EXTREMES ON THE DISCOUNTED AGGREGATE CLAIMS IN A TIME DEPENDENT RISK MODEL

Alexandru V. Asimit[1]

Andrei L. Badescu[2]

Department of Statistics
University of Toronto
100 St. George St.
Toronto, Ontario, Canada M5S 3G3

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Abstract

This paper presents an extension of the classical compound Poisson risk model for which the inter-claim time and the forthcoming claim amount are no longer independent random variables. Asymptotic tail probabilities for the discounted aggregate claims are presented when the force of interest is constant and the claim amounts are heavy tail distributed random variables. Furthermore, we derive asymptotic finite time ruin probabilities, as well as asymptotic approximations for some common risk measures associated with the discounted aggregate claims. A simulation study is performed in order to validate the results obtained in the free interest risk model.

Keywords: Compound Poisson risk model, Dependence, Discounted aggregate loss, Subexponential distribution, Value-at-Risk.

[1]E-mail: vali@utstat.toronto.edu

[2]Corresponding Author. Telephone: 416-946-7582; Fax: 416-978-5133;
E-mail: badescu@utstat.toronto.edu

1 Introduction

We assume the classical Poisson risk process in the presence of a constant force of interest, denoted by δ. The claim amounts $X_i, i = 1, 2, \ldots$ are non-negative independent and identically distributed (i.i.d.) random variables (rv's) with common distribution function (df) F. The claim arrival process $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$, with $W_i,\ i = 1, 2, \ldots$ denoting the i^{th} inter-claim time. We denote by $S_\delta(t)$ the discounted aggregate claim process over a finite time horizon $(0, t]$ with

$$S_\delta(t) = \sum_{i=1}^{N(t)} X_i e^{-\delta\sigma_i}, \tag{1}$$

where $\sigma_i = \sum_{k=1}^{i} W_k$ represents the i^{th} arrival time.

Several papers provide the asymptotic tail probability of the discounted aggregate claim process over a finite time horizon under the assumption of independence between the claim amounts and the inter-claim times. Assuming a constant force of interest, Tang (2005b, 2007) and Wang (2008) derive asymptotic results for both the classical compound Poisson and the renewal risk models. Ladoucette and Teugels (2006) study a similar problem for a free interest risk model assuming a general claim arrival process.

In this paper, we relax the independence assumption between the claim amounts and the inter-claim times. Hence, we assume a certain dependence structure (see Assumption 1 in Section 2) in which the distribution of the inter-claim times will affect the distribution of the forthcoming claim amounts. As will be seen in Section 2, the structural form of our dependence is quite general, different types of copulas falling under its umbrella.

Several authors assume different types of dependent risk models. A certain semi-Markov dependence structure is considered by Albrecher and Boxma (2004, 2005). Using random walk techniques, Albrecher and Teugels (2006) derive explicit exponential estimates for infinite and finite time ruin probabilities under a dependence structure described by a copula, when the claim sizes are light tailed. Another type of dependence structure that falls under our assumption is considered by Boudreault *et al.* (2006). The present work is not meant to be an exhaustive treatment on all possible dependence structures, but rather an attempt to assess the impact of dependence over independence, via asymptotic tail probability evaluation.

The paper is organized as follows. Section 2 describes in more details the type of dependence assumed, together with a brief review of some well-known results. Asymptotic forms for the tail probability of the discounted aggregate claims as well as for the finite time ruin probabilities are provided in the main section of the paper, Section 3. In Section 4, motivated by a risk management application, we derive asymptotic formulas for several risk measures associated with the discounted aggregate claims. Numerical illustrations are given in the end of this section.

2 Preliminaries

Our goal is to consider a model that relaxes the usual assumption of independence between the length of the i^{th} inter-claim time W_i and the i^{th} claim amount X_i. The underlying dependence structure assumed in this paper is given by the following assumption that holds asymptotically:

Assumption 1 *The bivariate random vectors (W_i, X_i), $i = 1, 2, \ldots$ are mutually independent and identically distributed. Moreover, there exists a function $g(\cdot)$ such that*

$$\Pr\left(X_1 > x \mid_{W_1=w}\right) \sim \Pr(X_1 > x) g(w), \text{ as } x \to \infty,$$

holds for all $w \in (0, t)$.

