfaces, or hypersurfaces. A way to represent the solutions is to compute a parameterization of the varieties, which can be accomplished using a Gröbner basis. However, this is computationally very expensive. The reader can find a nice discussion about this in [7].

Numerical Algebraic Geometry (NAG), on the other hand, cleverly uses another approach in which the solutions are represented as Witness Sets. We start with the fact that the number of points at which an irreducible component, say of dimension d in $\mathbb{C}^n$, of a system of polynomial equations intersects a random linear space of dimension $c = n - d$ is equal to the degree of the component. For example, a cubic curve in $\mathbb{C}^3$ intersects with a two-dimensional random hyperplane in three points. These intersection points are called Witness points. A *random* c-dimensional linear space means the solution set of $d = n - c$ linear equations with random coefficients. Here, each linear equation is of the form $\sum_{j=1}^{n} C_j x_j = C_0$ where C_k is a random complex number. After computing the Witness points, one can slide this random linear space around to obtain as many points on the component as needed. This yields a *parameterization* of the component, called the Witness sets representation.

Now, it should be clear how the NPHC method plays an important role here. Algebraically, we add d linear equations to the system of equations and then compute the isolated solutions of the combined system using the NPHC method.

The number of isolated solutions gives the number of Witness points for dimension d, i.e., the degree of the union of components of dimension d. One can do this for all d between 0 and n to compute the Witness points for all dimensions. We note that, in practice, a sequence of homotopies, called a cascade [53, 54], is used to compute the Witness points for all dimensions.

Given the Witness points for the components of dimension d, one needs to partition these points into Witness points for the irreducible components. This is done by the so-called monodromy algorithm. The basic approach is to consider the Witness points as the linear space is moved around and then back to the original linear space. In this move, some of the Witness points return to the same point. However, if a Witness point returns to a different point, these two points must be on the same component and thus are grouped together. This process is repeated until each group is verified to be the set of Witness points for an irreducible component. Since the full technical description of this verification step of monodromy is beyond the scope of this article, we refer the reader to [29] for more details.

A simple example to consider is that of the affine twisted cubic curve which is given by the equations:

$$x^2 - y = 0,\ x^3 - z = 0. \tag{4.17}$$

Running Bertini, for example, on these equations yields that the irreducible decomposition of the variety contains one component of dimension 1 and degree 3, as expected. It also gives a random hyperplane that was used to slice the variety so that we can then get as many points as we like on this positive dimensional component.

4.1 The Master Space

First we apply NAG to a model where the results are at least partially known, so that we can have a bench-marking to our NAG set up. In [51, 52], we introduced the concept of the "master space" of supersymmetric gauge theories, which controls the combined mesonic and baryonic branches of the vacuum moduli space. Computationally, this is the space of F-flatness, that is, the Jacobian ideal of the superpotential and can be construed as a baryonic fibration over the mesonic moduli space. In string theory, in the situation of a single D3-brane probing an affine toric Calabi-Yau threefold, the resulting world-volume physics is a $U(1)^g$ quiver gauge theory; here, the master space is a toric variety of dimension $g+2$, whose Kähler quotient by the $g-1$ independent $U(1)$-actions is the Calabi-Yau threefold.

Though in the toric case, polytope and plethystic techniques can simplify the computation of the master space, direct attack thereon is still needed in general. Once again, for large number of fields, recasting the Jacobian ideal into

standard bases may become prohibitively difficult and numerical methods can become a more natural choice in quickly extracting the dimension and irreducible components of the master space.

