OPTIMAL REINSURANCE IN THE PRESENCE OF COUNTERPARTY DEFAULT RISK

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Abstract. The optimal reinsurance arrangement is identified whenever the reinsurer counterparty default risk is incorporated in a one-period model. Our default risk model allows the possibility for the reinsurer to fail paying in full the promised indemnity, whenever it exceeds the level of regulatory capital. We also investigate the change in the optimal solution if the reinsurance premium recognises or not the default in payment. Closed form solutions are elaborated when the insurer's objective function is set via some well-known risk measures. It is also discussed the effect of reinsurance over the policyholder welfare. If the insurer is Value-at-Risk regulated, then the reinsurance does not increase the policyholder's exposure for any possible reinsurance transfer, even if the reinsurer may default in paying the promised indemnity. Numerical examples are also provided in order to illustrate and conclude our findings. It is found that the optimal reinsurance contract does not usually change if the counterparty default risk is taken into account, but one should consider this effect in order to properly measure the policyholders's exposure. In addition, the counterparty default risk may change the insurer's ideal arrangement if the buyer and seller have very different views on the reinsurer's recovery rate.

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1. Introduction

Two parties are involved in a standard reinsurance contract: the *insurer*, *cedent*, *insurance buyer*, or even simpler, *buyer*, who has an interest in transferring part of its risk to the *reinsurer*, also known as *insurance seller*, or even simpler *seller*. Let $X \geq 0$ be the total amount that the insurer is liable to pay during the duration of the insurance contract, with distribution function denoted by $F(\cdot)$ and survival function $\bar{F}(\cdot) = 1 - F(\cdot)$. In addition, the right end-point $x_F := \inf\{z \in \Re : F(z) = 1\}$ of the loss distribution could be either finite or infinite, even though the finiteness assumption would be more realistic. The reinsurance seller agrees to pay, $R[X]$, the amount by which the entire loss exceeds the insurer amount, $I[X]$. Thus, $I[X] + R[X] = X$. There are many possible reinsurance arrangements, which depend on the particular choice of the insurer and reinsurer sharing the premiums and underwritten risks. For example, the liabilities are shared in a fixed proportion under *proportional* reinsurance and therefore $I[X] = cX$, where $c \in [0, 1]1$ is a constant. Another common arrangement is the *stop-loss* reinsurance contracts, for which the buyer retained loss is limited to a fixed amount, M, known as *retention limit*. The net amount paid by the insurer is therefore given by $\min\{X, M\} := X \wedge M$.

There is a vast literature on identifying the optimal risk transfer contract between two insurance companies within a one-period setting. The first attempts are attributed to Borch (1960) and Arrow (1963) where the expected utility is maximised. Further extensions have been developed for various decision criteria that depend on the risk measure choice (for example, see Heerwaarden *et al.*, 1989, Young, 1999, Kaluska, 2001 and 2005, Verlaak and Beirlant, 2003, Kaluszka and Okolewski, 2008, Ludkovski and Young, 2009). Two commonly used in practice risk measures, *Value-at-risk (VaR)* and *Expected Shortfall (ES)*, are considered by Cai *et al.* (2008), Cheung (2010) and Chi and Tan (2011). The classical risk model setting has been successfully studied in the literature by Centeno and Guerra (2010) and Guerra and Centeno (2008 and 2010), where a natural choice for optimisation is the maximisation of the adjustment coefficient.

The classical approach of finding the optimal risk sharing ignores the possibility of default in payment that the risk transfer initiator is exposed to, known as the *counterparty default risk*. This can be viewed as a special case of the *background risk*, under the additive background risk assumption. This setting has been investigated by Dana and Scarsini (2007) via the expected utility maximisation. The paper of Biffis and Millossovich (2012) is related to the latter work and analyses the effect of counterparty default risk under some economic constraints. Bernard and Ludkovski (2012) deals with the same problem, but the loss given default is considered loss-dependent. Our approach is different, in the sense that the insurer prefers a risk measure when making a decision of sharing the liability. In addition, the default is assumed to be endogenous as it has been seen in Biffis and Millossovich (2012), and the seller's available assets are given by its regulatory capital. That is, the loss of basic own funds which the insurer would incur if the insurance seller defaults, known as the *loss given default*, is assumed to be proportional to the excess of the indemnity over the reinsurer level of capital requirements. Like any other counterparty default risk model, there are pros and cons for our choice, and we believe that our model is sufficiently rich to provide an understanding of the change in the optimal arrangement if the buyer incorporates the reinsurer chance to default.

As it has been anticipated, the reinsurer's default in payment is assumed to occur whenever the indemnity exceeds the reinsurer capital. Motivated by the *Solvency II* regulatory requirements developed within the European insurance industry, where the risk exposures are measured via VaR, we assume that the seller operates in an environment that is VaR regulated. The latter assumption enables us to identify the solution of an infinite dimensional optimisation problem by imposing mild and economically sound restrictions for the set of possible risk transfers. Alternatively, one may commit to a specific class of reinsurance contracts, such as focusing only on the stop-loss arrangements, which allows one to solve standard finite dimensional optimisation problems.

The VaR of a generic loss variable Z at a confidence level a, $VaR_a(Z)$, represents the minimum amount of capital that makes the insurance company to be solvent at least $a\%$ of the time. The mathematical formulation is then given by

$$VaR_a(Z) := \inf\{z \geq z_0 : \Pr(Z \leq z) \geq a\},$$

where $z_0 := \sup\{z \in \Re : \Pr(Z \leq z) = 0\}$ represents the left end point of the distribution of Z. By convention, $\inf \varnothing = +\infty$ is true.

In the absence of default risk, the indemnity is $R[X]$, otherwise it is given by

$$\tilde{R}[X;\delta] := R[X] \wedge VaR_\beta\big(R[X]\big) + \delta\Big(R[X] - VaR_\beta\big(R[X]\big)\Big)_+,$$

where $\delta \in [0,1]$ represents the *recovery rate* used to calculate the loss given default. By definition, $(z)_+ = \max\{z, 0\}$. Note that the seller is assumed to be VaR-regulated, and for example, $\beta = 99.5\%$ whenever Solvency II is in force. Thus, the *probability of default* is $1-\beta$, i.e. the insolvency probability allowed by the regulator. Obviously, the no-default scenario is recovered if we set $\delta = 1$.

