Submission to the DTA2012 Special Issue:

Effects of countdown displays in public transport route choice under severe overcrowding.

Corresponding Author: Ms Valentina Trozzi,
ValentinaTrozzi@tfl.gov.uk
044(0)20
Transport Planning – London Underground.
55 Broadway, London SW1H 0BD (UK)

Prof Guido Gentile,
Università di Roma "La Sapienza"
Via Eudossiana 18, 00184 Roma (Italy)

Dr Ioannis Kaparias,
City University London
Northampton Square, London EC1V 0HB (UK)

Prof Michael G. H. Bell
Sydney University
Newtown Campus, The University of Sydney, NSW 2006 (Australia)

ABSTRACT

The paper presents a route choice model for dynamic assignment in congested, i.e. overcrowded, transit networks where it is assumed that passengers are supported with real-time information on carrier arrivals at stops. If the stop layout is such that passenger congestion results in First-In-First-Out (FIFO) queues, a new formulation is devised for calculating waiting times, total travel times and route splits. Numerical results for a simple example network show the effect of information on route choice when heavy congestion is observed. While the provision of information does not lead to a remarkable decrease in total travel time, with the exception of some particular instances, it changes the travel behaviour of passengers that seem to be more averse to queuing at later stages of their journey and, thus, prefer to interchange at less congested stations.

KEYWORDS

Public Transport; Dynamic Assignment; Online Information; Passengers' queues.

Submission to the DTA2012 Special Issue:

Effects of countdown displays in public transport route choice under severe overcrowding.

1 INTRODUCTION

It has been largely acknowledged in the last decades that urban sustainable development needs to overcome the dependence on the private car (Newman and Kenworthy, 1999, European Commission, 2009) and requires a modal shift towards public transport, as it performs better than private transport with regard to the six sub-objectives for sustainability developed by May in 2001 (unpublished, cited by Black *et al.*, 2002). In this context, much hope is invested in Advanced Traveller Information Systems (ATIS). Indeed, although information provision cannot directly decrease private car use, it can produce time savings – either when tracking and comparing travel options or when planning and deciding – and thus can enhance the quality of service, which in turn contributes to persuading people to switch modes.

In order to evaluate the potential benefit brought about by ATIS in terms of total travel time savings and congestion relief on the public transport network, new route choice models for transit assignment are needed, which are capable of representing the travel choices of passengers assisted by information systems and highlight any change in the distribution of flows across the network with respect to the case where no ATIS is in place, especially if the system is subject to recurring overcrowding.

Consequently, this paper proposes a dynamic route choice model for transit assignment to densely connected networks where congestion results in passengers First-In-First-Out (FIFO) queues at the stops and where travellers are supported with real-time information on vehicle arrivals, for example through countdown displays.

In densely connected urban networks, following Nguyen and Pallottino (1988) and Spiess and Florian (1989), it is assumed that passengers would not select the shortest single itinerary to destination, but would rather choose a bundle of potentially optimal paths, formally known as *hyperpaths* or *travel strategies*, and then would follow one specific path of their hyperpath depending on events occurring while they are waiting at the stop, namely what is the first *attractive line* (Nguyen and Pallottino, 1988) that they can board.

Moreover, as in (Hickman and Wilson, 1995, Gentile *et al.*, 2005), it is assumed that real-time information changes the travel behaviour in such a way that travellers would not get on a carrier only because it is the first of their choice set that becomes available at the stop, but would board it only if its remaining travel time to destination is shorter than the sum of waiting time plus travel time upon boarding for subsequent services. An important innovation with respect to (Hickman and Wilson, 1995, Gentile *et al.*, 2005) is that the proposed model acknowledges that recurrent overcrowding can result in passengers' queues at transit stops and, in the context of commuting trips, it is assumed that travellers do not make their travel choices only considering the average values of frequencies and in-vehicle travel times, but also considering congestion levels for the different lines of their choices. In other words the proposed model assumes that the users know by previous travel experience how many vehicles of the same line they have to wait, on average, because of insufficient capacity on-board.

First applications to a small example network seem to suggest that, if real time information is provided, route choices tend to be more conscious in the sense that passengers would be more prone to wait for a subsequent service or select slower lines in order to avoid transfers at crowded stations.

The rest of the paper is organised as follows. The next section presents the background of the study, while the methodology is explained in Section 3. The solution algorithm is detailed in Section 4. Finally, in Section 5 a numerical example is presented and conclusions are drawn in Section 6.

2 BACKGROUND

Transit assignment aims at describing and predicting the choices of public transport users, depending on the assumptions made about travellers' behaviour, congestion effects, and the level of service supplied by the transport system.

For example, in networks with highly frequent services it is assumed that travellers do not time their arrival at stops with the lines' schedule and, when making their travel choices, they only consider average frequencies and in-vehicle travel times (this is the main assumtpion of frequency-based models). In such a setting, transit assignment models can be developed considering a *strategy-based* (or *hyperpath-based*) route choice model, as in (Spiess and Florian, 1989). Starting from the origin, the travel strategy involves the iterative sequence of walking to a public transport stop or to the destination, selecting the set of *attractive lines* (Nguyen and Pallottino, 1988) to board and, for each of them, the stop where to alight. If two or more attractive lines are available at the origin/transfer stop, then the best option is to board the first one approaching (Spiess, 1983, Spiess, 1984).

The result of such a choice is a set of simple itineraries that can diverge, only at stops, along the routes of the attractive lines (Bouzaiene-Ayari *et al.*, 2001), and the realisation of the same travel strategy may change, from day to day, due to 'micro-level' events such as what attractive line becomes available first at the stop, or what is the actual realisation of the waiting and in-vehicle time. Notwithstanding these uncertainties on the supply and the stochasticity of the waiting time, the classical application of the hyperpath paradigm allows for developing a *determisinstic* route choice model for transit assignment, where it is assumed that travel choices ultimately depend on the expected value of the total travel time and not on its actual realisation on a particular day. Despite some authors (Miller-Hooks and Mahmassani, 2000, Pretolani 2000, Yang and Miller-Hooks, 2004) have also applied hyperpaths to model explicitly the effect of day-to-day variations of travel times on route choice and on its en-trip adaptations, such extensions are not considered here, while the original formulation of travel strategies for deterministic route choice in networks with uncertanties is.

Furthermore, when the usual assumptions that no congestion occurs, and that the only information available to passengers is what line arrives first, do not hold true, the traditional strategy-based assignment models are not suitable to represent the behaviour of passengers that travel in densely connected transit networks. Consequently in the last two decades many works have been proposed to investigate either the effect of passenger queues at the stop or the effect of countdown displays, while the combination of the two problems has't been largely investigated yet.

2.1 Congestion and capacity constraints

While recurring passenger congestion is one of the main problems faced by large-city transit networks, in the literature there does not seem to be any broad agreement on how this phenomenon should be modelled.

The vast majority of research works carried out in this context focuses on static transit assignment and the effects of overcrowding are modelled by means of the *effective frequency*, with or without capacity constraints (De Cea and Fernandez, 1993, Cominetti and Correa, 2001, Cepeda *et al.*, 2006), *fail-to-board probability* (Kurauchi *et al.*, 2003), *attractivity threshold* (Leurent and Benezech, 2011), or by micro-simulation (Teklu, 2008).

However, even when capacity constraints are considered, static models can only yield average results (in terms of flows and travel time estimation) for the entire analysis period, and cannot reproduce the formation and dispersion of passenger queues at stops nor their dynamic effects on route choice. This drawback is partially overcome by Schmöcker *et al.* (2008), who develop a *quasi-dynamic* strategy based assignment that reproduces dynamic variations in the Level of Service (LoS) caused by passenger congestion. On the other hand, while in their route choice model it is assumed that the anticipated value of delays increases the expected total travel time to destination, the effect of congestion on passengers' distribution among attractive lines is disregarded.

