CITY UNIVERSITY LONDON

COMPARING THE RISKS OF DIVERSE METHODS OF ELECTRICITY GENERATION USING THE J-VALUE FRAMEWORK

by

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Volume I

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Declaration

The author declares that this thesis and the work presented in it is his own and has been generated by him as the result of his own original research.

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Abstract

This thesis presents and extends the J-value framework for assessing expenditure on risk mitigation, and then applies the method in a comparative risk assessment of UK electricity generating systems.

The thesis is split into two volumes. The first volume contains part one, in which the J-value framework is introduced and developed. The loss of life expectancy is a key parameter in the framework, and general risk models for calculating this parameter are developed in terms of exposures and responses. Specific examples of radiation and pollution models are also presented. The "Hazard Elimination Premium" is also introduced as a useful common metric for risk comparisons.

Part one also contains an assessment of the uncertainty of the J-value and its input parameters and it is found that the J-value has an internal accuracy of around 3%, but that other, context dependant parameters can degrade this accuracy. A sensitivity analysis of the J-value framework also found that the J-value was reasonably robust against random variation of the input parameters as well as against the use of simplifying assumptions used in the development of the J-value.

The second volume contains parts two and three. Part two describes the comparative risk analysis of the electricity generating systems. The analysis is carried out on nuclear, coal, natural gas, onshore wind and offshore wind. The analysis assesses human mortality impacts arising from the current and future plants over the sixty year period from 2010 to 2070 for the entire fuel chain. The results indicate that nuclear generally has the lowest impacts, while gas, onshore and offshore wind have indicative impacts that are about an order of magnitude greater, although the estimates for both wind technologies carry considerable uncertainty. Coal power was found to present high impacts compared with the other technologies, mainly as a result of pollution emissions. Total nuclear impacts were found to be sensitive to assumptions regarding the use of collective dose and the assumptions which are then used to calculate impacts. For the most pessimistic case, when world exposures are taken, total nuclear impacts increase by about an order of magnitude, which would render the risks from nuclear generation comparable with those from gas and wind generation.

Part three presents the conclusions, further work, bibliography and appendices.

Nomenclature

List of Roman Symbols

Symbol	Meaning	Units
A	Assets	£
A_P	Productivity constant	
a	Age	year
a_{rec}	Recruitment age	year
a_{ret}	Retirement age	year
B	Cost of risk mitigation system	£
B_0	Risk-neutral maximum reasonable spend on risk mitigation system	£
b	Constant exposure rate	additional deaths/year
b_a	Normalised cost of risk mitigation system	
b_{coll}	Collective exposure rate	additional man-deaths/year
b_i	Discrete value of normalised cost of risk mitigation system	
b_{max}	Maximum normalised reasonable spend on risk mitigation system	
$b(x)$	Exposure rate at time x	additional deaths/year
$b_{tot}(x)$	Total individual exposure	
C	Cost of accident	£
$c(a)$	Earnings per year at age a	£/year
c_a	Normalised cost of accident	
c_T	Total dose risk coefficient for radiation exposures	Sieverts^{-1}
D	Difference in expected utilities	
D_a	Number of deaths at age a	
D_f	Linearised discount factor	
$D(t)$	Probability of dying before age t	
$D(u_1,u_2\|\varepsilon)$	Difference in initial and final utility at given risk aversion ε	
d_a	Number of life table deaths at age a	
$d_r(x)$	Annual radiation dose	Sieverts/year
E	Emission rate	μgs^{-1}
$\hat{E}_a$	Number of deaths calculated from survival proababilities based on specific model	
$E(u_1)$	Initial expected utility	
$E(u_2)$	Final expected utility	
e_a	Life expectancy at discrete age a	year
F	Expected remaining free time	year
$F(a)$	Expected remaining free time at age a	year
f	Average free time fraction	

f_0	Optimal free time fraction chose by society as a whole	
$f_d(t)$	Probability density for death	year^{-1}
f_{male}	Fraction of population that is male	
$f_M(y)$	Probability density that the excess mortality resulting from a given exposure occurs at time y	year^{-1}
$f_T(\tau)$	Total probability density for death at time τ	year^{-1}
G	GDP per person	£/year
G_C	National GDP	£/year
$g(b_a, \varepsilon)$	Derivative of reluctance to invest	
$g(x)$	Probability density for death at time x from given exposure	year^{-1}
$g_d(t\|a)$	Probability density function for death at age t given survival to age a.	year^{-1}
g_w	fraction of time spent working for average person in work	
$g_w(t)$	Fraction of time spent working for average person of age, t, and in work	
H	Population entropy	
H_T	Total man-hours worked in all populations	hours
$H_w(t)$	Total man-hours worked at age t	hours
$h(a)$	Hazard rate at age a	year^{-1}
$h_w(t)$	Individual hours worked at age t	hours
J	Judgement value	
$J_p(x)$	Jump function for response to exposure	
J_T	Total judgement value	
J_2	Second judgement value	
K	Capital investment per person	£
K_C	National capital investment	£
k	Expected number of accidents as used in the Poisson distribution	
k_{rad}	Distributed radiation risk coefficient	year^{-1}
k_{poll}	Pollution risk coefficient	$\mu g^{-1}m^3$
k_1	Constant	
k_2	Constant	
L_C	National labour supply	man-year
l_a	Number of life-tables survivors to age a	
m_a	Discrete central rate of mortality at age a	
m_a^{male}	Male central rate of mortality at age a	
m_a^{female}	Female central rate of mortality at age a	
m_{rlow}	Low value of risk multiplier	

