Tail Dependence Measure for Examining Financial Extreme Co-movements

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Abstract. Modeling and forecasting extreme co-movements in financial market is important for conducting stress test in risk management. Asymptotic independence and asymptotic dependence behave drastically different in modeling such co-movements. For example, the impact of extreme events is usually overestimated whenever asymptotic dependence is wrongly assumed. On the other hand, the impact is seriously underestimated whenever the data is misspecified as asymptotic independent. Therefore, distinguishing between asymptotic independence/dependence scenarios is very informative for any decision-making and especially in risk management. We investigate the properties of the limiting conditional Kendall's tau which can be used to detect the presence of asymptotic independence/dependence. We also propose nonparametric estimation for this new measure and derive its asymptotic limit. A simulation study shows good performances of the new measure and its combination with the coefficient of tail dependence proposed by Ledford and Tawn (1996, 1997). Finally, applications to financial and insurance data are provided.

Keywords and phrases: Asymptotic dependence and independence; Copula; Extreme co-movement; Kendall's tau; Measure of association.

1. Introduction

An important task in risk management is to understand the reliability of the proposed model in the presence of adverse scenarios, known as stress testing. For example, the assessment of the capital adequacy in banking and insurance industries is based on quantifying the impact of extreme events on the solvability of financial and insurance conglomerates. Harmonized regulatory methodologies, such as the implementation of stress testing, have been imposed in the banking industry (known as Basel III; see, Basel Committee on Banking Supervision, 2010), and insurance industry within the European Union (known as Solvency II; see, European Commission, 2009) and in Switzerland (known as Swiss Solvency Test; see, Swiss Solvency Test, 2006). It is generally accepted that Extreme Value Theory provides the appropriate technology to address the quantitative side of the problem (see for example, Aragones *et al.*, 2001 and Longin, 2010). Since multiple sources of risks are competitive contributors to the calculations of the level of capital requirements, a holistic approach is to characterize such co-movements of extremes and then to effectively extrapolate data into tail region, which can naturally be done under the umbrella of Multivariate Extreme Theory as explained below.

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be independent and identically distributed random vectors with distribution function F and marginal distributions F_1 and F_2, i.e. $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$. Bivariate Extreme Value Theory assumes that there are constants $a_n > 0, c_n > 0, b_n \in R, d_n \in R$ such that

$$\lim_{n\to\infty} \mathbb{P}\Big(a_n \Big(\max_{1\le i\le n} X_i - b_n \Big) \le x, c_n \Big(\max_{1\le i\le n} Y_i - d_n \Big) \le y \Big) = G(x, y), \tag{1.1}$$

for all continuous points (x, y) of G. In this case, G is called an *extreme value distribution* and F is said to belong to the *domain of attraction of* G. It follows from (1.1) that the following dependence convergence holds:

$$\lim_{t\to 0} t^{-1}\Big\{1-F\big((1-F_1)^-(tx),(1-F_2)^-(ty)\big)\Big\} = -\log G\big((-\log G_1)^-(x), (-\log G_2)^-(y)\big) := l(x,y) \tag{1.2}$$

for all $x, y \ge 0$, where $G_1(x) = G(x, \infty)$, $G_2(y) = G(\infty, y)$ and $(\cdot)^-$ denotes the left continuous inverse function. Here, $l(x, y)$ is called the tail dependence function (see Huang, 1992). It is easy to check that $l(ax, ay) = al(x, y)$ for all $a, x, y \ge 0$ and $x \vee y \le l(x, y) \le x + y$. This homogeneous property has been employed to extrapolate data into a tail region so that extreme events can be predicted (for details, see for example, de Haan and Ferreira, 2006). However, when $l(x, y) = x + y$, equation (1.2) implies that

$$\lim_{t\to 0} t^{-1}\mathbb{P}(1 - F_1(X_1) < tx, 1 - F_2(Y_1) < ty) = 0, \tag{1.3}$$

which makes extrapolation, i.e. statistical inference, impossible for concomitant extreme sets. In this case, F is said to have the asymptotic independence property, and a different convergence rate condition in (1.3) is needed for predicting joint extreme events. In other words, extreme value condition (1.1) is not enough for predicting extreme events in case of asymptotic independence. If the limit in (1.3) is not identical to zero, then F is said to have the *asymptotic dependence* property. It is known that a bivariate normal distribution with correlation coefficient less than 1 is asymptotically independent, i.e. (1.3) holds (for details, see Sibuya, 1960).

Estimation of multivariate extreme becomes possible if the presence of asymptotic dependence/independece is known, and therefore, distinguishing between the two properties plays an important role in predicting extreme events. A mathematical formulation of this problem is made in Ledford and Tawn (1996, 1997), where the coefficient of tail dependence, $0 < \eta \le 1$, is introduced by assuming that

$$\mathbb{P}(1 - F_1(X_1) \le t, 1 - F_2(Y_1) \le t) = t^{1/\eta} s(t), \tag{1.4}$$

where $s(t)$ is a slowly varying function, i.e. $\lim_{t\to 0} s(tx)/s(t) = 1$ for all $x > 0$. Note that $0 < s(t) \le 1$ for all $0 \le t \le 1$ due to the facts that $0 < \eta \le 1$ and

$$\mathbb{P}(1 - F_1(X_1) \le t, 1 - F_2(Y_1) \le t) \le \mathbb{P}(1 - F_1(X_1) \le t) = t,$$

provided that F_1 is continuous, which is the case since both marginal distributions are assumed to be continuous throughout this paper. Under condition (1.4), when $\eta = 1$ and $\lim_{t\to 0} s(t) = c \in (0, 1]$, the asymptotically dependent property holds, while either $\eta < 1$ or $\eta = 1$ and $\lim_{t\to 0} s(t) = 0$ implies asymptotic independence. Therefore, η and the limit behavior of function $s(t)$ can be used to distinguish between asymptotic dependence and asymptotic independence. Nonparametric inference for η can be found in Peng (1999) and Draisma *et al.* (2004). Recently, Goegebeur and Guillou (2012) considered an asymptotically unbiased estimator for η in the case of $\eta < 1$, i.e. asymptotic independence. Nonparametric tests for the tail dependence function and asymptotic dependence are available in Einmahl, de Haan and Li (2006) and Hüsler and Li (2009).

It is known that testing asymptotic dependence is extremely challenging due to limited observations in the tail region, and so it is always desirable to have some alternative measures and competitive statistical methods. Our proposal appeals to a robust measure of association that is appealing to a wide audience, and we find that most of the extreme scenarios are characterized by our method in order to elaborate an alternative way to characterize the asymptotic independence and asymptotic dependence. In factual terms, we investigate the relationship between tail dependence and the conditional version of a classical measure of association, namely Kendall's tau. While estimating the univariate extreme events has become a standard procedure, dealing with multivariate extreme events is a more complicated problem, and it is of general interest in many papers with particular focus on financial and insurance applications (see for example, Frees and Valdez, 1998 and Breymann *et al.*, 2003).

Some useful background is now provided for a reader that is less familiar with the justifications we made. Dependence or association is fully characterized by the copula due to the Sklar's Theorem (for example, see Sklar, 1959), and for a bivariate random vector, (X_1, Y_1), is given by the joint distribution function of $\Big(F_1\big(X_1\big), F_2\big(Y_1\big)\Big)$, whenever the marginal distribution functions are continuous. Since (1.4) concerns the upper tail dependence, it is natural to study the survival copula

$$C(x, y) := \mathbb{P}\big(1 - F_1(X_1) \le x, 1 - F_2(Y_1) \le y\big). \tag{1.5}$$

Although the dependence is fully described by its copula or survival copula, it is sometimes difficult to explain the chosen model. The problem becomes more acute when extreme events are concerned. Instead of fully exploring the associated copula, a practical methodology is to focus on some measures of association that provide sufficient information to understand which model would be more appropriate. There are various measures of association proposed in the literature, and one of them is the Kendall's tau which is closely related to tail dependence and is defined as

$$\tau = \mathbb{P}\big((U_1 - U_2)(V_1 - V_2) > 0\big) - \mathbb{P}\big((U_1 - U_2)(V_1 - V_2) < 0\big),$$

where $U_i = 1 - F_1(X_i)$ and $V_i = 1 - F_2(Y_i)$ for $i = 1, 2$. It is well-known that this measure is scale-invariant, and therefore robust, marginal-free whenever the marginal distributions are continuous, and is based on the concept of concordance and discordance (for more details, see Nelsen, 2006). As a result of such appealing properties, Kendall's tau has been found useful in various fields, such as risk management (see McNeil *et al.*, 2005). However, if one is interested in evaluating the strength of dependence in the lower tail of (U_i, V_i) (i.e., the upper tail of (X_i, Y_i)), when concomitant extreme events are plausible, then the conditional Kendall's tau is more sound, which is defined as follows:

$$\tau(u) = \mathbb{P}\big((U_1 - U_2)(V_1 - V_2) > 0 | U_1, U_2, V_1, V_2 \le u\big) - \mathbb{P}\big((U_1 - U_2)(V_1 - V_2) < 0 | U_1, U_2, V_1, V_2 \le u\big). \quad (1.6)$$

Study of conditional Kendall's tau for a fixed level u is relatively known in the literature (see Venter, 2001 and Gijbels *et al.*, 2011). However, it remains unknown whether there exists some relationship between the limit of this conditional measure and asymptotic dependence, and how to estimate the limit.

In the next section, we shall show that $\theta^\tau := \lim_{u \to 0} \tau(u)$ are positive for a subclass of asymptotic dependence and non-positive for a subclass of asymptotic independence. We found that all well-known examples indicate a positive limit for the case of asymptotic dependence. It is known that testing for asymptotic dependence against asymptotic independence becomes quite challenging when η is close to one. Since $\theta^\tau > 0$ may be a bit far away from zero in case of asymptotic dependence, testing for $\theta^\tau = \theta_0$ against $\theta^\tau \le 0$ becomes much easier in the case of asymptotic dependence, where θ_0 is a given positive value. That is, intervals of θ^τ are useful in distinguishing asymptotic dependence from asymptotic independence. On the other hand, when the data has the asymptotic independence property, a test based on θ^τ is less efficient than a test based on η since θ^τ may be zero, while the true value of η, say η_0, is less than one, which can be used to effectively test for $\eta = \eta_0$ against $\eta = 1$. In other words, an interval of η is quite informative when the data has the asymptotic independence property. Given the above arguments, we argue that interval estimation of $\theta^\tau + \eta$ can be effective in distinguishing between asymptotic dependence and asymptotic independence since $\theta^\tau + \eta$ is larger than one in case of asymptotic dependence and less than one in case of asymptotic independence. Similar phenomena appeared in Doksum and Samarov (1995) for nonparametric regression and in Zhang *el a.* (2011) for testing independence.

We organize this paper as follows. Some nonparametric estimators for the limit of this conditional measure and its asymptotic distribution are derived in Section 2. A set of examples, a simulation study and some empirical analyses are given in Sections 3, 4 and 5, respectively. Finally, all technical proofs are relegated in Section 6.

2. Main Results

A summary of our initial assumptions needed to develop our results is that $\{(X_i, Y_i)\}_{i=1}^n$ are independent and identically distributed with distribution function F, continuous marginal distribution functions F_1 and F_2, and survival copula C as defined in (1.5).

2.1. **Conditional Kendall's tau.** First, we derive the limits of the conditional Kendall's tau defined in (1.6) by assuming the following multivariate regular variation, which has been found useful in characterizing tail behavior of a random vector. Some recent references on multivariate regular variation are Basrak *et al.* (2002), Hua and Joe (2011, 2013), and Mikosch and Wintenberger (2014).
We define $h(x,y) = \frac{\partial^2}{\partial x \partial y} H(x,y)$, $H_1(x,y) = \frac{\partial}{\partial x} H(x,y)$, $H_2(x,y) = \frac{\partial}{\partial y} H(x,y)$, $H_{11}(x,y) = \frac{\partial}{\partial x} H_1(x,y)$ and $H_{22}(x,y) = \frac{\partial}{\partial y} H_2(x,y)$, whenever the partial derivatives exist.

Assumption 2.1. *There exist a constant $\delta > 0$ and a function $H(x,y)$ such that $C(u,u) > 0$ for all $u \in (0,\delta)$ and*

$$H(x,y) := \lim_{u \downarrow 0} \frac{C(ux,uy)}{C(u,u)}$$

for all $(x,y) \in \mathcal{D} := [0,1]^2$. In addition, $H(x,y)$ is continuous on $\{(x,y) : xy = 0\}$.

Theorem 2.1. *Under Assumption 2.1, we have*

$$\theta^\tau = 4 \int_0^1 \int_0^1 H(x,y) dH(x,y) - 1. \tag{2.1}$$

Remark 2.1. *The above limit in (2.1) is indeed a proper Kendall's tau, which measures the association between two random variables with joint distribution function given by H. Moreover H has continuous marginals, hence one can extract the associated copula, C_H, as a result of Sklar's Theorem, and (2.1) can be rewritten as follows:*

$$\theta^\tau = 4 \int_{\mathcal{D}} C_H(x,y)\, dC_H(x,y) - 1 = 1 - 4 \int_{\mathcal{D}} \frac{\partial}{\partial x} C_H(x,y) \frac{\partial}{\partial y} C_H(x,y)\, dx\, dy$$

(see Theorems 5.1.1. and 5.1.5 of Nelsen, 2006). Finally, if H admits partial derivatives, then one may show that

$$\theta^\tau = 1 - 4 \int_{\mathcal{D}} H_1(x,y) H_2(x,y)\, dx\, dy.$$

Note that Assumption 2.1 implies that the next weak convergence

$$\mu_u(\cdot) := \mathbb{P}\big((U/u, V/u) \in \cdot | U, V \le u\big) \xrightarrow{w} \mu(\cdot) \tag{2.2}$$

holds on $\mathcal{D}$ as $u \to 0$, where the (probability) measure μ is given by $\mu\big([0,x] \times [0,y]\big) := H(x,y)$. In addition, $H(x,y)$ is a homogeneous function with an order larger than or equal to one (see de Haan and Resnick, 1979 and Resnick, 1987). Next, we show that the limit of the conditional Kendall's tau is positive for a subclass of asymptotic dependence and non-positive for a subclass of asymptotic independence as follows:

Assumption 2.2. *There exist a constant $c \in [0,1]$ and an $\eta \in (0,1]$ such that*

$$H(ax, ay) = a^{1/\eta} H(x,y) \quad \text{and} \quad \lim_{u \downarrow 0} u^{-1} C(u,u) = c \in [0,1]$$

for all $a > 0$ and $(x,y) \in \mathcal{D}$.

Assumption 2.3. $H(x,y) = \sum_{i=1}^{m} c_i x^{\alpha_i} y^{\beta_i}$ *for some positive $c_i's$ and some nonnegative $\alpha_i', s, \beta_i' s$ with $\alpha_i + \beta_i = 1/\eta$ for $i = 1, \cdots, m$ and $\sum_{i=1}^{m} c_i = 1$.*

We first investigate the properties of a bivariate distribution function $H : \mathcal{D} \to [0,1]$, for which all first and second partial derivatives exist, satisfying the homogeneity property

$$H(tu, tv) = tH(u, v) \text{ for all } t > 0 \text{ and } (u, v) \in \mathcal{D}. \tag{2.3}$$

Let $\mathcal{H}$ be the collection of all such H. Define $\mathcal{F}(\xi)$, for $0 < \xi < 1$, the set of all pairs (f_X, f_Y) of density functions on $(0,1)$ such that both f_X and f_Y are non-increasing (hence almost everywhere differentiable) and

$$\int_0^x f_X(u)\, du \geq x, \quad \int_0^y f_Y(v)\, dv \geq y, \quad \lim_{x\to 1} f_X(x) = \xi, \quad \lim_{y \to 1} f_Y(y) = 1 - \xi.$$

We also define $\mathcal{F} = \bigcup_{0<\xi<1} \mathcal{F}(\xi)$. The next proposition shows that there is a one-to-one correspondence between $\mathcal{H}$ and $\mathcal{F}$.

Proposition 2.1. *i) Let $H \in \mathcal{H}$ and define $f_X(x) = H_1(x,1)$, $f_Y(y) = H_2(1,y)$, $h(x,y) = H_{12}(x,y)$. Then, $(f_X, f_Y) \in \mathcal{F}$ and for all $(x,y),(u,v) \in \mathcal{D}$ we have*

$$h(x,y) = -\frac{x}{y^2} f'_X\left(\frac{x}{y}\right) I_{x<y} - \frac{y}{x^2} f'_Y\left(\frac{y}{x}\right) I_{y<x} \text{ and } H(u,v) = vF_X\left(\frac{u}{v}\right) I_{u<v} + uF_Y\left(\frac{v}{u}\right) I_{v\leq u}. \tag{2.4}$$

ii) Let $(f_X, f_Y) \in \mathcal{F}$. Define $h(x,y)$ by (2.4) and $H(u,v) = \int_0^u \int_0^v h(x,y)\, dy\, dx$. Then, H is a bivariate distribution function with marginal densities f_X and f_Y and satisfies (2.3).