Additionally, the majority of strategy-based assignment models assume that, if travel demand exceeds the supplied capacity, queuing passengers do not respect any boarding priority. The assumption is usually accepted when modelling passenger flows in rail and/or underground networks because large platforms allow travellers to mingle and, thus, it is though that who arrives last might be 'lucky' and board the first approaching carrier despite congestion, while other passengers can be 'unlucky' and keep waiting even if they arrived before. However, when overcrowding is very severe the priority of those who are closer to the edge of the platform is usually respected and, thus, a model based on a First-In-First-Out (FIFO) queuing mechanism would seem more appropriate. Additionally, for bus systems (where boarding is generally allowed only from front doors) the stop layout is usually designed to allow passengers queuing in a FIFO fashion.

Unfortunately, models based on the FIFO queuing assumption have proved to be very complex to develop and, to the best of the authors' knowledge, all existing attempts (Gendreau, 1984, Bouzaïene-Ayari, 1988, Bouzaïene-Ayari *et al.*, 2001, Leurent and Benezech, 2011) share the *stability condition* (passengers waiting at a stop would consider an attractive set that is never completely saturated, in the sense that, at least for one of the attractive lines, passengers can board the first vehicle coming, Bouzaïene-Ayari *et al.*, 2001) which implies the following two shortcomings:

- as congestion increases, more (and hence 'worse') lines are included in the attractive set; and
- if all lines are congested, passengers would rather walk than keep waiting (even if frequencies are high, so that the extra waiting time due to congestion is, anyhow, short).

A schedule-based approach has also been applied by some authors (Hamdouch and Lawphongpanich, 2008, Hamdouch *et al.*, 2011), who have extended an existing dynamic strategy-based model for traffic assignment with time-expanded network (Hamdouch *et al.*, 2004) to public transport systems. This approach has the advantage that the dynamic assignment reduces to a static assignment on the time-expanded network and, in this setting, it is possible to accurately represent the build-up and dissipation of passenger queues at stops. On the other hand, the very concept of travel strategy is changed because passengers know and trust the service time-table (this is one of the basic assumptions of schedule-based models) and can precisely select their best travel option; however, it is uncertain if they will be able to board/sit when congestion occurs.

2.2 Effects of countdown displays in networks with uncertainties

The effects of *way-side* (Grotenhuis *et al.*, 2007) travel information systems, such has Variable Message Signs (VMS), has been widely investigated in *traffic* networks, and the hyperpath paradigm has also been used to model drivers re/routing as consequence of real-time travel information received by means of VMS (Ukkusuri and Patil, 2007, Gao et al., 2010, Gao 2012) in stochastic road networks.

Also for public transport users the support of way-side information systems, for example count-down displays, can reduce uncertainties and, thus, affect their route choice. Nevertheless, for transit networks the topic has been studied less extensively than for private traffic networks. The few existing exceptions include Hickman and Wilson (1995) and Gentile *et al.* (2005).

The authors recognize that when count-down displays are installed at transit stops the route choice behaviour described in the seminal works on hyperpaths/travel strategies ceases to be rational. Instead, it is reasonable to assume that travellers use countdown displays in order to minimise their expected total travel time to the destination and when a vehicle approaches the stop, a waiting passenger does not board it simply because it is the first attractive line arriving, but instead compares its expected travel time to the destination upon boarding with the expected total travel times of later arrivals.

The authors only consider uncongested scenarios and acknowledge the fact that the travel time savings produced by countdown displays do not seem to be remarkable (Gentile *et al.*, 2005). On the other hand, as it will be clarified in the following sections, it is plausible to assume that in case of severe overcrowding, the provision of information may change the behaviour of public transport users and, thus, help in relieving congestion phenomena.

Consequently, in this paper the combined effect of queues and real-time travel information is investigated and a model is proposed, which may be exploited to assess if count down displays can help in relieving congestion.

3 METHODOLOGY

3.1 Problem definition

The provision of real-time information through countdown displays brings about some important demand-side effects in transit networks that are affected by recurrent congestion, as discussed here.

Depending on the design of the stop, two important sub-cases of FIFO queues may appear: either the stop is designed to have physically separate queues for each line; or passengers arriving at the stop join a single, mixed queue regardless of their attractive line set.

The first instance is very common in coach terminals. In this case, should congestion occur and no real-time information be available, passengers cannot behave strategically because they must join one specific queue as soon as they reach the stop. It may then be difficult to change queue in order to take advantage of events occurring while they are waiting (e.g. if another line arrives first). Consequently, the stop has to be modelled as a *group of separate stops*, each of which is served by one line only. However, if countdown displays are available and passengers have sufficient experience to predict how many vehicles will pass before being able to board each line, travel behaviour in the case of separate queues can also be modelled as strategic. Indeed, the information 'anticipates' the event of a vehicle arrival to the moment when the user reaches the stop; hence, the optimal travel strategy comes true in the moment when the traveller actually chooses which line to board, taking into account the length of the different queues. In other words, if information is provided, this case can be treated as if there were a single 'mixed' queue.

The second type of stop layout (single, 'mixed' FIFO queue) is more common in urban public transport networks. If congestion occurs, users arriving at the stop join the queue and board the first line of their attractive set that becomes available. However, if no real-time information is provided and regular services are available, it is possible that passengers would change their attractive set while they wait, as described by Billi *et al.* (2004) and Noekel and Wekeck (2007). On the other hand, if information is provided, an attractive-set structuring can be modelled more easily also in the presence of regular services because it can be assumed that passengers know the line they will board as soon as they reach the stop.

Consequently, in such a setting, the route choice can always be modelled by extending the results of Hickman and Wilson (1995) and Gentile *et al.* (2005) to a dynamic scenario where congestion phenomena are considered.

3.2 Network formalisation and basic notation

The transit network, which comprises a set of lines $\Im \subseteq \aleph$ ($\aleph$ is the set of natural integers), together with the pedestrian network is represented by a directed *hypergraph* (Gallo *et al.*, 1993) $HG = \{N, A\}$, where $N = \{i \mid i = 1, 2, \ldots, n\}$ is the node set and $A = \{a \mid a = 1, 2, \ldots, m\}$ is the hyperarc set. The generic hyperarc a is univocally identified by its initial, or *tail*, node $TL_a \in N$ and its final, or *head*, node(s) $HD_a \subset N$, that is $a = (TL_a, HD_a)$. The number of nodes included in the head of the hyperarc is called *cardinality* ($|HD_a|$), and hyperarcs with cardinality equal to one are also called *proper arcs* (Nguyen *et al.*, 1998) or, simply, "arcs".

The sets of nodes and arcs, as illustrated in Figure 1, are constructed as follows:

N^P: pedestrian nodes;

N^C: centroid nodes, including all passenger origins and destination ($N^C \subseteq N^P$);

N^S: stop nodes;

N^B: boarding nodes;

N^A: alighting nodes;

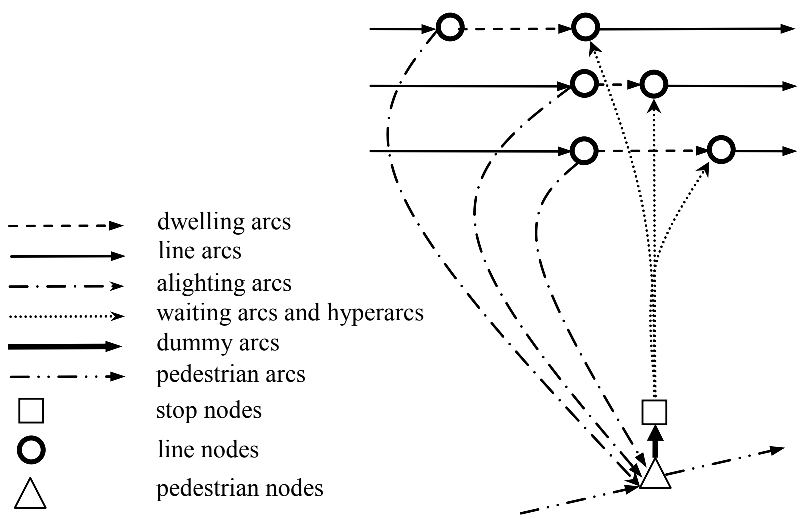


FIGURE 1: Representation of a stop in the hypergraph

A^P: pedestrian arcs, represent walking time. For each $a \in A^P$ its tail and head belong to the pedestrian node set: $TL_a, HD_a \in N^P, \ \forall \ a \in N^P$;

A^L: line arcs, represent in-vehicle travel time.

