ASYMPTOTIC TAIL PROBABILITIES FOR LARGE CLAIMS REINSURANCE OF A PORTFOLIO OF DEPENDENT RISKS

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Abstract

We consider a dependent portfolio of insurance contracts. Asymptotic tail probabilities of the ECOMOR and LCR reinsurance amounts are obtained under certain assumptions about the dependence structure.

Keywords: Archimedean copula, Dependence, ECOMOR and LCR reinsurance, Tail probability

1 Introduction

Insurance companies often use reinsurance as a mechanism for sharing risk, particularly when there is the possibility of catastrophic losses. Two appealing reinsurances are ECOMOR (excédent du coût moyen relatif) and LCR (largest claims reinsurance). Under ECOMOR, the reinsurer pays the sum of the exceedances of the l largest claims over the $l+1$st largest claim. Under LCR, the reinsurer pays the sum of the l largest claims. ECOMOR and LCR treaties were proposed by Thépaut (1950) and Ammeter (1964), respectively.

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We consider a portfolio of n similar insurance contracts. The associated loss random variables $X_i, i = 1, \ldots, n$ are assumed to be dependent and identically distributed with common distribution function $F = 1 - \bar{F}$ and dependence structure given by a suitable copula. Let $X_{1,n} \geq \ldots \geq X_{n,n}$ be the corresponding upper order statistics. Then the reinsurance amounts under ECOMOR and LCR are given by

$$E_l = \sum_{i=1}^{l} (X_{i,n} - X_{l+1,n}),$$

and

$$L_l = \sum_{i=1}^{l} X_{i,n}.$$

The purpose of this paper is to establish the asymptotic tail probabilities of the reinsurance amount under ECOMOR and LCR for a portfolio of dependent insurance contracts. This may be quite useful for risk management purposes, as it allows one to determine high quantiles of the reinsurance amount and therefore enables one to obtain capital amounts that will be adequate with high probability. This can also be done by performing a simulation study. However, to estimate high quantiles, a very large number of simulations are required, and since multivariate outcomes must be generated, the computations may be very time consuming.

2 Preliminaries

Let Y_i, $i = 1, 2, \ldots$ be a sequence of independent random variables with common distribution F, and let M_n be the maximum of $Y_1, \ldots, Y_n$. If there exist constants a_n, b_n and a random variable Z with nondegenerate df G such that $a_n M_n + b_n$ converges weakly to Z, then F is in the *maximum domain of attraction* of G and we write $F \in \mathrm{MDA}(G)$. Moreover, by the Fisher-Tippett theorem (see, for example, Embrechts *et al.*, 1997), G belongs to the type of the distribution

$$H_\xi(x) = \begin{cases} \exp\left\{-(1+\xi x)^{-1/\xi}\right\}, & 1+\xi x > 0, \quad \xi \neq 0 \\ \exp\{-e^{-x}\}, & -\infty < x < \infty, \quad \xi = 0 \end{cases}.$$

H_ξ is known as the *generalized extreme value distribution.* For $\alpha > 0$, $\Phi_\alpha(x) := H_{1/\alpha}(\alpha(x-1))$ is the standard Fréchet distribution, $\Psi_\alpha(x) := H_{-1/\alpha}(\alpha(x+1))$ is the standard Weibull distribution, and $\Lambda(x) := H_0(x)$ is the standard Gumbel distribution.

The dependence structure associated with the distribution of a random vector can be characterized in terms of a *copula.* An n-dimensional copula is a multivariate df defined on $[0,1]^n$ with uniformly distributed marginals. Due to Sklar's Theorem (see Sklar, 1959), if $X_1, \ldots, X_n$ has a joint distribution function with continuous marginals, then there exists a unique copula, C, such that

$$\Pr(X_1 \le x_1, \ldots, X_n \le x_n) = C\big(\Pr(X_1 \le x_1), \ldots, \Pr(X_n \le x_n)\big).$$

Similarly, the *survival copula*, $\widehat{C}$, is defined as the copula relative to the joint survival function and satisfies

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \widehat{C}\big(\Pr(X_1 > x_1), \ldots, \Pr(X_n > x_n)\big).$$

A well-known class of copulas is the Archimedean class. By definition, an *Archimedean copula* C is given by

$$C(u_1, \ldots, u_n) = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi(u_i)\right),$$

where $\varphi : [0,1] \mapsto [0,\infty)$ is its generator. Some regularity conditions are necessary to ensure that C is a valid copula (see Kimberling, 1974 and Nelsen, 1999, ch. 4).