$m_{r.max}$	Maximum risk multiplier	
N	Number of people affected by protection system	
N_C	Number of people in a country	
N_{Pop}	Total size of a given population	
N_{py}	Annual person-years worked	
n_a	Mid-year population at age *a*	
$n(a)$	Size of population at age *a*	
$n_w(t)$	Number of people working at age *t*	
O	Electrical energy output	Gigawatt-year (GWa)
p_L	Price of labour	£/year
$p(a)$	Population density at age *a*	year^{-1}
$p_{sw}(t\|a)$	Probability for being employed at age *t* given survival to age *a*	year^{-1}
p_w	Average probability of being in work for all persons of working age	
$p_w(t)$	Probability for being employed at age *t*	year^{-1}
$p^{(y)}{}_{\lambda}$	Probability density of *y* accidents occurring with frequency λ	
p_1	Initial no-accident probability	
p_2	Final no-accident probability	
Q	Life-quality index	
Q_f	Life-quality index in terms of income and free time fraction	
$Q_{f,d}$	Discounted life-quality index in terms of income and free time fraction	
$\overline{Q_f}$	Constant value of life-quality index on an indifference curve	
Q_X	Life-quality index in terms of income and life expectancy	
$\overline{Q_X}$	Constant value of life quality index on an indifference curve	
Q_1	Version of life-quality index	
Q_2	Version of life-quality index	
q	Elasticity parameter	
q_a	Probability of death at age *a*	
$R(a)$	Expected utility for individual of age, *a*	
R_r	Restoration requirement	
$R_r(a)$	Restoration requirement at age *a*	
R_{120A}	Reluctance to invest	
r	Net discount rate	year^{-1}
r_d	Discount rate	year^{-1}
r_g	Growth rate	year^{-1}
$S(a)$	Survival probability to age *a*	
$S(t\|a)$	Survival probability to age *t* given	

	survival to age *a*.	
T	Random age of death	year
T_R	Release Period	year
t	Age, time	year
t_{av}	Average age in a population	year
t_{av}^2	Average square age in a population	year^2
t_{av}^3	Average cubed age in a population	year^3
$t_{a+.ave}$	Average age of those above age *a*	year
$t_{w.av}$	Average working age	year
$U(G)$	Utility of income, *G*	
$u_0(\varepsilon)$	Initial utility at risk aversion ε	
$V_D(x_d)$	Value of a delaying a fatality by x_d years	£
$V_p(a)$	Value of temporarily preventing a fatality for someone of age *a*	£
V_p	Value of temporarily preventing a fatality for someone of unknown age	£
$V_{p.av}$	Average value of temporarily preventing a fatality	£
$W(a)$	Cumulative hazard rate at age *a*	
w	work-time fraction	
w_0	Optimal work-time fraction chosen by society as a whole.	
X	Average life expectancy	year
X_d	Average discounted life expectancy	year
$X(a)$	Life expectancy at age *a*	year
$X_d(a)$	Discounted life expectancy at age *a*	year
x	Time	year
x_d	Discounted delayed time until death	year
Y	Random number of accidents	
y	Time elapsed since induction	year
y_w	Work-life expectancy	year
$y_w(a)$	Work-life expectancy at age *a*	year
z_p	Normal quantile function	
$z_w(t\|a)$	Fraction of time someone of age, *a*, can expect to be working at age, *t*	