$\forall\ a \in A^L$: $TL_a \in N^B$, $HD_a \in N^A$;

A^D: dwelling arcs, representing the time a bus spends at a stop while passengers alight/board.

$\forall\ a \in A^D$: $TL_a \in N^A$, $HD_a \in A^B$;

A^Z: dummy arcs, are introduced for algorithmic purposes. They do not have a physical meaning, but represent a graphic connection between the transit network and the pedestrian network.

$\forall\ a \in A^Z$: $TL_a \in N^P$, $HD_a \in N^S$;

A^A: alighting arcs, represent the time that passengers need to disembark.

$\forall\ a \in A^A$: $TL_a \in N^A$, $HD_a \in N^P$;

A^H: waiting hyperarcs (Billi *et al.*, 2004), These represent the total expected waiting time for a specific set of *attractive lines* serving a stop: $A^H \subseteq \{(i, j)$: $i \in N^S$, $J \subseteq N^B$, $j \in J\}$. Each waiting hyperarc $h \in A^H$ is univocally identified by a singleton *tail* (TL_h), which is a stop node, and by a set *head* (HD_h) of boarding nodes. Therefore, the waiting hyperarc can be indicated as $h = \{(TL_h, j)$: $j \in HD_h\}$ and it can also be regarded as a set of 'branches', or simple *waiting arcs a*, each of which has the same tail node of h ($TL_a = TL_h$) and a head node belonging to the head set of h ($HD_a \in HD_h$). Moreover, the head node of a branch of a hyperarc h ($a \in h$) is associated with one particular line (L_{HDa}) among those who share the stop represented by $TL_a = TL_h$.

FS_i: forward star of node i, i.e. the set of arcs sharing the same head node i.

$FS_i = \{a \in A|\ HD_a = i\}$;

BS_i: backward star of node i, i.e the set of arcs sharing the same tail node i.

$BS_i = \{a \in A|\ TL_a = i\}$

HFS_i : *hyper-forward star* of node $i \in N^S$, i.e. the set of hyperarcs sharing the same stop tail i: $HFS_i = \{h \in A^H$: $TL_h = i\}$

In order to represent time-dependent travel times, waiting times, etc., the following dynamic variables are also introduced with reference to the generic $a \in h$ and $h \in A^H$:

$\varphi_a(\tau)$: *instantaneous frequency* (instantaneous flow of carriers) of the line L_{HDa} evaluated at the stop node corresponding to TL_a at time τ,

$\kappa_a(\tau)$: *congestion parameter*, expressed as the total number of vehicle arrivals that passengers are unable to board at time τ (because of capacity constraints) before they board the line L_{HDa};

$w_{h,d}(\tau)$: *expected waiting time* for passengers directed towards destination d, who reach the stop TL_h at time τ and considering the set of attractive lines represented by $h \in A^H$;

$w_{a|h,d}(\tau)$: *conditional expected waiting time.* This is the expected time before boarding the line L_{HDa} associated with $a \in h$ for passengers, directed towards destination d, who reach the stop TL_a at time τ; its value depends on the set of attractive lines considered, which is represented by $h \in A^H$;

$t_{a|h,d}(\tau)$: *conditional boarding time* on the line L_{HDa} for passengers, directed towards destination d, who reach the stop TL_a at time τ – namely $t_{a|h}(\tau) = \tau + w_{a|h}(\tau)$, and its value depends on the set of attractive lines considered, which is represented by $h \in A^H$;

$p_{a|h,d}(\tau)$: *diversion probability* (Cantarella, 1997) at time τ for passengers directed towards destination d: ratio of passengers that board line L_{HDa} to those whose set of attractive lines is represented by $h \in A^H$;

$\mathrm{PDF}_a(w_a, \tau)$: probability distribution function (PDF) of the waiting time before boarding line L_{HDa} at time τ,

$\overline{\mathrm{CDF}}_a(w_a, \tau)$: survival function of the waiting time before boarding line L_{HDa} at time τ. The survival function indicates the probability that the variable is greater than a certain value and it can be regarded as the opposite of the cumulative distribution function (CDF) for the same stochastic variable, namely $\overline{\mathrm{CDF}}_a(w_a, \tau) = 1 - \mathrm{CDF}_a(w_a, \tau)$.

It should be noticed here that, although diversion probabilities, conditional waiting and conditional boarding time depend on the specific destination considered, the subscript d is neglected in the following in order to improve readability.

Moreover, with reference to the generic proper arc $a \in HG \backslash \{A^H\}$ and $i \in N$, the following variables are also defined:

$c_a(\tau)$: *travel time* of arc a for users entering it at time τ;

$t_a(\tau)$: *exit time* from arc a for users entering it at time τ – namely, $t_a(\tau) = \tau + c_a(\tau)$;

$t_a^{-1}(\tau)$: *entry time* to the arc a for users exiting it at time τ;

$g_{i,d}(\tau)$: *total travel time* from node i to destination $d \in N^C$ at time τ;

$g^*_{i,d}(\tau)$: *minimum total travel time* from node i to destination $d \in N^C$ at time τ.

3.3 Formulation

In a dynamic setting, the results of Hickman and Wilson (1995) and Gentile *et al.* (2005) are extended to obtain a time-dependent expression for the travel cost of the minimal hyperpath from every node to the destination:

$$g_{i,d}(\tau)=\begin{cases}0, \text{ if } i=d\\ \min\limits_{a\in FSi}\left(c_a(\tau)+g_{HDa,d}\left(t_a(\tau)\right)\right), \text{ if } i\notin N^S\\ \min\limits_{h\in HFS_i}\left(w_h(\tau)+\sum\limits_{a\in h}p_{a|h}(\tau)\cdot g_{HDa,d}\left(t_{a|h}(\tau)\right)\right), \text{ if } i\in N^S\end{cases} \tag{1}$$

where:

$$p_{a|h}(\tau)=\int_0^{+\infty}\text{PDF}_a(w,\tau)\prod_{\substack{a'\in h,\\ a'\neq a}}\overline{\text{CDF}}_{a'}(w,\tau)\,dw \tag{2}$$

$$w_{a|h}(\tau)=\frac{1}{p_{a|h}(\tau)}\int_0^{+\infty}w\cdot\text{PDF}_a(w,\tau)\prod_{\substack{a'\in h,\\ a'\neq a}}\overline{\text{CDF}}_{a'}(w,\tau)\,dw \tag{3}$$

$$w_h(\tau)=\sum_{a\in h}p_{a|h}(\tau)\cdot w_{a|h}(\tau) \tag{4}$$

For each possible intermediate stop node *i*, $g_{i,d}(\tau)$ is fully defined when PDF_a and $\overline{\text{CDF}}_a$' are known; on the other hand the optimality of a travel strategy depends on the correct selection of the attractive set. Thus, the definition PDF_a and $\overline{\text{CDF}}_a$', and the method of selection of the attractive set are core problems in the development of the new route choice model, and will be considered in detail next.

PDFs and CDFs of the waiting times

The major assumption of the model is that in the context of commuting trips, if congestion leads to the formation of FIFO queues, passengers have a good estimate of the average number of vehicles of the same line that they must let go before being able to board (Trozzi *et al.*, 2013).