We will exemplify with $\mathcal{N} = 1$ gauge theories for a D3-brane on the Abelian orbifold $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$. This is a toric case which had been approached by both direct and combinatorial methods for some low values of (m, n) in [51], and hence provide a good point of reference. Another advantage we have here is that the Jacobian ideal is always "square", in the sense that the number of variables is equation to the number of vanishing equations. To summarize, the gauge theory of interest is a quiver theory with mn nodes, a total of $3mn$ bifundamental fields $\{X_{i,j},\ Y_{i,j},\ Z_{i,j}\}$ from node i to node j (with (i, j) defined modulo (k, m) and with superpotential

$$W = \sum_{i=0}^{k-1} \sum_{j=0}^{m-1} X_{i,j} Y_{i+1,j} Z_{i+1,j+1} - Y_{ij} X_{i,j+1} Z_{i+1,j+1} \ . \tag{4.18}$$

In Table 2, we provide the complete set of results for the systems up to $(m, n) = (3, 3)$. In [51], the top dimensional components of these systems were computed using toric variety methods. Our results agree with the results in this reference. In addition, we can now compute the complete irreducible decomposition of all these systems hence already for this simple set of systems we have got new and complete results.

For example, with $(m, n) = (2, 2)$, we arrive at a master space which is of dimension 16, in agreement with [51]. We further find that the variety reduces into 4 components: a dimension 6 piece of degree 14 and 3 linear pieces (i.e., degree 1) of dimension 4.

One can also go further than these systems. For example, (4,1): 1 component of dimension 6 and degree 8; 33 components of dimension 4 and degree 1, (5,1): 1 component of dimension 7 and degree 16; 131 components of dimension 5 and degree 1, and (6,1): 1 component of dimension 8 and degree 32; 473 components of dimension 6 and degree 1. For bigger systems such as (4,2), the decomposition space is more interesting, i.e., it has 1 component of dimension 10 and degree 584, 9 components of degree 1 and dimension 8, 8 components of degree 3 and dimension 8, 8 components of degree 4 and dimension 8, 24 components of degree 9 and dimension 8, and 8 components of degree 2 and dimension 6.

For (5,2), there is 1 component of degree 3632 and dimension 12, 17 component of degree 1 and dimension 10, 10 components of degree 3 and dimension 10, 40 components of degree 4 and dimension 10, 40 components of degree 9 and dimension 10, 40 components of degree 16 and dimension 10, 80 components of degree 27 and dimension 10, 130 components of degree 2 and dimension 8, 20 components of degree 6 and dimension 8, and 20 components of degree 8 and dimension 8. We can go on for bigger and bigger systems, however, we would rather stop here and move towards a more difficult system next.

$m\backslash n$	1	2	3
1	NA	$(1;4;2\|6)$, $(1;2;1\|6)$	$(1;5;4\|9)$, $(7;3;1\|9)$
2	$(1;4;2\|6)$, $(1;2;1\|6)$	$(1;6;14\|12)$ $(3;4;1\|12)$	$(1;8;92\|18)$, $(5;6;1\|18)$ $(6;6;3\|18)$
3	$(1;5;4\|9)$, $(7;3;1\|9)$	$(1;8;92\|18)$ $(5;6;1\|18)$, $(6;6;3\|18)$	$(1;11;1620\|27)$, $(6;9;1\|27)$, $(27;9;2\|27)$, $(36;9;7\|27)$, $(27;7;1\|27)$

Table 2: The master space F^b for $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ as explicit varieties for several values of m and n. $(p;q;r|d)$ means p components of dimension q and degree r contained in d-dimensional complex space.

4.2 Supersymmetric Quantum Chromodynamics

We now move to a more difficult problem of the familiar example of pure sQCD with gauge group $SU(N_c)$, N_f number of fundamental flavours of quarks Q and anti-quarks $\widetilde{Q}$, and no superpotential, so that the matter content is summarized as:

	GAUGE SYMMETRY $SU(N_c)$	GLOBAL SYMMETRY $SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$	$U(1)_Q$	$U(1)_{\widetilde{Q}}$
Q_a^i	$\overline{\square}$	$\square$	$\mathbf{1}$	1	$\frac{N_f-N_c}{N_f}$	1	0
$\widetilde{Q}_i^a$	$\square$	$\mathbf{1}$	$\overline{\square}$	-1	$\frac{N_f-N_c}{N_f}$	0	1

(4.19)