List of Greek Symbols

Symbol	Meaning	Units
α_1	Constant	
β	Constant	
γ	Constant	
δb_i	Step size for normalised cost of protection system	
$\delta c(x)$	Increase in concentration levels	μg.m^{-3}
δ_{dis}	Discrimination limit	
δG	Maximum reasonable change in a	£/year

	person's income as a result of spending on a health and safety scheme that will extend his life	
δG_N	Maximum reasonable change in a group of *N* people's income as a result of spending on a health and safety scheme	£/year
$\delta h_{abs}(t\|a)$	Absolute change in hazard rate at age *t* given survival to age *a*.	year^{-1}
$\delta h_{rel}(t\|a)$	Relative change in hazard rate at age *t* given survival to age *a*.	year^{-1}
δV_N	Maximum reasonable spend on a protection system for *N* people who will experience a gain in life expectancy of δX_d	£
$\delta \hat{V}_N$	Actual spend on protection system.	£
$\delta W(t\|a)$	Change in cumulative hazard rate at age *t* given survival to age *a*	
$\delta \hat{W}$	Actual spend on risk protection system that protects against physical and financial risks	£
δX_{coll}	Collective loss of life expectancy	man-year
δX_d	Change in average discounted life expectancy	year
$\delta X_d(a)$	Change in average discounted life expectancy at age *a*	year
δZ_R	Maximum reasonable spend on financial risk mitigation systems	£
$\delta \hat{Z}$	Actual spent on financial risk mitigation system	£
$\delta\varepsilon$	Step size for risk aversion	
$\delta\chi(a)$	Change in random life to come at age *a*	year
ε	Risk aversion coefficient	
ε_{max}	Maximum risk aversion	
ε_{pp}	Permission point	
η_f	Elasticity of free time fraction with respect to income	
η_{MU}	Elasticity of marginal utility with respect to income	
η_X	Elasticity of life expectancy with respect to income	
θ	Share of wages in the GDP	
$\Lambda(x)$	Number of deaths at time *x*	
λ	Hazard rate when deaths are exponentially distributed	year^{-1}
v	Deposition velocity	ms^{-1}

$v_d(x_d)$	Value of a discounted life-year	£
v_{ave}	Average value of a life-year	£
π_1	Initial accident probability	
π_2	Final accident probability	
ρ	Population density	persons/m^3
$\rho_{f,g}$	Correlation coefficient between parameters *f* and *g*	
σ_f	Standard deviation for parameter *f*	units of *f*
τ	Age	year
$\phi_0(y)$	Response function	year^{-1}
χ	Random life to come when age is unknown	year
$\chi(a)$	Random life to come at age *a*	year
χ^2_{k-1}	Chi-square test statistic with $k - 1$ degrees of freedom	
$\Phi^{-1}(p)$	Inverse normal cumulative distribution at value *p*	
$\psi_0(x)$	Prolonged response function	
$\psi_1(x)$	Integrated prolonged response function	
$\psi_2(x)$	Twice integrated prolonged response function	
Ω	Duration of long exposure	year
ω_1	Time to start of response to exposure	year
ω_2	Time to end of response to exposure	year

List of Abbreviations

COE	Compensation of Employees	£/year
GDP	Gross Domestic Product	£/year
MI	Mixed Income	£/year
MRS	Marginal rate of substitution	
RR	Relative risk	
VODLY	Value of a discounted life-year	£
VODLYA	Average value of a discounted life-year	£
VTPF	Value of temporarily preventing a fatality	£

Chapter 1 Introduction

1.1 Statement of Problem

The purpose of the research contained in this thesis is to use the J-value framework to assess and compare the risks from diverse methods of electricity generation in the UK.

1.2 Aims and Objectives

The aims of this research are:

1. Validate the J-value framework as a suitable and robust tool for risk assessment and analysis.
2. Compare, in a consistent manner, the risks posed by various electricity generating systems in the UK using the J-value framework.

It is intended that these aims will be achieved through the following objectives:

1. Extending the existing framework by incorporating more general risk models in the loss of life expectancy calculations, and conducting uncertainty and sensitivity analyses.
2. Use the J-value framework to develop a common metric that can be used to compare the risks from electricity generating systems on a consistent basis, i.e. in such a manner that does not bias the results towards any particular electricity generating system.
3. Develop a framework for the comparative risk analysis that will incorporate all relevant risks involved in the generation of electricity for each system in a manner that will ensure a fair and valid comparison.

1.3 Structure

To achieve the aims and objectives set out above, it has been necessary to separate the comparative risk analysis from the development of the J-value framework. The thesis thus has three parts. Part one is the valuation of health and safety, in which the J-value is presented and developed. The first chapter in part one considers the historical context and existing literature in this field. The subsequent chapters then describe in detail the concepts and methods used in deriving the J-value, and develop them further. Areas in which the existing framework is developed further include:

- A new derivation of the J-value through consideration of the trade-offs made at an individual and societal level.
- Generalised relative and absolute risk models of the loss of life expectancy following any given exposure and response pattern. This model is also applied to the specific case of pollution risks.
- A more rigorous treatment of the measurement and estimation procedures for the parameters used in the J-value framework, including an assessment of the tolerances to be placed on each parameter.
- Introduction of the concept of a "Hazard Elimination Premium", which is the maximum reasonable amount to spend to completely eliminate a hazard. The HEP is used extensively in the second part of the thesis.
- A sensitivity analysis of the J-value framework, in which the robustness of the J-value given the initial assumptions and uncertainty of some of the input parameters is assessed.