In this setting, the waiting time before boarding is a stochastic variable, whose value depends on the assumption made about service regularity. For example, if the basic hypotheses about carrier and passenger arrivals (Nguyen and Pallottino, 1988, Spiess and Florian, 1989) are not changed, the total waiting time before boarding may be modelled as an Erlang-distributed stochastic variable with parameters $\kappa_a(\tau)$ and $\varphi_a(\tau)$, such that:

$$\text{PDF}_a(w,\tau)=\begin{cases}\dfrac{\varphi_a(\tau)^{\kappa_a(\tau)}\cdot\exp\left(-\varphi_a(\tau)\cdot w\right)\cdot w^{\left[\kappa_a(\tau)-1\right]}}{\left[\kappa_a(\tau)-1\right]\,!}, \text{ if } w\geq 0\\ 0, \qquad \text{otherwise}\end{cases} \tag{5}$$

Alternatively, when regular services with constant headways are considered, the waiting time before the first arrival is uniformly distributed and, therefore, the PDF of the total waiting time can be expressed as in equation (6).

$$\mathrm{PDF}_a(w,\tau)=\begin{cases}\varphi_a(\tau), & \text{if } \dfrac{[\kappa_a(\tau)-1]}{\varphi_a(\tau)}\le w<\dfrac{\kappa_a(\tau)}{\varphi_a(\tau)}\\ 0, & \text{otherwise}\end{cases} \tag{6}$$

The definition of survival functions is not as straightforward as the definition of PDF_a. This is because some stops can be shared by regular and irregular services. For example, this can be the case for large bus terminals, where there are some lines whose routes run in segregated lanes (where the absence of interaction with private car traffic and/or road works enhances the service regularity) and there are also some other lines that are subject to service irregularity because their routes do not run in segregated lanes.

For this reason, the definition of equations (2) and (3) is articulated into two different subcases, depending on whether the line considered for the evaluation of its diversion probability and conditional expected waiting time has constant or exponentially distributed headways.

For example, if L_{HDa} is a service with constant headways, $\mathrm{PDF}_a(w,\ \tau)$ is expressed by means of equation (5). Moreover, if:

$$\beta_{a'}=w+\frac{\kappa_a(\tau)-1}{\varphi_a(\tau)}+g_{HDa,d}-g_{HDa',d} \tag{7}$$

then $\overline{\mathrm{CDF}}_a{}'\left(w+g_{HDa,d}-g_{HDa',d},\ \tau\right)$ is expressed as in equation (8) if $L_{HDa'}$ is a service with exponentially distributed headways; while if $L_{HDa'}$ is a service with constant headways, $\overline{\mathrm{CDF}}_a{}'\left(w+g_{HDa,d}-g_{HDa',d},\ \tau\right)$ is expressed as in equation (9).

$$\overline{\mathrm{CDF}}_{a'}\left(w+g_{HDa,d}-g_{HDa',d},\ \tau\right)=\sum_{j=0}^{\kappa_{a'}(\tau)}\frac{\varphi_{a'}(\tau)^{\kappa_{a'}(\tau)-j}\cdot e^{-\varphi_{a'}\cdot\beta_{a'}}\cdot\beta_{a'}{}^{[\kappa_{a'}(\tau)-j]}}{\left(\kappa_{a'}(\tau)-j\right)!} \tag{8}$$

$$\overline{\mathrm{CDF}}_{a'}\left(w+g_{HDa,d}-g_{HDa',d},\ \tau\right)=\begin{cases}1, & \beta_{a'}<\dfrac{\kappa_{a'}(\tau)-1}{\varphi_a{}'(\tau)}\\ \displaystyle\int_{\beta_{a'}}^{\frac{\kappa_{a'}(\tau)}{\varphi_a{}'(\tau)}}\varphi_a{}'(\tau), & \dfrac{\kappa_{a'}(\tau)-1}{\varphi_a{}'(\tau)}<\beta_{a'}<\dfrac{\kappa_{a'}(\tau)}{\varphi_a{}'(\tau)}\\ 0, & \beta_{a'}<\dfrac{\kappa_{a'}(\tau)}{\varphi_a{}'(\tau)}\end{cases} \tag{9}$$

On the other hand, in the case where L_{HDa} is a service with exponentially distributed headways, then $\mathrm{PDF}_a(w,\ \tau)$ is expressed by means of equation (6), while $\overline{\mathrm{CDF}}_a{}'\left(w+g_{HDa,d}-g_{HDa',d},\ \tau\right)$ is expressed by equations (8) and (9) for irregular and regular services respectively, where $\beta_{a'}$ is defined as:

$$\beta_{a'}=w+g_{HDa,d}-g_{HDa',d} \tag{10}$$

Attractive set

In general, the above expressions of the diversion probabilities and expected waiting times can be applied to any hyperarc $h \in HFS_i$. However, only a specific *waiting hyperarc* is associated with the set of lines that are mostly convenient to board, at time τ, in order to reach the destination in the minimum time.

The lines to be included in the waiting hyperarc (or, equivalently, in the attractive set) generally depend on the time τ when the set is evaluated and can be determined by solving a combinatorial problem. At least for the static case, the problem of determining the attractive set can be simplified because it is counter-intuitive to exclude a line from the choice set if it has a shorter remaining travel time than any other line already included in the set. Therefore, a greedy approach may be applied (Spiess and Florian, 1989, Nguyen and Pallottino, 1988, Chriqui and Robillard, 1975) by processing the lines in ascending order of their travel time upon boarding and the progressive calculation of the values of $p_{a|h}$, w_h, and $g_{i,d}$ is stopped as soon as the addition of the next line increases the value of $g_{i,d}$. At this point, the cost is minimal and the set of lines corresponds to the *attractive set*.

The correctness of the greedy method, in the static case, depends on the shape of the waiting time PDF (exponential). While this does not hold in the dynamic scenario, a greedy procedure is suggested anyhow for the application of the proposed model to real-scale networks, where the solution of the full combinatorial problem may become computationally intractable.

4 THE ALGORITHM

As mentioned in the introduction, the proposed route choice model should be embedded in a full dynamic transit assignment procedure. Consequently, a solution algorithm is needed to perform the shortest time-dependent many-to-one (hyper)path search for *every* possible arrival/departure time.

To this end, the Decreasing Order of Time (DOT) method, presented by Chabini (1998) and having been analytically proven to be the most efficient solution method for the all-to-one search for every possible arrival time, is extended to the time-dependent shortest hyperpath problem. It should be noted here that although the proposed model has a continuous time representation, a discrete-time representation is adopted for its numerical solution.

The main idea is to divide the analysis period P = [0, T] into Θ time intervals, such that $AP = \{\tau^0, \tau^1, ..., \tau^\theta, ..., \tau^{\Theta-1}\}$, with $\tau^0 = 0$ and $\tau^{\Theta-1} = T$, and to replicate the network along the time dimension, forming a time-expanded hypergraph HG_T, where nodes and (hyper)arcs have an explicit time dimension and are, respectively, called *vertices* and (hyper)*edges*. If time intervals are short enough to ensure that the exit time of a generic edge $t_a(\tau^\theta)$ is not earlier than the next interval $\tau^{\theta+1}$, for $\tau \leq \Theta$-2, it is ensured that the network is cycle-free and the *vertex* chronological ordering is equivalent to the topological one. Thus, HG_T is scanned starting from the last temporal layer to the value assumed for $\tau = \tau^0$ and, within the generic layer, no topological order is respected. When a generic vertex (i, τ^{θ}) is visited, its forward star is scanned in order to set the minimal travel cost to destination and the successive edge by means of equation (1). In fact, at this point of the algorithm, not only the costs of the edges $a = \left(\left(TL_a, \tau^{\theta}\right), \left(HD_a, t_a(\tau^{\theta})\right)\right)$ of the forward star, but also the minimal costs from every vertex $\left(HD_a, t_a(\tau^{\theta})\right)$ to destination are known. If the examined vertex represents a stop node in the time-expanded hypergraph, then the successive edge corresponds to a hyperarc of the hypergraph *HG* and it is determined by means of the greedy procedure detailed in Section 4.1.

By assumption the network behaves as static outside the analysis period, therefore for departure time intervals greater than or equal to Θ-1 the computation of the shortest hyperpath is equivalent to a static procedure and is calculated according the algorithm by Spiess and Florian (1989).