The motivation behind our assumption is given by the fact that under its premises we can study in a unified way a wide class of dependence structures defined in terms of a copula. A two-dimensional copula is a bivariate distribution function defined on $[0, 1]^2$ with uniformly distributed marginals. Due to Sklar's Theorem (see Sklar, 1959), if G is a joint df with continuous marginals G_1 and G_2 respectively, there exists a unique copula, C, given by

$$C(G_1(x), G_2(y)) = G(x, y), \ (x, y) \in Dom(G).$$

A more formal definition and examples of copulas are given in Nelsen (1999). Some examples of copulas that satisfy Assumption 1 are given below, for additional ones see Asimit and Jones (2007).

Example 1 *Ali-Mikhail-Haq*

$$C(u,v)=\frac{uv}{1-\theta(1-u)(1-v)},\ \theta\in[-1,1],$$

with $g(w)=1+\theta(1-2e^{-\lambda w})$.

Example 2 *Clayton*

$$C(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta},\ \theta\in(0,\infty),$$

with $g(w)=(1+\theta)(1-e^{-\lambda w})^{\theta}$.

Example 3 *Fréchet*

$$C(u,v)=\theta_1\max\{u+v-1,0\}+(1-\theta_1-\theta_2)uv+\theta_2\min\{u,v\},\ 0\le\theta_1,\theta_2,\ \theta_1+\theta_2\le 1,$$

with $g(w)=1-\theta_1-\theta_2$.

Example 4 *Gumbel-Barnett*

$$C(u,v)=uv\exp\{-\theta\log u\log v\},\ 0<\theta\le 1,$$

with $g(w)=1-\theta-\theta\log(1-e^{-\lambda w})$.

Example 5 *Marshall-Olkin*

$$C(u,v)=\min\{u^{1-\theta_1}v,uv^{1-\theta_2}\},\ 0<\theta_1,\theta_2<1,$$

with $g(w)=1-\theta_1$.

In the reminder of the section we give a brief review of some well-known definitions and results that will be used in the derivation of our main results.

There are many characterizations of heavy-tailed distributions, but one of the most popular families is the class $\mathcal{S}$ of sub-exponential distributions. By definition, a non-negative random variable X with df F belongs to $\mathcal{S}$ and we write $F \in \mathcal{S}$ if

$$\lim_{x\to\infty} \frac{\Pr(X_1 + X_2 > x)}{\Pr(X > x)} = 2, \tag{2}$$

where X_1 and X_2 are independent copies of X. A well-known subclass of $\mathcal{S}$ is the set of regularly varying df's. By definition, a random variable X with df F belongs to $RV_{-\alpha}$ and we write $F \in RV_{-\alpha}$ if

$$\lim_{x\to\infty} \frac{\Pr(X > xy)}{\Pr(X > x)} = y^{-\alpha},\ \alpha > 0. \tag{3}$$

For more details on heavy-tailed distributions, we refer the reader to Bingham *et al.* (1987) and Embrechts *et al.* (1997).

For a homogeneous Poisson process, recall that the probability density function of the inter-claim times, $\boldsymbol{W} = (W_1, \ldots, W_n)$, conditioned on the number of events by time t is

$$f_{\boldsymbol{W}|N(t)=n}(\boldsymbol{w}) = \frac{n!}{t^n}, \tag{4}$$

on $D_n := \{\boldsymbol{w} = (w_1, \ldots, w_n) \in (0,t)^n : \sum_{i=1}^n w_i < t\}$ (see, for example, Embrechts *et al.*, 1997, p. 187).

3 Main results

This section provides the main results of the paper for both the free interest and constant force of interest risk models. First the asymptotic tail probabilities of the discounted aggregate claims are obtained. Using these results, in the following subsection we derive limiting results for the finite time ruin probabilities.