An important concept that is crucial to establishing the main results of this paper is vague convergence. Let $\{\mu_n, n \ge 1\}$ be a sequence of measures on a locally compact space $\mathbb{E}$ with countable base. Then μ_n *converges vaguely* to some measure μ (written $\mu_n \xrightarrow{v} \mu$) if for all bounded continuous functions f with compact support we have

$$\lim_{n\to\infty} \int_{\mathbb{E}} f \, d\mu_n = \int_{\mathbb{E}} f \, d\mu.$$

A thorough background on vague convergence is given by Kallenberg (1983) and Resnick (1987).

3 Main Results

Throughout this paper it is assumed that the common df $F = 1 - \bar{F}$ has positive support and infinite right endpoint. For ease of exposition, we first assume that the survival copula, which describes the dependence among portfolio risks, is a member of the Archimedean class. This setup is used by Wüthrich (2003) and Alink *et al.* (2004 and 2005) in order to characterize the asymptotic tail behavior for a sum of dependent random variables. A similar problem is discussed by Albrecher *et al.* (2006), Barbe *et al.* (2006) and Kortschak and Albrecher (2007), when a more general dependence structure is assumed. Since the ECOMOR and LCR reinsurances are linear combinations of the order statistics, studying the asymptotic tail probability for the losses associated with these reinsurance treaties is closely related to the aforementioned problem.

We make the additional assumption that the generator φ of the survival copula is regularly varying at 0 with index $-\alpha$ ($\varphi \in RV_{-\alpha}^0$). That is,

$$\lim_{t \uparrow 0} \frac{\varphi(tx)}{\varphi(t)} = x^{-\alpha},$$

for any positive x. For more details on regular variation, we refer the reader to Bingham *et al.* (1987).

The Clayton copula is an example of an Archimedean copula with generator, $\varphi(u) = u^{-\alpha} - 1$, which satisfies the property $\varphi \in RV_{-\alpha}^0$. This copula has the form

$$C(u_1, \ldots, u_n) = \left(1 - n + \sum_{i=1}^{n} u_i^{-\alpha}\right)^{-1/\alpha},$$

where $\alpha > 0$.

Our assumption that the individual loss df F has infinite right endpoint implies that only $F \in MDA(\Phi_\beta)$ or $F \in MDA(\Lambda)$ may hold. We consider these two cases in turn.

3.1 Results for F in MDA of Fréchet

If $F \in MDA(\Phi_\beta)$ and $\varphi \in RV_{-\alpha}^0$, then for any positive $x_1, \ldots, x_l$ with $1 \leq l \leq n$,

$$\lim_{t \to \infty} \frac{\Pr(X_1 > tx_1, \ldots, X_l > tx_l)}{\bar{F}(t)} = \left(\sum_{i=1}^{l} x_i^{\alpha\beta}\right)^{-1/\alpha}, \tag{1}$$

provided that $0 < \alpha < \infty$ (see Alink *et al.* 2004).