Proposition 2.1 allows us to identify a sharp lower bound for θ^τ, which is given as Theorem 2.2.

Theorem 2.2. *Under Assumptions 2.1 and 2.2, if $\eta = 1$, $c > 0$, and $\frac{\partial^2}{\partial x^i \partial y^j} H(x,y)$ exists for all $(x,y) \in \mathcal{D}$, $i,j = 0,1,2$ and $i+j = 2$, then $\theta^\tau \geq -\frac{1}{2} + \frac{1}{\log(2/c)}$. Therefore, $\theta^\tau > 0$ if $c > 2e^{-2}$.*

Theorem 2.3. *If Assumption 2.3 holds, then $\lim_{u\downarrow 0} \tau(u) \leq 0$.*

Remark 2.2. *It is clear that asymptotic dependence holds under Assumptions 2.1 and 2.2 with $\eta = 1$ and $c > 0$. Although Theorem 2.2 gives a lower bound on c to ensure a positive limit for the conditional Kendall's tau, a study of some common copulas indicates the limit is positive for all $c \in (0,1]$ in the case of asymptotic dependence (see Section 3 below). Therefore it remains interesting to find a subclass of $\mathcal{H}$, which includes all $c \in (0,1]$ and gives a positive limit.*

Remark 2.3. *Note that $H(x,y) \leq \min\{x,y\}/c$ for all $(x,y) \in \mathcal{D}$ due to the fact that $C(ux, uy) \leq u\min\{x,y\}$, where c is defined in Assumption 2.2. If Assumption 2.3 holds with $\eta = 1$ and $c > 0$ given in Assumption 2.2, then $\sum_{i=1}^m c_i (y/x)^{\beta_i} \leq c^{-1}$ and $\sum_{i=1}^m c_i (x/y)^{\alpha_i} \leq c^{-1}$ for all $(x,y) \in \mathcal{D}$, which can not be true by taking either x or y small enough. Therefore, Assumption 2.3 does imply the asymptotic independence. Whenever the limiting function H is not absolutely continuous, Example 3.4 with $\alpha = \beta \in (0,1)$ from Section 3 illustrates that $\lim_{u\downarrow 0} \tau(u)$ may be positive for the case of asymptotic independence. Although we conjecture that $\lim_{u\downarrow 0} \tau(u) \leq 0$ for the case of asymptotic independence when $H(x,y)$ is absolutely continuous with second order partial derivatives, Theorem 2.3 only shows that this is true for a subclass of asymptotic independence, as defined in Assumption 2.3.*

Remark 2.4. *Example 3.4 with $\alpha = \beta \in (0,1)$ from Section 3 has some positive mass along the diagonal line $y = x$, which gives a positive value for $\lim_{u\downarrow 0} \tau(u)$ for this situation of asymptotic independence. However, if one slightly modifies the definition of Kendall's tau as follows*

$$\begin{aligned}\tilde{\tau}(u) \quad &= \mathbb{P}\big((U_1 - U_2)(V_1 - V_2) > 0, U_1 \neq V_1, U_2 \neq V_2 | U_1, U_2, V_1, V_2 \leq u\big) \\ &-\mathbb{P}\big((U_1 - U_2)(V_1 - V_2) < 0, U_1 \neq V_1, U_2 \neq V_2 | U_1, U_2, V_1, V_2 \leq u\big),\end{aligned}$$

then it can be shown that $\theta^\tau \leq 0$ for this example. Obviously, this modification does not affects the limit of the original definition of conditional Kendall's tau when C has a continuous density.

2.2. **Estimation procedure.** Theorems 2.2 and 2.3 show that the limit of conditional Kendall's tau may give a good insight on whether the underlying distribution is asymptotically independent or asymptotically dependent. Hence, estimating the limit is useful in applying Extreme Value Theory to predict extreme co-movements in financial markets.

Define $\hat{F}_1(x) = \frac{1}{n+1}\sum_{i=1}^n I(X_i \leq x)$, $\hat{F}_2(y) = \frac{1}{n+1}\sum_{i=1}^n I(Y_i \leq y)$, $\hat{U}_i = 1 - \hat{F}_1(X_i)$, $\hat{V}_i = 1 - \hat{F}_2(Y_i)$, and put $\theta^\tau = \lim_{u\downarrow 0} \tau(u)$. Then, we propose to estimate θ^τ by

$$\hat{\theta}^\tau(k) = \frac{\sum_{1\leq i<j\leq n} sgn\Big\{\big(\hat{U}_i - \hat{U}_j\big)\big(\hat{V}_i - \hat{V}_j\big)\Big\} I\Big(\max\big(\hat{U}_i, \hat{U}_j, \hat{V}_i, \hat{V}_j\big) \leq k/n\Big)}{\sum_{1\leq i<j\leq n} I\Big(\max\big(\hat{U}_i, \hat{U}_j, \hat{V}_i, \hat{V}_j\big) \leq k/n\Big)},$$

where $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$. The following theorem shows the consistency of the proposed estimator.

Theorem 2.4. *Under Assumption 2.1, $k = k(n) \to \infty$, $k/n \to 0$ and $nC\left(\frac{k}{n}, \frac{k}{n}\right) \to \infty$ as $n \to \infty$, we have $\hat{\theta}^\tau(k) \xrightarrow{p} \theta^\tau$ as $n \to \infty$.*

As usual in Extreme Value Theory, if one is interested in deriving the asymptotic limit of $\hat{\theta}^\tau(k)$, a rate of convergence in (1.5) is needed, which controls the asymptotic bias of the studied estimator. Here, we employ the following second order condition.

Assumption 2.4. *There exist a regular variation $A(u) \to 0$ with index $\tilde{\rho} \geq 0$, i.e. $\lim_{u\to 0} A(ux)/A(u) = x^{\tilde{\rho}}$ for $x > 0$, functions $Q(x,y)$ and $q(x,y)$ such that*

$$\lim_{u\downarrow 0} \frac{\frac{C(ux,uy)}{C(u,u)} - H(x,y)}{A(u)} = Q(x,y) \quad \textit{and} \quad \lim_{u\downarrow 0} \frac{\frac{u^2 C_{12}(ux,uy)}{C(u,u)} - H_{12}(x,y)}{A(u)} = q(x,y) \tag{2.5}$$

for all $(x,y) \in \mathcal{D}$ and uniformly on $\{(x,y) : x^2 + y^2 = 1\}$, where H_{12} and C_{12} are the densities of H and C, respectively.

Remark 2.5. *The second condition in (2.5) implies the first one when some mild integrability conditions are satisfied.*

Theorem 2.5. *Under Assumption 2.4, $\lim_{u\downarrow 0} u^{-1}C(u,u) = c \in [0,1]$,*

$$k = k(n) \to \infty,\ nC\left(\frac{k}{n}, \frac{k}{n}\right) \to \infty \ \textit{ and } \ \sqrt{nC\left(\frac{k}{n}, \frac{k}{n}\right)}A\left(\frac{k}{n}\right) \to \lambda \in (-\infty, \infty)$$

as $n \to \infty$, we have

$$\sqrt{nC\left(\frac{k}{n},\frac{k}{n}\right)}\{\hat{\theta}^\tau(k)-\theta^\tau\} \xrightarrow{d} N\left(\lambda b_\tau, \sigma_\tau^2\right) \tag{2.6}$$

as $n \to \infty$, where

$$b_\tau = 4\int_0^1\int_0^1 Q(s,t)H_{12}(s,t)\,dtds + 4\int_0^1\int_0^1 H(s,t)q(s,t)\,dtds,$$

$$\sigma_\tau^2 = 4\{\sigma_1^2-(\theta^\tau)^2+\sigma_2^2+\sigma_3^2+2\sigma_2\sigma_3 c\}, \tag{2.7}$$

with

$$\begin{cases} \sigma_1^2 & = 16\int_0^1\int_0^1 H^2(x,y)\,dH(x,y) - 16\int_0^1\int_0^1 H(x,1)H(x,y)\,dH(x,y) \\ & \quad -16\int_0^1\int_0^1 H(1,y)H(x,y)\,dH(x,y) + 8\int_0^1\int_0^1 H(x,y)\,dH(x,y) \\ & \quad +8\int_0^1\int_0^1 H(x,1)H(1,y)\,dH(x,y) - \frac{1}{3} \\ \sigma_2 & = \sqrt{c}(2\int_0^1 H(1,t)H_{12}(1,t)\,dy - H_1(1,1)) \\ \sigma_3 & = \sqrt{c}(2\int_0^1 H(s,1)H_{12}(s,1)\,ds - H_2(1,1)). \end{cases} \tag{2.8}$$

Remark 2.6. *When $C(u,u)=d_1u^{1/\eta}$ and $A(u)=d_2u^{\tilde{\rho}}$, a theoretical optimal k for $\hat{\theta}^\tau(k)$ can be chosen to minimize the asymptotic mean squared error $b_\tau^2A^2\left(\frac{k}{n}\right)+\frac{\sigma_\tau^2}{nC\left(\frac{k}{n},\frac{k}{n}\right)}$, which gives the optimal choice of k as*

$$k_0^\tau = \left(\frac{\sigma_\tau^2}{2\eta b_\tau^2 d_2^2 d_1\tilde{\rho}}\right)^{1/(2\tilde{\rho}+1/\eta)} n^{(1/\eta-1+2\tilde{\rho})/(1/\eta+2\tilde{\rho})}.$$

Remark 2.7. *A consistent estimator for σ_τ^2 can be obtained by replacing c, $H(x,y)$ and $H_{12}(x,y)$ in (2.7) and (2.8) by*

$$\hat{c} = \frac{1}{m}\sum_{i=1}^n I\left(1-\hat{F}_1(X_i)\frac{m}{n}, 1-\hat{F}_2(Y_i)\le\frac{m}{n}\right),$$

$$\hat{H}(x,y) = \frac{1}{m\hat{c}}\sum_{i=1}^n I\left(1-\hat{F}_1(X_i)\le\frac{m}{n}x, 1-\hat{F}_1(Y_i)\le\frac{m}{n}y\right),$$

$$\hat{H}_{12}(x,y)=\sum_{i=1}^n\frac{I\left(1-\hat{F}_1(X_i)\le\frac{m}{n}x, 1-\hat{F}_1(Y_i)\le\frac{m}{n}y\right)}{m\hat{c}}\,G\left(\frac{\frac{n}{m}(1-\hat{F}_1(X_i))-x}{q}\right)G\left(\frac{\frac{n}{m}(1-\hat{F}_2(Y_i))-y}{q}\right),$$

respectively, where $m=m(n)\to\infty$, $m/n\to 0$ as $n\to\infty$, G is a smooth distribution function and $q=q(n)>0$ is the bandwidth satisfying that $q\to 0$ and $qm\to\infty$ as $n\to\infty$. One can also use the corresponding estimators in Draisma et al. (2004). In the simulation study, we employ the bootstrap method to estimate the asymptotic variance. Theoretical justification of the proposed bootstrap method can be shown in a similar way to Peng and Qi (2008).

Remark 2.8. *The usual approach to construct confidence intervals for θ is to choose $k=o(k_0^\tau)$ so that the asymptotic bias is negligible, where k_0^τ is the theoretical optimal choice given in Remark 2.6. Motivated by the choice of sample fraction for the Hill estimator in terms of coverage probability in Cheng and Peng (2001), we propose to choose $k=O\left(n^{(1/\eta-1+\tilde{\rho})/(1/\eta+\tilde{\rho})}\right)$ for interval estimation of θ^τ based on the asymptotic limits of $\hat{\theta}^\tau(k)$.*

Remark 2.9. *As argued in the introduction, when the data is asymptotically independent, θ^τ may be zero, hence the interval may not be effective in distinguishing the asymptotic independence from the asymptotic dependence. In this case, one may use the quantity $\theta^\tau+\eta$. For estimating $\theta^\tau+\eta$, one can easily combine $\hat\theta^\tau$ with the estimator $\hat\eta$ for η proposed in Draisma et al. (2004), and the asymptotic distribution of $\hat\theta^\tau+\hat\eta$ can be derived by using expansions as given in the proof of Theorem 2.5 and those in Draisma et al. (2004), but we skip these derivations. For constructing an interval for $\theta^\tau+\eta$ based on the normal approximation of $\hat\theta^\tau+\hat\eta$, we simply employ the bootstrap method as we do in Section 5.*

3. Examples

This section shows that some well-known copulas satisfy the conditions from Theorems 2.2 and 2.3 for which the limit of the conditional Kendall's tau is also derived. If C^* is a copula with corresponding survival copula C defined in (1.5), then $C(u,v)=C^*(1-u,1-v)+u+v-1$ for all $(u,v)\in\mathcal{D}$.

Example 3.1. *Consider the Gumbel copula $C^*(u,v)=\exp\big\{-\big((-\log u)^\alpha+(-\log v)^\alpha\big)^{1/\alpha}\big\}$ where $\alpha\in(1,\infty)$. Then, Assumption 2.2 holds with $\eta=1$, $c=2-2^{1/\alpha}$ and $cH(x,y)=x+y-(x^\alpha+y^\alpha)^{1/\alpha}$. Figure 3.1 below plots the values of θ^τ against different α, which shows that the limit is positive. It is easy to show that $H_1(x,1)$ increases in α for $x\in(0,1]$ and so is the limit of conditional Kendall's tau. By $\lim_{\alpha\to1}H_1(x,1)=\frac{\ln(1+x)}{2\ln 2}$ and $\ln(1+x)\le\frac{x}{\sqrt{1+x}}$ for $x>0$, we have*

$$\begin{aligned}
\lim_{u\to0}\theta^\tau &\ge 4\int_0^1\lim_{\alpha\to1}xH_1^2(x,1)dx-1\\
&= 1-4\int_0^1\lim_{\alpha\to1}H_1(x,1)H_1(1,x)dx\\
&= 1-\int_0^1\frac{\ln(1+x)\ln(1+x^{-1})}{(\ln 2)^2}dx\\
&\ge 1-\int_0^1\frac{\frac{x}{\sqrt{1+x}}\frac{x^{-1}}{\sqrt{1+x^{-1}}}}{(\ln 2)^2}dx\\
&\ge 1-\frac{1}{\sqrt{6}(\ln 2)^2}\\
&> 0.
\end{aligned}$$

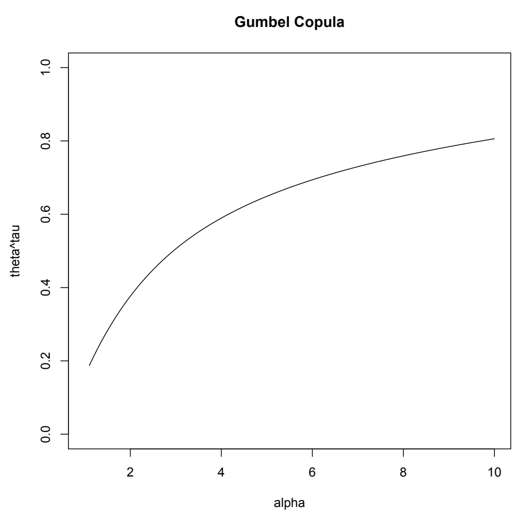

Figure 3.1. The limit of conditional Kendall's tau is plotted against parameter for Gumbel copula from Example 3.1.