3.1 Discounted aggregate claims

The following result is a slight generalization of a classical property of subexponential distributions (see for example Lemma 1.3.5 in Embrechts *et al.* 1997) and it will help us to prove our main results .

Lemma 1 *Let $X, Y_1, Y_2, \ldots$ be a sequence of independent non-negative rv's, with G the df of X, $G \in \mathcal{S}$. In addition, it is assumed that there exists a constant M such that $\Pr(Y_i > x) \le M \Pr(X > x)$ holds for all $x > 0$ and any $i = 1, 2, \ldots$. Then, for any $\epsilon > 0$ there exists $A < \infty$ such that*

$$s_n := \sup_{x>0} \frac{\Pr\left(\sum_{i=1}^{n} Y_i > x\right)}{\Pr(X > x)} \le A(1+\epsilon)^n,$$

holds for any integer n.

Proof. Let $Z_1, Z_2, \ldots$ be a sequence of non-negative i.i.d. rv's such that $P(Z_1 \le x) = \max\{0, 1 - M\bar{G}(x)\}$, with $\bar{G}(x) = 1 - G(x)$. Clearly, for any integer i we have

$$\Pr(Y_i > x) \le \Pr(Z_1 > x) \le M\bar{G}(x), \ x > 0, \tag{5}$$

which yields

$$\Pr\left(\sum_{i=1}^{n} Y_i > x\right) \le \Pr\left(\sum_{i=1}^{n} Z_i > x\right), \ x > 0. \tag{6}$$

Due to Theorem 1 from Cline (1986), the df of Z_1 is subexponential, since $G \in \mathcal{S}$ and $\Pr(Z_1 > x) \sim M\bar{G}(x)$. The latter together with equations (5), (6) and Lemma 1.3.5 from Embrechts *et al.* (1997) complete the proof.■

We are now able to state the main result for the free interest model.

Theorem 1 *Consider the free interest ($\delta = 0$) compound Poisson model such that $F \in \mathcal{S}$. If Assumption 1 is satisfied for any $t \in (0, T)$, then*

$$\Pr(S_0(T) > x) \sim K_0 \Pr(X_1 > x), \ x \to \infty, \tag{7}$$

where

$$K_0 = \lambda \int_0^T g(w) e^{-\lambda w} \big(1 + \lambda(T - w)\big) \, dw.$$

Proof. By conditioning on the number of claims and inter-claim times by time T and using (4), we obtain

$$\begin{aligned}
&\Pr\left(S_0(T) > x\right) \\
&= \sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \int_{D_n} \Pr\left(\sum_{i=1}^{N(T)} X_i > x \,|_{\boldsymbol{W}=\boldsymbol{w},\, N(T)=n}\right) \Pr\left(\boldsymbol{W} = \boldsymbol{w} \,|_{N(T)=n}\right)\, d\boldsymbol{w} \\
&= \sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \int_{D_n} \Pr\left(\sum_{i=1}^{n} X_i > x \,|_{\boldsymbol{W}=\boldsymbol{w}}\right) \frac{n!}{T^n}\, d\boldsymbol{w}. \qquad (8)
\end{aligned}$$

Using the fact that $Y_i \overset{d}{=} X_i \,|_{W_i=w_i}$ are independent and $F \in \mathcal{S}$, we can apply Lemma 1 since for any integer i the following holds

$$\Pr\left(Y_i > x\right) \leq \frac{\Pr(X_1 > x)}{\inf\limits_{0 \leq t \leq T} \Pr(W_i = w_i)} = \frac{e^{\lambda T}}{\lambda} \Pr(X_1 > x),\ x > 0.$$