4.1 Time-dependent shortest hypertree algorithm for every possible arrival time

Beyond variables already specified, the algorithm also includes:

- θ time interval index;
- θInt: time interval length;
- d: destination node;
- i: generic node;
- FS_i: set of arcs belonging to the forward star of node i;
- HFS_i: set of hyperarcs belonging to the hyper-forward star of node i;
- $a = (i, j)$: generic arc and/or branch of hyperarc $a \in h$;
- h: generic hyperarc;
- $suc(i, \tau^{\theta})$: successor arc and/or hyperparc of the generic node i at time interval τ^{θ};
- $c_a(\tau^{\theta})$: generalised travel time on arc a at time interval τ^{θ}, $a \in A \backslash \{A^H\}$;
- $\varphi_a(\tau^{\theta})$: instantaneous frequency corresponding to the line associated with arc a at time interval τ , $a \in FS_i$, $i \in N^S$;
- $t_a(\tau^{\theta})$: exit time from arc a for users entering it at time interval τ^{θ};
- $t_a^{-1}(\tau^{\theta})$: *entry time* to the arc a for users exiting it at time τ^{θ} ;
- $\kappa_a(\tau^{\theta})$: congestion parameter at time interval τ^{θ} for the line L_{HDa} associated with the arc $a \in FS_i$, $i \in N^S$;
- $p_{a|h}(\tau^{\theta})$: diversion probability at time interval τ^{θ}
- $w_{a|h}(\tau^{\theta})$: conditional expected waiting time at time interval τ^{θ};
- $w_h(\tau^{\theta})$: waiting time at node $i = TL_h$ at time interval τ^{θ};
- $g_{i,d}(\tau^{\theta})$: current travel cost from generic node i to destination d at time interval τ^{θ};
- $g_{i,d,h}(\tau^{\theta})$: current travel cost from stop node i to destination d at time interval τ^{θ} if considering the attractive line represented by hyprarc h;

- $g^*_{i,d}(\tau^\theta)$: minimum travel cost from generic node i to destination d at time interval τ^θ;
- $g^*_{i,d}{}^{\text{stat}}$: minimum travel cost from generic node i to destination d at time interval $\tau^\theta \geq \tau^{\Theta-1}$;

The pseudo-code of the solution algorithm for the time-dependent all-to-one shortest hyperpath problem for every possible arrival time is, hence, detailed:

Step 0 (SSHP – Initialisation): $\forall\ i \in N \setminus \{d\}$

Calculate $g^*_{is}(\tau^{\Theta-1}) = \text{g}s^{*\ \text{stat}}_{is}$

$\theta \in [0, \Theta\text{-}2]$

Set $g^*_{d,d}(\tau^\theta) = 0$, $suc(d,\ \tau^\theta) = \varnothing$

$i \in N \setminus \{d\}$

Set $g^*_{i,d}(\tau^\theta) = \infty$

Step 1 (Calculate hyperpath travel time): $\theta \in [0, \Theta\text{-}2]$

$i \in N \setminus \{d\}$

If $i \in N^S$,

Apply the greedy procedure to define the set of attractive lines and calculate the travel cost

$g^*_{i,d}(\tau^\theta) = g_{i,d,h}(\tau^\theta)$ and $suc(i,\ \tau^\theta) = h$

Else if $i \notin N^S$, $a \in FS_i$

If $[\![c_a(\tau^\theta) / \theta\text{Int}]\!] \geq 1$

$t_a(\tau^\theta) = [\![c_a(\tau^\theta) / \theta\text{Int}]\!] + \tau^\theta$

Else

$t_a(\tau^\theta) = \tau^\theta + 1$

$g_{i,d}(\tau^\theta) = c_a(\tau^\theta) + g_{HD_a,d}(t_a(\tau^\theta))$

If $g^*_{i,d}(\tau^\theta) > g_{i,d}(\tau^\theta)$

$g^*_{i,d}(\tau^\theta) = g_{i,d}(\tau^\theta)$ and $suc(i,\ \tau^\theta) = a$

The greedy-like procedure invoked in Step 1 of the solution algorithm requires that once a stop node i is reached, all lines L_{HDa}, $a \in FS_i$, are sorted in increasing order of travel time upon boarding ($g_{HD_a,d}$). In general, $g_{HD_a,d}$ should be evaluated for each line L_{HDa}, at the conditional boarding time $t_{a|h}(\tau^\theta)$ and this value, in turns, does not only depend on the particular line L_{HDa} considered, but also on what other lines are included in the choice set (hyperarc h).

Because at this stage the attractive hyperarc has not been determined yet, the following hyperarcs are defined:

$$h_l = a_l, \qquad l = \{1, 2, .., n\} \tag{11}$$

and lines are sorted according to the following criterion:

$$g_{HDa_1,d}\left(t_{a_1|h_1}(\tau)\right) \leq g_{HDa_2,d}\left(t_{a_2|h_2}(\tau)\right) \leq \ldots \leq g_{HDa_n,d}\left(t_{a_n|h_n}(\tau)\right), \qquad n = |FS_i| \tag{12}$$

The rest of the greedy-type procedure adopted follows as normal: one line at a time is added to the attractive set and the calculation is stopped as soon as the addition of the next line increases the value of $g_{i,d,h}$.

Step 1.0 (Initialisation): $\forall\, a \in FS_i,\ a \in A^W$

Set $h_l = a_l$, according to equation (11)

Sort $a_l \in FS_i$, according to equation (12)

Set $h := a_1$

Calculate $w_{a_1|h}(\tau^\theta)$ with equation (3)

If $\llbracket w_{a_1|h}(\tau^\theta) / \theta\mathrm{Int} \rrbracket \geq 1$

$t_{a_1|h}(\tau^\theta) = \llbracket w_{a_1|h}(\tau^\theta) / \theta\mathrm{Int} \rrbracket + \tau^\theta$

Else

$t_{a_1|h}(\tau^\theta) = \tau^\theta + 1$

Calculate $w_h(\tau^\theta)$ with equation (4)

$g_{i,d,h}(\tau^\theta) := w_h(\tau^\theta) + g_{HDa_1,d}(t_{a_1|h_1}(\tau^\theta))$

$l := 2$

Step 1.1 (Updating h): While ($l \leq n$) and $g_{HDa_l,d}(t_{a_l|h_l}(\tau^\theta)) < g_{i,d,h}(\tau^\theta)$ do:

$h := h \cup \{a_l\}$

$\forall\, a \in h$

Calculate $p_{a|h}(\tau^\theta)$ with equation (2)

Calculate $w_{a|h}(\tau^\theta)$ with equation (3)

Calculate $w_h(\tau^\theta)$ with equation (4)

If $\llbracket w_{a|h}(\tau^\theta) / \theta\mathrm{Int} \rrbracket \geq 1$

$t_{a|h}(\tau^\theta) = \llbracket w_{a|h}(\tau^\theta) / \theta\mathrm{Int} \rrbracket + \tau^\theta$

Else

$t_{a|h}(\tau^\theta) = \tau^\theta + 1$

$$g_{i,d,h}(\tau^\theta) = w_h(\tau^\theta) + \sum_{a \subseteq h} p_{a|h}(\tau^\theta) \cdot g_{HD_a|h}\left(t_{a|h}(\tau^\theta)\right)$$

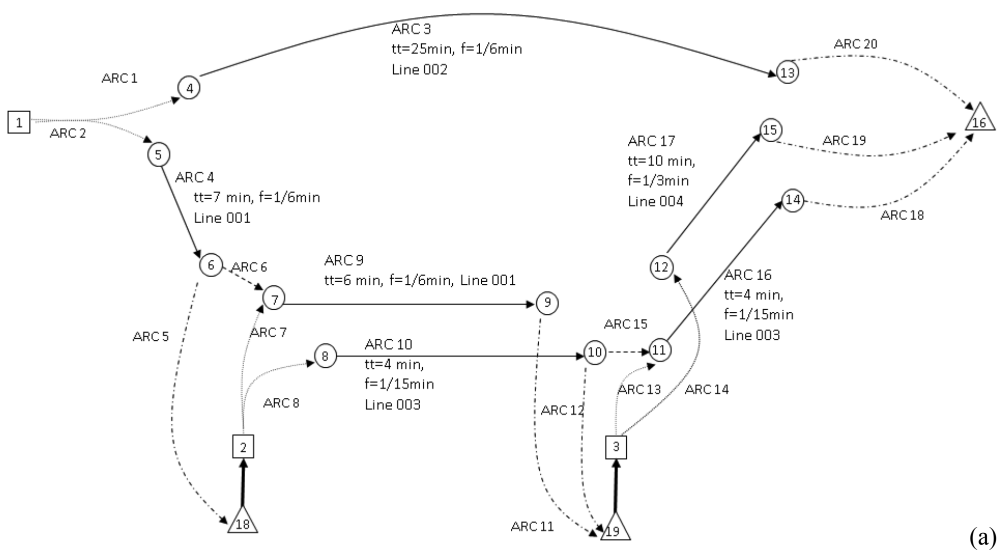
$l := l + 1$

5 NUMERICAL EXAMPLE

A numerical example is presented in order to show the effects of queues on passenger route choice, when information about actual waiting times is provided at transit stops. The example network is the same used by Spiess and Florian (1989) in their seminal work on optimal travel strategies in static networks, and is depicted in Figure 2a.