The J-value has been recently extended by Thomas et al (2009, 2010) [190], [191], [192] to include mitigation of financial risks in addition to physical risks. These concepts come together to form a "total judgement value", or J_T-value. The model behind this extension is shown, and the computational methods employed to calculate some of its outputs are also presented. Part one then concludes with some example calculations.

The second part of the thesis applies the methods laid out in part one in a comparative risk analysis of UK electricity generating systems. The analysis is carried out on five electricity generating systems in the UK: nuclear, coal, natural gas, onshore wind and offshore wind, and uses the hazard elimination premium to compare each technology on an equal footing. This section opens with a literature review, before discussing the technical procedures of the report, such as scope and the assumed boundaries of the assessed systems. This is followed by the analysis of risks from nuclear, fossil fuels, and the wind technologies. Part two concludes with the overall results, comparisons with other studies and a discussion of the significance and limitations of the results.

The third and final part of the thesis considers the overall conclusions, and whether the aims and objectives have been met in answering the research problem. Areas requiring further work are also identified and discussed. Part three also contains the bibliography and appendices.

Part 1 Valuing Health and Safety

Individuals have always traded risks to their health and life in order to obtain other benefits. These trades reflect how the individual values his or her life. In a modern democratic society, it is necessary to make decisions about public safety that invariably affects the health and the wealth of many individuals. There is now widespread consensus that any such method used to aid the decision making process regarding public safety should reflect as far as is possible the preferences which the individuals in a society place upon their safety. Any such method must be fully consistent in the way that risks are valued, and should also be transparent. Currently the most widespread method used for valuing risks are stated preference techniques used to elicit an individual's willingness to pay (WTP) for a given risk reduction. The advantages and disadvantages of this method have been summarised in the preceding section. The purpose of this thesis is to describe a relatively new technique for valuing risks known as the "J-value" method, developed by Thomas et al (2006) [182], [183], and (2009) [188].

The J-value method values risks by using the Life Quality Index (LQI), which is an indicator for measuring the development of nations, and was developed by Pandey, Nathwani and Lind (1997) [137], (2004) [157] and (2006) [158], as a means to test the efficiency of risk management decisions. The central postulate of the LQI methodology is that the two primary determinants of an individual's quality of life is how much free time he can expect to enjoy from now on, and how much he will have available to spend over this period. The relative importance of these two factors is then determined by using labour market data to analyse society's preferences for how it allocates its time. It is assumed that an individual can choose how much time he wishes to work for, and accordingly how much free time he has. The more importance he places upon his free time, the less time he will spend in work. Conversely, if his preferences are for more money available for consumption, he will spend more time in work. Thus, the proportion of time which the average individual will choose to spend in work from now on can be used to weight the two factors appropriately. A value for risk can then be inferred by insisting that any decision that changes a society's average life expectancy and income (measured by the GDP per person) must at least preserve the initial LQI, and preferably increase it, i.e. the change in the LQI must not be negative. If a protection system is known to afford a given increase in life expectancy to a group of individuals, then the constraint on the change in the LQI places an upper bound on the amount of money that should be spent on implementing the scheme. This maximum value can then be taken as representing the societal cost of risk. If the actual cost of the protection system is known, then the J-value is the ratio of this cost to the societal cost. The J-value is therefore a dimensionless positive number. J-values of less than unity indicate that the protection system costs less than the maximum theoretical cost of risk, and so represent good value for money. Implementing these schemes will result in an increased LQI. J-values greater than unity indicate that the cost of the protection system is greater than the theoretical maximum, and hence should not be implemented. The J-value can be seen to be a scale on which safety projects and risk policies may be judged. The scale is universal, in the sense that it is not specific to any single industry, and all the input parameters are fully objective quantities, most of which are derived from reliable national and actuarial statistics. The J-value, being a single dimensionless number, is also transparent and easily interpreted.

The J-value framework has also been extended recently (2010) [192] to include financial risks to assets. This is formulated around an expected utility model, which can be used to determine objectively the risk preferences of the individual or organisation facing the risk, which can then be used to determine the maximum reasonable spend on eliminating the risk.