Example 3.2. *Consider the t copula*

$$C^*(u,v)=\int_{-\infty}^{t_\nu^-(u)}\int_{-\infty}^{t_\nu^-(v)}\frac{1}{2\pi(1-\rho^2)^{1/2}}\left\{1+\frac{x^2-2\rho xy+y^2}{\nu(1-\rho^2)}\right\}^{-(\nu+2)/2}dxdy,$$

where $|\rho|<1$, $\nu>0$ and t_ν denotes the distribution function of a t distribution with ν degrees of freedom. Let (U_1^*,V_1^*) be a bivariate random vector with distribution C^*. Since $t_\nu^-(1-s)\sim ds^{-1/\nu}$ for some constant $d>0$ as $s\to 0$, we have

$$\begin{aligned}
&\lim_{s\to 0}\frac{1-C^*(1-su,1-sv)}{s}\\
=\;& u\lim_{s\to 0}\mathbb{P}\big(V_1^*\le 1-sv|U_1^*=1-su\big)+v\lim_{s\to 0}\mathbb{P}\big(U_1^*\le 1-su|V_1^*=1-sv\big)\\
=\;& u\lim_{s\to 0}\mathbb{P}\big(t_\nu^-(V_1^*)\le t_\nu^-(1-sv)|t_\nu^-(U_1^*)=t_\nu^-(1-su)\big)\\
&+v\lim_{s\to 0}\mathbb{P}\big(t_\nu^-(U_1^*)\le t_\nu^-(1-su)|t_\nu^-(V_1^*)=t_\nu^-(1-sv)\big)\\
=\;& u\lim_{s\to 0}t_{\nu+1}\left(\frac{t_\nu^-(1-sv)-\rho t_\nu^-(1-su)}{\sqrt{1-\rho^2}}\left(\frac{\nu+1}{\nu+\big(t_\nu^-(1-su)\big)^2}\right)^{1/2}\right)\\
&+v\lim_{s\to 0}t_{\nu+1}\left(\frac{t_\nu^-(1-su)-\rho t_\nu^-(1-sv)}{\sqrt{1-\rho^2}}\left(\frac{\nu+1}{\nu+\big(t_\nu^-(1-sv)\big)^2}\right)^{1/2}\right)\\
=\;& ut_{\nu+1}\left(\frac{\big((v/u)^{-1/\nu}-\rho\big)\sqrt{\nu+1}}{\sqrt{1-\rho^2}}\right)+vt_{\nu+1}\left(\frac{\big((u/v)^{-1/\nu}-1\big)\sqrt{\nu+1}}{\sqrt{1-\rho^2}}\right).
\end{aligned}$$

Consequently, Assumption 2.2 holds with $\eta=1$, $c=2-2t_{\nu+1}\left(\sqrt{\frac{(1-\rho)(\nu+1)}{1+\rho}}\right)$ and

$$cH(x,y)=x\left\{1-t_{\nu+1}\left(\frac{\big((y/x)^{-1/\nu}-\rho\big)\sqrt{\nu+1}}{\sqrt{1-\rho^2}}\right)\right\}+y\left\{1-t_{\nu+1}\left(\frac{\big((x/y)^{-1/\nu}-\rho\big)\sqrt{\nu+1}}{\sqrt{1-\rho^2}}\right)\right\}.$$

Figure 3.2 below plots the values of θ^τ against various ρ and ν, which shows that the limit is indeed positive.

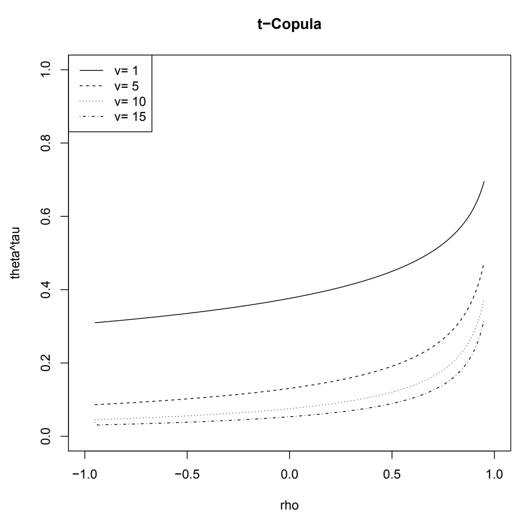

FIGURE 3.2. The limit of conditional Kendall's tau is plotted against parameters for t copula from Example 3.2.

Example 3.3. *Consider the elliptical copula $Z \overset{d}{=} GAU$, where $G > 0$ is a random variable with a survival function, $\bar{G}(\cdot)$, that satisfies $\bar{G}(tx)/\bar{G}(t) \sim x^{-\alpha}$ as $t \to \infty$ for all $x > 0$, A is a deterministic 2×2 matrix with $AA^T = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $|\rho| < 1$, U is uniformly distributed on $\{z \in \Re^2 : z^T z = 1\}$ and independent of G. Put*

$$\lambda(x, y) = \frac{x \int_{g\left((x/y)^{1/\alpha}\right)}^{\pi/2} (\cos\phi)^\alpha \, d\phi + y \int_{g\left((x/y)^{-1/\alpha}\right)}^{\pi/2} (\cos\phi)^\alpha \, d\phi}{\int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \, d\phi}, \tag{3.1}$$

where $g(t) = \arctan\left((t-\rho)/\sqrt{1-\rho^2}\right)$. Then it follows from Klüppelberg et al. (2008) that Assumption 2.2 holds with $\eta = 1$, $c = \lambda(1,1)$ and $H(x,y) = \lambda(x,y)/\lambda(1,1)$. Figure 3.3 below plots the values of θ^τ against various ρ and α, which shows that the limit is indeed positive. A rigorous verification goes as follows. First it is easy to check that

$$\begin{cases} g(t) + g(t^{-1}) = \arccos\rho, \quad \textit{for} \quad t > 0 \\ cos(g(t)) = (1 + (\frac{t-\rho}{\sqrt{1-\rho^2}})^2)^{-\frac{1}{2}} = \sqrt{(1-\rho^2)^{\frac{1}{2}} g'(t)} = t^{-1}\cos(g(t^{-1})) \quad \textit{for} \quad t > 0 \end{cases} \tag{3.2}$$

where g' is the derivative of g with respect to t. Taking the partial derivatives of $\lambda(x,y)$, by (3.2), it follows that

$$\begin{cases} \frac{\partial}{\partial x}\lambda(x,y) = \left\{ \int_{g((\frac{x}{y})^{\frac{1}{\alpha}})}^{\pi/2} (\cos\phi)^\alpha d\phi \right\} \times \left\{ \int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha d\phi \right\}^{-1}, \\ \frac{\partial}{\partial y}\lambda(x,y) = \left\{ \int_{g((\frac{x}{y})^{-\frac{1}{\alpha}})}^{\pi/2} (\cos\phi)^\alpha d\phi \right\} \times \left\{ \int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha d\phi \right\}^{-1}. \end{cases} \tag{3.3}$$

Define $D(t,\rho) = \sqrt{1-\rho^2} \int_{g(t)}^{\pi/2} (\cos\phi)^\alpha d\phi$ for $t > 0$, then $D(t,\rho)$ is strictly decreasing in t and has the following properties:

$$\begin{cases} D'(t,\rho) = \frac{d}{dt} D(t,\rho) = -(\cos(g(t)))^{\alpha+2}, \\ D(t,\rho) = \int_t^\infty (\cos(g(s)))^{\alpha+2} ds = \int_0^{t^{-1}} s^\alpha (\cos(g(s)))^{\alpha+2} ds, \\ D(t,\rho) < D(0^+,\rho) = \lim_{t\to 0^+} D(t,\rho) < \infty, \\ D(t,\rho) > D(\infty,\rho) = \lim_{t\to\infty} D(t,\rho) = 0. \end{cases} \tag{3.4}$$

Further

$$H_1(x,1) = H_2(1,x) = \frac{D(x^{\frac{1}{\alpha}},\rho)}{2D(1,\rho)}. \tag{3.5}$$

Since the elliptical copula is symmetric, we also have $H_1(1,x) = H_2(x,1)$. Put them into (2.1) we have

$$\begin{aligned} \lim_{u\downarrow 0} \tau(u) &= 4\int_0^1 \int_0^1 H(x,y) dH(x,y) - 1 \\ &= 2\int_0^1 x H_1^2(x,1)dx + 2\int_0^1 y H_2^2(1,y)dy - 1 \\ &= 4\int_0^1 x H_1^2(x,1)dx - 1 \\ &= 4(\tfrac{1}{2} - \int_0^1 H_1(x,1)H_2(x,1)dx) - 1 \\ &= 1 - 4\int_0^1 H_1(x,1)H_1(1,x)dx \\ &= 1 - \int_0^1 \frac{D(x^{\frac{1}{\alpha}},\rho)D(x^{-\frac{1}{\alpha}},\rho)}{D^2(1,\rho)} dx. \end{aligned} \tag{3.6}$$

Hence, to show the limit is positive, it is equivalent to show that

$$\int_0^1 D(x^{\frac{1}{\alpha}},\rho)D(x^{-\frac{1}{\alpha}},\rho)dx < D^2(1,\rho).$$

which is sufficiently implied by

$$D(t,\rho)D(t^{-1},\rho) < D^2(1,\rho) \quad \textit{for} \quad 0<t<1. \tag{3.7}$$

By (3.4), for $0<t<1$ *we have*

$$\begin{aligned} D(t,\rho)D(t^{-1},\rho) &= \Big(D(1,\rho)+\textstyle\int_t^1(\cos(g(s)))^{\alpha+2}ds\Big)\Big(D(1,\rho)-\int_1^{t^{-1}}(\cos(g(s)))^{\alpha+2}ds\Big) \\ &= D^2(1,\rho)+\Big(\textstyle\int_t^1(\cos(g(s)))^{\alpha+2}ds-\int_1^{t^{-1}}(\cos(g(s)))^{\alpha+2}ds\Big)D(1,\rho) \\ &\quad -\textstyle\int_t^1(\cos(g(s)))^{\alpha+2}ds\int_1^{t^{-1}}(\cos(g(s)))^{\alpha+2}ds. \end{aligned} \tag{3.8}$$

Put $a=\int_t^1(\cos(g(s)))^{\alpha+2}ds$ *and* $b=\int_1^{t^{-1}}(\cos(g(s)))^{\alpha+2}ds=\int_t^1 s^{\alpha}(\cos(g(s)))^{\alpha+2}ds$, *let* a',b' *be the derivatives of functions* a *and* b *with respect to* t. *It follows that* $a>b>0$ *and* $a'<b'<0$, *and thus (3.7) is equivalent to*

$$D(1,\rho) < \frac{ab}{a-b}, \tag{3.9}$$

and taking the derivative of the left side of (3.9), we have

$$\frac{d}{dt}\Big(\frac{ab}{a-b}\Big) = \frac{a^2b'-a'b^2}{(a-b)^2} > \frac{a^2a'-a'a^2}{(a-b)^2} = 0. \tag{3.10}$$

Therefore,

$$\begin{aligned} \tfrac{ab}{a-b} &\geq \frac{\int_0^1(\cos(g(s)))^{\alpha+2}ds\int_1^{\infty}(\cos(g(s)))^{\alpha+2}ds}{\int_0^1(\cos(g(s)))^{\alpha+2}ds-\int_1^{\infty}(\cos(g(s)))^{\alpha+2}ds} \\ &= \frac{D(0,\rho)-D(1,\rho)}{D(0,\rho)-2D(1,\rho)}D(1,\rho) \\ &> D(1,\rho), \end{aligned} \tag{3.11}$$

which implies the limit of conditional Kendall's tau is positive.

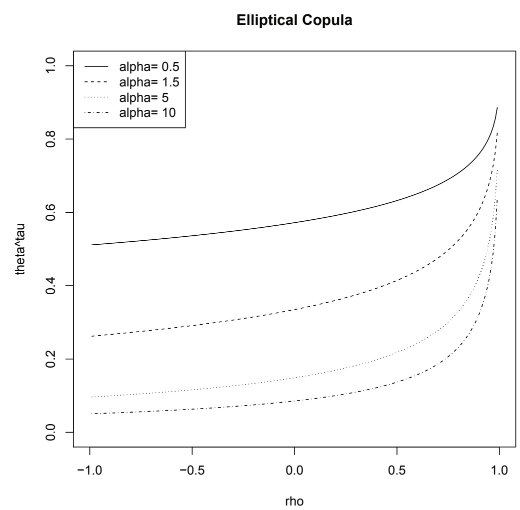

FIGURE 3.3. The limit of conditional Kendall's tau is plotted against parameters for elliptical copula from Example 3.3.

Example 3.4. *Assume that the survival copula is given by the Marshall-Olkin copula. That is, we have*

$$C(u,v)=\begin{cases} u^{1-\alpha}v & \textit{if} \quad u^{\alpha}\geq v^{\beta}, \\ uv^{1-\beta} & \textit{if} \quad u^{\alpha}<v^{\beta}, \end{cases} \tag{3.12}$$

where $0<\alpha,\beta<1$. Simple calculations yield that Assumption 2.1 holds with

$$H(x,y)=\begin{cases} xy^{1-\beta} & \text{if} \quad \alpha>\beta,\\ x^{1-\alpha}y & \text{if} \quad \alpha<\beta,\\ xy\big(\max\{x,y\}\big)^{-\alpha} & \text{if} \quad \alpha=\beta. \end{cases}$$

Therefore, Assumption 2.3 holds with $\eta=\big(2-\min\{\alpha,\beta\}\big)^{-1}$, $m=1$, and $\theta^{\tau}=0$ for $\alpha\neq\beta$. When $\alpha=\beta$, $\eta=(2-\alpha)^{-1}$, $H(x,y)$ has a positive mass along the line $y=x$ and Assumption 2.3 does not hold. In this case, some straightforward computations lead to $\mathbb{P}(U=V\le z)=\frac{\alpha}{2-\alpha}z^{2-\alpha}$ for $0\le z\le 1$, $\theta^{\tau}=\frac{4}{4-2\alpha}-1=\frac{\alpha}{2-\alpha}>0$, where (U,V) has the distribution $C(u,v)$ given in (3.12).

Example 3.5. *Consider the bivariate normal copula*

$$C^*(u,v)=\int_{-\infty}^{\Phi^-(u)}\int_{-\infty}^{\Phi^-(v)}\frac{1}{2\pi(1-\rho^2)^{1/2}}\exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}dydx,\quad |\rho|<1,$$

where Φ denotes the distribution function of the standard normal random variable. Then, it follows from Example 2.1 of Draisma et al. (2004) or Theorem 5.3 of Juri and Wüthrich (2003) that Assumption 2.3 holds with $H(x,y)=(xy)^{1/(1+\rho)}$ and $\eta=(1+\rho)/2$. Thus, Assumption 2.3 holds with $m=1$, and $\theta^{\tau}=0$. Interestingly, a more general result can be found for the class of elliptical copulas, as defined in Example 3.3, where $\bar G(\cdot)$ satisfies $\bar G\big(t+a(t)x\big)/\bar G(t)\sim e^{-x}$ and $a(ty)/a(t)\sim y^{-\alpha}$ as $t\to\infty$ for all $x\in\Re$ and $y>0$. In has been shown in Asimit and Jones (2007) that $H(x,y)=(xy)^{1/2\eta}$ where $\eta=\big(2/(1+\rho)\big)^{(\alpha-1)/2}$. Note that the Gaussian copula is a special case of this last result and it holds with $\alpha=-1$, which confirms the earlier finding. Once again, Assumption 2.3 holds with $m=1$, and $\theta^{\tau}=0$.

Example 3.6. *Consider the Farlie-Gumbel-Morgenstern copula*

$$C^*(u,v)=uv\big\{1+\xi(1-u)(1-v)\big\}\quad \text{with} \quad \xi\in[-1,1].$$

Simple computations yield that Assumption 2.1 holds with

$$H(x,y)=\begin{cases} xy & \text{if } \xi\in(-1,1],\\ \frac{xy(x+y)}{2} & \text{if } \xi=-1. \end{cases}$$

Hence, Assumption 2.3 holds with $(\eta,m)=(1/2,1)$ for $\xi\in(-1,1]$ and $(\eta,m)=(1/3,2)$ for $\xi=-1$. Further, $\theta^{\tau}=0$ for $\xi\in(-1,1]$, and $\theta^{\tau}=-\frac{1}{18}$ for $\xi=-1$.

4. Simulation study

In this section, we examine the finite sample behavior of the proposed estimator $\hat\theta^{\tau}(k)$ for estimating the limit of conditional Kendall's tau by drawing $1,000$ random samples with size $n=1000$ from Examples 3.2, 3.5 and 3.6 given in Section 3. For estimating the asymptotic variance of $\hat\theta^{\tau}(k)$ we simply employ the bootstrap method with $1,000$ re-samples. Based on these random samples, we have estimators $\hat\theta_{\tau}^{(i)}(k)$ and the corresponding bootstrap variance estimator $\sigma^{(i)}(k)$ for $i=1,\cdots,1000$. In Figures 4.1–4.4 we plot the estimator $\frac{1}{1000}\sum_{i=1}^{1000}\hat\theta_{\tau}^{(i)}(k)$, the bias $\frac{1}{1000}\sum_{i=1}^{1000}(\hat\theta_{\tau}^{(i)}(k)-\theta^{\tau})$, the mean squared error $\frac{1}{1000}\sum_{i=1}^{1000}(\hat\theta_{\tau}^{(i)}(k)-\theta^{\tau})^2$ and the ratio of asymptotic variance to its bootstrap estimator

$$\sum_{i=1}^{1000}\left(\hat\theta_{\tau}^{(i)}(k)-\frac{1}{1000}\sum_{j=1}^{1000}\hat\theta_{\tau}^{(j)}(k)\right)^2\Big/\sum_{i=1}^{1000}\sigma^{(i)}(k)$$

against $k = 21, \cdots, 300$. These figures show that the estimator and its bootstrap variance estimator work well for k around 150. Without doubt, more research on choosing the tuning parameter k in estimating θ^τ, $\theta^\tau + \eta$, and corresponding bias reduced estimators is needed in the near future.