The generators of the gauge invariant operators consists of the mesons and baryons:

$$\begin{array}{ll} M_j^i = Q_a^i \widetilde{Q}_j^a & \text{(mesons) ;} \\ B^{i_1\ldots i_{N_c}} = Q_{a_1}^{i_1} \ldots Q_{a_{N_c}}^{i_{N_c}} \epsilon^{a_1\ldots a_{N_c}} & \text{(baryons) ;} \\ \widetilde{B}_{i_1\ldots i_{N_c}} = \widetilde{Q}_{i_1}^{a_1} \ldots \widetilde{Q}_{i_{N_c}}^{a_{N_c}} \epsilon_{a_1\ldots a_{N_c}} & \text{(antibaryons) .} \end{array} \tag{4.20}$$

It is a standard fact that the dimension of the (classical) vacuum moduli space $\mathcal{M}_{(N_f,N_c)}$ of the theory is

$$\dim \mathcal{M}_{(N_f,N_c)} = \begin{cases} N_f^2 \ , & N_f < N_c \ ; \\ 2N_cN_f - (N_c^2-1) \ , & N_f \geq N_c \ . \end{cases} \tag{4.21}$$

More specifically, one can explicitly obtain the algebraic variety [49] and prove they are, in fact, all affine Calabi-Yau varieties (the reader is referred to Table 2 in [49]); for example, $\mathcal{M}_{(N_f,N_c)}$ is simply the affine space $\mathbb{C}^{N_f^2}$ when $N_f < N_c$,

$N_f \backslash N_c$	1	2	3	4
1	(1; 2; 2|3)	-	-	-
2	(1; 4; 6|8)	(1; 5; 2|6)	-	-
3	(1; 6; 20|15)	(1; 9; 14|15)	(1; 10; 3|11)	-
4	(1; 8; 70|24)	(1; 13; 132|28)	(1; 16; 115|24)	(1; 17; 4|18)

Table 3: The irreducible decomposition of the moduli space of sQCD theories with N_f fermions and N_c bosons (after elimination from Macaulay 2). Here, $N_f \geq N_c$, and the irreducible decomposition is presented in the form: (no.components; dimension; degree | total dim).

and, for $N_f = N_c$, it is a complete intersection, with the Hilbert series given as

$$g^{N_f=N_c}(t) = \frac{1-t^{2N_c}}{(1-t^2)^{N_c^2}(1-t^{N_c})^2} \ . \tag{4.22}$$

In general, however, the vacuum is quite involved. Primary decomposition to find the irreducible branches of the moduli space, as was experimented in [49], is prohibitively difficult, even for small values of N_f and N_c. It was conjectured, by working over the coefficient fields of the rationals or number fields of finite characteristic, that $\mathcal{M}_{(N_f,N_c)}$ is actually irreducible. Now, we can properly check the components by working over the complex numbers, as one should. For example, take $(N_f, N_c) = (3, 3)$, we readily find that there is only one component, of complex dimension 10 and degree 3. For $(N_f, N_c) = (4, 3)$, there is again only one component of dimension 16 and degree 115. A full study of the vacuum structure requires algebraic elimination, a subject which we will address systematically in a forthcoming publication [55].

For now, we can extract information about the full mesonic moduli space by allowing Gröbner techniques to perform the elimination, such as using Macaulay2, and then use our numerical methods to perform the hard step of primary decomposition. We tabulate some results in Table 3. In [49], only the top components of these systems using Macaulay2 were obtained and conjectured that the results were irreducible. In our case, we can now get the complete irreducible decomposition, not only the top dimension, of these systems. Hence our results yield that the conjecture was correct. We could have easily been able to go beyond the (4,4) systems, but the limiting factor was Macaulay2 itself which failed to obtain the eliminated ideals for the bigger systems for us.