Now, as a result of our assumptions, the random variables $X_1, \ldots, X_n$ are exchangeable. Therefore,

$$\begin{aligned} &\Pr(X_{1,n} > tx_1, \ldots,\ X_{l,n} > tx_l) \qquad (2)\\ &= \sum_{(k_1,\ldots,k_l)\in A_l} \frac{n!}{k_1!\cdots k_l!\,(n-k_1-\cdots-k_l)!}\ \Pr\Big(\{X_1,\ldots,X_{k_1} > tx_1\},\\ &\qquad \{tx_2 < X_{k_1+1},\ldots,X_{k_1+k_2} \le tx_1\},\cdots,\{X_{k_1+\ldots+k_l+1},\ldots,X_n \le tx_l\}\Big), \end{aligned}$$

for any $x_1 > \ldots > x_l$, where $A_l = \{(k_1, \ldots, k_l) : i \le k_1 + \ldots + k_i \le n, i = 1, \ldots, l\}$. Each term on the right-hand side of (2) can be expressed as a linear combination of joint survival probabilities. This fact combined with (1) allows us to conclude that there exists a positive function f_l such that

$$\Pr(X_{1,n} > tx_1, \ldots, X_{l,n} > tx_l) \sim \bar{F}(t) f_l(x_1, \ldots x_l),\ t \to \infty. \qquad (3)$$

Under more general assumptions for which the exchangeability property does not hold, a similar but even more cumbersome relationship to that in (2) can be obtained.

Now, relation (3) implies that

$$\frac{\Pr\left((X_{1,n}/t, \ldots, X_{l,n}/t) \in \cdot\right)}{\Pr(X_1 > t)} \xrightarrow{v} \mu_l(\cdot),$$

holds on $[0,\infty]^l \setminus \{\mathbf{0}\}$ where the measure μ_l is given by

$$\mu_l\big((x_1,\infty] \times \cdots \times (x_l,\infty]\big) := f_l(x_1, \ldots, x_l). \qquad (4)$$

We now have the essential development for the main results of this subsection, which are stated in the following theorem.

Theorem 1 *Let $(X_1, \ldots, X_n)$ be a positive random vector with an Archimedean survival copula for which the generator satisfies $\varphi \in RV^0_{-\alpha}$ with $\alpha \in (0,\infty)$. In addition, the marginals are identically distributed with df $F \in MDA(\Phi_\beta)$. For $l = 1, \ldots, n-1$, the asymptotic tail probability for E_l, the reinsurance amount under an ECOMOR treaty, is given by*

$$\Pr(E_l > t) \sim C_{EF}(l,\alpha,\beta)\ \bar{F}(t) \text{ as } t \to \infty,$$

where

$$C_{EF}(l,\alpha,\beta) = \mu_{l+1}\left(\boldsymbol{x} : \sum_{i=1}^{l} x_i - l x_{l+1} \geq 1, x_1 \geq \cdots \geq x_{l+1} \geq 0\right),$$

with μ_l *defined by (4).*

For $l = 1,\ldots,n$*, the asymptotic tail probability for* L_l*, the reinsurance amount under an LCR treaty, is given by*

$$\Pr(L_l > t) \sim C_{LF}(l,\alpha,\beta)\,\bar{F}(t) \text{ as } t \to \infty,$$

where

$$C_{LF}(l,\alpha,\beta) = \mu_l\left(\boldsymbol{x} : \sum_{i=1}^{l} x_i \geq 1, x_1 \geq \cdots \geq x_l \geq 0\right).$$

It should be noted that in order to obtain these results, we used the fact that each measure μ_l contributes zero mass to $\bigcup_{i=1}^{l}\{x_i = \infty\}$.

3.2 Result for F in MDA of Gumbel

As in the Fréchet case, the first step is to establish the joint tail extreme behavior. It is well-known (see, for example, Embrechts *et al.*, 1997) that if $F \in MDA(\Lambda)$, then there exists a positive, measurable function $a(\cdot)$ such that

$$\lim_{t\to\infty} \frac{\bar{F}(t + xa(t))}{\bar{F}(t)} = e^{-x}, \tag{5}$$

for any real x. Once again, we assume that $\varphi \in RV^0_{-\alpha}$, which gives that

$$\lim_{t\to\infty} \frac{\Pr(X_1 > t + x_1 a(t),\ldots,X_l > t + x_l a(t))}{\bar{F}(t)} = \left(\sum_{i=1}^{l} e^{\alpha x_i}\right)^{-1/\alpha}, \tag{6}$$

for any real $x_1,\ldots x_l$ with $1 \leq l \leq n$ (see Alink *et al.* 2004).