Chapter 3 describes the conceptual foundations of the J-value method in depth, and shows how the J-value can be derived based on considerations of the trade-offs individuals make between their free time and income, and the trade-off between safety spend and life expectancy improvement. Chapters 4 to 6 then introduce the methods and techniques required for calculation of the actuarial parameters: the life expectancy; the change in life expectancy and the work-life expectancy. It is also shown how the latter parameter can be used in calculating the work-time fraction: a key parameter in the J-value framework. Chapter 7 describes how the J-value can be used to infer common metrics of the value of life, namely the value of temporarily preventing a fatality (VTPF), and the value of a discounted life-year (VODLY), and also introduces the “Hazard Elimination Premium” (HEP), which will be used extensively in part 2 of this thesis. Chapter 8 presents the measurements of all the necessary input parameters to the J-value, and also provides an assessment of the tolerance limits of the J-value. In chapter 9 a sensitivity analysis is performed to assess the robustness of the J-value to the underlying assumptions. Chapter 10 gives an introduction to the J_2 and J_T-values, and describes how the maximum reasonable spend on financial risks can be determined. Finally, chapter 11 presents some example calculations, demonstrating the general nature and applicability of the J, J_2 and J_T-value methods.

Chapter 2 Historical Context and Existing Literature

The valuation of health and safety schemes, proposals or policies must also reflect the value to be placed on physical risk, and consequently, the value placed on human lifespan. In this section, some of the historical and more recent literature of such valuations will be reviewed. Particular focus will be given to the various methodologies that have been used to value these risks. It is common practice to express risk valuations in terms of how much should be spent on avoiding one statistical fatality, a measure commonly known as the "value of a statistical life" or the "value of preventing a fatality". However, the latter term is somewhat misleading, as preventing a fatality is in the long run impossible – all individuals will eventually die. It is for this reason that, for the purposes of this thesis, the term "Value of Temporarily Preventing a Fatality" (VTPF) will be used. Although there are many ways to calculate the VTPF, one of the most common methods is the following: if it has been determined that each member of a population of size N is willing to pay £v to eliminate a risk that has a probability of $1/N$ of killing each member, then an amount totalling £Nv is willing to be spent on eliminating a risk that is expected to kill one person. Therefore, the VTPF = £Nv. The VTPF is usually an input into health and safety decision making. However, this is not the case in J-value analysis – the risk valuation technique that is the main concern of this thesis – where the VTPF is an output that can be calculated if so required.

The earliest known valuations of human life can be found in the Babylonian Code of Hammurabi (ca. 1,700 BCE) and the Book of Leviticus of the Hebrew Bible (ca. 1,400 BCE). The former decreed compensation values to be paid by a man that assaulted or killed another individual, which were based on the relative social status between the offender and the victim. For example, if one man accidentally killed another man as the result of an argument, then the offender should pay half a mina to the victim's family if the victim was a freeborn man, or one third of a mina, if the man had been a slave but was now free. Using extremely crude calculation methods, the VTPF for the free born man is £206, whilst the VTPF for the former slave is £137, in 2011 prices [91]. In the Book of Leviticus, values were assigned to consecrated individuals based upon the individual's productive value to society, with males of ages between 20 and 60 being deemed the most valuable, at 50 shekels of silver. Females of these ages were valued at thirty shekels. This would mean a VTPF of £412, and £247 respectively, using the same calculations as before. Individuals outside this age group had lower valuations.

The first formal research into the value of life came some three thousand years later, but used largely the same methods of valuation. The method of valuing human life in terms of an individual's future productivity and earnings came to be known as the "human capital" method. Some of the first authors to investigate this method were Adam Smith in 1776 [176], and Ernst Engel in 1883 [74]. A more in depth historical review of human life valuation is provided by Dublin and Lotka (1930) [68], who also provide a calculation of a VTPF using this approach. They calculate the net future earnings of an individual to be approximately $9,802, in 1930 prices, or a VTPF of about £82,000 in 2011 prices. This approach suffers from some serious ethical problems, such as the zero value of retirees or those who do not work. Children are also assigned a relatively small valuation, due to the traditional economic method of discounting future earnings. According to Schulze (1980) [174], the early attempts at applying this method to value health and safety programs:

"Have given economists a "black eye" for supposedly advocating that individual human lives could be valued as the lost economic productivity associated with a shortened life span"

These problems have meant that there have been relatively few modern attempts at valuing physical risk using this method, the most notable being Rice (1967) [169], who used this approach to value the cost to society of illness, disability and death. A follow up to this study was published ten years later by Cooper and Rice (1976) [41]. Lave and Seskin (1970) [127] have also used this method to value the societal cost of air pollution.

The human capital approach is an example of one methodology that has been used as a procedure for valuing mortality risks in a consistent manner. Another important methodology that is now widely used is the "willingness to pay" (WTP) method. At the foundation of this method is the belief that public sector decisions regarding how to mitigate risks to society should reflect the degree to which the individuals are willing to pay to do so. Precisely how much an individual is willing to pay must be determined through techniques that can be classed as either “revealed preference” or “stated preference”.