Indeed, one could combine our present course of study with that of the previous subsection. Indeed, the standard Yukawa term to sQCD for $N_f > N_c$, in the above notation, is

$$W = \sum_{i,j=1}^{N_f} \sum_{a,b=1}^{N_f - N_c} \epsilon_{ab} M_{ij} Q_a^i \tilde{Q}_b^j \ . \tag{4.23}$$

$N_f \backslash N_c$	1	2	3	4
1	(1; 0; 1|2)	-	-	-
2	(1; 0; 1|4)	(1; 5; 4|8)	-	-
3	(1; 0; 1|6)	(1; 7; 6|12)	(2; 11; 18|18)	-
4	(1; 0; 1|8)	(1; 9; 8|16)	(2; 14; 32|24)	(2;19;88|32), (1;20;320|32)

Table 4: The master space of sQCD theories with N_f fermions and N_c bosons. Here, $N_f \geq N_c$, and the irreducible decomposition is presented in the form: (no.components; dimension; degree | total dim).

Here, M_{ij} is the meson condensate of the quark-anti-quark. This superpotential can be considered as being generated by Seiberg duality from one with pure matter content. For example, at $(N_f, N_C) = (3, 2)$, we find the master space to be of dimension 9, with 3 components, one at degree 1 and two at degree 7. The complete results are shown in Table 4 and we see non-trivial primary components to the master space.

4.3 Instanton Moduli Spaces

The study of Yang-Mill instantons is, undoubtedly, another important subject in the investigation of gauge theories. Ever since their discovery by Belavin, Polyakov, Schwartz and Tyuplin [56] as well as the construction by Atiyah, Drinfeld, Hitchin and Manin (ADHM) on self-dual solutions [57] in the 1970's, the parameter, or moduli, space of these instanton solution has been of great attention to physicists and mathematicians alike. The geometry of the moduli space can be quite involved. Indeed, whereas the ADHM construction gives the moduli space for the classical Lie groups, that for the exceptional ones still remain a mystery.

With the embedding of the construction into string theory [58] by Witten, Douglas and Moore, one could realize many cases as the vacuum moduli space of the supersymmetric (quiver) theories which we have introduced above. Recently, the algebraic geometry of the one-instanton moduli space was analyzed along the line of our present thought in [59], calculating, specifically, the Hilbert series of the space. To clarify notation, we shall let k G-instantons signify instantons of (classical) gauge group G and with winding number k. Let us take Figure 7 of the said paper as the illustrative starting point. Here, the moduli space of k $SU(N)$ instantons is given as the $\mathcal{N} = 1$ vacuum moduli space of the "flower quiver", with the field content: $\phi^{(i=1,2)}$ and Φ charged as $k \times k$ matrix fields under $U(k)$, X_{21} charged as $N \times k$ and X_{12} as $k \times N$ matrix fields respectively, all obeying the superpotential:

$$W = X_{21} \cdot \Phi \cdot X_{12} + \epsilon_{\alpha\beta}\phi^{(\alpha)} \cdot \Phi \cdot \phi^{(\alpha)} \ . \tag{4.24}$$

Since the case of $k = 1$ was considered in great detail in [59], let us move on the much more complicated and unsolved problem of, say, $k = 2$. We will focus on the Higgs branch by setting Φ to have zero vacuum expectation value. The input data to the moduli space is as follows.

Gauge invariants:

$$\begin{array}{l} \left(\mathrm{Tr}(\phi^{(1)})^a(\phi^{(2)})^b\right)_{0<a+b\leq 2} \ ; \\ \left(\sum_{i=1}^{2} X_{12}^{i,j_1} X_{21}^{j_2,i}\right)_{j_1,j_2=1,\ldots,N} \ ; \\ \left(\sum_{i_1,i_2=1}^{2} X_{12}^{i_1,j_1}\phi^{(\alpha)}_{i_1,i_2} X_{21}^{j_2,i_2}\right)_{j_1,j_2=1,\ldots,N,\alpha=1,2} \end{array}$$

F-Terms:

$$0 = \partial_{\text{fields}}\left|_{\Phi=0}\left[\mathrm{Tr}\, X_{21}\cdot\Phi\cdot X_{12} - \mathrm{Tr}\,\Phi[\phi^{(1)},\phi^{(2)}]\right]\right. . \tag{4.25}$$

Let us be more specific and take, for example, at $N = 2$. First, the master space is easy to determine. It is a dimension 12 variety of degree 16, defined by 4 quadrics in $\mathbb{C}^{16}$. The Hilbert series is

$$g(t;\mathcal{F}^{\flat}_{(k,N)=(2,2)}) = \frac{(1+t)^4}{(1-t)^{12}} \qquad PE^{-1}[g](t) = 16t - 4t^2 \ . \tag{4.26}$$

From the termination of the plethystic logarithm [60], we see that it is, as also suggested by the dimension, a complete intersection.