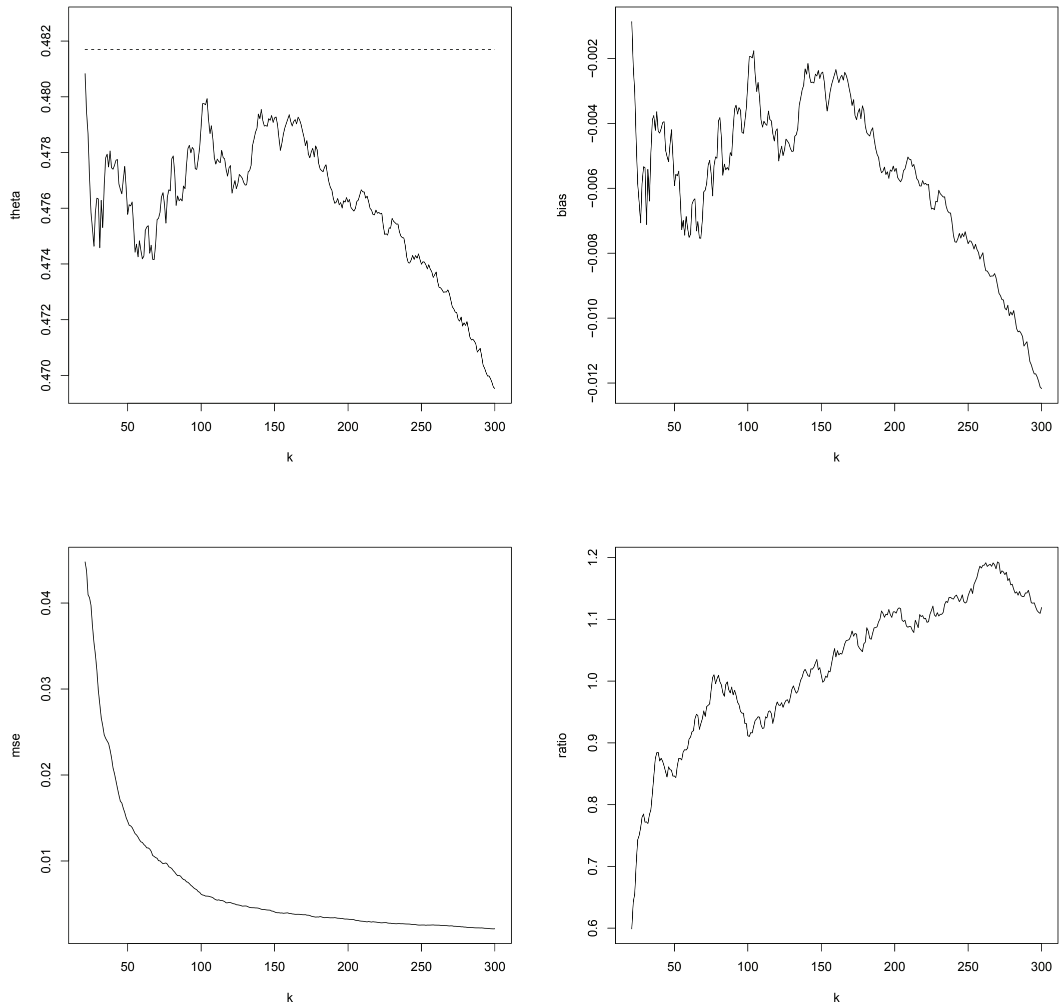

FIGURE 4.1. The estimator $\hat{\theta}^\tau(k)$, its bias, mean squared error and ratio of asymptotic variance to the bootstrap estimator are plotted against $k = 21, \cdots, 300$ for t copula with $\rho = 0.5$ and $\nu = 1$ given in Example 3.2 of Section 3.

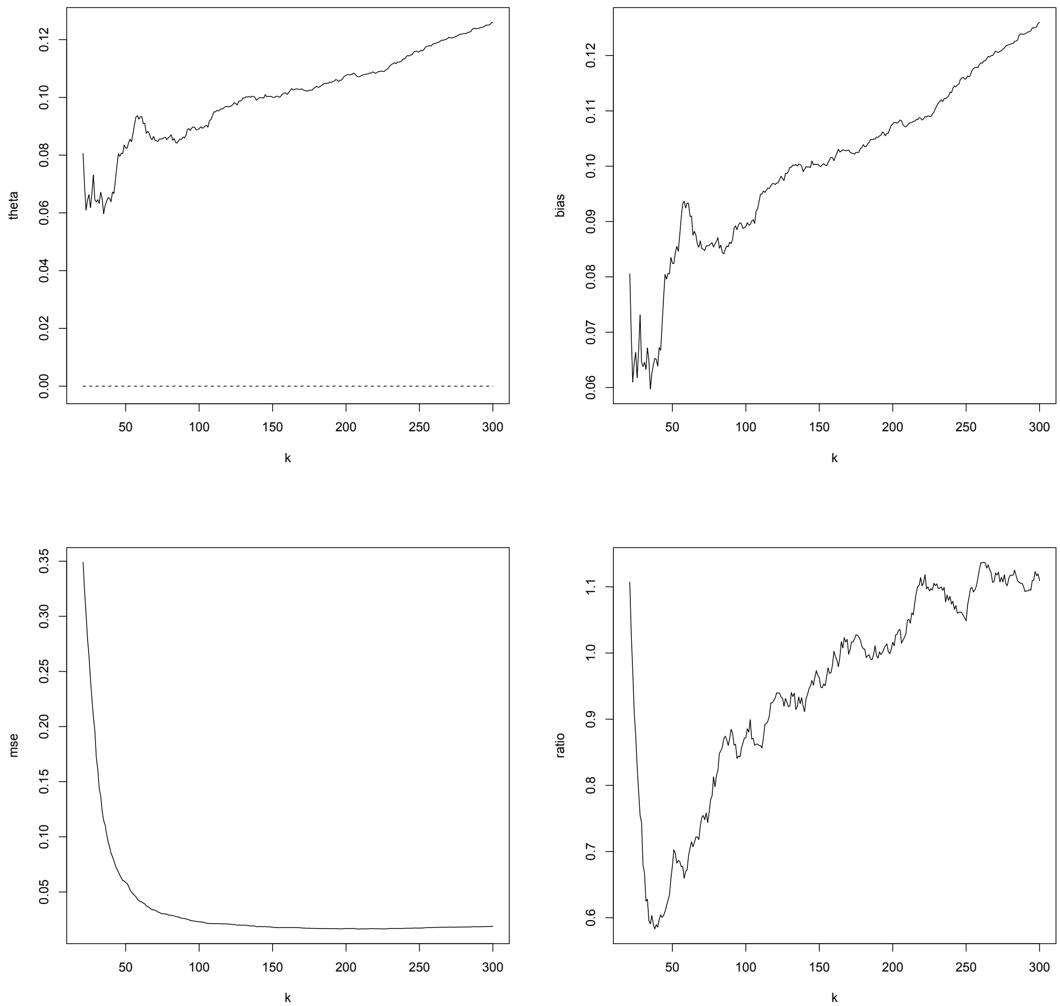

FIGURE 4.2. The estimator $\hat{\theta}^{\tau}(k)$, its bias, mean squared error and ratio of asymptotic variance to the bootstrap estimator are plotted against $k = 21, \cdots, 300$ for normal copula with $\rho = 0.5$ given in Example 3.5 of Section 3.

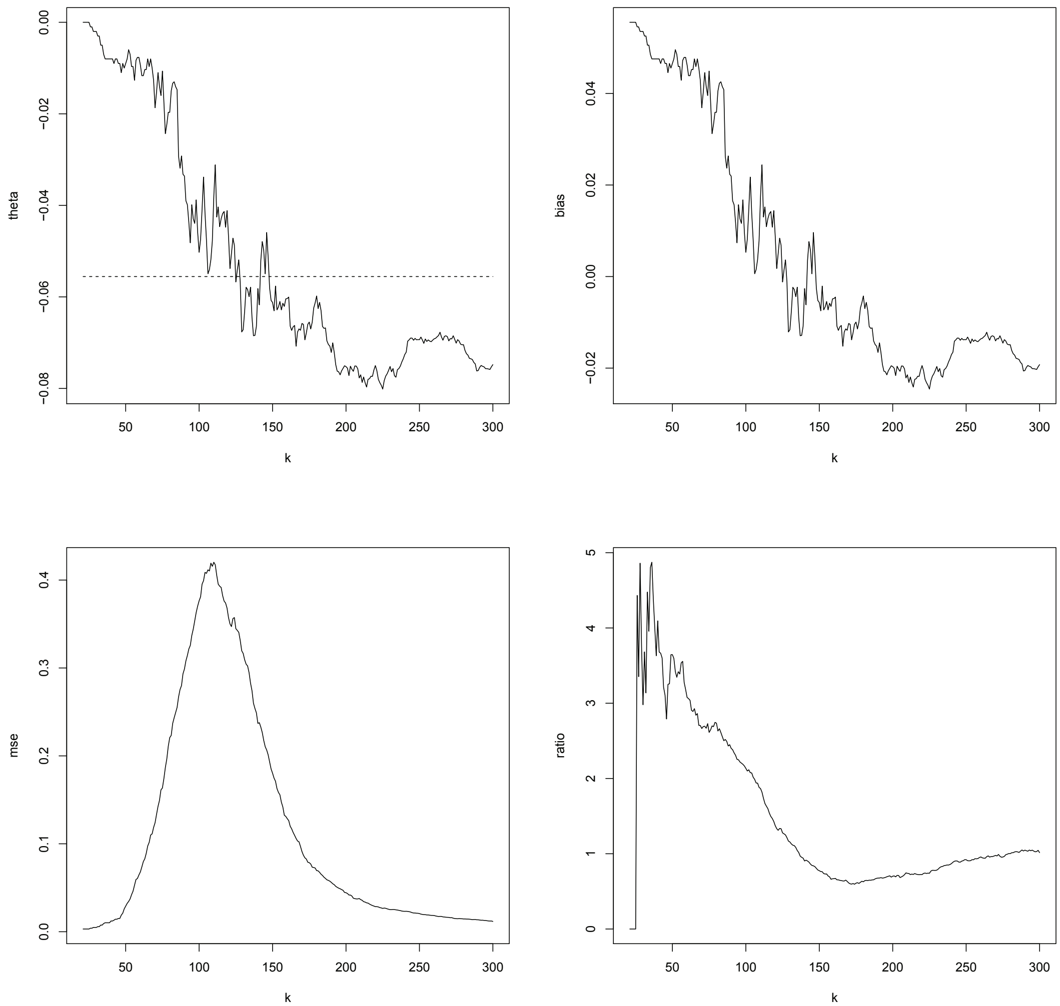

FIGURE 4.3. The estimator $\hat{\theta}^{\tau}(k)$, its bias, mean squared error and ratio of asymptotic variance to the bootstrap estimator are plotted against $k = 21, \cdots, 300$ for Farlie-Gumbel-Morgenstern copula with $\xi = -1$ and given in Example 3.6 of Section 3.

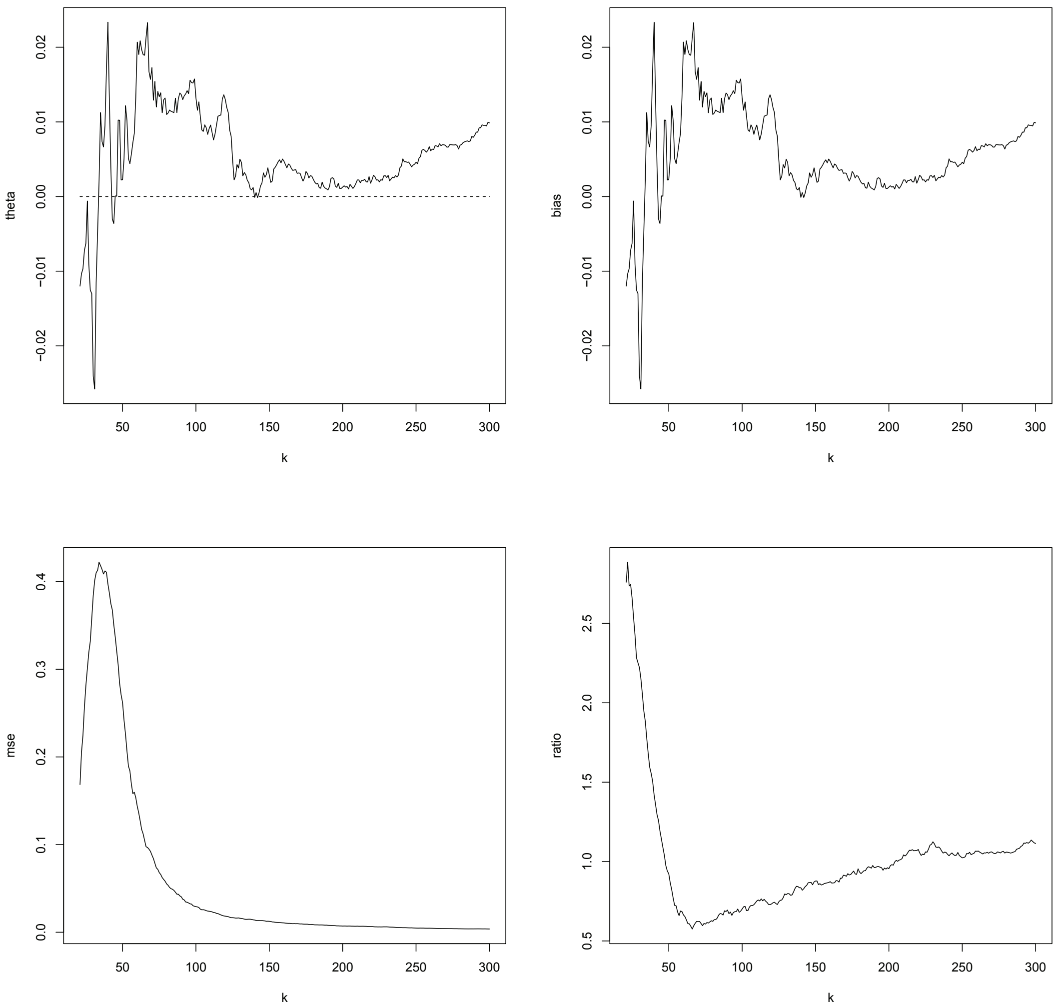

Figure 4.4. The estimator $\hat{\theta}^{\tau}(k)$, its bias, mean squared error and ratio of asymptotic variance to the bootstrap estimator are plotted against $k = 21, \cdots, 300$ for Farlie-Gumbel-Morgenstern copula with $\xi = 1$ and given in Example 3.6 of Section 3.

5. Real data analysis

In this section, we analyze the tail dependence of the following three data sets by estimating $\eta, \theta^\tau, \theta^\tau + \eta$ by $\hat\eta(k), \hat\theta^\tau(k), \hat\theta^\tau(k) + \hat\eta(k)$, respectively, where $\hat\eta(k)$ is the Hill estimator based on the largest k order statistics of

$$\Big\{T_i = \min\{\frac{n+1}{n+1-R_i^X}, \frac{n+1}{n+1-R_i^Y}\}\Big\}_{i=1}^n$$

with R_i^X being the rank of X_i among $X_1, \cdots, X_n$ and R_i^Y being the rank of Y_i among $Y_1, \cdots, Y_n$. More details on $\hat\eta(k)$ can be found in Draisma *et al.* (2004). For constructing confidence intervals for η, θ^τ, $\theta^\tau + \eta$ via corresponding estimators, we simply employ the bootstrap method with $1,000$ replications.