Stated preference techniques involve eliciting an individual’s WTP by direct questioning, and can be further sub-divided into the “contingent valuation” (CV) method and the “choice experiment” (CE) method. The CV method involves simply asking a representative sample of individuals how much they would be willing to pay to reduce a particular risk, whilst the CE method involves indirectly deducing an individual’s WTP by presenting him with a series of hypothetical alternative scenarios, which the individual then orders in terms of his preference. This preference ordering then allows the experimenter to determine the individual’s marginal rate of substitution (MRS) between risk and wealth, which can then be used to determine the individual’s WTP for a given risk reduction. Beattie et al (1998) [16] published a report that tested the consistency of the CV method, finding that the results were dependent upon the way in which the questions were asked. Carthy et al (1999) [29] published a follow up study that sought to improve the consistency of the results by using a CE method instead, eventually concluding that a VTPF for road fatalities of £1 million was most appropriate (about £1.3 million in 2011 prices). The CV and CE approaches have also been employed by various UK regulatory bodies to determine safety policy. In a report for the UK Health and Safety Executive (HSE), Chilton et al (2000) [32] used both the CV and the CE approaches to establish a WTP “tariff” for risks in different contexts – those from roads and other public transport, fires, hazardous substances in the workplace, nuclear power, genetically modified organisms and sport and leisure. The HSE then commissioned a follow up study, published by Burton et al (2001) [22] following the Ladbroke Grove rail accident of October 1999, in order to assess how individual attitudes towards risk changed following a major accident. The procedures used in this study were essentially the same as in the previous one. A report by Covey et al (2008) [44] for The Rail Safety and Standards Board also used the CE approach to determine how to value risks that involved multiple fatalities, track worker fatalities, child and adult trespasser fatalities, and adult suicides.

Stated preference techniques have the advantage that they can be used to estimate the value of any type of risk. There are, however, a number of drawbacks. These include the tendency for the respondents to give inconsistent answers. For example, as briefly mentioned above, the same question can elicit different responses, depending on how the question was asked. This is known as the “framing effect”. Such studies also usually have to resort to “trimming”, whereby respondent’s answers are removed from the sample if the experimenter judges them to be either inconsistent or not representative of the sample as a whole. This process violates the ethical and democratic principle that all individual’s preferences should be accounted for with equal weight, and also undermines the fundamental principle that the VTPF should reflect the willingness to pay of society. Perhaps the most severe drawback of the stated preference technique is that there is little reason to suspect that an individual’s preferences for safety, when elicited in an isolated environment devoid of the vast array of factors that are confronted in everyday life, will be representative of how the individual makes decisions about his safety in reality.

Revealed preference techniques involve inferring the individual’s WTP for safety from his or her behaviour. The two most popular methods of doing so are the “compensating wage” method and the “avertive behaviour” method. The compensating wage method, which is the most widely used of all WTP methods, uses data from the labour market to assess the wage differentials for jobs with varying health and safety risks. It assumes that employees understand the nature and magnitude of the risks involved, and make informed choices that reflect their preferences for physical risk. Viscusi and Aldy (2003) [198] published a comprehensive review of compensating wage studies, showing that there was quite a large disparity in the VTPF, from around £3 million to £55 million, in 2011 prices. Avertive behaviour methods use price data of various risk reducing items, such as smoke detectors and seatbelts to determine WTP. It is assumed that the cost of buying one extra item is equal to the value of the associated risk reduction. Viscusi (1993) [197] reviewed seven such studies that inferred a value of risk from cigarette smoking, property prices in less polluted areas, and prices of inherently safer automobiles. The VTPF calculated using this method ranged from £0.6 million to £4 million, in 2011 prices. The advantages of the revealed preference techniques are that they use fairly reliable data, which accounts for the behaviour of many individuals, and much of which is freely available. The techniques also reflect to some degree decisions based on real-world choices, as opposed to the isolated decisions elicited by the stated preference techniques discussed above. The disadvantages of these techniques are that the assumptions regarding wage differentials being caused by differing levels of safety, and the price of a risk reducing item being equal to the value of the risk, are implausible. Clearly, many factors can affect wage levels and prices. The assumption that employees make considered decisions about whether to take a job based only on wage and safety considerations is also doubtful. The difficulties of these assumptions are borne out by the large range of the VTPF calculated in this manner.