This is an ideal with 21 generators in 33 variables: 16 variables in the X and ϕ fields together with 17 auxiliary variables. It can be easily shown that this ideal has a unique irreducible component of dimension 12 in $\mathbb{C}^{33}$, but computing its degree and the actual moduli space via elimination are already quite overwhelming for standard computer algebra packages. Bertini, in just under 24 hours using 200 processors in parallel, found that the degree of this irreducible component of dimension 12 is 20364. We will report on computing the moduli space via numerical elimination in a forthcoming paper.

5 A Few Words About NAG Packages

Although it is not at all our intention to compare the performances of different packages in this paper, we convey what the important computations are that different packages can do more efficiently than others. There are so far three independently packages for NAG: Bertini, HOM4PS2 and PHCPack.

Bertini is a general purpose package for NAG, i.e., it can find all the stationary points of a 0-dimensional ideal and also find irreducible decomposition of positive dimensional ideals efficiently. It also has many other Numerical Algebraic Geometry implementations such as a membership test, finding multiplicities of isolated

singular solutions, facility to specify user-defined homotopy, etc. In particular, Bertini automatically constructs Total Degree and 2-Homogeneous Homotopies. It utilizes adaptive multiprecision, and a parallel version of Bertini is also publicly available.

HOM4PS2 is regarded as having the fastest path-tracker [36]. It also has both Total Degree and Polyhedral Homotopies. HOM4PS2 uses MixedVol-2.0 which is a very efficient package to compute the BKK bound. On the other hand, the applications of the current version of HOM4PS2 are limited: for the 0-dimensional case, it only takes one of the two homotopies as its only input parameter along with the equations themselves and gives out all the isolated real and complex solutions. For the positive dimensional systems, it also takes in only two of the homotopies as its input parameter and the dimension of which one wants to get the irreducible decomposition of along with the equations, and returns the irreducible decomposition of that particular dimension. The parallel version of HOM4PS2 is under development and will be soon publicly available [61].

PHCPack is a general purpose solver like Bertini. It does quite a few things, such as computing many different upper bounds on the number of solutions in addition to the CBB, 2HomBB, and BKK root counts, it also can preprocess the system before running the path-tracker on it, one can supply a user-defined homotopy as well, one can choose different numerical precision and parameters manually if one wants to, etc. It runs interactively and one can chose different options in its menu-based interface. The parallel version of PHCpack is also publicly available.

All these packages have their own advantages. It is a goal of the authors of the current paper to write a separate article on demonstrating various features of these packages specifically to the string phenomenologists and particle physicists, and write a Mathematica interface for these packages in the same fashion as Stringvacua.

6 Conclusion and Prospects

With the advances in computer algebra and algorithmic geometry, we have witnessed an increasing trend over the last decade wherein such a paradigm has been gaining importance in theoretical physics, especially in gauge and string theories. Not only can hitherto unthinkable problems such as stabilizing highly complicated potentials or scanning through huge classes of string vacua become feasible, but also new geometrical quantities which characterize gauge theories such as Hilbert series or topological invariants of vacuum moduli spaces can be calculated to elucidate the physics.

One great hurdle to this approach of computational algebraic geometry is that the central method involved is finding the Gröbner basis of polynomial ideals, a highly non-parallelizable and exponential growth algorithm. In this paper, we

have demonstrated many instances where numerical algebraic geometry, especially using homotopy continuation methods, can supplement this short-coming when addressing certain quantities. Though by no means a replacement for the powers of the Gröbner techniques due to its symbolic nature, the numerical methods presented in this paper are very useful and often computationally less expensive. In addition, the numerical methods are parallelizable making it a spectacularly robust tool to address the string phenomenology related problems.