For the scope of this example, the analysis morning peak period [07:30–09:30] is divided in one-minute intervals. In order to fully consider the effect of queues and information, frequencies and in-vehicle travel times are assumed to stay equal to the values depicted in Figure 2a, and all lines are irregular, with exponentially distributed headways.

By contrast, it is assumed that since 08:00 a queue arises at stop node 3, such that passengers wishing to board line arc 17 have to wait for the second arrival of the corresponding transit Line 004. Also, from 08:30 onwards, a queue arises at stop node 1 and passengers wanting to board Line 001 or Line 002 have to wait for the second carrier. Before 08:00 and from 09:30 onwards there is no passenger congestion, so the problem can be considered static and the optimal travel strategy from each node to destination (node 16) is depicted in Figure 2b, where in bold are represented values calculated without considering the effect of countdown displays.



(a)

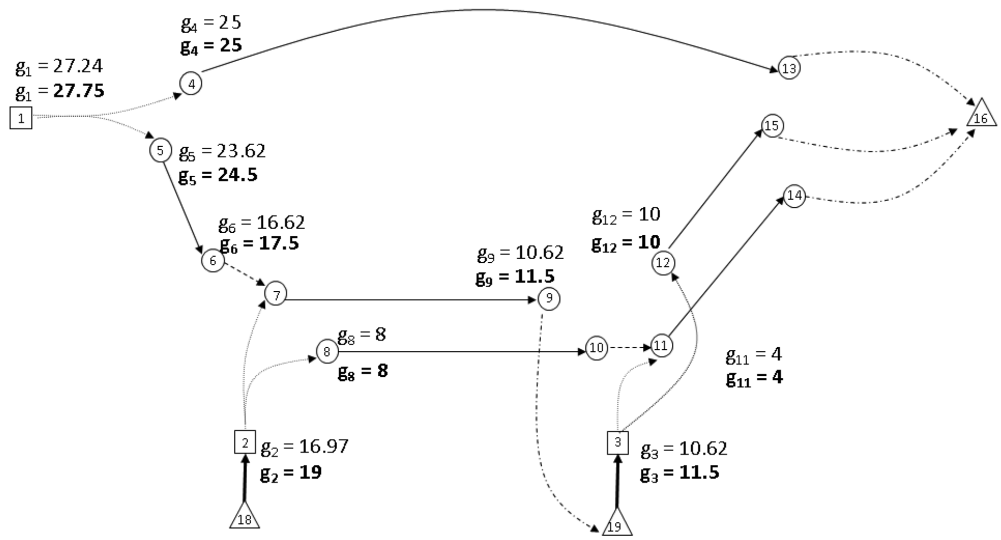


FIGURE 2

(a): hypergraph representation of the example network with in-vehicle travel times (tt) and average frequencies (f) of each line

(b): travel times to destination (node 16) outside the analysis period, expressed in minutes. In bold are the values calculated without considering the effect of countdown displays

The effects of congestion at a stop with a *mixed* FIFO queue are shown in Figure 3 for the case where information is provided (a) and not provided (b). If information is provided and a *mixed* queue arises at stop 3, passengers that have boarded Line 001 at stop 1 prefer to alight at stop 2 rather than staying on board. The behaviour is perfectly rational because, should they stay on board (i.e. the dwelling arc 6 of Figure 2a is included in the optimal strategy), they would necessarily alight at stop 3 and experience, there, the queuing delay due to oversaturation. Interestingly, if no real-time bus departure information is provided the optimal travel strategy is to stay on-board, as depicted in Figure 3b. Therefore it could be inferred that when information mitigates the uncertainty, due to service irregularity, the expectation of congestion, further down along the trip, seems to influence local choices more than the waiting time at the current location. On the other hand, in case of full uncertainty (irregular services and no additional information) the decision tends to be more myopic and to consider mainly the local delay.

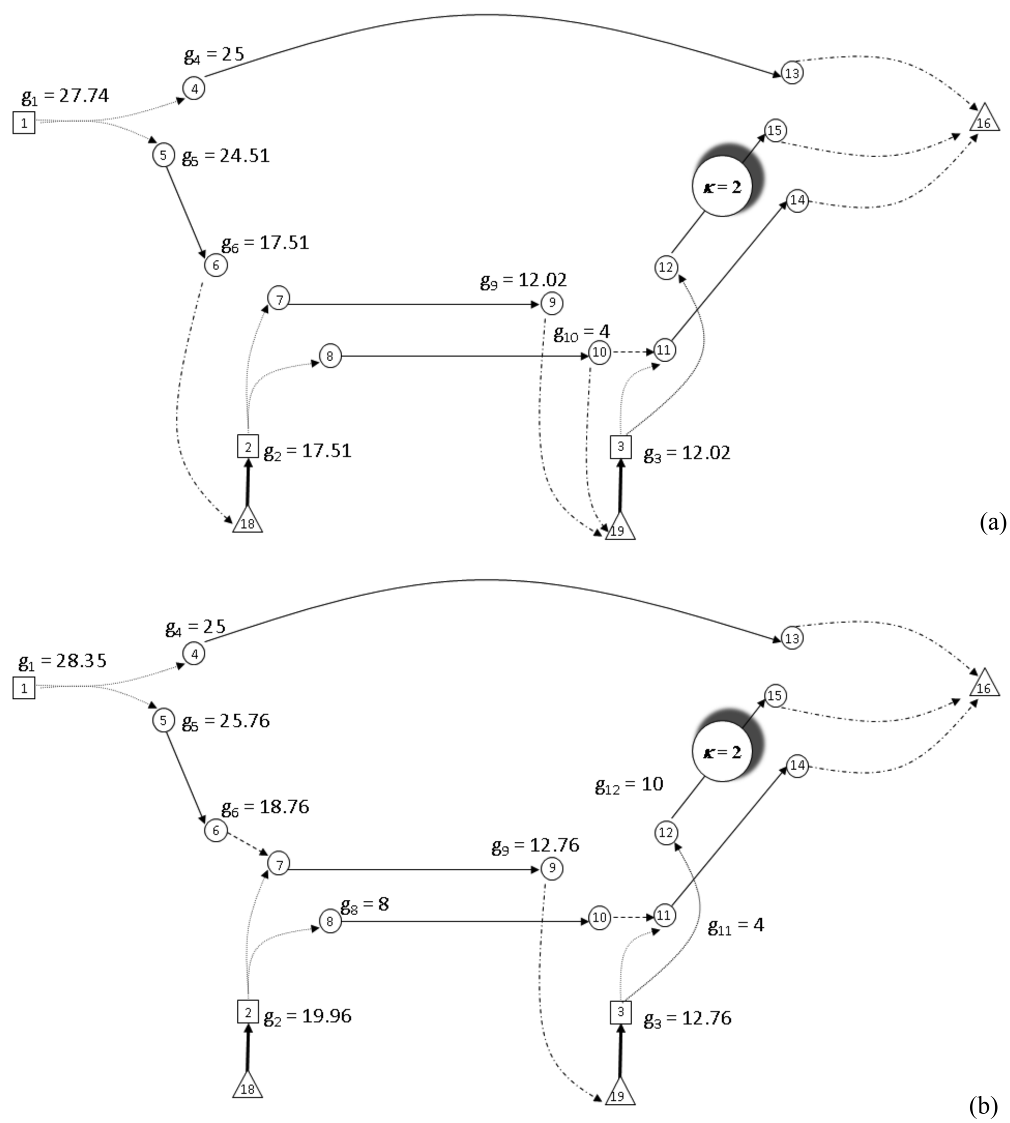


FIGURE 3

(a): travel times in minutes from each node to destination (node 16) when countdown displays are available at each stop and passenger queues are '*mixed*'