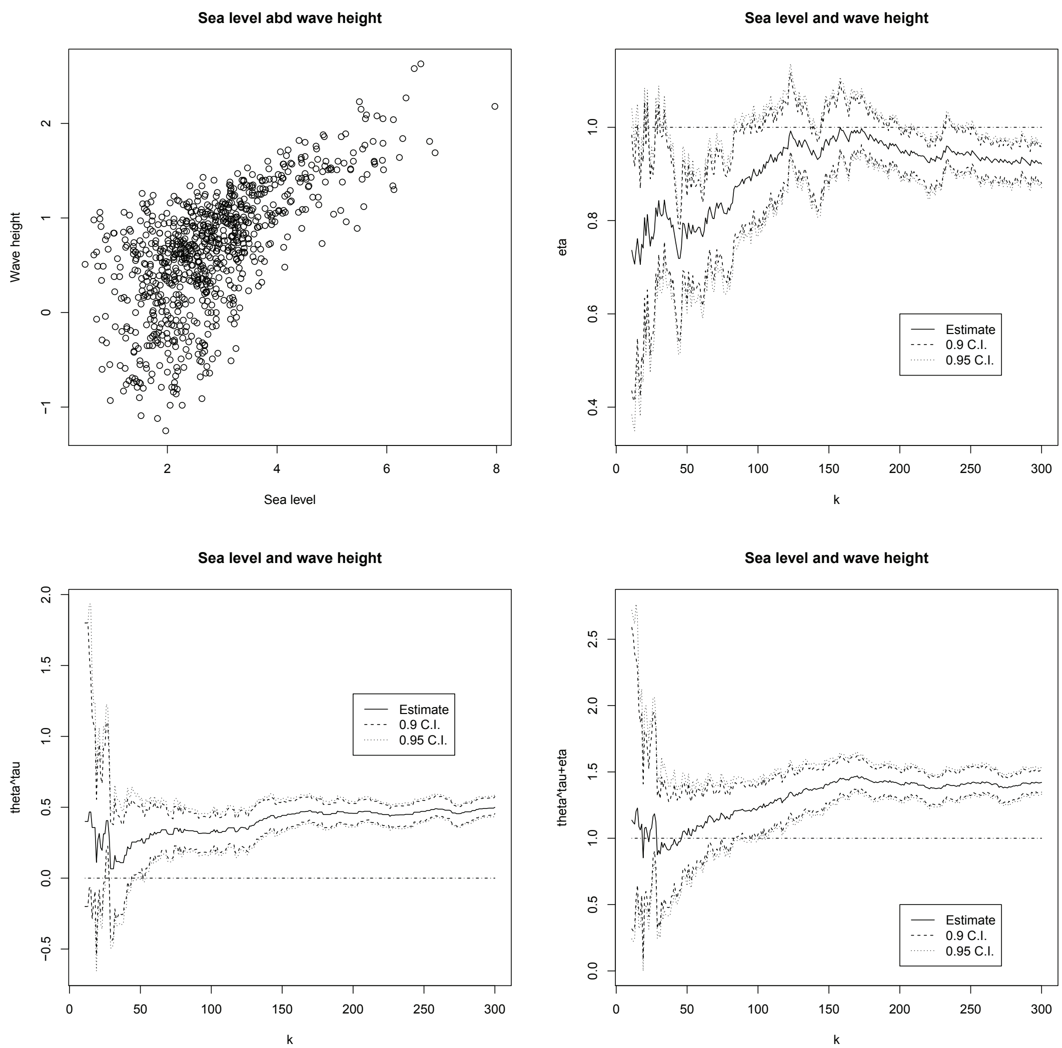

Figure 5.1. Sea level and wave height. Estimators $\hat\eta(k)$, $\hat\theta^\tau(k)$, $\hat\theta^\tau(k) + \hat\eta(k)$, and their intervals with level 0.9 and 0.95 are plotted against k.

First, we consider the sea level and wave height measured at the Eierland station, 20 km off the Dutch coast from 1979 through 1991; see the left upper panel in Figure 5.1. The right upper panel depicts the $\hat\eta(k)$ and its intervals, which may suggest asymptotic independence by looking at k near 50 as argued in

Draisma *et al.* (2004). However, the left lower panel may well suggest $\theta^\tau > 0$ by looking at the range of $50 < k < 100$, i.e., the data set is asymptotically dependent. The right lower panel do not claim that $\theta^\tau + \eta < 1$, i.e. asymptotic independence, even when one chooses a smaller k. Therefore, it is reasonable to assume asymptotic dependence and so it is recommended to employ the asymptotic dependent classical Extreme Value Theory to predict extreme co-movements.

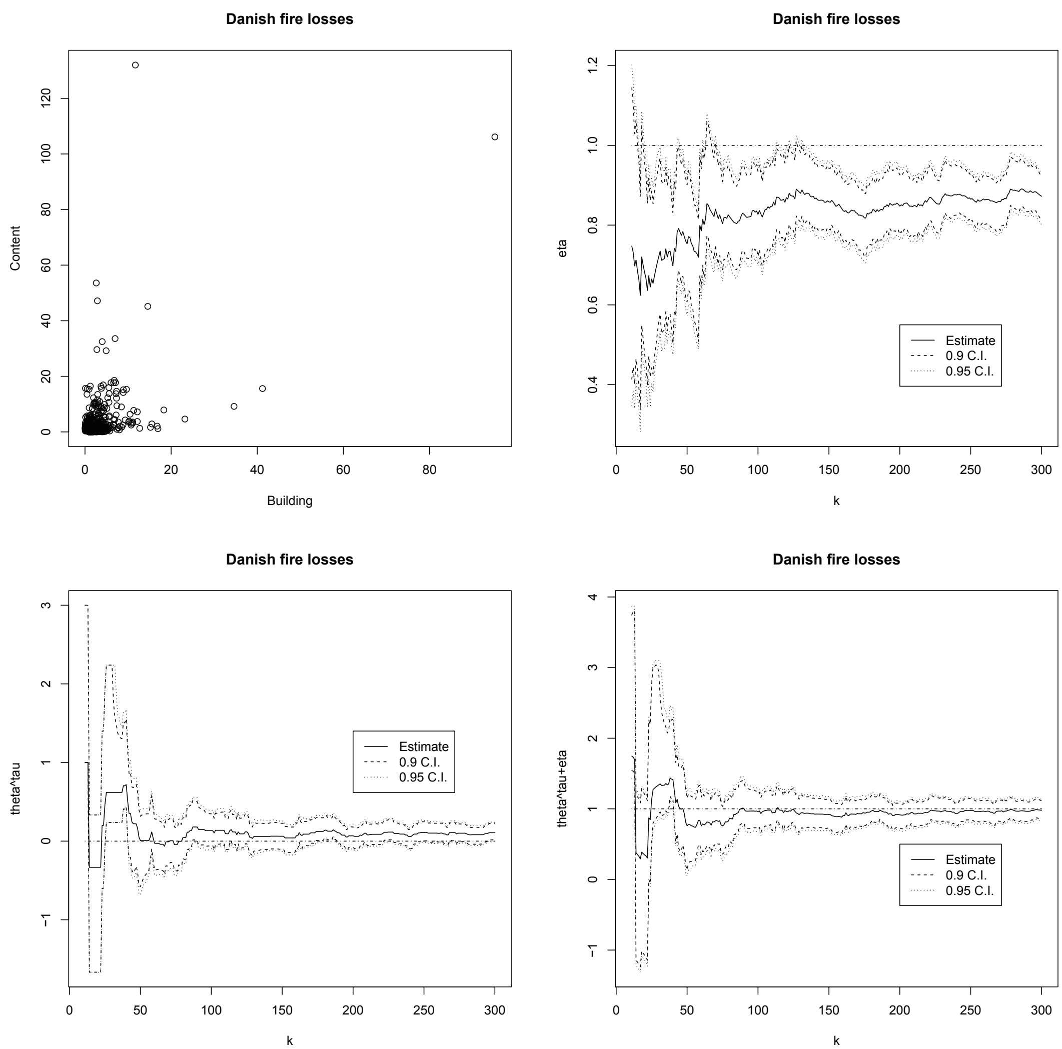

FIGURE 5.2. Danish fire losses. Estimators $\hat{\eta}(k)$, $\hat{\theta}^\tau(k)$, $\hat{\theta}^\tau(k) + \hat{\eta}(k)$, and their intervals with level 0.9 and 0.95 are plotted against k.

Next, we consider the non-zero losses to building and content in the Danish fire insurance claims; see the left upper panel in Figure 5.2. This data set is available at *www.ma.hw.ac.uk/∼mcneil/*, which comprises 2,167 fire losses over the period 1980 to 1990. The right upper panel may prefer $\eta < 1$, i.e., asymptotic independence. However, the lower panels can neither claim asymptotic independence nor asymptotic dependence. Therefore one may claim asymptotic independence for this data set. On the other hand, given the fact that distinguishing asymptotic behavior is extremely challenging, one has to take a caution of making the claim of asymptotic independence since this claim is not confirmed by the two new measures $\hat{\theta}^\tau(k)$ and $\hat{\theta}^\tau(k) + \hat{\eta}(k)$.

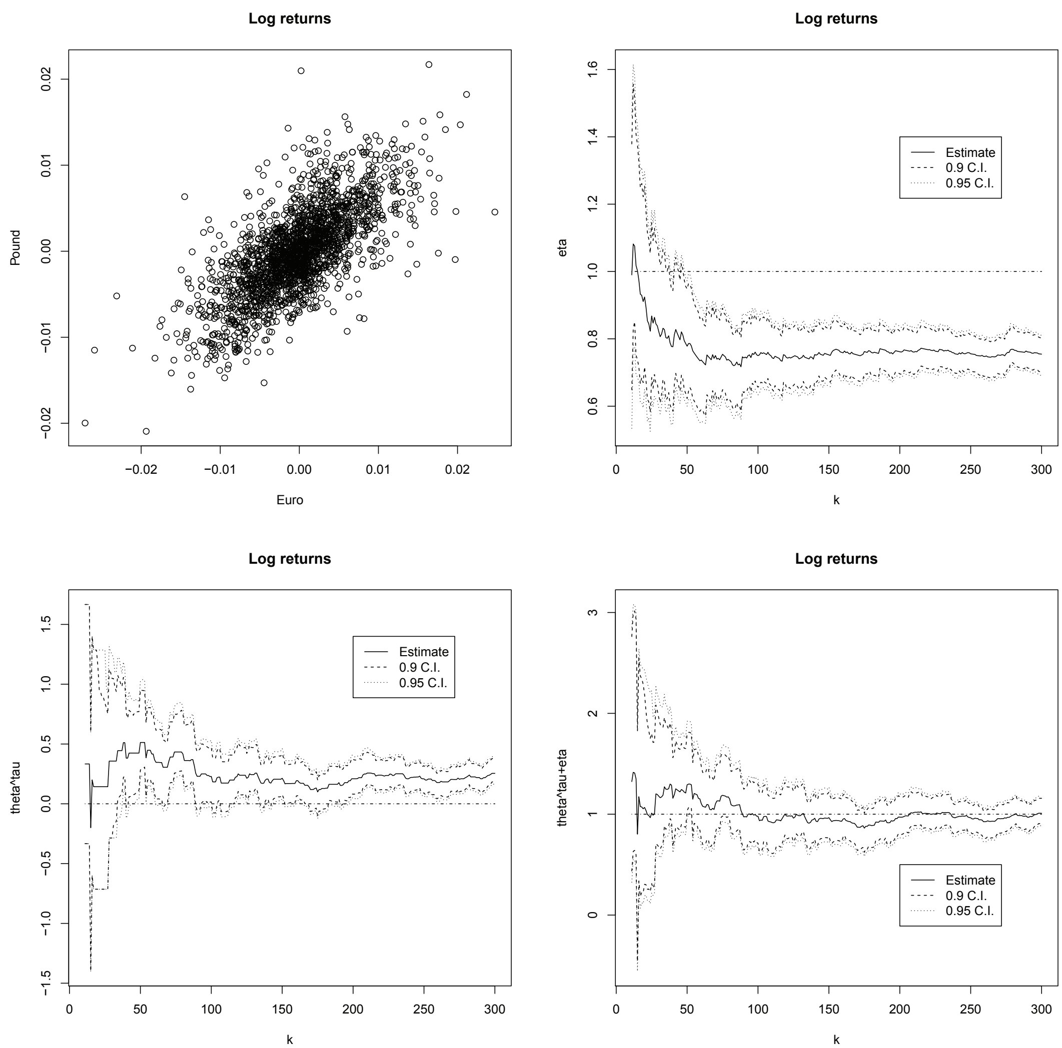

FIGURE 5.3. Log returns of exchange rates. Estimators $\hat{\eta}(k)$, $\hat{\theta}^\tau(k)$, $\hat{\theta}^\tau(k) + \hat{\eta}(k)$, and their intervals with level 0.9 and 0.95 are plotted against k.

Finally, we consider the log-returns of the exchange rates between Euro and US dollar and those between British pound and US dollar from January 3, 2000 until December 19, 2007; see the left upper panel in Figure 5.3. The right upper panel may well suggest $\eta < 1$, i.e., asymptotic independence. The left lower panel may prefer $\theta^\tau > 0$, i.e., asymptotic dependence. The right lower panel can neither claim asymptotic independence nor asymptotic dependence. Therefore, it remains cautious to claim the asymptotic behavior for this data set, which calls for more effective methods.

In summary, the proposed new measure of tail dependence and its combination with the coefficient of tail dependence are useful in distinguishing between asymptotic dependence and asymptotic independence, so as to ensure a sound application of multivariate Extreme Value Theory to the study of extreme co-movements in financial markets and so to predicting extreme events.

6. Proofs

Proof of Theorem 2.1. Since

$$\begin{aligned}
&\mathbb{P}\big(U_1 > U_2, V_1 > V_2 | \max(U_1, U_2, V_1, V_2) \le u\big) + \mathbb{P}\big(U_1 > U_2, V_1 < V_2 | \max(U_1, U_2, V_1, V_2) \le u\big) \\
&= \mathbb{P}\big(U_1 > U_2 | \max(U_1, U_2, V_1, V_2) \le u\big) = \tfrac{1}{C^2(u,u)} \int_0^u C(t,u) \mathbb{P}(U_1 \in dt, V_1 \le u) \\
&= \tfrac{1}{2},
\end{aligned}$$

it follows from (1.6) that

$$\begin{aligned}
\tau(u) &= 2\mathbb{P}\big(U_1 > U_2, V_1 > V_2 | \max(U_1, U_2, V_1, V_2) \le u\big) - 2\mathbb{P}\big(U_1 > U_2, V_1 < V_2 | \max(U_1, U_2, V_1, V_2) \le u\big) \\
&= 4\mathbb{P}\big(U_1 > U_2, V_1 > V_2 | \max(U_1, U_2, V_1, V_2) \le u\big) - 1.
\end{aligned} \tag{6.1}$$

Next, we define the following probability measure

$$\nu_u(\cdot) := \mathbb{P}\Big(\big(U_1/u, V_1/u, U_2/u, V_2/u\big) \in \cdot | U_1, U_2, V_1, V_2 \le u\Big)$$

on $\mathcal{E} := [0,1]^4$. Thus, due to equation (2.2) and the independence assumption between (U_1, V_1) and (U_2, V_2), we have that

$$\nu_u(\cdot) \stackrel{w}{\to} \nu(\cdot) \tag{6.2}$$

holds on $\mathcal{E}$ as $u \to 0$, where the measure ν is given by

$$\nu\big([0,x_1] \times [0,y_1] \times [0,x_2] \times [0,y_2]\big) := H(x_1,y_1)H(x_2,y_2).$$

Let $A := \{0 \le x_2 < x_1 \le 1, 0 \le y_2 < y_1 \le 1\}$. Therefore, relation (6.2) leads to

$$\mathbb{P}\big(U_1 > U_2, V_1 > V_2 | \max(U_1, U_2, V_1, V_2) \le u\big) = \nu_u(A) \to \nu(A) = \int_{\mathcal{D}} H(x,y)\, dH(x,y), \text{ as } u \downarrow 0 \tag{6.3}$$

as long as $\nu(\partial A) = 0$, which remains to justify. Note that

$$\begin{aligned}
\nu(\partial A) &\le \nu\big(x_1 = x_2, y_1 \ge y_2\big) + \nu\big(x_1 \ge x_2, y_1 = y_2\big) \\
&+ \nu\big(x_1 \ge x_2, y_1 \ge y_2,\ x_1 y_1 = 0 \text{ or } x_1 = 1 \text{ or } y_1 = 1\big) \\
&+ \nu\big(x_1 \ge x_2, y_1 \ge y_2,\ x_2 y_2 = 0 \text{ or } x_2 = 1 \text{ or } y_2 = 1\big).
\end{aligned}$$

The first two terms are equal to zero since no mass is put by the measure ν over the lines $x_1 = x_2$ and $y_1 = y_2$ due to the independence between (U_1, V_1) and (U_2, V_2). The last two terms are also negligible and due to symmetry, it is sufficient to justify only one of them. Denote $B = \big\{(x_1, y_1) : x_1 y_1 = 0 \text{ or } x_1 = 1 \text{ or } y_1 = 1\big\}$ and note that

$$\nu\big(x_1 \ge x_2, y_1 \ge y_2,\ x_1 y_1 = 0 \text{ or } x_1 = 1 \text{ or } y_1 = 1\big) \le \int_B \mu\big(dx_1, dy_1\big) = 0,$$

since H continuous on $\{(x,y) : xy = 0\}$ (due to Assumption 2.1) and the fact that $\mu(x_1 = 1) = \mu(y_1 = 1) = 0$, where the measure μ is defined in (2.2). The later is true, since otherwise we find a contradiction as follows

$$\begin{aligned}
\mu(x_1 \geq y_1 > 0) &\geq \mu\left(\bigcup_{q\in Q\bigcap(0,1]} \{x_1 = q, y_1 \leq q\}\right) \\
&= \sum_{q\in Q\bigcap(0,1]} \mu\big(\{x_1 = q, y_1 \leq q\}\big) \\
&= \mu\big(\{x_1 = 1\}\big) \sum_{q\in Q\bigcap(0,1]} q^a = \infty,
\end{aligned}$$

where $a \geq 1$ is the homogeneous order of H. Therefore, (2.1) follows from equations (6.1) and (6.3). $\square$

Proof of Proposition 2.1. i) Clearly, $\int_0^u f_X(x)\,dx = H(u,1) \geq H(u,u) = uH(1,1) = u$. Similarly, one may get the mirror result for f_Y. Differentiating (2.3) with respect to t in the case $\eta = 1$, we have $H_1(tu,tv) + H_2(tu,tv) = H(u,v)$, and therefore, $f_X(1) + f_Y(1) = 1$ is true. Now, differentiating (2.3) with respect to u (respectively v), we have

$$H_1(tu,tv) = H_1(u,v) \quad \text{(respectively } H_2(tu,tv) = H_2(u,v)\text{)}.$$

Let us first look at the case $x < y$. By setting $v = 1$, $t = y$, $u = x/y$ in the above equation, we have

$$H_1(x,y) = H_1\left(\frac{x}{y},1\right) = f_X\left(\frac{x}{y}\right),$$

and in turn, differentiating with respect to y gives $h(x,y) = -\frac{x}{y^2} f_X'\left(\frac{x}{y}\right)$. Note that the left-hand side of the latter equation is a bivariate density function, and thus, it is non-negative. In addition, it follows that $f_X' \leq 0$. The same procedure can be applied in the case $y < x$ in order to justify (2.4).