Two classes of problems, arising from a myriad of physical situations such as effective potentials in string and M-theory compactifications and supersymmetric vacua of gauge theories, are particularly amenable to our method: finding critical points of zero-dimensional ideals and irreducible decomposition of higher-dimensional polynomial ideals. The former is a typical problem in vacuum stabilization problems and the latter, in extracting branches of the moduli space of vacua. We have shown many concrete examples where these are beyond current computational powers using Gröbner bases but are fairly quickly done, in a highly parallelizable fashion, using numerical homotopy.

We are clearly only touching the surface of a reservoir of great utility. The two classes of problems we have focused on in this paper, already of diverse applications, are only two of the many quantities and techniques which have recently emerged to be crucial in the study of gauge and string theories. For example, the computation of the Hilbert series of a polynomial ideal, a problem seemingly of interest only to pure algebraic geometry, has turned out to enumerate supersymmetric BPS spectra of operators [60]. Schubert calculus is another fascinating area coming from enumerative geometry where the NAG method can be directly and efficiently applied. A new homotopy meant to deal with the Schubert calculus problems has already been worked out. Currently, methods for numerically obtaining the Hilbert series are being developed. As another example, systematic elimination of variables in the so-called syzygy problem has turned to be important in the study of moduli space of vacua in gauge theories and give explicit defining equations of the vacuum manifold [3]. This, too, is being developed. Indeed, there are several international collaborations which are engaged in the combing over the plethora of string vacua using computer search. Most of these ultimately reduce to manipulation of polynomial ideals and are thus limited in parallelizability and running time by the Gröbner paradigm. The possibilities of numerically computing such sophisticated quantities as Hilbert series and ranks of (co-)kernels of polynomial maps should offer a new outlook.

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A 2-Homogeneous Homotopy

As seen above, the variables in a polynomial system can often be partitioned into two groups. The key is a refined notion of the degree of a polynomial. To illustrate, consider the polynomial $f(y,z) = 1 + 2y - 5z + 3y^2z$. Clearly, the degree of f is three. However, y has a maximum degree of 2 and z only appear linearly in f. That is, if α and β are random constants and

$$g(y) = f(y,\beta) \text{ and } h(z) = f(\alpha,z), \tag{A.27}$$

the degree of g is 2 while the degree of h is 1. We say the **bidegree** of f is $(2,1)$.

The notion of the bidegree of $f(y,z)$ naturally extends to the case when y and z are sets of variables. That is, the bidegree of $f(y,z)$ is (a,b) if the $\deg g = a$ and $\deg h = b$ where g and h are defined by (A.27) with α and β random vectors of the appropriate size.

For a system $P(x) = P(y,z)$ of m polynomials in m complex variables, where y is k complex variables and z is $m-k$ complex variables, let (a_i,b_i) be the bidegree of $P_i(y,z)$. When each P_i is nonconstant, the polynomial

$$R(s,t) = \prod_{i=1}^{m}(a_i s + b_i t) \tag{A.28}$$

is homogeneous of degree m. The 2-Homogeneous Bézout Theorem (see [29] for more details) states that the number of isolated solutions of $P(x) = 0$ in $\mathbb{C}^m$ is bounded above by the coefficient of the $s^k t^{m-k}$ term of R. This coefficient is the 2-Homogeneous Bézout bound (2HomBB).

For example, consider the system

$$\begin{aligned} f_1(y,z) &= 3y - 2z + 4z^3 = 0, \\ f_2(y,z) &= 1 - yz = 0, \end{aligned} \tag{A.29}$$

where y and z are complex variables. The CBB yields that this system has at most $3 \cdot 2 = 6$ isolated solutions in $\mathbb{C}^2$. The bidegree of $f_1(y,z)$ and $f_2(y,z)$ is $(1,3)$ and $(1,1)$, respectively, with the polynomial $R(s,t) = (s+3t)(s+t) = s^2 + 4st + 3t^2$. Since the coefficient of st in R is 4, the 2HomBB yields that (A.29) has at most 4 isolated solutions in $\mathbb{C}^2$.