Another method of valuing physical risk that has been developed recently is based on the Life Quality Index (LQI) method, first developed in 1997 by Nathwani, Lind and Pandey [137], [157]. The LQI is a summary indicator that can be used to measure the development of a nation, based on its Gross Domestic Product (GDP) per person, and its average life expectancy. By insisting that any protection system at least maintains the initial LQI, a maximum reasonable cost for the system can be determined. This cost is then the societal value of the given risk reduction. The calculation involves using labour market data to infer how individuals prefer to distribute their time between working, in which income is raised, and leisure, in which the income is consumed. In this sense, the LQI method can be seen to be a revealed preference technique for determining the societal WTP for risk reductions.

More recently, the LQI method has been expanded by Thomas et al in 2006, [182], [183] who introduced the "J-value method" for use in risk management and assessment, and which is the central concern of this thesis. The J-value is the ratio of the actual cost of a given risk reduction scheme, to the maximum cost of the risk given by the LQI method, and is therefore dimensionless. A J-value of less than unity indicates that the risk reduction scheme costs an acceptable amount, and should therefore be implemented, whilst a J-value of greater than unity indicates that the scheme is too expensive, and would impact society's quality of life adversely. This method can also be used to calculate a VTPF of £2.5 million in 2011 prices, and with a 2.5% per annum discount rate. This method has been used to value and assess risks from a diverse range of sources, such as railway protection systems, the cost-effectiveness of drugs, and radioactivity abatement systems. Much of the initial J-value research centred around radiation protection, in which the exposure to radiation and subsequent mortality response was stochastically modelled in order to determine the loss of life expectancy from a given exposure to ionising radiation, see Thomas et al (2006) [184], (2007) [185] and (2009) [186], [187].

Further recent developments of the J-value method include an extension of the method to include valuation of environmental risks (2010) [192], and an analysis of the tolerance of the J-value(2010) [123]. The main advantages of the J-value method are that the input parameters are objective, being estimated from actuarial or national statistics. The method is also transparent, the output being a simple dimensionless number that is easy to interpret. It is also consistent, offering a simple scale by which risks can be assessed. The disadvantages of the method are that it only values mortality risks, and cannot be used to assess morbidity, or non-fatal risks. Nor does the method account for the pain or suffering which may be experienced over the individual's remaining lifespan, for example, by using "Quality Adjusted Life-Years" (QALYs) that are used in health economics.

The various methods of valuing mortality risks are summarised in Table 1.

Method	Examples of Major Publications	VTPF (2011 £)	Advantages	Disadvantages
Human Capital	Dublin and Lotka [68] Rice, [169] Cooper and Rice, [41]	~82,000	Can be easily calculated from labour market data.	Severe ethical problems. Those who do not work have no value.
WTP – Stated Preference	Beattie et al, [16] Carthy et al, [29] Chilton et al, [32]	1,300,000	Can be used to value any type of risk.	Vulnerable to framing effects. The practice of "trimming" raises ethical issues. The answers of the respondents are out of everyday context and may therefore not be representative of true preferences.
WTP – Revealed Preference	Viscusi, [197] Viscusi and Aldy, [198]	600,000 – 55,000,000	Uses reliable labour market data that accounts for large numbers of people. Data accounts for behaviours in everyday context.	Assumption about the wage differential reflecting the risk level is implausible. Assumption that the price of a risk reducing commodity is equal to the value of the risk is also implausible.
LQI/ J-Value	Pandey and Nathwani, [157][158] Thomas et al, [182][183]	2,600,000	Input parameters are objective. National and actuarial data is used that accounts for millions of people. Output is transparent.	Does not account for morbidity risks or QALYs.

Table 1 Summary of literature on valuation of mortality risks.

Chapter 3 Conceptual Foundations of the J-Value

3.1 The Life Quality Index

It is impossible to determine each and every factor required to ensure that the highest quality of life may be enjoyed by all individuals. There are a vast amount of variables that influence an individual's welfare, and exactly what is entailed by a high quality of life is entirely subjective. Any rational analysis of such a complex and indeterminate concept must attempt to make an appropriate simplification by identifying the key factors which underlie the concept of quality of life. It is postulated that the quality of life of an individual can be distilled into two fundamental factors: how long an individual can expect to live from now on, and how much the individual has available to spend, both on life's necessities and on its luxuries. The first of these factors is encapsulated in the life expectancy, X, which is measured in years. This factor may be distilled further by recognising that individuals generally enjoy their life during time that they are free to dispose of as they wish, in contrast to time that is spent working.