The seller and buyer may have different beliefs about the recovery rate, but it is likely that the insurer to be more pessimistic than the reinsurer. In this paper, it is assumed that $0 \leq \delta_1 \leq \delta_2 \leq 1$, where δ_1 and δ_2 are the recovery rates of the buyer and seller, respectively. Let $\mathbf{P}\big(\tilde{R}[X;\delta_2]\big)$ be the reinsurance premium if the seller incorporates the default risk. Therefore, the total insurer loss becomes

$$\tilde{\mathbf{L}}[X;\delta_1,\delta_2] \;\; := \;\; X - \tilde{R}[X;\delta_1] + \mathbf{P}\big(\tilde{R}[X;\delta_2]\big).$$

A large class of such risk measures is given by

$$\varphi(Z) := \int_0^1 VaR_s(Z)\Phi(s)\,ds = \int_0^\infty g\big(\Pr(Z>z)\big)\,dz - \int_{-\infty}^0 \Big(g\big(\Pr(Z>z)\big) - 1\Big)\,dz, \qquad (1.1)$$

where $\Phi(s) = g'(1-s)$. This class is known as the *distorted* (see Wang and Young, 1998 and Jones and Zitikis, 2003) and *spectral* (see Acerbi, 2002) class of risk measures, respectively. Note that the distorted function $g : [0,1] \to [0,1]$ is assumed to be non-decreasing and concave such that $g(0) = 0$ and $g(1) = 1$. Therefore, $g(\cdot)$ is differentiable almost everywhere, but not necessarily differentiable on $[0,1]$. Consequently, using the usual derivative $g'(\cdot)$, whenever it exists, does not change the representation from 1.1 (for further details, see Dhaene *et al.*, 2012).

The previously-mentioned class includes the well-known ES risk measure, which has various representations in the literature (see Acerbi and Tasche, 2002). We only refer to the next definition:

$$ES_\alpha(Z) := \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(Z)\,ds = VaR_\alpha(Z) + \frac{1}{1-\alpha}\mathbf{E}\big(Z - VaR_\alpha(Z)\big)_+.$$

Interestingly, this risk measure is a special case of the Haezendonck-Goovaerts class, which was introduced many years ago by Haezendonck and Goovaerts (1982). Further details can be found in Bellini and Rosazza Gianin (2012), Goovaerts *et al.* (2004 and 2012) and the references therein.

We aim to identify the optimal arrangement that reduces the seller's risk as much as possible, where the risk is evaluated via VaR or a distorted risk measure. VaR and ES are standard tail risk measures used in practice to set technical provisions and capital requirements, and therefore, it is natural to believe that both are good choices for the insurance company to base its decision. That is, we intend to minimise $\varphi_I\big(\tilde{\mathbf{L}}[X;\delta_1,\delta_2]\big)$ over a set of feasible reinsurance contracts, where $\varphi_I(\cdot)$ represents a measure of the risk taken by the insurer. In order to avoid potential moral hazard issues related to the reinsurance arrangement, the set of feasible contracts is given by

$$\mathcal{F} := \big\{R(\cdot) : I(x) = x - R(x) \text{ } and \text{ } R(x) \text{ } are \text{ } non-decreasing \text{ } functions\big\}.$$

Note that $R \in \mathcal{F}$ implies that I and R are 1-Lipschitz functions, i.e. $|I(y) - I(x)| \leq |y - x|$ and $|R(y) - R(x)| \leq |y - x|$ are true for all $x, y \geq 0$.

The premiums are usually assumed to be positively loaded, and therefore it is expected to have that $\mathbf{P}\big(\tilde{R}[X;\delta_2]\big) \geq \mathbf{E}\big(\tilde{R}[X;\delta_2]\big)$. A common insurance pricing is the *expected value* principle, $\mathbf{P}\big(\tilde{R}[X;\delta_2]\big) = (1+c)\mathbf{E}\big(\tilde{R}[X;\delta_2]\big)$, where $c > 0$ is known as the *security loading factor*. Our results are given for a more general pricing method, where the seller prices the premium based on a distorted risk measure. That is, $\mathbf{P}\big(\tilde{R}[X;\delta_2]\big) = (1+c)\varphi_R\big(\tilde{R}[X;\delta_2]\big)$, where $\varphi_R(\cdot)$ is a distorted risk measure. Besides its general formulation, distorted risk measure have proved to be valuable choices for insurance pricing (see for example, Wang, 2000).

The rest of the paper is organised as follows. Section 2 investigates the VaR-based decisions, while Section 3 evaluates optimal arrangements based on distorted risk measures. Section 4 explains the effect of reinsurance over the policyholder welfare. The last section provides some numerical examples that are meant to illustrate our main results and conclude the paper.

2. VaR-based Optimal Reinsurance Contract

The current section describes the optimal choice for the buyer if VaR measures its perception about risk. In other words, we assume that $\varphi_I = VaR_\alpha$. As a result of the translation invariance property of this risk measure, our optimisation problem is reduced to

$$\min_{R\in\mathcal{F}} \Big\{VaR_\alpha\big(X - \tilde{R}[X;\delta_1]\big) + (1+c)\varphi_R\big(\tilde{R}[X;\delta_2]\big)\Big\}. \tag{2.1}$$

Throughout this paper, two heavily used notations are given by $t^* = \sup\big\{t \geq 0 : g_R(t) \leq 1/(1+c)\big\}$ and $t_1^* = \sup\big\{t \geq 0 : g_R(t) \leq \delta_1/\delta_2(1+c)\big\}$, where the $g_R(\cdot)$ always refers to the reinsurer's distorted function used to settle the reinsurance premium. In addition, it is implicitly assumed that $\sup\emptyset = 0$.

In order to simplify our presentation, the following notation is made:

$$SL(\theta) := \big(X - VaR_{1-\theta}(X)\big).$$

Our proofs from this section follow a two-stage procedure developed in Asimit *et al.* (2013 b). The main result is given in Theorem 2.1, and it illustrates the optimal choice if the buyer and seller have different views on the reinsurer's default.

Theorem 2.1. *Let $0 \le \delta_1 \le \delta_2 \le 1$. The VaR-based optimal decision that minimises the insurer total loss from 2.1 is given by*

$$R^*[X] = \begin{cases} \big(SL(t^*) - SL(1-\alpha\wedge\beta)\big)_+, & \alpha\le\beta \text{ or } \beta<\alpha\le 1-t_1^* \\ \big(SL(t^*) - SL(1-\alpha)\big)_+, & 1-t_1^*<\beta<\alpha \\ \big(SL(t^*) - SL(1-\beta)\big)_+ + SL(t_1^*) - SL(1-\alpha), & \beta\le 1-t_1^*<\alpha \end{cases}. \tag{2.2}$$

Note 2.1. *The seller's optimal risk increases if δ_1 and δ_2 increases or decreases, respectively. This can be explained by the fact that the insurer has more confidence in buying reinsurance once its perception on reinsurer's default improves, which is the case for high values of δ_1. In turn, lower δ_2's reduce the reinsurance premium, and therefore the buyer has access to less expensive reinsurance.*

Note 2.2. *Theorem 2.1 recovers many existing results in the literature. The case in which $\delta_1 = 1$, and $\delta_2 = 1$ or $\beta = 1$, recovers the insurer's optimal VaR-based arrangement for various choices of reinsurance premium when the default is not taken into account, that had been previously found by Cheung et al. (2011), Chi and Tan (2011) and Asimit et al. (2013 b), among others.*