Suppose that $u \leq v$. Now,

$$\begin{aligned}
H(u,v) &= \int_0^u dx \int_0^v dy\, h(x,y) \\
&= -\int_0^u dx \left\{\int_0^x dy\, \frac{y}{x^2} f_Y'\left(\frac{y}{x}\right) + \int_x^v dy\, \frac{x}{y^2} f_X'\left(\frac{x}{y}\right)\right\} \\
&= -\int_0^u dx \left\{\int_0^1 dw\, w f_Y'(w) + \int_{\frac{x}{v}}^1 dz\, f_X'(z)\right\} \\
&= -\int_0^u dx \left\{[w f_Y(w)]_0^1 - \int_0^1 dw\, f_Y(w) + f_X(1) - f_X\left(\frac{x}{v}\right)\right\} \\
&= -\int_0^u dx \left\{(1-\xi) - 1 + \xi - f_X\left(\frac{x}{v}\right)\right\} \\
&= \int_0^{\frac{u}{v}} dw\, v f_X(w) \\
&= vF_X\left(\frac{u}{v}\right).
\end{aligned}$$

Again, the same procedure can be applied for $u > v$, and thus part i) is justified.

ii) The function h is certainly non-negative, since f_X and f_Y are non-increasing functions. In addition, the integration procedure to derive H from h has been accomplished above. Moreover, it is elementary to check that $H(u,1) = F_X(u)$ and $H(1,v) = F_Y(v)$. Finally, part ii) is concluded due to

$$H(tu,tv) = I_{tu<tv}\, tvF_X\left(\frac{tu}{tv}\right) + I_{tu\geq tv}\, tuF_Y\left(\frac{tv}{tu}\right) = tH(u,v).$$

□

Proof of Theorem 2.2. Since $H(tx,ty)=tH(x,y)$, by taking derivatives with respect to t at both sides, we have $xH_1(tx,ty)+yH_2(tx,ty)=H(x,y)$, i.e., $txH_1(tx,ty)+tyH_2(tx,ty)=tH(x,y)=H(tx,ty)$, which implies that

$$xH_1(x,y)+yH_2(x,y)=H(x,y)\quad\text{for all}\quad (x,y)\in\mathcal{D}. \tag{6.4}$$

By taking the derivative with respect to x in (6.4), one may show

$$xH_{11}(x,y)+yh(x,y)=0\quad\text{for all}\quad (x,y)\in\mathcal{D}. \tag{6.5}$$

Similarly, $yH_{22}(x,y)+xh(x,y)=0$ holds for all $(x,y)\in\mathcal{D}$. By (6.4), we can write

$$\int_0^1\int_0^1 H(x,y)h(x,y)\,dxdy=\int_0^1\int_0^1 xH_1(x,y)h(x,y)\,dxdy+\int_0^1\int_0^1 yH_2(x,y)h(x,y)\,dxdy. \tag{6.6}$$

It follows from (6.5) that

$$\begin{aligned}
&\int_0^1\int_0^1 xH_1(x,y)h(x,y)\,dxdy && (6.7)\\
=\;&\int_0^1\int_0^y xH_1(x,y)h(x,y)\,dxdy+\int_0^1\int_0^x xH_1(x,y)h(x,y)\,dydx\\
=\;&\int_0^1\int_0^1 xyH_1(xy,y)h(xy,y)y\,dxdy+\int_0^1\int_0^1 xH_1(x,xy)h(x,xy)x\,dydx\\
=\;&\int_0^1\int_0^1 xyH_1(x,1)h(x,1)\,dxdy+\int_0^1\int_0^1 xH_1(1,y)h(1,y)\,dydx\\
=\;&\frac{1}{2}\int_0^1 xH_1(x,1)h(x,1)\,dx+\frac{1}{2}\int_0^1 H_1(1,y)h(1,y)\,dy && (6.8)\\
=\;&\frac{1}{2}\int_0^1 xH_1(x,1)h(x,1)\,dx+\frac{1}{4}H_1^2(1,1)\\
=\;&-\frac{1}{2}\int_0^1 x^2H_1(x,1)H_{11}(x,1)\,dx+\frac{1}{4}H_1^2(1,1)\\
=\;&-\frac{1}{4}\int_0^1 x^2\,dH_1^2(x,1)+\frac{1}{4}H_1^2(1,1)\\
=\;&\frac{1}{2}\int_0^1 xH_1^2(x,1)\,dx.
\end{aligned}$$

Following the same steps as above, we can show that

$$\int_0^1\int_0^1 yH_2(x,y)h(x,y)\,dxdy=\frac{1}{2}\int_0^1 yH_2^2(1,y)\,dy. \tag{6.9}$$

Now, Theorem 2.1 together with relations (6.6)–(6.9) yield

$$\theta^\tau=2\int_0^1 xH_1^2(x,1)\,dx+2\int_0^1 yH_2^2(1,y)\,dy-1. \tag{6.10}$$

Note that

$$H(x,1)=\lim_{u\to 0}\frac{C(ux,u)}{cu}\le\frac{x}{c}\ \text{ and }\ H(x,1)\ge H(x,x)=xH(1,1)=x. \tag{6.11}$$

The first step is to find a decreasing density function f with support $(0,1)$ and an associated distribution function F in such a way as to minimize the objective function

$$J = \int_0^1 x f^2(x)\, dx$$

subject to the constraints that $c^{-1}x \geq F(x) \geq x$ for all $0 \leq x \leq 1$ (due to (6.11)) and that $\lim_{x\to 1} f(x) = \xi$, where $c \leq 1$ and $\xi \in (0,1)$ are constants. We regard this as a problem of finding the minimal-cost trajectory from $x = 0, F = 0$ to $x = 1, F = 1$, which we approach by a Dynamic Programming argument. Denote by $V(x,F)$ the following minimum

$$V(x,F) = \inf_{f \in \mathcal{F}} \left\{ \int_x^1 y f^2(y)\, dy \text{ subject to } \int_x^1 f(y)\, du = 1 - F \right\}.$$

Suppose we are starting from position (x_0, F_0). Further, consider a strategy which sets $f(x) = u$ for $x_0 \leq x < x_0 + h$ and uses the optimal strategy for $x_0 + h \leq x \leq 1$. The cost of this strategy is

$$\int_{x_0}^{x_0+h} x u^2\, dx + V(x_0 + h, F_0 + uh) = x_0 u^2 h + V(x_0, F_0) + hV_1(x_0, F_0) + uhV_2(x_0, F_0) + o(h).$$

If we choose u optimally, we now have an optimal strategy from x_0 to 1; in other words,

$$V(x_0, F_0) = \inf_{u \in \mathcal{A}(x_0,F_0)} \left\{ V(x_0, F_0) + (x_0 u^2 + V_1(x_0, F_0) + uV_2(x_0, F_0))h + o(h) \right\},$$

where V_1 and V_2 represent the partial derivatives of V and where $\mathcal{A}(x_0, F_0)$ represents the set of values u is permitted to take. This consists of $[0, f(x_0)]$ if (x_0, F_0) is in the interior of the accessible region, $[1, f(x_0)]$ if it is on the right-hand boundary, $[0, c^{-1}]$ if on the left-hand boundary.
As we let $h \to 0$, it can be seen that

$$\inf_{u \in \mathcal{A}(x_0,F_0)} \left\{ x_0 u^2 + V_1(x_0, F_0) + uV_2(x_0, F_0) \right\} = 0,$$

which is the optimality equation.
Minimizing over u, the optimal value u^* satisfies $u^*(x_0, F_0) = -\frac{1}{2x_0} V_2(x_0, F_0)$, as long as $u^* \in \mathcal{A}(x_0, F_0)$, in which case we conclude that $V_1(x_0, F_0) = \frac{1}{4x_0} V_2^2(x_0, F_0)$.
Let f be a feasible strategy and denote by V^f the associated value function $V^f(x_0, F_0) = \int_{x_0}^1 x f^2(x)\, dx$. If V^f satisfies the optimality equation and the associated boundary conditions, then f is the optimal strategy and $V = V^f$. Our approach, then, is to display the optimal strategy and to check that the optimality equation and boundary conditions are satisfied.
Define $k = -(1-\xi)/\log(c\xi)$ and we show now that the optimal trajectory starting from $(0,0)$ is

$$\begin{array}{llll} f(x) = 1/c & \text{and} & F(x) = x/c & \text{if } x < ck, \\ f(x) = k/x & \text{and} & F(x) = k + k\log x - k\log(ck) & \text{if } ck \leq x \leq k/\xi, \\ f(x) = \xi & \text{and} & F(x) = 1 - \xi(1-x) & \text{if } k/\xi < x \leq 1. \end{array} \tag{6.12}$$

Let D denote the triangular region bounded below by $F = x$ and above by $F = x/c$ and $F = 1 - \xi(1-x)$. D therefore represents the set of points which are accessible from $(0,0)$ and from which $(1,1)$ is accessible without violating the restrictions. We divide D into sub-regions as follows:

- A is the region bounded below by $F = x$ and above by the curve $F = 1 + \xi \log x$.
- B is the region bounded above by $F = x/c$, below by $F = x$ and to the right by the curve $F = k - k\log(ck) + k\log x$, where $k = -(1-\xi)/\log(c\xi)$.

- $C = D \cap (A \cup B)^c$.

In order to fully justify (6.12), the following claims will be shown:

(i) For $(x_0, F_0) \in A$, the trajectory which minimizes J, and the associated optimal value function, are $1 - F(x) = (1 - F_0)\frac{\log x}{\log x_0}$ and $V(x_0, F_0) = \frac{(1-F_0)^2}{-\log x_0}$, respectively;

(ii) For $(x_0, F_0) \in B$, the optimal strategy is to follow the trajectory $F(x) = F_0 + \frac{x_L}{c}\log\left(\frac{x}{x_0}\right)$ until it hits the point $(x_L, x_L/c)$, after which it follows the trajectory presented in (6.12). In addition, x_L is the solution of the equation

$$x_L = cF_0 + x_L \log(x_L/x_0), \tag{6.13}$$

and the optimal value function in region B is given by

$$V(x_0, F_0) = \frac{x_L^2}{c^2}\log\left(\frac{x_L}{x_0}\right) - \frac{x_L^2}{2c^2} + k(1-\xi) + \frac{1}{2}\xi^2.$$

(iii) For $(x_0, F_0) \in C$, the optimal strategy is to follow the trajectory $F(x) = F_0 + \xi x_U \log\left(\frac{x}{x_0}\right)$ until it hits the point $\big(x_U, 1 - \xi(1 - x_U)\big)$, after which it follows the trajectory presented in (6.12). In addition, x_U is the solution of the equation $\xi x_U + 1 - F_0 - \xi = \xi x_U \log(x_U/x_0)$, and the optimal value function in region C is given by $V(x_0, F_0) = \xi^2 x_U^2 \log\left(\frac{x_U}{x_0}\right) + \frac{1}{2}\xi^2(1 - x_U^2)$.

First of all, claim (i) does not claimed that the strategy is optimal. This is because the natural trajectory from (x_0, F_0) to $(1,1)$, which is the one given in (6.12), arrives at $(1,1)$ with $f(1-) > \xi$. In order to fit the criteria for acceptable trajectories, a small adjustment is required in the region of 1 so that $f(1-) = \xi$. The scale of the adjustment can be as small as desired, but it means that there is no optimal strategy, only a collection of ϵ-optimal strategies for any ϵ.

We first show claim (i). We begin by verifying that V and the proposed strategy satisfy the optimality equation. Note that

$$\frac{\partial V}{\partial F_0} = -2\frac{1 - F_0}{-\log x_0}, \quad \frac{\partial V}{\partial x_0} = \frac{(1-F_0)^2}{(-\log x_0)^2} \cdot \frac{1}{x_0},$$

so that $V_2^2 = 4xV_1$, as required. One can check that $\frac{dF}{dx}\big|_{x=x_0} = -\frac{1}{2x_0}\frac{\partial V}{\partial F_0}$. f^* is non-increasing, since it takes the form constant$/x$.

Finally, we need to check that the optimal value of f is at least equal to 1 when (x_0, F_0) lies on the lower boundary of A, i.e., when $F_0 = x_0$. In this case $f^* = \frac{1-x_0}{-x_0 \log x_0} = y^{-1}(e^y - 1)$ if we write $x = e^{-y}$. Since we know that $e^y > 1 + y$, this is fine.

The proof of claim (ii) is less straightforward, as the quantity x_L, which features in the statement of the optimal strategy, is defined by an implicit equation (6.13). However, we have

$$\frac{1}{c}\frac{\partial x_L}{\partial F_0} = 1 + \frac{1}{c}\log(x_L/x_0)\frac{\partial x_L}{\partial F_0} + \frac{1}{c}\frac{\partial x_L}{\partial F_0}, \quad \text{so that} \quad \frac{\partial x_L}{\partial F_0} = -\frac{c}{\log(x_L/x_0)},$$

and

$$\frac{1}{c}\frac{\partial x_L}{\partial x_0} = \frac{1}{c}\log(x_L/x_0)\frac{\partial x_L}{\partial x_0} + \frac{1}{c}\frac{\partial x_L}{\partial x_0} - \frac{x_L}{cx_0}, \quad \text{so that} \quad \frac{\partial x_L}{\partial x_0} = \frac{x_L/x_0}{\log(x_L/x_0)},$$

Now, $\frac{\partial V}{\partial F_0} = 2\frac{x_L}{c^2}\log\left(\frac{x_L}{x_0}\right)\frac{\partial x_L}{\partial F_0} = -2\frac{x_L}{c}$ and $\frac{\partial V}{\partial x_0} = 2\frac{x_L}{c^2}\log\left(\frac{x_L}{x_0}\right)\frac{\partial x_L}{\partial x_0} - \frac{x_L^2}{c^2 x_0} = \frac{x_L^2}{c^2 x_0}$, and it is apparent that the optimality equation is satisfied. In addition, f is decreasing over this range and, at $x = x_0$, $\frac{dF}{dx}\big|_{x=x_0} = \frac{x_L}{cx_0} = -\frac{1}{2x_0}\frac{\partial V}{\partial F_0}$. On the lower boundary, where $x_0 = F_0$, we need to show that $f^* \geq 1$. But

$f^* = x_L/(cx_0)$, and $c < 1$, $x_0 \le x_L$, so that is fine. On the upper boundary, where $F_0 = x_0/c$, x_L is by definition equal to x_0, and $-\frac{1}{2x_0}V_2 = 1/c$, as required.