In the same manner as the previous subsection, we have

$$\Pr(X_{1,n} > t + x_1 a(t),\ldots,X_{l,n} > t + x_l a(t)) \sim \bar{F}(t)\, g_l(x_1,\ldots x_l), \tag{7}$$

where g_l is a positive function.

Now, relation (7) implies that

$$\frac{\Pr\Big(\big((X_{1,n} - t)/a(t),\ldots,(X_{l,n} - t)/a(t)\big) \in \cdot\Big)}{\Pr(X_1 > t)} \xrightarrow{v} \nu_l(\cdot),$$

holds on $(-\infty, \infty]^l$ where the measure ν_l is given by

$$\nu_l\big((x_1, \infty] \times \cdots \times (x_l, \infty]\big) := g_l(x_1, \ldots, x_l). \tag{8}$$

Now, we are able to give the main result from this subsection, which is only for the LCR reinsurance. This is stated as Theorem 2.

Theorem 2 *Let $(X_1, \ldots, X_n)$ be a positive random vector with an Archimedean survival copula for which the generator satisfies $\varphi \in RV^0_{-\alpha}$ with $\alpha \in (0, \infty)$. In addition, the marginals are identically distributed with df $F \in MDA(\Lambda)$. For $l = 1, \ldots, n$, we have*

$$\Pr(L_l > lt) \sim C_{LF}(l, \alpha, \beta)\ \bar{F}(t) \text{ as } t \to \infty,$$

where

$$C_{LG}(l, \alpha) = \nu_l \left(\boldsymbol{x} : \sum_{i=1}^{l} x_i \geq 0, x_1 \geq \cdots \geq x_l \right),$$

with ν_l defined by (8).

Two more remarks are useful in understanding Theorem 2. First, note that each measure ν_l contributes zero mass to $\bigcup_{i=1}^{l} \{x_i = \infty\}$. Second, ν_l has no mass on regions around $-\infty$. This is obvious for $l = 1$, so we consider the case in which $l > 1$. It is sufficient to check that

$$\lim_{M \to \infty} \nu_l \left(\mathbf{x} : \sum_{i=1}^{l} x_i \geq 0, x_1 \geq \cdots \geq x_{l-1} \geq -M \geq x_l \right) = 0. \tag{9}$$

In doing so, we first mention that the following clearly holds

$$\begin{aligned} \Pr(X_{1,n} > t) &= \binom{n}{1} \Pr(X_1 > t) - \cdots + (-1)^{n+1} \binom{n}{n} \Pr(X_1 > t, \ldots, X_n > t) \\ &\sim \Delta \bar{F}(t), \text{ as } t \to \infty, \end{aligned} \tag{10}$$

where the last step is due to (6) and Δ is a positive constant. Combining (5) and (10), we have

$$\begin{aligned} &\nu_l \left(\mathbf{x} : \sum_{i=1}^{l} x_i \geq 0, x_1 \geq \cdots \geq x_{l-1} \geq -M \geq x_l \right) \\ &\leq \lim_{t \to \infty} \frac{\Pr\left(X_{1,n} > t + a(t) \frac{M}{l-1}\right)}{\bar{F}(t)} = \Delta e^{-M/(l-1)}, \end{aligned}$$

which leads to (9).

3.3 Examples

In this subsection, examples for the limiting constants from Theorems 1 and 2 are given. In order to avoid long computations, a portfolio consisting of $n = 3$ insurance contracts is considered. First, the Fréchet case is explored. From (2), we have

$$\begin{aligned} \Pr(X_{1,3} > tx_1, X_{2,3} > tx_2) &= \Pr(X_1, X_2, X_3 > tx_1) + 3\Pr(X_1, X_2 > tx_1, X_3 \leq tx_2) \\ &\quad + 3\Pr(X_1, X_2 > tx_1, tx_2 < X_3 \leq tx_1) \\ &\quad + 3\Pr(X_1 > tx_1, tx_2 < X_2, X_3 \leq tx_1) \\ &\quad + 6\Pr(X_1 > tx_1, tx_2 < X_2 \leq tx_1, X_3 \leq tx_2), \end{aligned}$$

for any $x_1 > x_2 > 0$. Otherwise,

$$\Pr(X_{1,3} > tx_1, X_{2,3} > tx_2) = \Pr(X_1, X_2, X_3 > tx_2) + 3\Pr(X_1, X_2 > tx_2, X_3 \leq tx_2).$$