Proof. Note first that $x - \tilde{R}(x;\delta_1)$ and $R(x)$ are non-decreasing and continuous functions. Therefore,

$$VaR_s\big(X - \tilde{R}[X;\delta_1]\big) = VaR_s\big(X\big) - \tilde{R}\Big(VaR_s\big([X]\big);\delta_1\Big) \text{ and } VaR_s\big(R[X]\big) = R\Big(VaR_s\big(X\big)\Big), \tag{2.3}$$

are true for all $0 < s < 1$, due to the co-monotonic property (for details, see Dhaene *et al.*, 2002 a and b, Denuit *et al.*, 2005). Similarly,

$$VaR_s\big(\tilde{R}[X];\delta_1\big) = R\big(VaR_s(X)\big) \wedge R\big(VaR_\beta(X)\big) + \delta_1\Big(R\big(VaR_s(X)\big) - R\big(VaR_\beta(X)\big)\Big)_+. \tag{2.4}$$

Let us assume that $\alpha \le \beta$. Thus, as a result of 1.1, 2.3 and 2.4, the objective function from 2.1 becomes

$$VaR_\alpha(X) - R\big(VaR_\alpha(X)\big) - (1+c)\delta_2 R\big(VaR_\beta(X)\big)\int_\beta^1 \Phi_R(s)\,ds +$$
$$(1+c)\left(\int_0^\beta R\big(VaR_s(X)\big)\Phi_R(s)\,ds + (1+\delta_2)\int_\beta^1 R\big(VaR_s(X)\big)\Phi_R(s)\,ds\right)$$

Now, using the two-stage procedure elaborated in Asimit *et al.* (2013 b), the first step is to solve

$$\begin{cases} \min\limits_{R\in\mathcal{F}} \displaystyle\int_0^\beta R\big(VaR_s(X)\big)\Phi_R(s)\,ds + (1+\delta_2)\int_\beta^1 R\big(VaR_s(X)\big)\Phi_R(s)\,ds \\ \text{subject to} \quad R\big(VaR_\alpha(X)\big) = \xi_1, \;\; R\big(VaR_\beta(X)\big) = \xi_2 \end{cases}, \tag{2.5}$$

where $(\xi_1,\xi_2) \in \mathcal{D} = \big\{0 \le \xi_2 - \xi_1 \le VaR_\beta(X) - VaR_\alpha(X), 0 \le \xi_1 \le VaR_\alpha(X)\big\}$ are some parameters. Proposition 2.1 from Asimit *et al.* (2013 b) shows that 2.5 is solved by

$$R^*[X;\xi_1,\xi_2] := \big(X - VaR_\alpha(X) + \xi_1\big)_+ \wedge \xi_1 + \big(X - VaR_\beta(X) + \xi_2 - \xi_1\big)_+ \wedge (\xi_2 - \xi_1).$$

Note that $\big(R^*[X;\xi_1,\xi_2] - R^*(VaR_\beta(X);\xi_1,\xi_2)\big)_+ = 0$. Thus, the latter and 1.1 imply that the second step in solving 2.5 is to minimise on $\mathcal{D}$

$$H(\xi_1,\xi_2) := -\xi_1 + (1+c)\left(\int_{VaR_\alpha(X)-\xi_1}^{VaR_\alpha(X)} + \int_{VaR_\beta(X)-(\xi_2-\xi_1)}^{VaR_\beta(X)}\right) g\big(\Pr(X>x)\big)\,dx.$$

Clearly, $\mathcal{D}$ is a simple region with respect to ξ_2. In addition, $H(\cdot)$ is non-decreasing with respect to ξ_2, and therefore we have that

$$H(\xi_1,\xi_2) \geq H(\xi_1,\xi_1) = h(\xi_1) = -\xi_1 + (1+c)\int_{VaR_\alpha(X)-\xi_1}^{VaR_\alpha(X)} g\big(\Pr(X>x)\big)\,dx. \tag{2.6}$$

Now, $h^{'}(\xi_1) = -1 + (1+c)g\Big(\bar{F}\big(VaR_\alpha(X)-\xi_1\big)\Big)$, which implies that

$$h^{'}(\xi_1) \leq 0 \Leftrightarrow \bar{F}\big(VaR_\alpha(X)-\xi_1\big) \leq t^* \Leftrightarrow VaR_\alpha(X)-\xi_1 \geq VaR_{1-t^*}(X).$$

Therefore, the function from 2.6 is minimised at $VaR_\alpha(X) - VaR_{1-t^*}(X)$ if $1-t^* < \alpha$, which replicates 2.2 in this case. Similarly, $H(\cdot)$ attains its global minimum at $(0,0)$, whenever $1-t^* \geq \alpha$. These complete the $\alpha \leq \beta$ case.

The mirror case $\beta < \alpha$ is now investigated, where the objective function becomes

$$\begin{aligned}
& VaR_\alpha(X) - R\big(VaR_\alpha(X)\big) + (1-\delta_1)\Big(R\big(VaR_\alpha(X)\big) - R\big(VaR_\beta(X)\big)\Big) - \\
& (1+c)\delta_2 R\big(VaR_\beta(X)\big)\int_\beta^1 \Phi_R(s)\,ds + \\
& (1+c)\left(\int_0^\beta R\big(VaR_s(X)\big)\Phi_R(s)\,ds + (1+\delta_2)\int_\beta^1 R\big(VaR_s(X)\big)\Phi_R(s)\,ds\right).
\end{aligned}$$

As before, the first step is to solve

$$\begin{cases} \displaystyle\min_{R\in\mathcal{F}} \int_0^\beta R\big(VaR_s(X)\big)\Phi_R(s)\,ds + (1+\delta_2)\int_\beta^1 R\big(VaR_s(X)\big)\Phi_R(s)\,ds \\ \quad \text{subject to} \quad R\big(VaR_\alpha(X)\big) = \xi_1, \;\; R\big(VaR_\beta(X)\big) = \xi_2 \end{cases}, \tag{2.7}$$

where $(\xi_1,\xi_2) \in \mathcal{E} = \big\{0 \leq \xi_1 - \xi_2 \leq VaR_\alpha(X) - VaR_\beta(X), 0 \leq \xi_2 \leq VaR_\beta(X)\big\}$ are some parameters. Thus, 2.7 is solved by

$$R^*[X;\xi_1,\xi_2] := \big(X - VaR_\beta(X) + \xi_2\big)_+ \wedge \xi_2 + \big(X - VaR_\alpha(X) + \xi_1 - \xi_2\big)_+ \wedge (\xi_1 - \xi_2),$$

as a result of Proposition 2.1 from Asimit *et al.* (2013 b). The second stage problem is the minimisation on $\mathcal{E}$ of

$$\begin{aligned}
H(\xi_1,\xi_2) &:= (1+c)\int_0^{x_F} g\Big(\Pr\left(R^*[X;\xi_1,\xi_2] > x\right)\Big)\,dx - \delta\xi_1 - (1-\delta)\xi_2 \\
&= (1+c)\left(\int_{VaR_\beta(X)-\xi_2}^{VaR_\beta(X)} + \delta_2\int_{VaR_\alpha(X)-(\xi_1-\xi_2)}^{VaR_\alpha(X)}\right) g\big(\Pr(X>x)\big)\,dx - \delta\xi_1 - (1-\delta)\xi_2.
\end{aligned}$$