(b): travel times in minutes from each node to destination (node 16) when countdown displays are not available at each stop and passenger queues are '*mixed*'

The effects of congestion at a stop with a *separate* FIFO queues (e.g. bus terminals), are shown in Figure 4, where it is assumed that stop 1 has such a layout. If *no* countdown displays are available and congestion occurs, as soon as passengers arrive at the stop, they have to join either the queue for boarding Line 001 or the queue for boarding Line 002. Consequently, they cannot take advantage of events taking place while they are waiting at the stop and no travel strategy is possible. In this scenario, a rational passenger will compare the total travel time of boarding Line 001 (12' expected waiting time + 25' travel time upon boarding = 37'), the total travel time of boarding Line 002 (12’ expected waiting time + 24.5’ travel time upon boarding = 36.5’) and will choose the second option, as in Figure 4a.

By contrast, if information is provided at stop 1, the route choice can be *strategic* also in case of passenger congestion, as explained in Section 2, and will result in the hypertree depicted in Figure 4b. Because in this case the provision of real-time information allows for a travel strategy, the decrease in total travel time is quite substantial and, with reference to the *o-d* pair 1-16, it accounts for 11.35% of the total travel time, while in the first instance (no congestion) the reduction is only of 0.5 minutes (1.8%), and in the second instance (08:30-09:00) it is only of 0.51 minutes (1.9%).

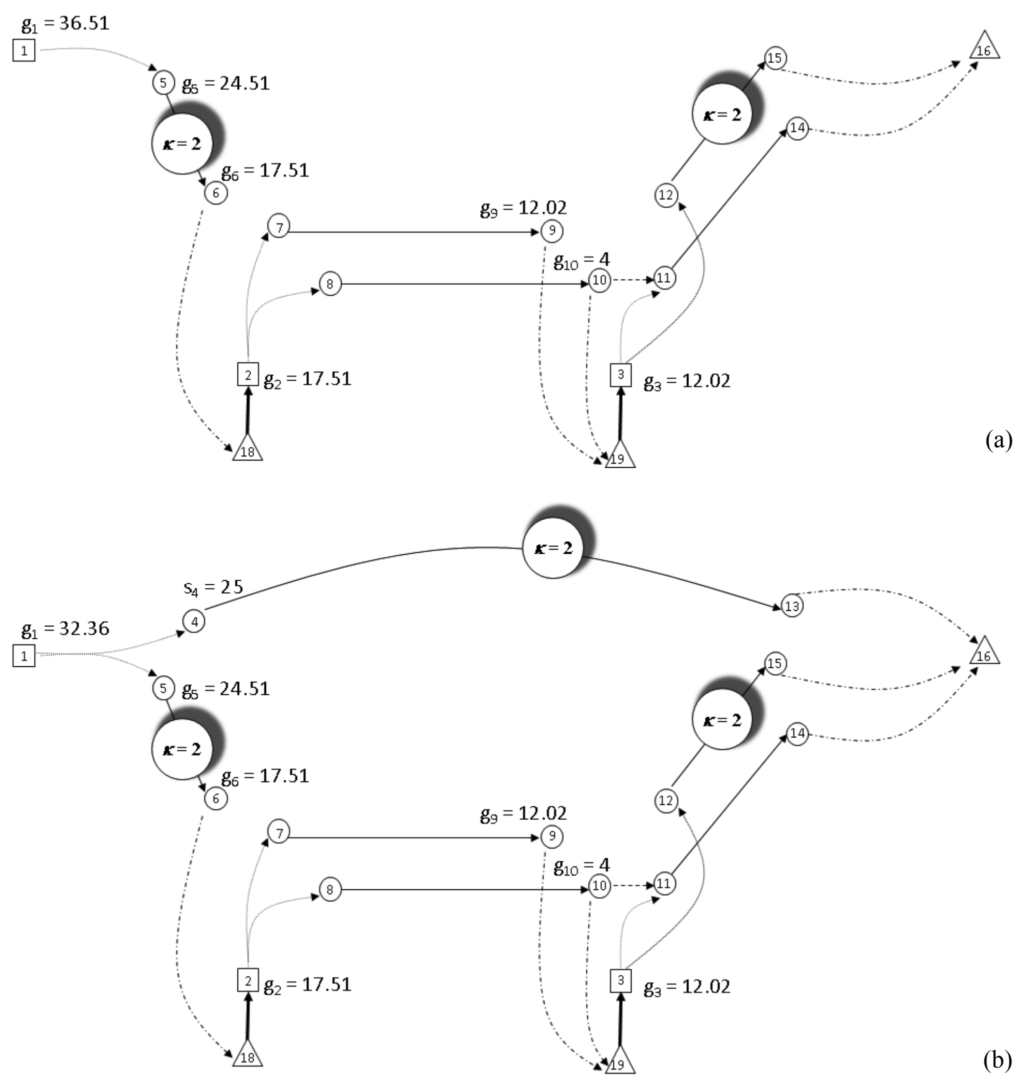


FIGURE 4

(a): travel times in minutes from each node to destination (node 16) when countdown displays are not available at each stop. The passenger queues at stop 1 are *separate* and ‘*mixed*’ at stop 3

(b): travel times in minutes from each node to destination (node 16) when countdown displays are available at each stop. The passenger queues at stop 1 are *separate* and ‘*mixed*’ at stop 3 and passenger queues are ‘*mixed*’.

6 CONCLUSIONS

In this paper a time-dependent route choice model and algorithm have been presented to assess the effects of cont down displeys under sever overcrowding.

Assuming that congestion can be represented by a First-In-First-Out (FIFO) queue of passengers at transit stops, it has been shown that the route choice model independently developed by Hickman and Wilson (1995) Gentile *et al.* (2005) can also be applied to time-dependent, congested scenarios, provided that the selection method for the attractive set and the waiting times' probability distribution function (PDF) and survival function $(\overline{\mathrm{CDF}})$ are changed in accordance with the new hypotheses. The presence of real-time information at stops ensures that the model can describe route choice both in case of *separate* or '*mixed*' queues. Moreover, the different adaptive behaviours considered by Billi *et al.* (2004) and Noekel and Wekeck (2007) in case of regular services can be disregarded.

The proposed model cannot devise an exact solution for services with an intermediate degree of regularity because in this case it is usually assumed that the PDF of the waiting time before the first carrier arrives follows an Erlang distribution, which cannot be convoluted. On the other hand, the model represents a step forward with respect of those usually applied for representing route choice in congested scenarios because it can handle easily both the case of perfectly irregular services (i.e. lines with exponentially distributed headways, this is the case usually considered in models with capacity constriants) and perfectly regular services (i.e. lines with constant headways), for which an exact solution is devised.

Finally, it should be highlighted here that the application envisaged for the proposed route choice model is dynamic transit assignment and not passenger routing. This is for two main reasons. First, the congestion parameter $\kappa_a(\tau)$ can only be evaluated by means of a queuing model embedded in a full assignment procedure, for example like the one presented in (Trozzi *et al.*, 2013).