To construct a 2-Homogeneous Homotopy, we need to describe how to construct the start system $Q(x) = Q(y,z)$. The polynomials $P_i(y,z)$ and $Q_i(y,z)$ must have the same bidegree, namely (a_i, b_i). One way to construct such a polynomial is to take

$$Q_i(y,z) = \left(\prod_{j=1}^{a_i} L_{i,j}(y)\right) \cdot \left(\prod_{j=1}^{b_i} M_{i,j}(z)\right) \tag{A.30}$$

where $L_{i,j}(y)$ and $M_{i,j}(z)$ are random linear polynomials. The number of solutions of $Q(x) = Q(y,z) = 0$ is the 2HomBB of $P(x) = P(y,z)$, and the solutions themselves can be computed using linear algebra.

B Polyhedral Homotopy

As mentioned above, polynomial equations arising in real-life problems are sparse in terms of the number of monomials which appear. In general, CBB becomes an upper bound on the number of solutions for such cases. We are looking for a tighter bound that takes the sparsity of the system into account. There has been a huge amount of work done on related issues using resultants and algebro-geometric methods, but the most important result for us is Bernstein's theorem. In order to state it clearly, we remind the reader of some standard notions of Laurent Polynomials, Newton Polytopes, and Mixed Volume.

A Laurent polynomial allows negative exponents for the monomials, so no variable is allowed to be zero. Hence, multiplication of any Laurent polynomial by a monomial does not change the root count in $(\mathbb{C}^*)^m = (\mathbb{C}/\{0\})^m$. Formally, let $S_i \subset \mathbb{Z}^m$ be a set of vectors whose elements are the exponents of the monomials of the ith polynomial. S_i is called the support of the ith polynomial. Then a polynomial (say, ith polynomial) of the form $f_i(x) = \sum_{\alpha \in S_i} c_{i,\alpha} x^\alpha$ is called a Laurent polynomial. Here, $c_{i,\alpha} \in \mathbb{C}$ are the coefficients of the monomial x^α with $x \in (\mathbb{C}^*)^m$.

Next, a set of points is called a convex set if for every pair of points within the set (or more formally, the mathematical object made by the set), every point on the straight line segment that joins them is also within the set. The convex hull of a set X is the minimal convex set containing X. We note that the convex hull of support S_i of a polynomial, say $Q_i = \text{conv}(S_i)$, is called the **Newton polytope** of $f_i(x)$.

For example, consider a two-variable system

$$\begin{aligned} f_1(x,y) &= 1 + ax + bx^2y^2 = 0, \\ f_2(x,y) &= 1 + cx + dy + exy^2 = 0, \end{aligned} \tag{B.31}$$

where x and y are complex variables and a, b, c, d, e are complex coefficients. The CBB of this system is $4 \cdot 3 = 12$, i.e., there can be a maximum of 12 isolated solutions for this system in $\mathbb{C}^2$. The 2HomBB, using $\{x\} \times \{y\}$ is 6. Now, the supports of these equations are $S_1 = \{(0,0),(1,0),(2,2)\}$ and $S_2 = \{(0,0),(1,0),(0,1),(1,2)\}$ respectively. The Newton polytope for $f_1(x,y)$ is $Q_1 = \text{conv}(S_1) = \{(0,0),(1,0),(1,1),(2,2)\}$ and for $f_2(x,y)$ it is $Q_2 = \text{conv}(S_2) = \{(0,0),(1,0),(0,1),(1,1),(1,2)\}$.