Recall that $\mathcal{E}$ is simple region with respect to ξ_1. One may get that

$$\frac{dH}{d\xi_1} = (1+c)\delta_2 g\Big(\bar{F}\big(VaR_\alpha(X) - \xi_1 + \xi_2\big)\Big) - \delta_1,$$

which is non-positive for some $\xi_2 \leq \xi_1 \leq VaR_\alpha(X) - VaR_\beta(X) + \xi_2$, if and only if

$$\bar{F}\big(VaR_\alpha(X) - \xi_1 + \xi_2\big) \leq t_1^* \Leftrightarrow VaR_\alpha(X) - \xi_1 + \xi_2 \geq VaR_{1-t_1^*}(X). \tag{2.8}$$

Firstly, we assume $\alpha \leq 1 - t_1^*$. Equation 2.8 shows that $\frac{dH}{d\xi_1}$ is always positive, and thus

$$H(\xi_1,\xi_2) \geq H(\xi_2,\xi_2) = (1+c)\int_{VaR_\beta(X)-\xi_2}^{VaR_\beta(X)} g\big(\Pr(X>x)\big)\,dx - \xi_2. \tag{2.9}$$

This subcase can be further investigated by following the same steps as given after relation 2.6, and one may get that the global solution is attained at $\xi_2^* = \big(VaR_\beta(X) - VaR_{1-t^*}(X)\big)_+$, which concludes the $\beta < \alpha \le 1 - t_1^*$ scenario.

Secondly, the $1 - t_1^* < \beta$ subcase is investigated. Recall that $\xi_1 - \xi_2 \le VaR_\alpha(X) - VaR_\beta(X)$ should hold, and therefore, 2.8 is always satisfied. Thus,

$$H(\xi_1, \xi_2) \ge H\big(\xi_2 + VaR_\alpha(X) - VaR_\beta(X), \xi_2\big)$$

holds for all $(\xi_1, \xi_2) \in \mathcal{E}$. The latter function and the right hand side of relation 2.9 differ from a constant, and as a result, the global minimum are both attained at $\xi_2^* = \big(VaR_\beta(X) - VaR_{1-t^*}(X)\big)_+$, which recovers 2.2 for this subcase.

Thirdly, it is further assumed that $\beta \le 1 - t_1^* < \alpha$. For any fixed ξ_2, we have

$$\begin{array}{ll} \frac{dH}{d\xi_1} \le 0 \text{ if } \xi_2 & \le \xi_1 \le \xi_2 + VaR_\alpha(X) - VaR_{1-t_1^*}(X) \\ \frac{dH}{d\xi_1} > 0 \text{ if } \xi_2 + VaR_\alpha(X) - VaR_{1-t_1^*}(X) & < \xi_1 \le \xi_2 + VaR_\alpha(X) - VaR_\beta(X) \end{array},$$

which yields that $H(\xi_1, \xi_2) \ge H\big(\xi_2 + VaR_\alpha(X) - VaR_{1-t_1^*}(X), \xi_2\big)$ for all $(\xi_1, \xi_2) \in \mathcal{E}$. Once again, the latter function and the right hand side of relation 2.9 differ from a constant, and thus, both reach its global minimum at $\xi_2^* = \big(VaR_\beta(X) - VaR_{1-t^*}(X)\big)_+$. Thus, the global minimum of $H(\cdot)$ is attained at (ξ_1^*, ξ_2^*) with $\xi_1^* = \xi_2^* + VaR_\alpha(X) - VaR_{1-t_1^*}(X)$, which recovers 2.2 for our final subcase. The proof is now complete. $\square$

3. Optimal Reinsurance Contract Based on Distortion Risk Measures

In this section, the optimal choice for the buyer is investigated if a general concave distortion risk measure, $\varphi_I(\cdot)$, is adopted. In addition, the same premium principle assumption as given in Section 2 is in force. Therefore, the optimal reinsurance problem is given by:

$$\min_{R \in \mathcal{F}} \Big\{ \varphi_I\big(X - \tilde{R}[X;\delta_1]\big) + (1+c)\varphi_R\big(\tilde{R}[X;\delta_2]\big) \Big\}, \tag{3.1}$$

where $\varphi_I(\cdot)$ and $\varphi_R(\cdot)$ are given by (1.1) with the function $g(\cdot)$ is replaced by distortion functions $g_I(\cdot)$ and $g_R(\cdot)$ respectively.

To solve 3.1, the method described in Cui *et al.* (2012) is used by expressing the objective function as integrals taken under an identifiable measure of a function that depends on the survival function of the underlying loss random variable X. For each $R \in \mathcal{F}$, we first define $\ell_R := \lim_{x \to \infty} R(x)$ and $R^{-1} : [0, \ell_R) \to [0, \infty)$, where

$$R^{-1}(t) := \inf\{x : R(x) > t\}, \quad t \in [0, \ell_R).$$

Obviously, $R(x) > t$ if and only if $x > R^{-1}(t)$. The image measure on $[0, \infty)$ of the Lebesgue measure λ on $[0, \ell_R)$ under the map R^{-1} is denoted as $\lambda_R(\cdot)$, i.e.

$$\lambda_R(B) := \lambda\big\{t \in [0, \ell_R) : R^{-1}(t) \in B\big\}$$

for any Borel measurable subset B of $[0, \infty)$. Note that, from now on, $\lambda(\cdot)$ always represents the Lebesgue measure on the real line. The first step in solving the optimisation problem defined in 3.1 is given in the next lemma.

Lemma 3.1. *For each $R \in \mathcal{F}$, the objective function from 3.1 is given by*

$$\varphi_I\big(X - \tilde{R}[X;\delta_1]\big) + (1+c)\varphi_R\big(\tilde{R}[X;\delta_2]\big) = \varphi_I(X) + \int_{[0,\infty)} \Theta\big(\Pr(X > t)\big)\ \lambda_R(dt),$$

where

$$\Theta(u) := (1-\delta_1)g_I(u)1_{\{u \le 1-\beta\}} - g_I(u) + (1+c)g_R(u) - (1+c)(1-\delta_2)g_R(u)1_{\{u \le 1-\beta\}}, \quad u \in [0,1].$$

Proof. Fix $R \in \mathcal{F}$. Since the risk measure $\varphi_I(\cdot)$ is cash invariant, comonotonic additive and positively homogeneous (for details, see Denuit *et al.* 2005), one can derive as in 2.3 and 2.4 that

$$\begin{aligned} &\varphi_I\Big(X - \tilde{R}[X;\delta_1] + (1+c)\varphi_R\big(\tilde{R}[X;\delta_2]\big)\Big) \\ &\quad = \varphi_I(X) + (1-\delta_1)\varphi_I\Big(\Big(R[X] - R\big(VaR_\beta(X)\big)\Big)_+\Big) - \varphi_I\big(R[X]\big) + (1+c)\varphi_R\big(\tilde{R}[X;\delta_2]\big). \end{aligned} \tag{3.2}$$