This implies that for any $\epsilon > 0$ there exists $A > 0$ such that

$$\sup_{x>0} \frac{\Pr\left(\sum_{i=1}^{n} X_i > x \,|_{\boldsymbol{W}=\boldsymbol{w}}\right)}{\Pr(X_1 > x)} \leq A(1+\epsilon)^n,$$

holds for any positive integer n. Now, the latter and the fact that

$$\sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \int_{D_n} A(1+\epsilon)^n \frac{n!}{T^n}\, d\boldsymbol{w} = \sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} A(1+\epsilon)^n < \infty, \qquad (9)$$

allow us to apply the Dominated Convergence Theorem in (8), which together with Assumption 1, Theorem 1 from Cline (1986) and relation (4) yield

$$\begin{aligned}
\lim_{x\to\infty} \frac{\Pr(S_0(T) > x)}{\Pr(X_1 > x)} &= \sum_{n=1}^{\infty} e^{-\lambda T} \lambda^n \int_{D_n} \sum_{i=1}^{n} g(w_i)\, d\boldsymbol{w} \\
&= \sum_{n=1}^{\infty} e^{-\lambda T} \lambda^n n \int_0^T g(w) \frac{(T-w)^{n-1}}{(n-1)!}\, dw. \qquad (10)
\end{aligned}$$

Due to Pratt's Lemma (see Pratt, 1960), one can interchange the summation and integral in (10), which completes the proof. ■

The model with positive force of interest requires the claim amount distribution to be regularly varying. The result is stated in the following theorem.

Theorem 2 *Consider the compound Poisson model with constant force of interest (*$\delta > 0$*) and* $F \in RV_{-\alpha}$*. If Assumption 1 is satisfied for any* $t \in (0, T)$*, then*

$$\Pr(S_\delta(T) > x)) \sim K_\delta \Pr(X_1 > x),\ x \to \infty, \qquad (11)$$

where

$$K_\delta = \sum_{n=1}^{\infty} e^{-\lambda T} \lambda^n \int_{D_n} \sum_{i=1}^{n} g(w_i) e^{-\alpha\delta \sum_{j=1}^{i} w_j} \, d\boldsymbol{w}.$$

Proof. By using the same reasoning as in the proof of Theorem 1 we obtain

$$\Pr(S_\delta(T) > x) = \sum_{n=1}^{\infty} e^{-\lambda T} \lambda^n \int_{D_n} \Pr\left(\sum_{i=1}^{n} X_i e^{-\delta \sum_{j=1}^{i} W_i} > x \,|_{\boldsymbol{W}=\boldsymbol{w}} \right) d\boldsymbol{w}. \quad (12)$$

Let $Y_i \overset{d}{=} X_i e^{-\delta \sum_{j=1}^{i} W_i} \,|_{\boldsymbol{W}=\boldsymbol{w}}$. The random variables Y_i are independent and $\Pr(Y_i > x) \leq \Pr(X_1 > x) e^{\lambda T}/\lambda$ holds for any integer i and all $x > 0$. This allows us to apply Lemma 1, which yields that for any $\varepsilon > 0$ there exists a positive constant A such that

$$\sup_{x>0} \frac{\Pr\left(\sum_{i=1}^{n} Y_i > x \right)}{\Pr(X_1 > x)} \leq A(1+\epsilon)^n.$$

The latter and (9) give the sufficient conditions for applying the Dominated Convergence Theorem in (12), which completes the proof .■

3.2 Ruin probability

The asymptotic behavior of the finite time ruin probability under the classical Poisson model with independence between the inter-claim times and the claim amounts is analyzed in detail by Tang (2005b). This work is extended within the class of renewal risk models in Tang (2005a, 2007) and Wang (2008). Asymptotic formulas for finite/infinite time ruin probabilities with a certain dependence structure between claim sizes and inter-claim times are also obtained by Albrecher and Teugels (2006) for the class of light-tailed claim amounts. In this subsection we extend some of the previous results obtained under the classical Poisson risk model, assuming heavy-tailed claim size distributions under the dependence structure defined in Assumption 1.