Straightforward computations together with (1) yield the following

$$f_2(x_1, x_2) = \begin{cases} (3^{-1/\alpha} - 3 \cdot 2^{-1/\alpha})x_1^{-\beta} + 6(x_1^{\alpha\beta} + x_2^{\alpha\beta})^{-1/\alpha} & \\ \qquad -3(x_1^{\alpha\beta} + 2x_2^{\alpha\beta})^{-1/\alpha}, & 0 < x_2 < x_1 \\ (3 \cdot 2^{-1/\alpha} - 2 \cdot 3^{-1/\alpha})x_2^{-\beta}, & 0 < x_1 \leq x_2 \end{cases} . \tag{11}$$

In a similar manner, if $F \in MDA(\Lambda)$ then (6) yields

$$g_2(x_1, x_2) = \begin{cases} (3^{-1/\alpha} - 3 \cdot 2^{-1/\alpha})e^{-x_1} + 6(e^{\alpha x_1} + e^{\alpha x_2})^{-1/\alpha} & \\ \qquad -3(e^{\alpha x_1} + 2e^{\alpha x_2})^{-1/\alpha}, & 0 < x_2 < x_1 \\ (3 \cdot 2^{-1/\alpha} - 2 \cdot 3^{-1/\alpha})e^{-x_2}, & 0 < x_1 \leq x_2 \end{cases} . \tag{12}$$

The measure $\mu_2\big((x_1, \infty] \times (x_2, \infty]\big) := f_2(x_1, x_2)$, and it follows from Theorem 1 that the respective constants for ECOMOR and LCR are

$$\begin{aligned} &C_{EF}(1, \alpha, \beta) \\ &= \mu_2\big((x_1, x_2) : x_1 - x_2 \geq 1, 0 \leq x_2 \leq x_1\big) \\ &= 6\beta \int_0^\infty t^{\alpha\beta - 1} \left\{ \left[t^{\alpha\beta} + (1+t)^{\alpha\beta}\right]^{-1-1/\alpha} - \left[2t^{\alpha\beta} + (1+t)^{\alpha\beta}\right]^{-1-1/\alpha} \right\} \, dt \end{aligned}$$

and

$$
\begin{aligned}
&C_{LF}(2,\alpha,\beta)\\
&= \mu_2\big((x_1,x_2): x_1+x_2\ge 1, 0\le x_2\le x_1\big)\\
&= \mu_2\big((x_1,x_2): x_1=x_2\ge 1/2\big)+\mu_2\big((x_1,x_2): x_1+x_2\ge 1, 0\le x_2< x_1\big)\\
&= f_2(1/2,1/2)\\
&\quad +6(1+\alpha)\beta^2\int_{1/2}^{1}\int_{1-s}^{s}(st)^{\alpha\beta-1}\left[\big(s^{\alpha\beta}+t^{\alpha\beta}\big)^{-2-1/\alpha}-\big(s^{\alpha\beta}+2t^{\alpha\beta}\big)^{-2-1/\alpha}\right]dt\,ds\\
&\quad +6(1+\alpha)\beta^2\int_{1}^{\infty}\int_{0}^{s}(st)^{\alpha\beta-1}\left[\big(s^{\alpha\beta}+t^{\alpha\beta}\big)^{-2-1/\alpha}-\big(s^{\alpha\beta}+2t^{\alpha\beta}\big)^{-2-1/\alpha}\right]dt\,ds\\
&= 3+3\cdot 2^{-1/\alpha}(2^{\beta}-1)+3^{-1/\alpha}(1-2^{\beta+1})\\
&\quad +6(1+\alpha)\beta^2\int_{1/2}^{1}\int_{1-s}^{s}(st)^{\alpha\beta-1}\left[\big(s^{\alpha\beta}+t^{\alpha\beta}\big)^{-2-1/\alpha}-\big(s^{\alpha\beta}+2t^{\alpha\beta}\big)^{-2-1/\alpha}\right]dt\,ds.
\end{aligned}
$$