The proof of claim (iii) is very similar to the proof of claim (ii). We have

$$\xi\frac{\partial x_U}{\partial F_0} - 1 = \xi\log(x_U/x_0)\frac{\partial x_U}{\partial F_0} + \xi\frac{\partial x_U}{\partial F_0}, \quad \text{so that} \quad \frac{\partial x_U}{\partial F_0} = -\frac{1}{\xi\log(x_U/x_0)},$$

and

$$\xi\frac{\partial x_U}{\partial x_0} = \xi\log(x_U/x_0)\frac{\partial x_U}{\partial x_0} + \xi\frac{\partial x_U}{\partial x_0} - \frac{\xi x_U}{x_0}, \quad \text{so that} \quad \frac{\partial x_U}{\partial x_0} = \frac{x_U/x_0}{\log(x_U/x_0)},$$

Now, $\frac{\partial V}{\partial F_0} = 2\xi^2 x_U \log\left(\frac{x_L}{x_0}\right)\frac{\partial x_U}{\partial F_0} = -2\xi x_U$ and $\frac{\partial V}{\partial x_0} = 2\xi^2 x_U\log\left(\frac{x_U}{x_0}\right)\frac{\partial x_U}{\partial x_0} - \xi^2\frac{x_U^2}{x_0} = \xi^2\frac{x_U^2}{x_0}$, and it is apparent that the optimality equation is satisfied. The checks on the boundaries proceed as before.

We have demonstrated the optimal strategy throughout the region D, and can therefore state that

$$\begin{aligned} V(0,0) &= \int_0^{ck} c^{-2}x\,dx + \int_{ck}^{k/\xi}\frac{k^2}{x}\,dx + \int_{k/\xi}^1 \xi^2 x\,dx \\ &= \frac{k^2}{2} - k^2\log(c\xi) + \frac{1}{2}(\xi^2 - k^2) \\ &= \frac{\xi^2}{2} - \frac{(1-\xi)^2}{\log(c\xi)}. \end{aligned}$$

This quantity represents the minimal value of $\int_0^1 xH_1^2(x,1)\,dx$ under the restrictions that $x \le H(x,1) \le x/c$ and $H_1(1-,1) = \xi$. For $\int_0^1 xH^2(1,x)\,dx$ we perform the same minimization, with the exception that ξ is replaced by $1-\xi$. This shows us that

$$\theta^\tau \ge -1 + 2\inf_{\xi\in(0,1)}\left\{\frac{\xi^2}{2} - \frac{(1-\xi)^2}{\log(c\xi)} + \frac{(1-\xi)^2}{2} - \frac{\xi^2}{\log(c(1-\xi))}\right\}.$$

The minimum occurs at $\xi = \frac{1}{2}$, giving a minimal value of

$$-1 + 2\left(\frac{1}{4} - \frac{1}{2\log(c/2)}\right) = -\frac{1}{2} - \frac{1}{\log(c/2)}.$$

□

Proof of Theorem 2.3. Clearly,

$$\begin{aligned} &\textstyle\int_0^1\int_0^1 H(x,y)h(x,y)\,dxdy \\ =\;& \textstyle\frac{1}{4}\sum_{i=1}^m\sum_{j=1}^m c_ic_j\alpha_j\beta_j\frac{1}{(\alpha_i+\alpha_j)(\beta_i+\beta_j)} \\ =\;& \textstyle\sum_{i=1}^m \frac{c_i^2}{4} + \sum_{i\ne j} c_ic_j\alpha_j\beta_j\frac{1}{(\alpha_i+\alpha_j)(\beta_i+\beta_j)} \\ =\;& \textstyle\frac{(\sum_{i=1}^m c_i)^2}{4} - \sum_{i\ne j}\frac{c_ic_j}{4} + \sum_{i\ne j} c_ic_j\alpha_j\beta_j\frac{1}{(\alpha_i+\alpha_j)(\beta_i+\beta_j)} \\ =\;& \textstyle\frac{1}{4} + \sum_{i\ne j} c_ic_j\frac{4\alpha_j\beta_j-(\alpha_i+\alpha_j)(\beta_i+\beta_j)}{4(\alpha_i+\alpha_j)(\beta_i+\beta_j)} \\ =\;& \textstyle\frac{1}{4} + \sum_{i\ne j} c_ic_j\frac{2\alpha_i\beta_i+2\alpha_j\beta_j-(\alpha_i+\alpha_j)(\beta_i+\beta_j)}{4(\alpha_i+\alpha_j)(\beta_i+\beta_j)} \\ =\;& \textstyle\frac{1}{4} + \sum_{i\ne j} c_ic_j\frac{(\alpha_i-\alpha_j)(\beta_i-\beta_j)}{4(\alpha_i+\alpha_j)(\beta_i+\beta_j)} \\ =\;& \textstyle\frac{1}{4} + \sum_{i\ne j} c_ic_j\frac{-(\alpha_i-\alpha_j)^2}{4(\alpha_i+\alpha_j)(\beta_i+\beta_j)} \\ \le\;& \textstyle\frac{1}{4}. \end{aligned}$$

Thus, the latter and Theorem 2.1 illustrate that $\theta^\tau \le 0$. □

Proof of Theorem 2.4. Put

$$\begin{cases} \theta_n = E\left\{sgn\big((U_1-U_2)(V_1-V_2)\big)I\big(\max(U_1,V_1,U_2,V_2)\le \frac{k}{n}\big)\right\}, \\ \tilde{h}(u_1,v_1,u_2,v_2) = sgn\big((u_1-u_2)(v_1-v_2)\big)I\big(\max(u_1,v_1,u_2,v_2)\le \frac{k}{n}\big)-\theta_n, \\ \tilde{h}_1(u_1,v_1) = E\left\{sgn\big((u_1-U_2)(v_1-V_2)\big)I\big(\max(u_1,v_1,U_2,V_2)\le \frac{k}{n}\big)\right\}-\theta_n, \\ S_{1n} = \sum_{i=1}^n \tilde{h}_1(U_i,V_i), \\ S_{2n} = \sum_{1\le i<j\le n}\left\{\tilde{h}(U_i,V_i,U_j,V_j)-\tilde{h}_1(U_i,V_i)-\tilde{h}_1(U_j,V_j)\right\}, \\ Z_n = \frac{2}{n(n-1)}\sum_{1\le i<j\le n}\left\{sgn\big((U_i-U_j)(V_i-V_j)\big)I\big(\max(U_i,V_i,U_j,V_j)\le\frac{k}{n}\big)-\theta_n\right\}. \end{cases}$$

Then it follows from the Hoeffding decomposition (Hoeffding (1948) or Lemma A from page 178 of Serfling (1980)) that

$$Z_n = \frac{2}{n}S_{1n} + \frac{2}{n(n-1)}S_{2n}. \tag{6.14}$$

In addition, Lemma A from page 183 of Serfling (1980) leads to

$$EZ_n^2 = \frac{4(n-2)}{n(n-1)}E\tilde{h}_1^2(U_1,V_1) + \frac{2}{n(n-1)}E\tilde{h}^2(U_1,V_1,U_2,V_2). \tag{6.15}$$

It is straightforward to check that

$$\theta_n/C^2\left(\frac{k}{n},\frac{k}{n}\right) \to \theta^\tau \tag{6.16}$$

and

$$\begin{aligned} \tilde{h}_1(u_1,v_1) \quad &= 2P\big(u_1>U_2, v_1>V_2, \max(u_1,v_1,U_2,V_2)\le\tfrac{k}{n}\big) \\ &\quad +2P\big(u_1<U_2, v_1<V_2, \max(u_1,v_1,U_2,V_2)\le\tfrac{k}{n}\big) \\ &\quad -P\big(\max(u_1,v_1,U_2,V_2)\le\tfrac{k}{n}\big)-\theta_n \\ &= 4P\big(u_1>U_2, v_1>V_2, \max(u_1,v_1,U_2,V_2)\le\tfrac{k}{n}\big) \\ &\quad -2P\big(u_1>U_2, \max(u_1,v_1,U_2,V_2)\le\tfrac{k}{n}\big)-2P\big(v_1>V_2, \max(u_1,v_1,U_2,V_2)\le\tfrac{k}{n}\big) \\ &\quad +P\big(\max(u_1,v_1,U_2,V_2)\le\tfrac{k}{n}\big)-\theta_n \\ &= 4C(u_1,v_1)I\big(\max(u_1,v_1)\le\tfrac{k}{n}\big)-2C\big(u_1,\tfrac{k}{n}\big)I\big(\max(u_1,v_1)\le\tfrac{k}{n}\big) \\ &\quad -2C\big(\tfrac{k}{n},v_1\big)I\big(\max(u_1,v_1)\le\tfrac{k}{n}\big)+C\big(\tfrac{k}{n},\tfrac{k}{n}\big)I\big(\max(u_1,v_1)\le\tfrac{k}{n}\big)-\theta_n. \end{aligned}$$

Thus, it follows from Assumption 2.1 that

$$
\begin{aligned}
&\frac{E\tilde{h}_1^2(U_1,V_1)}{C^3\left(\frac{k}{n},\frac{k}{n}\right)}\\
= \;& C^{-3}\left(\tfrac{k}{n},\tfrac{k}{n}\right) E\Bigg\{16C^2(U_1,V_1)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&+4C^2\left(U_1,\tfrac{k}{n}\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)+4C^2\left(\tfrac{k}{n},V_1\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&+C^2\left(\tfrac{k}{n},\tfrac{k}{n}\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)+\theta_n^2\\
&-16C(U_1,V_1)C\left(U_1,\tfrac{k}{n}\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&-16C(U_1,V_1)C\left(\tfrac{k}{n},V_1\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&+8C(U_1,V_1)C\left(\tfrac{k}{n},\tfrac{k}{n}\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&+8C\left(U_1,\tfrac{k}{n}\right)C\left(\tfrac{k}{n},V_1\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right) \qquad (6.17)\\
&-4C\left(U_1,\tfrac{k}{n}\right)C\left(\tfrac{k}{n},\tfrac{k}{n}\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&-4C\left(\tfrac{k}{n},V_1\right)C\left(\tfrac{k}{n},\tfrac{k}{n}\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&-2\theta_n\Bigg(4C(U_1,V_1)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)-2C\left(U_1,\tfrac{k}{n}\right)I\left(\max(U_1,V_1)\le \tfrac{k}{n}\right)\\
&-2C\left(\tfrac{k}{n},U_1\right)I\left(\max(V_1,U_1)\le \tfrac{k}{n}\right)+C\left(\tfrac{k}{n},\tfrac{k}{n}\right)I\left(\max(V_1,U_1)\le \tfrac{k}{n}\right)\Bigg)\Bigg\}\\
\to\;& 16\int_0^1\int_0^1 H^2(x,y)\,dH(x,y)-16\int_0^1\int_0^1 H(x,1)H(x,y)\,dH(x,y)\\
&-16\int_0^1\int_0^1 H(1,y)H(x,y)\,dH(x,y)+8\int_0^1\int_0^1 H(x,y)\,dH(x,y)\\
&+8\int_0^1\int_0^1 H(x,1)H(1,y)\,dH(x,y)-\tfrac{1}{3}
\end{aligned}
$$

and

$$
\frac{E\tilde{h}^2(U_1,V_1,U_2,V_2)}{C^2\left(\frac{k}{n},\frac{k}{n}\right)}=\frac{\theta_n}{C^2\left(\frac{k}{n},\frac{k}{n}\right)}+o(1)\to 1. \tag{6.18}
$$

By equations (6.15), (6.17) and (6.18), and the fact that $nC\left(\frac{k}{n},\frac{k}{n}\right)\to\infty$, we have

$$
E\left(Z_n/C^2\left(\frac{k}{n},\frac{k}{n}\right)\right)^2\to 0,
$$

which in turn implies that $Z_n/C^2\left(\frac{k}{n},\frac{k}{n}\right)\xrightarrow{p}0$. Hence, (6.16) allows us to conclude that

$$
\frac{2}{n(n-1)C^2\left(\frac{k}{n},\frac{k}{n}\right)}\sum_{1\le i<j\le n} sgn\big((U_i-U_j)(V_i-V_j)\big)I\left(\max(U_i,V_i,U_j,V_j)\le\frac{k}{n}\right)\xrightarrow{p}\theta^\tau. \tag{6.19}
$$

Denote $G_{n1}(x)=\frac{1}{n+1}\sum_{i=1}^n I(U_i\le x)$ and $G_{n2}(y)=\frac{1}{n+1}\sum_{i=1}^n I(V_i\le y)$. Note that

$$
\begin{aligned}
&sgn\big((\hat{U}_i-\hat{U}_j)(\hat{V}_i-\hat{V}_j)\big)I\big(\max(\hat{U}_i,\hat{V}_i,\hat{U}_j,\hat{U}_j)\le\tfrac{k}{n}\big)\\
=\;& sgn\big((U_i-U_j)(V_i-V_j)\big)I\left(\max(U_i,U_j)\le G_{n1}^-(\tfrac{k}{n}),\max(V_i,V_j)\le G_{n2}^-(\tfrac{k}{n})\right),
\end{aligned}
$$

$\frac{n}{k}G_{n1}^-\left(\frac{k}{n}\right)\xrightarrow{p}1$ and $\frac{n}{k}G_{n2}^-\left(\frac{k}{n}\right)\xrightarrow{p}1$. These properties, equation (6.19) and the continuity of H yield

$$
\frac{2}{n(n-1)C^2\left(\frac{k}{n},\frac{k}{n}\right)}\sum_{1\le i<j\le n} sgn\big((\hat{U}_i-\hat{U}_j)(\hat{V}_i-\hat{V}_j)\big)I\left(\max(\hat{U}_i,\hat{V}_i,\hat{U}_j,\hat{V}_j)\le\frac{k}{n}\right)\xrightarrow{p}\theta^\tau. \tag{6.20}
$$

Similarly, we can show that

$$
\frac{2}{n(n-1)C^2\left(\frac{k}{n},\frac{k}{n}\right)}\sum_{1\le i<j\le n} I\left(\max(\hat{U}_i,\hat{V}_i,\hat{U}_j,\hat{V}_j)\le\frac{k}{n}\right)\xrightarrow{p}1. \tag{6.21}
$$

Therefore, it follows from (6.20) and (6.21) that $\hat{\theta}^\tau(k)\xrightarrow{p}\theta^\tau$. $\square$

Proof of Theorem 2.5. It is worth mentioning that the current proof follows the same notations defined in the proof of Theorem 2.4. In addition, we define

$$\begin{aligned}\beta_{n1}(x,y) &= \tfrac{2}{n(n-1)C^2\left(\frac{k}{n},\frac{k}{n}\right)}\textstyle\sum_{1\le i<j\le n} \\ &\quad sgn\big((U_i-U_j)(V_i-V_j)\big) I\left(\max(U_i,U_j)\le \tfrac{k}{n}x\right) I\left(\max(V_i,V_j)\le \tfrac{k}{n}y\right)\end{aligned}$$

and

$$\beta_{n2}(x,y) = \frac{2}{n(n-1)C^2(\frac{k}{n},\frac{k}{n})}\sum_{1\le i<j\le n} I\left(\max(U_i,U_j)\le \frac{k}{n}x\right) I\left(\max(V_i,V_j)\le \frac{k}{n}y\right).$$

Now, Assumption 2.4 leads to

$$\begin{aligned}& A^{-1}\left(\tfrac{k}{n}\right)\left\{E\beta_{n1}(x,y) - 4\int_0^x\int_0^y H(s,t)H_{12}(s,t)\,dtds + H^2(x,y)\right\} \\ \to\ & 4\int_0^x\int_0^y Q(s,t)H_{12}(s,t)\,dtds + 4\int_0^x\int_0^y H(s,t)q(s,t)\,dtds - 2H(x,y)Q(x,y)\end{aligned} \tag{6.22}$$

and

$$A^{-1}\left(\frac{k}{n}\right)\left\{E\beta_{n2}(x,y) - H^2(x,y)\right\} \to 2H(x,y)Q(x,y). \tag{6.23}$$