Consider the classical Poisson insurance risk model with constant force of interest, for which the evolution of the surplus $U_\delta(t)$ is given by

$$U_\delta(t) = xe^{\delta t} + C_\delta(t) - e^{\delta t} S_\delta(t), \quad (13)$$

where x is the initial capital and $C_\delta(t) = \int_0^t e^{\delta(t-s)} \, dC(s)$ represents the accumulated amount of premiums at time t. We let $\{C(s)\}_{s\geq 0}$ with $C(0) = 0$ be a non-decreasing and right continuous stochastic process, denoting the total amount of premiums accumulated to time s. Furthermore, we define the time to ruin as

$$\tau(x) = inf\{t > 0 : U_\delta(t) < 0 || U_\delta(0) = x\} \tag{14}$$

and the associated finite time ruin probability by

$$\psi_\delta(x;T) = \Pr(\tau(x) \leq T). \tag{15}$$

Clearly,

$$\Pr\left(S_\delta(T) > x + e^{-\delta T} C_\delta(T)\right) \leq \psi_\delta(x;T) \leq \Pr\left(S_\delta(T) > x\right), \tag{16}$$

holds for $\delta \geq 0$. Since $C_\delta(T) < \infty$, we can use the long-tailed property of subexponential distributions (see for example, Lemma 1.3.5(a) in Embrechts *et al.* 1997) in (16), which leads to the following corollary of Theorems 1 and 2.

Corollary 1 *Consider the compound Poisson model with constant interest rate such that Assumption 1 is satisfied for any $t \in (0, T)$. In addition if $C_\delta(T) < \infty$, then*

a) if $\delta = 0$ and $F \in \mathcal{S}$ then $\psi_0(x;T) \sim K_0 \Pr(X_1 > x),\ x \to \infty$,

b) if $\delta > 0$ and $F \in RV_{-\alpha}$ then $\psi_\delta(x;T) \sim K_\delta \Pr(X_1 > x),\ x \to \infty$.

Remarks:

1. Recall that Assumption 1 is satisfied in the case of independence with $g \equiv 1$. This particular case yields that $K_0 = \lambda T$ and $K_\delta = \lambda(1 - e^{-\alpha\delta T})/\alpha\delta$, which recovers the results of Tang (2005b).

2. Asimit and Jones (2007) established that

$$\Pr\left(\max_{i=1,\ldots,N(T)} X_i > x\right) \sim K_0 \Pr(X_1 > x),\ x \to \infty.$$

This suggests that the main contribution of the asymptotic finite time ruin probability is given by the tail probability of the maximum claim of the process. Similarly, if $F \in RV_{-\alpha}$, the maximum among the discounted claim amounts at time T plays the same role

$$\Pr\left(\max_{i=1,\ldots,N(T)} X_i e^{-\delta\sigma_i} > x\right) \sim K_\delta \Pr(X_1 > x),\ x \to \infty.$$

4 Risk management application

Capital adequacy standards for banking organizations are summarized in the June 2004 Basel II Agreement, which creates an international standard for banking regulations. A comprehensive reference on this topic is McNeil *et al.* (2005). Similar to Basel II, the Solvency II project defines the regulatory requirements for insurance companies that operate in the European Union. A recent paper of Ronkainen *et al.* (2007) describes in more details the objective and current developments of the Solvency II framework. One of the main questions addressed in the first pillar is the determination of the capital requirements for an insurance company. Two solvency control levels of the capital are proposed. The higher level is a risk-based capital requirement, known as the *Solvency Capital Requirement* (SCR). The lower level is the *Minimum Capital Requirement* (MCR), which plays the role of a lower bound for the SCR. Several risk measures may be used in practice by regulators to determine the SCR. For example, the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) proposes a prudential standard of a 99.5% survival probability for a one-year horizon.

Motivated by the above mentioned problem, we further provide asymptotic approximations for some risk measures associated with the discounted aggregate claims. The paper is concluded with a simulation study that illustrates some of our results.

4.1 Risk Measures

One of the most popular risk measures used in practice is *Value-at-Risk* (VaR), see Jorion (2001). The VaR at confidence level p for a generic loss variable L represents the p-quantile, defined as

$$VaR_p(L) := \inf\{x \in \Re : \Pr(L > x) \leq 1 - p\}.$$

As described earlier, high quantile estimates are useful in order to get the SCR level, which is a Value-at-Risk based capital requirement. A simple approximation for the SCR can be obtained from Theorems 1 and 2 as follows

$$VaR_{1-p}(S_\delta(T)) \sim VaR_{1-p/K_\delta}(X_1), \text{ for } p \downarrow 0, \tag{17}$$

provided that the distribution function of $S_\delta(T)$ is continuous close enough in the right tail.