The measure $\nu_2\big((x_1,\infty]\times(x_2,\infty]\big):=g_2(x_1,x_2)$ and from Theorem 2 the limiting constant for LCR is

$$
\begin{aligned}
&C_{LG}(2,\alpha)\\
&= \nu_2\big((x_1,x_2): x_1+x_2\ge 0, x_1\ge x_2\big)\\
&= \nu_2\big((x_1,x_2): x_1=x_2\ge 0\big)+\nu_2\big((x_1,x_2): x_1+x_2\ge 0, x_1> x_2\big)\\
&= 3\cdot 2^{-1/\alpha}-2\cdot 3^{-1/\alpha}\\
&\quad +6(1+\alpha)\int_{0}^{\infty}\int_{-s}^{s}e^{\alpha(s+t)}\left[\big(e^{\alpha s}+e^{\alpha t}\big)^{-2-1/\alpha}-\big(e^{\alpha s}+2e^{\alpha t}\big)^{-2-1/\alpha}\right]dt\,ds.
\end{aligned}
$$

Numerical exemplifications of our main results are now considered for the LCR treaty. It is assumed that each marginal is a two-parameter Pareto distribution with df

$$F_{Pareto}(x;\beta,\gamma)=1-\left(1+\gamma\frac{x}{\beta}\right)^{-\beta},\ x\ge 0$$

in order to illustrate Theorem 1 and exponentially distributed for Theorem 2. In both cases, the expected value is set to 10,000, which implies that the Pareto parameters should satisfy $\gamma=\beta/((\beta-1)\times 10,000)$. We performed the calculations for $\beta=2,3,4,5$. For both the Pareto and exponential cases we considered $\alpha=2,3,5,7,9,10$. The following tables show the values of the asymptotic constants and the resulting quantiles at level 0.999.

Table 1: Asymptotic constants, $C_{LF}(2, \alpha, \beta)$

α	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$
2	8.6293	17.2031	34.3509	68.6358
3	8.5542	17.0840	34.1435	68.2577
5	8.4062	16.8037	33.5987	67.1870
7	8.3146	16.6248	33.2452	66.4851
9	8.2557	16.5087	33.0147	66.0263
10	8.2336	16.4651	32.9280	65.8535

Table 2: Quantile estimates of L_2 at 0.999 level

α	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$
2	$918,940$	$496,296$	$378,419$	$330,997$
3	$914,891$	$495,102$	$377,801$	$330,587$
5	$906,852$	$492,269$	$376,164$	$329,417$
7	$901,844$	$490,445$	$375,092$	$328,642$
9	$898,606$	$489,254$	$374,388$	$328,132$
10	$897,393$	$488,805$	$374,122$	$327,939$

Tables 1 and 2 show that, as α increases, the asymptotic constants $C_{LF}(2, \alpha, \beta)$ decrease. This makes the corresponding quantile decrease, which is expected since an increasing value of α results in a stronger dependence between the insurance contracts. Changing the value of α does not have a significant impact on the quantiles, but the sensitivity to β is quite apparent. This indicates that poor quantification of the tail index β may yield incorrect results. A heavier tail, which corresponds to a lower value of β, results in larger quantiles.