Second, notwithstanding the inherent uncertainty and stochasticity on the supply-side, the proposed deterministic model only considers *average* values of the waiting and in-vehicle travel time, independently from their actual realization on a particular day. In dynamic routing applications, this would lead to a distortion in the computation of travel times, as the following examples clarify. Consider a stop i, a set of attractive lines represented by hyperarc h and the attractive line L_{HDa} ($a \subseteq h$): on a specific day the actual realization of the waiting time before boarding L_{HDa} may be different than the conditional expected value $w_{a|h}(\tau)$ and thus those who have reached stop i at time τ will be subject to a different travel time upon boarding than $g_{HD_a|h}\left(t_{a|h}(\tau)\right)$. Similarly, if on a specific day the in-vehicle travel time on the first lag of the journey is different than the expected value, the passenger will experience at the transferring stop a queuing delay that is generally different than what expected.

While these (small) distortions would not allow an application of the proposed model for dynamic routing purposes, it can always be embedded into a dynamic deterministic transit assignment procedure where, in general, average traffic conditions and travel times are considered.

Hence, future work will concentrate on dynamic transit assignment applications to real-scale networks in order to fully evaluate the potential congestion relief brought about by countdown displays. Moreover, applications to medium-size networks will also be implemented to evaluate the impact of the proposed greedy heuristic for the selection of the line choice set.

REFERENCES

BILLI, C., GENTILE, G., NGUYEN, S., PALLOTTINO, S. & BARRETT, K. Rethinking the wait model at transit stops. Proceedings of TRISTAN V - Triennial Symposium on Transportation Analysis, 2004 2004.

BOUZAÏENE-AYARI, B. 1988. *Modélisation des arrêts multiples d'autobus pour les réseaux de transport en commun.*, École Polytechnique de Montréal.

BOUZAÏENE-AYARI, B., GENDREAU, M. & NGUYEN, S. 2001. Modeling Bus Stops in Transit Networks:A Survey and New Formulations. *Transportation Science,* 35**,** 304-321.

CEPEDA, M., COMINETTI, R. & FLORIAN, M. 2006. A frequency-based assignment model for congested transit networks with strict capacity constraints: characterization and computation of equilibria. *Transportation research Part Part B: Methodological,* 40**,** 437-459.

CHABINI, I. 1998. Discrete dynamic shortest path problems in transportation applications: complexity and algorithms with optimal running time. *Transportation Research Record: Journal of the Transportation Research Board,* 1645**,** 170-175.

CHRIQUI, C. & ROBILLARD, P. 1975. Common bus lines. *Transportation Science,* 9**,** 115-121.

COMINETTI, R. & CORREA, J. 2001. Common-Lines and Passenger Assignment in Congested Transit Networks. *Transportation Science,* 35**,** 250 - 267.

DE CEA, J. & FERNANDEZ, E. 1993. Transit Assignment for Congested Public Transport Systems: An Equilibrium Model. *Transportation Science,* 27**,** 133-147.

GALLO, G., LONGO, G., PALLOTTINO, S. & NGUYEN, S. 1993. Directed Hypergraphs and Applications. *Discrete Applied Mathematics,* 42**,** 177-201.

GAO, S., FREJINGER, E., BEN-AKIVA, M. 2010. Adaptive Route Choices in Risky Traffic Networks: A Prospect Theory Approach. *Transportation Research Part C: Emerging Technologies,* 18(5)**,** 727-740.

GAO, S. 2012. Modeling Strategic Route Choice and Real-Time Information Impacts in Stochastic and Time-Dependent Networks. *Intelligent Transportation Systems, IEEE Transactions on,* 13(3)**,** 1298-1311.

GENDREAU, M. 1984. *Étude approfondie d'un modéle d'équilibre pour l'affectation des passangers dans les réseaux de transport en commun.* PhD, Université de Montréal.

GENTILE, G., NGUYEN, S. & PALLOTTINO, S. 2005. Route Choice on Transit Networks with Online Information at Stops. *Transportation Science,* 39**,** 289-297

GROTENHUIS, J.-W., WIEGMANS, B., W., RIETVEL, P. 2007. The desired quality of integrated multimodal travel information in public transport: Customer needs for time and effort savings. *Transport Policy*, 14, 27-38.

HAMDOUCH, Y., HO, H. W., SUMALEE, A. & WANG, G. 2011. Schedule-based transit assignment model with vehicle capacity and seat availability. *Transportation Research Part B: Methodological,* 45**,** 1805-1830.

HAMDOUCH, Y. & LAWPHONGPANICH, S. 2008. Schedule-based transit assignment model with travel strategies and capacity constraints. *Transportation Research Part B: Methodological,* 42**,** 663-684.

HAMDOUCH, Y., MARCOTTE, P. & NGUYEN, S. 2004. A Strategic Model for Dynamic Traffic Assignment. *Networks and Spatial Economics,* 4**,** 291-315.

HICKMAN, M. D. & WILSON, N. H. M. 1995. Passenger travel time and path choice implications of real-time transit information. *Transportation Research Part C: Emerging Technologies,* 3**,** 211-226.

KURAUCHI, F., BELL, M. G. H. & SCHMÖCKER, J. D. 2003. Capacity Constrained Transit Assignment with Common Lines. *Journal of Mathematical Modelling and Algorithms,* 2**,** 309-327.

LEURENT, F. & BENEZECH, V. 2011. The Passenger Stock and Attractivity Threshold model for traffic assignment on a transit network with capacity constraint. *Transportation Research Board, 90th Annual Meeting.* Washington, D.C.

MILLER-HOOKS ,E.,D. & MAHMASSANI, H., S. 2000. Least Expected Time Paths in Stochastic, Time-Varying Transportation Networks. *Transportation Science*, 3(2), 198–215.

NGUYEN, S. & PALLOTTINO, S. 1988. Equilibrium traffic assignment for large scale transit networks. *European Journal of Operational Research,* 37**,** 176-186.

NGUYEN, S. & PALLOTTINO, S. 1989. Hyperpaths and shortest hyperpaths. *Combinatorial Optimization.* Berlin / Heidelberg: Springer

NGUYEN, S., PALLOTTINO, S. & GENDREAU, M. 1998. Implicit enumeration of hyperpaths in a logit model for transit networks. *Transportation Science,* 32**,** 54.

NOEKEL, K. & WEKECK, S. Choice models in frequency-based transit assignment. European Transport Conference 2007 2007.

PRETOLANI, D. 2000. A directed hypergraph model for random time dependent shortest paths. *European Journal of Operational Research*, 123 (2), 315-324.

SCHMÖCKER, J.-D., BELL, M. G. H. & KURAUCHI, F. 2008. A quasi-dynamic capacity constrained frequency-based transit assignment model. *Transportation Research Part B: Methodological,* 42**,** 925-945.

SPIESS, H. 1983. On optimal route choice strategies in transit networks. *In:* CENTRE DE RECHERCHE SUR LES TRANSPORTS, U. D. M. (ed.).

SPIESS, H. 1984. *Contributions á la téorie at aux outils de planification des résaux de transport urbain.* Ph.D. Thesis Ph.D. Thesis, Université de Monteréal.

SPIESS, H. & FLORIAN, M. 1989. Optimal strategies: A new assignment model for transit networks. *Transportation Research Part B: Methodological,* 23**,** 83-102.

TEKLU, F. 2008. A Stochastic Process Approach for Frequency-based Transit Assignment with Strict Capacity Constraints. *Networks and Spatial Economics,* 8**,** 225-240.

TROZZI, V., GENTILE, G., BELL, M. G. H. & KAPARIAS, I. 2013. Dynamic User Equilibrium in Public Transport Networks with Passenger Congestion and Hyperpaths. *Transportation Research Part B: Methodological,* in press doi: 10.1016/j.trb.2013.06.011.

UKKUSURI, S.V. & PATIL, G. 2007. Exploring user behavior in online network equilibrium problems. *Transportation Research Record: Journal of the Transportation Research Board*, 2029, 31-38.

YANG, B. & MILLER-HOOKS ,E.,D. 2004. Adaptive routing considering delays due to signal operations. *Transportation Research Part B: Methodological,* 38(5), 385–413.