As we observed before, VaR and ruin probability give the same asymptotic results. However, generally speaking, both risk measures fail to incorporate the severity of the extreme events. Alternative risk measures to VaR have been proposed in the literature, from which *Expected shortfall* (ES) is the most discussed. For a discussion regarding the advantage of this risk measure over VaR we refer the reader to Tasche (2002). The expected shortfall of a loss variable L with continuous df, at confidence level p, represents the average loss in the worst 100 $p\%$ cases, and is given by

$$ES_p(L) = E[L \mid L > VaR_p(L)].$$

Similar definitions of this risk measure and equivalence among them can be found in Acerbi and Tasche (2002).

In order to find an asymptotic result for the expected shortfall of the discounted aggregate loss, more assumptions regarding the tail behaviour for a subexponential distribution are necessary to be made. This can be achieved through Extreme Value Theory, for which some background is now given.

A df F is in the *maximum domain of attraction* of a non-degenerate df G, written as $F \in \mathrm{MDA}(G)$, if

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x),$$

where $a_n > 0$ and b_n are real numbers. Due to the Fisher-Tippett Theorem (see Gnedenko, 1943), G belongs to the type of one of the following three df's:

$$\begin{array}{lcllll}
\Phi_\alpha(x) & = & \exp(-x^{-\alpha}), & x > 0 \ \ (\alpha > 0) & \Rightarrow & G \text{ is of Fréchet type} \\
\Psi_\alpha(x) & = & \exp(-(-x)^{\alpha}), & x \leq 0 \ \ (\alpha > 0) & \Rightarrow & G \text{ is of Weibull type} \\
\Lambda(x) & = & 1 - \exp(-e^{-x}), & x \in \Re & \Rightarrow & G \text{ is of Gumbel type}
\end{array}$$

It is well known that a subexponential df can be only on the $MDA(\Phi_\alpha)$ or $MDA(\Lambda)$. If $F \in MDA(\Phi_\alpha)$ then F is $RV_{-\alpha}$ (see Resnick, 1987). Moreover, if $F \in MDA(\Lambda)$ then there exists a positive, measurable function $a(\cdot)$ such that

$$\lim_{x\to\infty} \frac{\bar{F}(x + ta(x))}{\bar{F}(x)} = e^{-t}, \tag{18}$$

for any real t (see, for example, Embrechts *et al.*, 1997). A characterization of subexponential df's that are in the $MDA(\Lambda)$ is provided by Goldie and Resnick (1988).

The connection between ES and VaR of the discounted aggregate loss, for high confidence levels, for the case that $F \in MDA(\Phi_\alpha)$ is given by

$$ES_p(S_\delta(T)) \sim \frac{\alpha}{\alpha - 1} VaR_p(S_\delta(T)),\ p \uparrow 1,$$

provided that $\alpha > 1$. The proof follows in a similar manner to the one of Theorem 3.1 of Alink *et al.* (2005). Due to (7) and (11), $F \in \mathcal{S} \bigcap MDA(\Lambda)$ implies that the df of $S_0(T)$ belongs to $MDA(\Lambda)$ and furthermore

$$ES_p(S_0(T)) \sim VaR_p(S_0(T)),\ p \uparrow 1.$$

Both results are also noted by McNeil *et al.* (see page 283, 2005). The latter is a consequence of Proposition 1, which is now given.