Table 3: Asymptotic constants, $C_{LG}(2, \alpha)$ and quantile estimates of L_2 at level 0.999

α	$C_{LG}(2, \alpha)$	Quantile
2	2.1367	$153,340$
3	2.1294	$153,272$
5	2.0983	$152,978$
7	2.0770	$152,774$
9	2.0630	$152,638$
10	2.0576	$152,586$

The asymptotic constant $C_{LG}(2,\alpha)$ and quantile from Table 3 exhibit the same behaviour as in Tables 1 and 2, regarding changes in the strength of dependence. As anticipated, the quantiles for the exponential case are smaller than the corresponding Pareto quantiles, due to the light-tail extreme behaviour of the exponential distribution.

4 Other Dependence Structures

In the previous section it was assumed that the survival copula is Archimedean, and some regularity conditions were imposed. The main purpose of this section is to extend those results.

4.1 Archimedean Copula

A natural question is how do the asymptotic results differ when the copula itself (rather than the survival copula) is assumed to be Archimedean? This can be done, but we give up some simplicity. In this case, we assume that the generator φ is regularly varying at 1. By definition, this means that for any positive x the following holds

$$\lim_{t\downarrow 0}\frac{\varphi(1-tx)}{\varphi(1-t)} = x^{\alpha},$$

and we write $\varphi \in RV_{\alpha}^{1}$. Furthermore, the index satisfies the condition that $\alpha \geq 1$ (see Juri and Wütrich, 2003). The Gumbel copula is an example of such a copula with regularly varying generator $\varphi(u) = (-\ln u)^{\alpha}$, which satisfies the latter property ($\varphi \in RV_{\alpha}^{1}$).

$$C(u_1,\ldots,u_n) = \exp\left(-\left[\sum_{i=1}^{n}(-\ln u_i)^{\alpha}\right]^{1/\alpha}\right),$$

where $\alpha \geq 1$.

Upon defining the joint tail extreme behavior, the same steps as in the case of the survival Archimedean copula are followed, where (1) and (6) are replaced respectively by

$$\lim_{t\to\infty}\frac{\Pr(X_1 > tx_1, X_2 > tx_2)}{\bar{F}(t)} = x_1^{-\beta} + x_2^{-\beta} - \left(x_1^{-\alpha\beta} + x_2^{-\alpha\beta}\right)^{1/\alpha},\ x_1, x_2 > 0,$$

in the Fréchet case, and

$$\lim_{t\to\infty} \frac{\Pr(X_1 > t + x_1 a(t), X_2 > t + x_2 a(t))}{\bar{F}(t)}$$
$$= \quad e^{-x_1} + e^{-x_2} - \left(e^{-\alpha x_1} + e^{-\alpha x_2}\right)^{1/\alpha}, \; -\infty < x_1, x_2 < \infty,$$

in the Gumbel case (see Juri and Wütrich, 2003) provided that $1 < \alpha < \infty$. For simplicity, the bivariate case has been considered, but the result can be extended to the multivariate case, which is more cumbersome.

4.2 Extension

All previous cases were done under the assumption of exchangeability, which simplifies the computations since we deal with order statistics. We recognize that this assumption may be questionable, but extensions can be made when it does not hold, though they are tedious.

Earlier we mentioned that the joint tail extreme behaviour is essential to characterize the tail probability for the ECOMOR and LCR reinsurances. In the case that the exchangeability property fails to hold we can still make the same characterization, provided that for any set $I \subseteq \{1, \ldots, n\}$ the following exist

$$\lim_{t\to\infty} \frac{\Pr(X_i > tx_i, i \in I)}{V(t)}, \; x_i > 0,$$

in the Fréchet case, and

$$\lim_{t\to\infty} \frac{\Pr(X_i > t + a(t)x_i, i \in I)}{V(t)}, \; -\infty < x_i < \infty,$$

for Gumbel, where $V(\cdot)$ is a positive-valued function.

5 Conclusions

In this paper, we provide a procedure to understand the tail behavior of the ECOMOR and LCR reinsurances for a portfolio of dependent insurance contracts. First, a specific dependence structure is considered. Namely, the survival copula is assumed to be Archimedean. This choice of dependence structure aids in giving closed form results, while the exchangeability between random variables simplifies the analysis. Finally, we note that our main results can be extended, provided that we control the limiting joint tail probabilities.

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