Note that the second term from the right hand side of 3.2 can be rewritten as

$$\begin{aligned} \varphi_I\Big(\big(R[X] - R(VaR_\beta(X))\big)_+\Big) &= \int_0^\infty g_I\Big(\Pr\big(R(X) > x + R(VaR_\beta(X))\big)\Big)\, dx \\ &= \int_{R(VaR_\beta(X))}^\infty g_I\Big(\Pr\big(R(X) > x\big)\Big)\, dx \\ &= \int_0^{\ell_R} 1_{\{x \ge R(VaR_\beta(X))\}} g_I\Big(\Pr\big(X > R^{-1}(x)\big)\Big)\, dx \\ &= \int_0^{\ell_R} 1_{\{R^{-1}(x) \ge VaR_\beta(X)\}} g_I\Big(\Pr\big(X > R^{-1}(x)\big)\Big)\, dx \\ &= \int_{[0,\infty)} 1_{\{t \ge VaR_\beta(X)\}} g_I\Big(\Pr\big(X > t\big)\Big)\ \lambda_R(dt) \\ &= \int_{[0,\infty)} 1_{\{\Pr(X>t) \le 1-\beta\}} g_I\Big(\Pr\big(X > t\big)\Big)\ \lambda_R(dt). \end{aligned}$$

Similarly, the third and fourth terms from the right hand side of 3.2 equal to

$$\varphi_I\big(R[X]\big) = \int_{[0,\infty)} g_I\Big(\Pr\big(X > t\big)\Big)\ \lambda_R(dt).$$

and

$$\begin{aligned} \varphi_R\big(\tilde{R}[X;\delta_2]\big) &= \int_{[0,\infty)} g_R\Big(\Pr\big(X > t\big)\Big)\ \lambda_R(dt) \\ &\quad -(1-\delta_2)\int_{[0,\infty)} 1_{\{\Pr(X>t) \le 1-\beta\}} g_R\Big(\Pr\big(X > t\big)\Big)\ \lambda_R(dt), \end{aligned}$$

respectively. Combining all the results, the lemma is fully justified. □

Lemma 3.1 shows that solving 3.1 is a special case of the following more general optimisation problem:

$$\min_{R \in \mathcal{F}} \int_{[0,\infty)} H(t)\ \lambda_R(dt), \tag{3.3}$$

where $H(\cdot)$ is any real-valued measurable function on $[0,\infty)$. The set of optimal solutions from 3.3 is given in Lemma 3.2, where $\mathcal{G}$ denotes the set of all measurable functions $f : [0,\infty) \to \mathbb{R}$ with values in $[0,1]$ almost everywhere with respect to the Lebesgue measure λ.

Lemma 3.2. *The set of all optimal solutions $R^* \in \mathcal{F}$ of Problem 3.3 is given by*

$$\left\{ R^*(x) = \int_0^x \left(1_{\{H(t)<0\}} + f(t)1_{\{H(t)=0\}}\right) dt \;\middle|\; f \in \mathcal{G} \right\}.$$

In particular, Problem 3.3 has a unique optimal solution if the set $\lambda\{x \mid H(x) = 0\} = 0$.

Proof. Recall that

$$R^{-1}(t) \in B \to t \in R(B) := \{R(x); x \in B\}$$

and $R(B)$, the image of a Borel measurable set under the Lipschitz map $R(\cdot)$, is Lebesgue measurable (see Lemma 3.6.3 of Bogachev, 2007), we obtain that $\lambda_R(B) \leq \lambda\big(R(B)\big)$ for any $R \in \mathcal{F}$ and Borel measurable subset B of $[0, \infty)$. Moreover, since $R(\cdot)$ is 1-Lipschitz, it follows that $\lambda\big(R(B)\big) \leq \lambda(B)$ (see Lemma 3.10.12 of Bogachev, 2007). Therefore, $\lambda_R(B) \leq \lambda(B)$, and hence, we can define the corresponding Radon-Nikodym derivative

$$0 \leq D_R := \frac{d\lambda_R}{d\lambda}$$

on $[0, \infty)$. Denote $A := \{D_R > 1\}$. If $\lambda(A) > 0$, then

$$\lambda_R(A) = \int_A D_R(t)\, dt > \lambda(A),$$

which is a contradiction. Thus, it can be concluded that $0 \leq D_R \leq 1$ λ-almost surely, and the objective function from 3.3 can be written as

$$\int_{[0,\infty)} H(t) D_R(t)\, dt.$$

It is now clear that in order to minimise the last integral, it is optimal to set $D_R(t)$ to 1 whenever $H(t) < 0$, and 0 whenever $H(t) > 0$. On the set $\{t : H(t) = 0\}$, the value of $D_R(t)$ can be set arbitrarily. Thus, the above integral is minimised at

$$D^*_R(t) = 1_{\{H(t)<0\}} + f(t)1_{\{H(t)=0\}}, \tag{3.4}$$

for any $f \in \mathcal{G}$. Now, it remains to show that an optimal solution $R^*(\cdot)$ of 3.3 can be recovered from $D^*_R(\cdot)$ as defined in 3.4, which is true due to the following relation

$$R^*(x) = R^*(x) - R^*(0) = \lambda_{R^*}\big([0, x)\big) = \int_{[0,x)} D^*_R(t)\, dt = \int_0^x \left(1_{\{H(t)<0\}} + f(t)1_{\{H(t)=0\}}\right) dt.$$

The proof is now complete. □

It is worth mentioning that Cui *et al.* (2012) provides an alternative way to solve 3.3. They first show that the density function of the measure $\lambda_R(\cdot)$ is equal to R, and then verify the optimality of the proposed solution by using of the 1-Lipschitz property of $R(\cdot)$. One advantage of our method is that we provide a constructive, more intuitive and transparent way to find out all the optimal solutions.

With Lemma 3.2 on hand, we are ready to solve 3.1. For the sake of notational simplicity, some mild additional assumption are necessary in order to display the optimal solutions. More specifically, the expected value principle is further assumed for the reason of being able to write down the solution in a general setting, but any other premium principle can be solved via Lemma 3.2. The same reason applies to the strictly increasing assumption of $F(\cdot)$. Recall that x_0 represents the left end point of the distribution of X, as it has been defined in Section 1. We will first deal with strictly convex distorted risk measures in Theorem 3.1, while later in Theorem 3.2, a non-strict distorted risk measure is considered.