Proposition 1 *Let L be a random variable with df F such that $\sup_x \{F(x) < 1\} = \infty$. If $F \in MDA(\Lambda)$ then $E(L \mid L > x) \sim x$ as $x \to \infty$.*

Proof. We first note that $\bar{F}(x) = o(1/x)$. This is stated in Resnick (1987, p.52), but for a formal proof one may look at the proof of Lemma 1.8 from the same reference. The latter together with integration by parts yields

$$E(L \mid L > x) = x + \int_x^\infty \frac{\bar{F}(t)}{\bar{F}(x)}\, dt. \tag{19}$$

Let $a(\cdot)$ be the auxiliary function of F as defined in (18). The change of variable $t = x + a(x)\xi$ in (19) gives

$$\int_x^\infty \frac{\bar{F}(t)}{\bar{F}(x)}\, dt = a(x) \int_0^\infty \frac{\bar{F}(x + a(x)\xi)}{\bar{F}(x)}\, d\xi \sim a(x), \tag{20}$$

where the last step is due to the Dominated Convergence Theorem. In order to see this, we may use (18), and the fact that $e^{-\xi} < 1/\xi(\xi+1)$, for $\xi > 2$. The final result is obtained by combining (19), (20), and the fact that $a(x) = o(1/x)$ (see Resnick, 1987, p.40) .■

4.2 Numerical Results

The purpose of this subsection is two-fold. First, using a simulation study we examine the accuracy of the asymptotic results obtained in Theorem 1 for the prudential standard level proposed by Solvency II. Second, we determine the SCR level which corresponds to the $\mathrm{VaR}_{99.5\%}$ and interpret how the results vary when one assumes dependence over independence.

It is assumed the marginal distribution of the claim sizes to be Weibull, given by

$$F_{X_1}(x) = 1 - \exp(-x^{1/\tau}),\ x \geq 0, \tau > 1.$$

As it is well-known, this distribution is subexponential with a non-regularly varying tail. For computational simplicity (in terms of execution times), we choose the dependence structure between claim amounts and inter-claim times to be given by the Ali-Mikhail-Haq copula (see Example 1), with values of θ equal to -0.9, -0.5, 0, 0.5 and 0.9. Recall that $\theta = 0$ corresponds to the case where the claim amounts and the inter-claim times are independent. Simple algebraic computations give the asymptotic constant of Theorem 1 to be

$$K_0(\theta) = \lambda T + \frac{\theta}{2}(e^{-2\lambda T} - 1).$$

Each analysis consists of $10,000,000$ simulations of the risk process with Poisson rate arrival $\lambda = 1$ and time horizon $T = 50$. This choice of parameters is arbitrary. Simulation studies for other parameter settings have been performed and the conclusion remains the same as the one obtained for this particular choice. For each simulation study, the values of $\Pr(S_0(T) > x)$ are calculated empirically for a threshold x such that $\Pr(X_1 > x)$ is $5 \times 10^{-4}, 10^{-4}$ and 5×10^{-5} respectively. The choice of these tail probabilities for the marginal claim size distribution is made in order to ensure a survival probability for $S_0(T)$ around 99.5%, which is the proposed value of Solvency II.

Several numerical results are illustrated for different values of the parameter τ in Table 1. The results are very good for values of $\tau \geq 6$. As the value of τ increases, the Weibull distribution is heavier tailed and the asymptotic formula obtained in Theorem 1 performs better. The opposite effect is encountered as the value of τ decreases.

Table 1: Estimated probability ratios, $\Pr(S_0(T) > x)/K_0 \Pr(X_1 > x)$.

$\tau = 6$					
x	$\theta = -0.9$	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$	$\theta = 0.9$
$\bar{F}_{X_1}^{-1}(5 \times 10^{-4})$	1.1604	1.1650	1.1677	1.1741	1.1801
$\bar{F}_{X_1}^{-1}(10^{-4})$	1.0791	1.0846	1.0917	1.0904	1.0933
$\bar{F}_{X_1}^{-1}(5 \times 10^{-5})$	1.0595	1.0640	1.0749	1.0713	1.0759

$\tau = 8$					
x	$\theta = -0.9$	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$	$\theta = 0.9$
$\bar{F}_{X_1}^{-1}(5 \times 10^{-4})$	1.0472	1.0502	1.0528	1.0587	1.0642
$\bar{F}_{X_1}^{-1}(10^{-4})$	1.0209	1.0241	1.0318	1.0306	1.0347
$\bar{F}_{X_1}^{-1}(5 \times 10^{-5})$	1.0126	1.0198	1.0303	1.0285	1.0325