A Minkowski sum of any two convex sets is defined as

$$Q_1 + Q_2 = \{q_1 + q_2 \colon q_1 \in Q_1, q_2 \in Q_2\}. \tag{B.32}$$

The Minkowski sum of two Newton polytopes corresponds to multiplying the corresponding polynomials algebraically. The m-dimensional volume of a simplex having vertices $v_0, v_1, \ldots, v_m$, is

$$\text{Vol}_m(\text{conv}(v_0, \ldots, v_m)) = \frac{1}{m!} |\det[v_1 - v_0, \ldots, v_m - v_0]|. \tag{B.33}$$

From there on, one can show that the m-dimensional volume $\text{Vol}_m(\lambda_1 Q_1 + \cdots + \lambda_m Q_m)$, where $0 \geq \lambda_i \in \mathbb{R}$, is a homogeneous polynomial of degree m in variables λ_i. The mixed volume of convex polytopes $Q_1, \ldots, Q_m$, denoted $M(Q_1, \ldots, Q_m)$, is defined as the coefficient of $\lambda_1 \cdots \lambda_m$ in $\text{Vol}_m(\lambda_1 Q_1 + \cdots + \lambda_m Q_m)$. It can be shown that

$$M(Q_1, \ldots, Q_m) = \sum_{i=1}^{m} (-1)^{m-i} \ \text{Vol}_m\Big(\sum_{j \in \Omega_i^m} Q_j\Big), \tag{B.34}$$

where the inner sum is a Minkowski sum of polytopes and Ω_i^m are the combinations of m-objects (i.e., m-dimensional geometrical objects made of m-simplices) taken i at a time. Moreover, the mixed volume is always an integer for a system of Laurent polynomials. For the case of two polynomials in two variables,

$$M(Q_1, Q_2) = \ \text{Vol}_2(Q_1 + Q_2) - \ \text{Vol}_2(Q_1) - \ \text{Vol}_2(Q_2). \tag{B.35}$$

For the system in (B.31),

$$\begin{aligned} \text{Vol}_2(Q_1) &= 1, \\ \text{Vol}_2(Q_2) &= \text{ area of parallelogram made by } \{(0,0),(1,0),(0,1),(1,1)\} \\ &+ \text{ area of triangle made by } \{(1,1),(1,0),(1,2)\} \\ &= 1 + \frac{1}{2} = \frac{3}{2}, \\ \text{Vol}_2(Q_1 + Q_2) &= \frac{13}{2}. \end{aligned} \tag{B.36}$$

Thus, the mixed volume for this system is 4. This is important because the Bernstein theorem and Bernstein-Khovanskii-Kushnirenko (BKK) theorem [37–39] says that for generic coefficients, the number of isolated solutions in $(\mathbb{C}^*)^m$ of a Laurent system is exactly equal to the mixed volume of this system counting with multiplicity. For any particular set of coefficients, this is an upper bound. This result is very interesting since one can get a generically sharp bound on the number of solutions in $(\mathbb{C}^*)^m$ of a polynomial system by knowing the exponent vectors of monomials. For the system in (B.31), there can be a maximum of 4 isolated solutions in $(\mathbb{C}^*)^2$, which is in contrast to the CBB and 2HomBB bound of 12 and 6, respectively, in $\mathbb{C}^2$.

To yield the BKK count for $\mathbb{C}^m$, one considers an extension of the mixed volume, called the *stable mixed volume* [1], which ensures that we have all necessary solutions in $\mathbb{C}^m$ [62–64]. A discussion on the stable mixed volume is beyond the scope of this article, although its calculation is similar to that of the mixed volume with some formal complications. However, it should be noted that a highly sophisticated implementation of an algorithm to calculate the mixed volume of a given system is MixedVol [35] which is transplanted in PHCpack and MixedVol-2.0 which is transplanted in HOM4PS2.

Now, after calculating the stable mixed volume of the original system $P(x) = 0$, we require a start system, $Q(x) = 0$, such that $Q(x)$ has the same stable mixed volume, where of course solutions of $Q(x) = 0$ should be known or can be obtained easily. This homotopy method is called Polyhedral Homotopy.

[1] Because of the highly technical nature of the stable mixed volume, the bound in $\mathbb{C}^m$ which has been commonly used and implemented in the community is the bound given in [62]. This bound is quite easy to state: Add a constant term to polynomials in the system which do not have constant term, and the mixed volume of the resulting augmented system serves as a bound in $\mathbb{C}^m$ for the original system. The "stable mixed volume" is a little bit more general then this "augmented mixed volume". We would like to thank T.Y. Li for clarifying this point.

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