Theorem 3.1. *Let $\Delta := \frac{g(1-\beta)}{1-\beta}$. Assume that $F(\cdot)$ is strictly increasing on (x_0, x_F), $g_I(\cdot)$ is strictly concave on $[0,1]$, and $g_R(x) = x$ for all $x \in [0,1]$. Then, the optimal decision that minimises the insurer total loss from 3.1 is given by the following:*

(1) *Suppose that $1 < 1+c < \Delta$. Let $\beta^* \in (1-\beta, 1)$ be the unique solution of the equation $(1+c)\beta = g_I(\beta)$.*
   (a) *If $(1+c)\delta_2/\delta_1 < \Delta$, then $R^*[X] = SL(\beta^*)$.*
   (b) *If $\Delta \leq (1+c)\delta_2/\delta_1 < g_I'(0)$, then $R^*[X] = SL(\beta^*) - SL(1-\beta) + SL(\tilde{\beta})$, where $\tilde{\beta} \in (0, 1-\beta]$ is the unique solution of the equation $(1+c)\delta_2\beta/\delta_1 = g_I(\beta)$.*
   (c) *If $g_I'(0) \leq (1+c)\delta_2/\delta_1$, then $R^*[X] \equiv 0$.*

(2) *Suppose that $\Delta \leq 1+c$.*
   (a) *If $(1+c)\delta_2/\delta_1 < g_I'(0)$, then $R^*[X] = SL(\tilde{\beta})$, where $\tilde{\beta} \in (0, 1-\beta]$ is the unique solution of the equation $(1+c)\delta_2\beta/\delta_1 = g_I(\beta)$.*
   (b) *If $g_I'(0) \leq (1+c)\delta_2/\delta_1$, then $R^*[X] \equiv 0$.*

Proof. Since all the cases can be proved in a similar manner, we only chose to show Case (1)(b). Lemma 3.1 and the fact that $g_R(u) = u$ for all u yield that 3.1 is equivalent to

$$\min_{R \in \mathcal{F}} \int_{[0,\infty)} \Theta_1\big(\bar{F}(t)\big)\ \lambda_R(dt),$$

where

$$\Theta_1(u) := (1-\delta_1) g_I(u) 1_{\{u \leq 1-\beta\}} - g_I(u) + (1+c)u - (1+c)(1-\delta_2) u 1_{\{u \leq 1-\beta\}}, \quad u \in [0,1].$$

Further, Lemma 3.2 implies that any optimal solution $R^* \in \mathcal{F}$ to this problem takes the form of

$$R^*[X] = \int_0^X \Big(1_{\{\Theta_1(\bar{F}(t))<0\}} + f(t) 1_{\{\Theta_1(\bar{F}(t))=0\}}\Big)\, dt$$

for some $f \in \mathcal{G}$. It only remains to identify the sets $\{\Theta_1(\bar{F}(t)) < 0\}$ and $\{\Theta_1(\bar{F}(t)) = 0\}$.

Keeping in mind the strictly concavity assumption of of $g_I(\cdot)$, it is easy to check that

$$\Theta_1(u) < 0 \Longleftrightarrow u \in (0, \tilde{\beta}) \cup (1-\beta, \beta^*) \quad \text{and} \quad \Theta_1(u) = 0 \Longleftrightarrow u = 0 \text{ or } \tilde{\beta} \text{ or } \beta^*.$$

Therefore, any optimal solution of 3.1 takes the form of

$$\begin{aligned}
R^*[X] &= \int_0^X \big(1_{\{\Theta_1(\bar{F}(t))<0\}} + f(t) 1_{\{\Theta_1(\bar{F}(t))=0\}}\big)\, dt \\
&= \int_0^X 1_{\{0<\bar{F}(t)<\tilde{\beta}\}}\, dt + \int_0^X 1_{\{1-\beta<\bar{F}(t)<\beta^*\}}\, dt + \int_0^X f(t) 1_{\{\bar{F}(t)=0 \text{ or } \tilde{\beta} \text{ or } \beta^*\}}\, dt \\
&= \int_0^X 1_{\{x_F > t > VaR_{1-\tilde{\beta}}(X)\}}\, dt + \int_0^X 1_{\{VaR_\beta(X) > t > VaR_{1-\beta^*}(X)\}}\, dt + 0 \\
&= \big(X - VaR_{1-\tilde{\beta}}(X)\big)_+ + \big(X - VaR_{1-\beta^*}(X)\big)_+ - \big(X - VaR_\beta(X)\big)_+,
\end{aligned}$$

where $f(\cdot)$ is any function from $\mathcal{G}$. Note that the last integral from the second step is zero due to the strictly increasing assumption $F(\cdot)$, which completes our proof. ∎

As it can see from the proof of Lemma 3.1, the strict concavity assumption of $g_I(\cdot)$ guarantees that the set $\{u \mid \Theta_1(u) = 0\}$ is finite. The latter together with the strictly increasing assumption of F suggest that the set $\{\Theta_1(\bar{F}(t)) = 0\}$ has zero Lebesgue measure and hence 3.1 is uniquely solved. If we relax our assumptions, an integral of $f \in \mathcal{G}$ may appear, but the Lemma 3.2 is still applicable for such cases. The next theorem provides the optimal arrangements whenever the buyer's risk appetite is evaluated by a non-strictly distorted risk measure, such as the ES. To simplify our notation, we denote by

$$I_f(\theta_1, \theta_2) := \int_{X \wedge VaR_{1-\theta_2}(X)}^{X \wedge VaR_{1-\theta_1}(X)} f(t)\, dt, \quad f \in \mathcal{G}, 0 < \theta_1 < \theta_2 \leq 1.$$

Moreover, whenever $\delta_1 \neq 0$, we define $1 + C := (1 + c)\delta_2/\delta_1$, so that $1 < 1 + c \leq 1 + C$.

Theorem 3.2. *Assume that $F(\cdot)$ is strictly increasing on (x_0, x_F), $\varphi_I = ES_\alpha$, and $g_R(x) = x$ for all $x \in [0, 1]$. Then, the optimal decision that minimises the insurer total loss from 3.1 is given by the following:*

(1) *Suppose that $\delta_1 \neq 0$ and $\alpha > \beta$.*
  (a) *If $1 < 1 + C \leq 1/(1 - \beta)$, then $R^*[X] = SL\big(1/(1 + c)\big)$.*
  (b) *If $1/(1 - \beta) < 1 + C < 1/(1 - \alpha)$, then*

$$R^*[X] = SL\big(1/(1 + C)\big) + \Big(SL\big(1/(1 + c)\big) - SL(1 - \beta)\Big)_+.$$

  (c) *If $1 + C = 1/(1 - \alpha)$, then for any $f \in \mathcal{G}$ we have*

$$R^*[X] = I_f(0, 1 - \alpha) + \Big(SL\big(1/(1 + c)\big) - SL(1 - \beta)\Big)_+.$$

  (d) *If $1 + C > 1/(1 - \alpha)$, then $R^*[X] = \Big(SL\big(1/(1 + c)\big) - SL(1 - \beta)\Big)_+$.*

(2) *Suppose that $\delta_1 \neq 0$ and $\alpha \leq \beta$.*
  (a) *If $1 + C \leq 1/(1 - \alpha)$, then $R^*[X] = SL\big(1/(1 + c)\big)$.*
  (b) *If $1 + C = 1/(1 - \alpha)$, then*

$$R^*[X] = \begin{cases} SL\big(1/(1 + c)\big) - SL(1 - \beta) + I_f(0, 1 - \beta) & \text{if } 1 + c < 1/(1 - \alpha) \\ I_f(0, 1 - \alpha) & \text{if } 1 + c = 1/(1 - \alpha) \end{cases}.$$