$\tau = 10$					
x	$\theta = -0.9$	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$	$\theta = 0.9$
$\bar{F}_{X_1}^{-1}(5 \times 10^{-4})$	1.0139	1.0174	1.0204	1.0257	1.0305
$\bar{F}_{X_1}^{-1}(10^{-4})$	1.0052	1.0094	1.0159	1.0148	1.0190
$\bar{F}_{X_1}^{-1}(5 \times 10^{-5})$	1.0016	1.0090	1.0199	1.0176	1.0211

Table 2: $VaR_{99.5\%}[S_0(50)]$ for Weibull claim amounts and Ali-Mikhail-Haq copula

	$\rho = -0.2$	$\rho = -0.1$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$
$\tau = 6$	613228	611741	610456	608398	606912
$\tau = 8$	5.2099×10^7	5.19306×10^7	5.17852×10^7	5.15525×10^7	5.13848×10^7
$\tau = 10$	4.42626×10^9	4.40838×10^9	4.39296×10^9	4.3683×10^9	4.35054×10^9

In Table 2 we present the desired SCR levels for which the VaR is exactly 99.5% for different Spearman's correlation coefficients, ρ. Note that for this particular dependence structure, the correlation coefficient varies in $[-0.27, 0.48]$ (see Nelsen 1999, page 139). From Table 2 one can observe that under this scenario the dependence structure does not heavily influence the SCR level.

Remark: In our numerical attempt, we also consider two other examples that assume Lognormal and Pareto distributed claim sizes. Under the Lognormal case, the results are similar to those obtained in the Weibull case. Under the Pareto case, with a df given by $F(x) = 1 - (1 + x)^{-\alpha}$, the asymptotic results were accurate for levels of survival probability higher than 99.5%, whenever $\alpha > 2$. We remind the reader that Pareto is a subexponential distribution with regularly varying tail. Simulation results show good results for a level of 99.5% and Pareto distributions with infinite second moment. For these reasons, we omit the presentation of the numerical results obtained in both cases.

As can be seen from Table 2, the dependence does not have a huge impact on the SCR level, when the underling dependence structure is given by the Ali-Mikhail-Haq copula. We now illustrate that the previous statement is not generally true. For this purpose, the Fréchet copula (see Example 3) is considered, with the Spearman's correlation coefficient given by $\rho = \theta_2 - \theta_1$. The values of the SCR level significantly

Table 3: $VaR_{99.5\%}[S_0(50)]$ for Weibull claim amounts and Fréchet copula

(θ_1, θ_2)	(0.5, 0)	(0.45, 0.15)	(0.35, 0.35)	(0.25, 0.55)	(0.2, 0.7)
ρ	-0.5	-0.3	0	0.3	0.5
$\tau = 6$	381750	325537	263398	192837	108648
$\tau = 8$	2.76931×10^7	2.23941×10^7	1.68843×10^7	1.11409×10^7	5.18436×10^6
$\tau = 10$	2.00893×10^9	1.54052×10^9	1.08232×10^9	6.43653×10^8	2.47383×10^8

change as the copula parameters change (see Table 3). This may be explained by the fact that the dependence structure introduced by the Fréchet copula is more flexible, which allows both extremal cases of positive and negative dependence to be attained. The independence scenario is obtained when $\theta_1 = \theta_2 = 0$, which corresponds to $\rho = 0$

in Table 2. Finally, note that upon permuting the values of (θ_1, θ_2), the values of the SCR level remain the same but in reverse order. For example, the SCR level for $\tau = 6$ and $(\theta_1, \theta_2) = (0.7, 0.2)$ corresponds to the one obtained in Table 3, when $\tau = 6$ and $(\theta_1, \theta_2) = (0.2, 0.7)$. Hence, at the same level of Spearman's correlation coefficient, the choice of the SCR level is highly influenced by the choice of the underlying dependence structure. This emphasizes that the correlation coefficient does not provide a complete description of the dependence between random variables.

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