By (6.14), (6.15), (6.17), (6.18) and the fact that $nC(\frac{k}{n},\frac{k}{n})\to\infty$, we have

$$\sqrt{nC\left(\frac{k}{n},\frac{k}{n}\right)}\{\beta_{n1}(1,1)-E\beta_{n1}(1,1)\} = \frac{2\sigma_1}{\sqrt{n}}\sum_{i=1}^n \frac{\tilde{h}_1(U_i,V_i)}{\sqrt{E\tilde{h}_1^2(U_1,V_1)}} + o_p(1), \tag{6.24}$$

where σ_1^2 is defined in (2.8). Similarly,

$$\sqrt{nC(\frac{k}{n},\frac{k}{n})}\{\beta_{n2}(1,1)-E\beta_{n2}(1,1)\} = \frac{2}{\sqrt{n}}\sum_{i=1}^n \frac{\hat{h}_1(U_i,V_i)}{\sqrt{E\hat{h}_1^2(U_1,V_1)}} + o_p(1), \tag{6.25}$$

where $\hat{h}_1(u_1,v_1) = I\big(\max(u_1,v_1)\le\frac{k}{n}\big) - C\big(\frac{k}{n},\frac{k}{n}\big)$. Using $H(1,1)=1$ and $Q(1,1)=0$, and (6.22)–(6.25), we have

$$\begin{aligned}& \sqrt{nC\big(\tfrac{k}{n},\tfrac{k}{n}\big)}\left\{\tfrac{\beta_{n1}(1,1)}{\beta_{n2}(1,1)} - \theta^\tau\right\} \\ =\ & \sqrt{nC\big(\tfrac{k}{n},\tfrac{k}{n}\big)}\left\{\tfrac{\beta_{n1}(1,1)-E\beta_{n1}(1,1)}{\beta_{n2}(1,1)} - \tfrac{(\beta_{n2}(1,1)-E\beta_{n2}(1,1))E\beta_{n1}(1,1)}{\beta_{n2}(1,1)E\beta_{n2}(1,1)}\right\} \\ & +\sqrt{nC\big(\tfrac{k}{n},\tfrac{k}{n}\big)}\left\{\tfrac{E\beta_{n1}(1,1)}{E\beta_{n2}(1,1)} - \theta^\tau\right\} \\ =\ & \tfrac{2\sigma_1}{\sqrt{n}}\sum_{i=1}^n\frac{\tilde{h}_1(U_i,V_i)}{\sqrt{E\tilde{h}_1^2(U_1,V_1)}} - \frac{2\theta^\tau}{\sqrt{n}}\sum_{i=1}^n\frac{\hat{h}_1(U_i,V_i)}{\sqrt{E\hat{h}_1^2(U_1,V_1)}} \\ & +\lambda\left\{4\int_0^1\int_0^1 Q(s,t)H_{12}(s,t)\,dtds + 4\int_0^1\int_0^1 H(s,t)q(s,t)\,dtds\right\} + o_p(1).\end{aligned} \tag{6.26}$$

Further, we have

$$
\begin{aligned}
&\sqrt{nC\left(\tfrac{k}{n},\tfrac{k}{n}\right)}\left\{\hat{\theta}(k)-\theta^{\tau}\right\}\\
=\;& \sqrt{nC\left(\tfrac{k}{n},\tfrac{k}{n}\right)}\Bigg\{\frac{\beta_{n1}\left(\frac{n}{k}G_{n1}^{-}\left(\frac{k}{n}\right),\frac{n}{k}G_{n2}^{-}\left(\frac{k}{n}\right)\right)}{\beta_{n2}\left(\frac{n}{k}G_{n1}^{-}\left(\frac{k}{n}\right),\frac{n}{k}G_{n2}^{-}\left(\frac{k}{n}\right)\right)}-4\int_0^{\frac{n}{k}G_{n1}^{-}\left(\frac{k}{n}\right)}\int_0^{\frac{n}{k}G_{n2}^{-}\left(\frac{k}{n}\right)}H(s,t)H_{12}(s,t)\,dtds\\
&+H^2\left(\tfrac{n}{k}G_{n1}^{-}\left(\tfrac{k}{n}\right),\tfrac{n}{k}G_{n2}^{-}\left(\tfrac{k}{n}\right)\right)\Bigg\}\\
&+4\sqrt{nC\left(\tfrac{k}{n},\tfrac{k}{n}\right)}\Big\{\int_0^{\frac{n}{k}G_{n1}^{-}\left(\frac{k}{n}\right)}\int_0^{\frac{n}{k}G_{n2}^{-}\left(\frac{k}{n}\right)}H(s,t)H_{12}(s,t)\,dtds\\
&-\int_0^1\int_0^1 H(s,t)H_{12}(s,t)\,dtds\Big\}\\
&-\sqrt{nC\left(\tfrac{k}{n},\tfrac{k}{n}\right)}\Big\{H^2\left(\tfrac{n}{k}G_{n1}^{-}\left(\tfrac{k}{n}\right),\tfrac{n}{k}G_{n2}^{-}\left(\tfrac{k}{n}\right)\right)-H^2(1,1)\Big\}\\
=\;& \sqrt{nC\left(\tfrac{k}{n},\tfrac{k}{n}\right)}\left\{\frac{\beta_{n1}(1,1)}{\beta_{n2}(1,1)}-\theta^{\tau}\right\}\\
&+4\sqrt{c}\sqrt{k}\left\{\tfrac{n}{k}G_{n1}^{-}\left(\tfrac{k}{n}\right)-1\right\}\int_0^1 H(1,t)H_{12}(1,t)\,dt\\
&+4\sqrt{c}\sqrt{k}\left\{\tfrac{n}{k}G_{n2}^{-}\left(\tfrac{k}{n}\right)-1\right\}\int_0^1 H(s,1)H_{12}(s,1)\,ds\\
&-2\sqrt{c}\sqrt{k}\left\{\tfrac{n}{k}G_{n1}^{-}\left(\tfrac{k}{n}\right)-1\right\}H_1(1,1)\\
&-2\sqrt{c}\sqrt{k}\left\{\tfrac{n}{k}G_{n2}^{-}\left(\tfrac{k}{n}\right)-1\right\}H_2(1,1)+o_p(1).
\end{aligned}
\tag{6.27}
$$

It is not difficult to find that

$$
\begin{cases}
E\left\{\frac{\tilde{h}_1(U_i,V_i)}{\sqrt{E\tilde{h}_1^2(U_1,V_1)}}\frac{\hat{h}_1(U_i,V_i)}{\sqrt{E\hat{h}_1^2(U_1,V_1)}}\right\}=\frac{\theta_n-\theta_nC\left(\frac{k}{n},\frac{k}{n}\right)}{\sigma_1C^2\left(\frac{k}{n},\frac{k}{n}\right)\left(1+o(1)\right)}\to\frac{\theta^{\tau}}{\sigma_1}\\
E\left\{\frac{\tilde{h}_1(U_i,V_i)}{\sqrt{E\tilde{h}_1^2(U_1,V_1)}}\frac{I\left(U_i\le\frac{k}{n}\right)-\frac{k}{n}}{\sqrt{k/n}}\right\}=\frac{\theta_n-\frac{k}{n}\theta_n}{\sigma_1C^2\left(\frac{k}{n},\frac{k}{n}\right)}\sqrt{\frac{n}{k}C\left(\frac{k}{n},\frac{k}{n}\right)}\{1+o(1)\}\to\frac{\theta^{\tau}\sqrt{c}}{\sigma_1}\\
E\left\{\frac{\tilde{h}_1(U_i,V_i)}{\sqrt{E\tilde{h}_1^2(U_1,V_1)}}\frac{I\left(V_i\le\frac{k}{n}\right)-\frac{k}{n}}{\sqrt{k/n}}\right\}=\frac{\theta_n-\frac{k}{n}\theta_n}{\sigma_1C^2\left(\frac{k}{n},\frac{k}{n}\right)}\sqrt{\frac{n}{k}C\left(\frac{k}{n},\frac{k}{n}\right)}\{1+o(1)\}\to\frac{\theta^{\tau}\sqrt{c}}{\sigma_1}\\
E\left\{\frac{\hat{h}_1(U_i,V_i)}{\sqrt{E\hat{h}_1^2(U_1,V_1)}}\frac{I\left(U_i\le\frac{k}{n}\right)-\frac{k}{n}}{\sqrt{k/n}}\right\}=\frac{C\left(\frac{k}{n},\frac{k}{n}\right)-\frac{k}{n}C\left(\frac{k}{n},\frac{k}{n}\right)}{C\left(\frac{k}{n},\frac{k}{n}\right)}\sqrt{\frac{n}{k}C\left(\frac{k}{n},\frac{k}{n}\right)}\{1+o(1)\}\to\sqrt{c}\\
E\left\{\frac{\hat{h}_1(U_i,V_i)}{\sqrt{E\hat{h}_1^2(U_1,V_1)}}\frac{I\left(V_i\le\frac{k}{n}\right)-\frac{k}{n}}{\sqrt{k/n}}\right\}=\frac{C\left(\frac{k}{n},\frac{k}{n}\right)-\frac{k}{n}C\left(\frac{k}{n},\frac{k}{n}\right)}{C\left(\frac{k}{n},\frac{k}{n}\right)}\sqrt{\frac{n}{k}C\left(\frac{k}{n},\frac{k}{n}\right)}\{1+o(1)\}\to\sqrt{c}\\
E\left\{\frac{I\left(U_i\le\frac{k}{n}\right)-\frac{k}{n}}{\sqrt{k/n}}\frac{I\left(V_i\le\frac{k}{n}\right)-\frac{k}{n}}{\sqrt{k/n}}\right\}=\frac{C\left(\frac{k}{n},\frac{k}{n}\right)-\left(\frac{k}{n}\right)^2}{k/n}\to c.
\end{cases}
$$

Consequently, using the Cramér-device, we can show that

$$
\begin{aligned}
&\left(\tfrac{1}{\sqrt{n}}\sum_{i=1}^n\frac{\tilde{h}_1(U_i,V_i)}{\sqrt{E\tilde{h}_1^2(U_1,V_1)}},\tfrac{1}{\sqrt{n}}\sum_{i=1}^n\frac{\hat{h}_1(U_i,V_i)}{\sqrt{E\hat{h}_1^2(U_1,V_1)}},\sqrt{k}\left(\tfrac{n}{k}G_{n1}^{-}\left(\tfrac{k}{n}\right)-1\right),\sqrt{k}\left(\tfrac{n}{k}G_{n2}^{-}\left(\tfrac{k}{n}\right)-1\right)\right)^T\\
=\;&\left(\tfrac{1}{\sqrt{n}}\sum_{i=1}^n\frac{\tilde{h}_1(U_i,V_i)}{\sqrt{E\tilde{h}_1^2(U_1,V_1)}},\tfrac{1}{\sqrt{n}}\sum_{i=1}^n\frac{\hat{h}_1(U_i,V_i)}{\sqrt{E\hat{h}_1^2(U_1,V_1)}},\right.\\
&\left.-\tfrac{1}{\sqrt{k}}\sum_{i=1}^n\left(I\left(U_i\le\tfrac{k}{n}\right)-\tfrac{k}{n}\right),-\tfrac{1}{\sqrt{k}}\sum_{i=1}^n\left(I\left(V_i\le\tfrac{k}{n}\right)-\tfrac{k}{n}\right)\right)^T+o_p(1)\\
\xrightarrow{d}\;& N(0,\Sigma)
\end{aligned}
\tag{6.28}
$$

as $n\to\infty$, where

$$
\Sigma=\begin{pmatrix}
1 & \frac{\theta^{\tau}}{\sigma_1} & -\frac{\theta^{\tau}\sqrt{c}}{\sigma_1} & -\frac{\theta^{\tau}\sqrt{c}}{\sigma_1}\\
-\frac{\theta^{\tau}}{\sigma_1} & 1 & -\sqrt{c} & -\sqrt{c}\\
-\frac{\theta^{\tau}\sqrt{c}}{\sigma_1} & -\sqrt{c} & 1 & c\\
-\frac{\theta^{\tau}\sqrt{c}}{\sigma_1} & -\sqrt{c} & c & 1
\end{pmatrix}.
$$

Therefore, it follows from equations (6.26)–(6.28) that (2.6) holds. $\square$

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References

[1] Aragones, J.R., Blanco, C. and Dowd, K. (2001). Incorporating stress tests into market risk modeling. *Derivatives Quarterly, Spring 2001, 44--49.*

[2] Asimit, A.V., Jones, B.L. (2007). Extreme behavior of bivariate elliptical distributions. *Insurance: Mathematics and Economics 41(1), 53–61.*

[3] Basel Committee on Banking Supervision (2010). Basel III: a global regulatory framework for more resilient banks and banking systems , available at `http://www.bis.org/publ/bcbs189.pdf`

[4] Basrak, B., Davis, R.A., and Mikosch, T. (2002). Regular variation of GARCH processes. *Stoch. Process. Appl. 99, 95–115.*

[5] Breymann, W., Dias, A. and Embrechts, P. (2003). Dependence structures for multivariate high frequency data in finance. *Quantative Finance 3(1), 1–14.*

[6] Cheng, S. and Peng, L. (2001). Confidence intervals for the tail index. *Bernoulli 7, 751–760.*

[7] De Haan, L. and Ferreira, A. (2006). Extreme Value Theory: An Introduction. *Springer-Verlag, New York.*

[8] De Haan, L. and Resnick, S.I. (1979). Derivatives of regularly varying functions in $\Re^d$ and domains of attraction of stable distributions. *Stochastic Processes and their Applications 8(3), 349–355.*

[9] Doksum, K. and Samarov, A. (1995). Nonparametric estimation of global functions and a measure of the explanatory power of covariates in regressions. *Ann. Statist. 23, 1443–1473.*

[10] Draisma, G., Drees, H., Ferreira, A. and de Haan, L. (2004). Bivariate tail estimation: dependence in asymptotic independence. *Bernoulli 10, 251–280.*

[11] Einmahl, J.H.J., de Haan, L. and Li, D. (2006). Weighted approximations to tail copula processes with application to testing the extreme value condition. *Ann. Statist. 34, 1987–2014.*

[12] European Commission (2009). Directive 2009/138/EC of the European parliament and of the council of 25 november 2009 on the taking-up and pursuit of the business of insurance and reinsurance (Solvency II), *Official Journal of the European Union L335.*

[13] Frees, E.W. and Valdez, E.A. (1998). Understanding relationships using copulas, *North American Actuarial Journal 2(1), 1–25.*

[14] Gijbels, I., Veraverbeke, N. and Omelka, M. (2011). Conditional copulas, association measures and their applications. *Computational Statistics & Data Analysis 55, 1919–1932.*

[15] Goegebeur, Y. and Guillou, A. (2012). Asymptotically unbiased estimation of the coefficient of tail dependence. *Scandinavian Journal of Statistics 40, 174–189.*

[16] Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Stat. 19, 293–325.*

[17] Hua, L. and Joe, H. (2011). Tail order and intermediate tail dependence of multivariate copulas. *Journal of Multivariate Analysis 102, 1454–1471.*

[18] Hua, L. and Joe, H. (2013). Intermediate tail dependence: a review and some new results. *In Li, H. and Li, X., editors, Stochastic Orders in Reliability and Risk: In Honor of Professor Moshe Shaked, Chapter 15.*

[19] Huang, X. (1992). Statistics of Bivariate Extreme Values. *Tinbergen Institute Research Series 22.*

[20] Hüsler, J. and Li, D. (2009). Testing asymptotic independence in bivariate extremes. *Journal of Statistical Planning and Inference 139, 990–998.*

[21] Juri, A. and Wüthrich, M. V. (2003). Tail dependence from a distributional point of view. *Extremes 6(3), 213–246.*

[22] Klüppelberg, C., Kuhn, G. and Peng, L. (2008). Semi-parametric models for the multivariate tail dependence function - the asymptotically dependent case. *Scandinavian Journal of Statistics 35, 701–718.*

[23] Ledford, A.W. and Tawn, J. (1996). Statistics for near independence in multivariate extreme values. *Biometrika 83, 169–187.*

[24] Ledford, A.W. and Tawn, J. (1997). Modelling dependence within joint tail regions. *J. Roy. Statist. Soc. Ser. B 59, 475–499.*

[25] Longin, F.M. (2010). From value at risk to stress testing: the extreme value approach. *Journal of Banking & Finance 24(7), 1097–1130.*

[26] McNeil, A.J., Frey, R. and Embrechts, P. (2005). Quantitative Risk Management: Concepts, Techniques, and Tools. *Princeton University Press.*

[27] Mikosch, T. and Wintenberger, O. (2014). The cluster index of regularly varying sequences with applications to limit theory for functions of multivariate Markov chains. *Probability Theory and Related Fields 159, 157–196.*

[28] Nelsen, R. B. (2006). An Introduction to Copulas, 2nd ed. *Springer-Verlag, New York.*

[29] Peng, L. (1999). Estimation of the coefficient of tail dependence in bivariate extremes. *Statistics and Probability Letters 43, 399—409.*

[30] Peng, L. and Qi, Y. (2008). Bootstrap approximation of tail dependence function. *Journal of Multivariate Analysis 99, 1807–1824.*

[31] Resnick, S.I. (1987). Extreme Values, Regular Variation and Point Processes. *Springer-Verlag, New York.*

[32] Serfling, R.J. (1980). Approximation Theorems of Mathematical Statistics. *John Wiley & Sons, INC.*

[33] Sibuya, M. (1960). Bivariate extreme statistics. *Ann. Inst. Statist. Math. 11, 195–210.*

[34] Sklar, A. (1959). Fonctions de répartion à n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris, 8, 229–231.*

[35] Swiss Solvency Test (2006). FINMA SST Technisches Dokument, available at `http://www.finma.ch/archiv/bpv/download/d/SST_technischesDokument_061002.pdf`

[36] Venter, G. (2001). Tails of copulas. *Proceedings ASTIN Washington.*

[37] Zhang, Z., Qi, Y. and Ma, X.W. (2011). Asymptotic independence of correlation coefficients with application to testing hypothesis of independence. *Electronic Journal of Statistics 5, 342–372.*