  (c) *If $1 + C > \frac{1}{1-\alpha}$, then for for any $f \in \mathcal{G}$ we have*

$$R^*[X] = \begin{cases} SL\big(1/(1 + c)\big) - SL(1 - \beta) & \text{if } 1 + c < 1/(1 - \alpha) \\ I_f(1 - \beta, 1 - \alpha) & \text{if } 1 + c = 1/(1 - \alpha) \\ 0 & \text{if } 1 + c > 1/(1 - \alpha) \end{cases}.$$

(3) *Suppose that $0 = \delta_1 < \delta_2$ and $\alpha > \beta$. Then, for any $f \in \mathcal{G}$ we have*

$$R^*[X] = \begin{cases} SL\big(1/(1 + c)\big) - SL(1 - \beta) & \text{if } 1 + c \leq 1/(1 - \beta) \\ 0 & \text{if } 1 + c > 1/(1 - \beta) \end{cases}.$$

(4) *Suppose that* $0=\delta_1<\delta_2$ *and* $\alpha\le\beta$. *Then, for any* $f\in\mathcal{G}$ *we have*

$$R^*[X]=\begin{cases} SL\big(1/(1+c)\big)-SL(1-\beta) & \text{if } 1+c<1/(1-\alpha)\\ I_f(1-\beta,1-\alpha) & \text{if } 1+c=1/(1-\alpha)\\ 0 & \text{if } 1+c>1/(1-\alpha)\end{cases}.$$

(5) *Suppose that* $0=\delta_1=\delta_2$ *and* $\alpha>\beta$. *Then, for any* $f\in\mathcal{G}$ *we have*

$$R^*[X]=\Big(SL\big(1/(1+c)\big)-SL(1-\beta)\Big)_+ + I_f(0,1-\beta).$$

(6) *Suppose that* $0=\delta_1=\delta_2$ *and* $\alpha\le\beta$. *Then, for any* $f\in\mathcal{G}$ *we have*

$$R^*[X]=\begin{cases} SL\big(1/(1+c)\big)-SL(1-\beta)+I_f(0,1-\beta) & \text{if } 1+c<1/(1-\alpha)\\ I_f(0,1-\alpha) & \text{if } 1+c=1/(1-\alpha)\\ I_f(0,1-\beta) & \text{if } 1+c>1/(1-\alpha)\end{cases}.$$

The proof is omitted because its technique is the same as that of Theorem 3.1.

4. Expected policyholder deficit

While the optimal reinsurance reflects the buyer's ideal choice to reduce its exposure, it is not known if reinsurance creates an advantage to the policyholder. Asimit *et al.* (2013 a) showed that some risk transfers within an insurance group may be detrimental to the policyholder if the counterparty default risk is not properly incorporated when setting the regulatory capital. In this section, we evaluate the reinsurance impact on policyholder's welfare by investigating the resulting *Expected Policyholder Deficit (EPD)*. The policyholder deficit is equal to the difference between nominal liabilities to policyholders and liabilities that will actually be paid, thus reflecting the reduction in the payoff received due to potential default. The policyholder deficit can also be seen as an asset transferred from policyholders to shareholders, reflecting the option of the latter to default on their obligations (see Butsic, 1994, Phillips *et al.*, 1998, and Myers and Read, 2001). Formally, the EPD for a generic downside risk Z and available assets c is defined by

$$EPD(Z;c)=E\,(Z-c)_+=\int_c^{x_F}P(Z>z)\,dz.$$

If the capital requirements are VaR_γ-based, as it is the case in Solvency II when $\gamma=99.5\%$, and the cedent chooses a reinsurance arrangement $R(\cdot)$ and believes that the reinsurer's recovery rate is δ, then the EPD becomes:

$$EPD[X,R;VaR_\gamma,\beta,\delta]:=EPD\Big(X-\tilde{R}[X;\delta];VaR_\gamma\big(X-\tilde{R}[X;\delta]\big)\Big),$$

where β represents the level that triggers the seller's default. Interestingly, we find that any possible reinsurance arrangement does not increase the EPD. In addition, the policyholder's exposure is optimally reduced if the tail risk is transfer by the insurance company. These findings are formalised in the next proposition.

Proposition 4.1. *Let* $0\le\delta\le1$ *and* $0\le a\le VaR_\gamma(X)$. *Then,*

a) $\max_{R\in\mathcal{F}} EPD[X,R;VaR_\gamma,\beta,\delta]=\int_{VaR_\gamma(X)}^{x_F}\bar{F}(x)\,dx$ *and is solved by* $R^*[X]=h(X;a)\wedge a$,

b) $\min_{R\in\mathcal{F}} EPD[X,R;VaR_\gamma,\beta,\delta]=(1-\delta)\int_{VaR_{\max\{\gamma,\beta\}}(X)}^{x_F}\bar{F}(x)\,dx$ *and is solved by*

$$R^*[X]=h(X;a)\wedge a+\big(X-VaR_\gamma(X)\big)_+,$$

where $h(\cdot;a)$ is a 1-Lipschitz function such that $h(0;a)=0$ and $h\big(VaR_\gamma(X);a\big)=a$.

Proof. The proof is based on the same technology illustrated in showing the results of Theorem 2.1. Since both optimisation problems can be proved in the same manner, only part a) is fully explained. We first rewrite the objective function as follows:

$$\begin{aligned}
&EPD[X,R;VaR_\gamma,\beta,\delta]\\
&\quad=(1-\gamma)\Big(ES_\gamma\big(X-\tilde{R}[X;\delta]\big)-VaR_\gamma\big(X-\tilde{R}[X;\delta]\big)\Big)\\
&\quad=\int_\gamma^1 VaR_s(X)\,ds-\int_\gamma^{\max\{\gamma,\beta\}}R\big(VaR_s(X)\big)\,ds-\delta\int_{\max\{\gamma,\beta\}}^1 R\big(VaR_s(X)\big)\,ds-\\
&\qquad(1-\delta)\big(1-\max\{\gamma,\beta\}\big)R\big(VaR_\beta(X)\big)-\\
&\qquad(1-\gamma)\bigg(I\big(VaR_\gamma(X)\big)+(1-\delta)\Big(R\big(VaR_\gamma(X)\big)-R\big(VaR_\beta(X)\big)\Big)_+\bigg).
\end{aligned}$$

As it has already been seen, the first step of our maximisation problem is to include the boundary conditions $R\big(VaR_\gamma(X)\big)=\xi_1$ and $R\big(VaR_\beta(X)\big)=\xi_2$ to the initial optimisation problem.

Firstly, if $\gamma\le\beta$, then the parameters (ξ_1,ξ_2) should satisfy $0\le\xi_2-\xi_1\le VaR_\beta(X)-VaR_\gamma(X)$ and $0\le\xi_1\le VaR_\gamma(X)$. In addition, the optimal solution is given by:

$$R^*[X;\xi_1,\xi_2]=h(X;\xi_1)\wedge\xi_1+\big(X-VaR_\beta(X)+\xi_2-\xi_1\big)_+\wedge(\xi_2-\xi_1),$$

where $h(\cdot;\xi_1)$ is a 1-Lipschitz function such that $h(0;\xi_1)=0$ and $h\big(VaR_\gamma(X);\xi_1\big)=\xi_1$. Thus, the objective function of the second step is to maximise

$$\begin{aligned}
&EPD\big[X,R^*[X;\xi_1,\xi_2];VaR_\gamma,\beta,\delta\big]\\
&\quad=(1-\gamma)\big(ES_\gamma(X)-VaR_\gamma(X)\big)-E\big[\tilde{R}^*[X;\xi_1,\xi_2]-\xi_1\big]_+\\
&\quad=(1-\gamma)\big(ES_\gamma(X)-VaR_\gamma(X)\big)-E\big[R^*[X;\xi_1,\xi_2]-\xi_1\big]_+\\
&\quad=(1-\gamma)\big(ES_\gamma(X)-VaR_\gamma(X)\big)-\int_{VaR_\beta(X)-(\xi_2-\xi_1)}^{VaR_\gamma(X)}\bar{F}(x)\,dx,
\end{aligned}$$

over all possible (ξ_1,ξ_2). Therefore, the condition $\xi_1=\xi_2$ is sufficient to describe the set of optimal solutions, i.e. $R^*[X;\xi_1,\xi_1]$, which completes the proof in this case.

Secondly, if $\gamma>\beta$, then $0\le\xi_1-\xi_2\le VaR_\gamma(X)-VaR_\beta(X)$ and $0\le\xi_1\le VaR_\gamma(X)$ should hold. The first step optimal solution becomes $R^*[X;\xi_1,\xi_2]=h_1(X;\xi_1,\xi_2)\wedge\xi_1$, where $h_1(0;\xi_1,\xi_2)=0$, $h\big(VaR_\beta(X);\xi_1,\xi_2\big)=\xi_2$ and $h\big(VaR_\gamma(X);\xi_1,\xi_2\big)=\xi_1$. Indeed, the later describes the set of optimal solutions due to the fact that

$$\begin{aligned}
EPD\big[X,R^*[X;\xi_1,\xi_2];VaR_\gamma,\beta,\delta\big]&=(1-\gamma)\big(ES_\gamma(X)-VaR_\gamma(X)\big)-E\big[\tilde{R}^*[X;\xi_1,\xi_2]-\xi_1\big]_+\\
&=(1-\gamma)\big(ES_\gamma(X)-VaR_\gamma(X)\big)\\
&=\int_{VaR_\gamma(X)}^{x_F}\bar{F}(x)\,dx,
\end{aligned}$$

for any (ξ_1, ξ_2). The proof is now complete. $\square$

We have tried to understand the effect of reinsurance over the policyholder's welfare whenever the buyer is ES-regulated, which still remains an open problem, and it could be possibly developed in the future. It would be interesting to capture this aspect that is of great interest whenever the primary insurer operates on the Swiss insurance market, but unfortunately the current available approaches are not helpful in deriving such results.

5. Numerical examples and conclusions

The main results of this paper concerned with the optimal reinsurance arrangement are summarised in Theorems 2.1, 3.1 and 3.2, and describe the insurer's ideal choice based on VaR, strictly convex distorted risk measures and ES, respectively. The effect of counterparty default risk has been analysed, which augment the existing literature on optimal reinsurance. This section is devoted to explore some plausible scenarios among the ones considered in these theorems, and understand the change in EPD as a result of reinsurance risk transfer.

Note that the EPD does not increase as a result of reinsurance, if the buyer is VaR_γ-regulated, as we assume from now on. The latter can be easily justified for a generic risk transfer, $R \in \mathcal{F}$. Clearly, $h(x) := x - \tilde{R}(x;\delta_1)$ is a 1-Lipschitz function, and therefore we have

$$|h(X) - h\big(VaR_\gamma(X)\big)| \le |X - VaR_\gamma(X)| \Rightarrow EPD[X, R; VaR_\gamma, \beta, \delta] \le E\big(X - VaR_\gamma(X)\big)_+.$$

Therefore, the EPD remains at the maximal possible level if all γ-worst possible outcomes are covered by the insurer, as it can be seen in Proposition 4.1. On the contrary, the EPD is reduced to its minimal level once all γ-worst possible outcomes are covered by the reinsurer.

To illustrate our results, let us consider a plausible parameters setting, $\beta = \gamma = 99.5\%$, as it is the case if both insurance players operate in the European market, where Solvency II Regime is currently implemented. It is likely to assume that the insurer would optimise its VaR decision at some confidence levels that are less than the regulatory one, i.e. $\alpha \le 99.5\%$. Thus, Theorem 2.1 reveals that the insurer's decision is exactly the same as in the case where the counterparty default risk is not taken into account. In addition, the EPD stays at its maximal possible level.

The ES_α optimal risk transfers are exemplified for some $\alpha \in \{70\%, 80\%, 90\%, 95\%\}$, and expected value principle with $c = 1$. Thus, $1+c < 1/(1-\alpha)$ is always true. Theorem 3.2 gives us the following two possible optimal reinsurance arrangements

$$R^*[X] = \begin{cases} SL\big(50\%)\big) - SL(0.5\%) & \text{if } 1 + C > 1/(1-\alpha) \\ SL\big(50\%\big) & \text{if } 1 + C \le 1/(1-\alpha) \end{cases},$$

where $C = 2\delta_2/\delta_1 - 1$. The $1 + C \le 1/(1-\alpha)$ case shows an optimal ES-based decision that is identical to the corresponding decision when the counterparty default risk is neglected, and Proposition 4.1 reveals that is optimal for policyholder(s) as well. Whenever $1 + C > 1/(1-\alpha)$, the ideal reinsurance contract changes as a result of taking into account the counterparty default risk, and the policyholder(s) have the maximal exposure as we can see from Proposition 4.1. Now, the recovery rates are assumed to belong from

$$\Big\{(\delta_1, \delta_2) : 0\% \le \delta_1 \le \delta_2 \le 100\%,\ \delta_1 \in \big\{50\%, 75\%, 100\%\big\}\Big\}.$$

Among all these possible combinations, only the case in which $\alpha = 70\%$ with $(\delta_1, \delta_2) = (50\%, 100\%)$ leads to the $1 + C > 1/(1-\alpha)$ case.

As a conclusion, the optimal reinsurance contract does not usually change if the counterparty default risk is taken into account, but one should consider this effect in order to properly measure the policyholders's exposure. Our numerical results show that the counterparty default risk may change the insurer's ideal arrangement only if the buyer and seller have very different views on the reinsurer's recovery rate.

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