For many people, the distinction between working time and free time is an arbitrary one, as people often engage in productive work even though they are not compelled to do so. Nevertheless, individuals will generally wish to retain flexibility over how they choose to spend their time. The productiveness of a society may be viewed as the result of a complex trade-off that each individual makes between working time and free time. In this trade-off the benefit gained from extra income obtained by working longer hours is balanced against the cost of loss of free time. This suggests that a more precise indicator of quality of life can be obtained by replacing the life expectancy with the remaining average free time, F, where:

$$F = (1-w)X \tag{3.1}$$

in which w is the average fraction of time spent working from now on. The amount available to an individual to spend on consumption can be represented by a summary measure of average income. This is taken as the Gross Domestic Product (GDP) per person, G (£/year). This figure is chosen for ethical reasons, namely that everyone within the nation is treated equally with regards to income. Thus, free time and average income are taken as being the two main inputs contributing to the single output of quality of life. In economic theory, inputs are related to outputs through a "production function", the most common of which is the Cobb-Douglas production function, (see e.g. Johansson (1991) [117]). If the output is denoted, Q_1, and represents a "life quality index" of an average person, then G and F are related to Q_1 by:

$$Q_1 = \alpha_1 G^{\beta} F^{\gamma} \tag{3.2}$$

where α_1, β and γ are dimensionless positive constants. A property of the Cobb-Douglas function is that any monotonic increasing function of Q_1 will also suffice as a life quality index. This property is then used to define a second life quality index, Q_2:

$$Q_2 = \left(\frac{Q_1}{\alpha_1}\right)^{1/\gamma} = G^q F = G^q (1 - w) X \tag{3.3}$$

where $q = \beta/\gamma$ is a dimensionless positive constant, and where equation (3.1) has been used in the last step. It may also be noted that the work time fraction is the complement of free time fraction, f:

$$f = (1 - w) \tag{3.4}$$

which allows equation (3.3) to be recast as:

$$Q = G^q fX \tag{3.5}$$

where Q is used instead of Q_2, as this is the most general form for the life quality index, and will be used in much of the following derivation. Equation (3.5) expresses three important considerations for an individual: how long he will live for, the fraction of his remaining time which is free for him to dispose of as he wishes, and the amount of money available to spend over this time. The potential for trade-offs between these three factors will now be considered. Firstly, it is assumed that free time fraction and life expectancy cannot be substituted. However, there are some very low values of f which would be associated with a reduced level of life expectancy due to overwork. This presumably is not an issue for most individuals. It therefore seems reasonable to assume that f and X are independent of one another. Two important trade-offs remain, however. These are the trade-off an individual can make between income and free time fraction, i.e. between G and f, and the trade-off between income and life expectancy, i.e. between G and X, which occurs when spending on a risk reducing protection scheme, or indeed, accepting compensation for a reduced life expectancy (for example via higher wages in a high risk job).

Consideration of these trade-offs leads to the concept of a maximum reasonable spend on safety and protection systems. This then allows a judgement or J-value to be assigned to such a system, which can be expressed as a single equation. Although the J-value has been derived before from different principles (e.g. see Thomas et al (2006a) [182]), the following is a new derivation based upon standard economic theory[2]. The independence of f and X means that the two tradeoffs described above can be considered separately, as will be done in the following sections.

3.2 The Trade-Off between Free Time Fraction and Income

In exploring the free time fraction-income trade-off, it is assumed that any such trade does not affect the individual's life expectancy. This means that a new life quality index, Q_f, can be formed by dividing the original life quality index, equation (3.5), by X, without loss of generality:

$$Q_f = \frac{Q}{X} = G^q f \tag{3.6}$$

This new life quality index is introduced in order that the features of the trade-off can be explored explicitly. It is apparent from equation (3.6) that it is possible for an

[2] Much of this chapter is based upon a paper published by Thomas, Jones and the present author, see Thomas, Jones and Kearns (2010) [189].

individual to exchange his income for free time, whilst still retaining his original life quality index. The set of values of *G* and *f* that will render a constant level of life quality, which will be denoted as $\overline{Q_f}$, is known as an "indifference curve", as it is assumed that the individual is indifferent to how his level of life quality is attained. The indifference curve must satisfy:

$$\overline{Q_f} = G^q f \tag{3.7}$$

which can be solved for *f* or *G*. Here it will be solved for *G*, to obtain:

$$G = \frac{\overline{Q_f}^{1/q}}{f^{1/q}} \tag{3.8}$$

One property of equation (3.8) is that there are an infinite number of indifference curves, with each one representing a different level of life quality. Also, none of these indifference curves intersect one another. The indifference curve is also convex, meaning that the function will always lie below a straight line drawn between any two points on the line. Convexity of indifference curves directly implies a diminishing marginal rate of substitution (MRS) of free time fraction for income. This is the amount of income that must be exchanged for a unit of free time fraction, and is given as:

$$MRS = -\frac{dG}{df} = \frac{\overline{Q_f}^{1/q}}{qf^{(1/q)+1}} = \frac{G}{qf} \tag{3.9}$$

Equation (3.9) clearly shows that the MRS diminishes with increasing levels of free time fraction. The implication of a diminishing MRS is that the higher the free time fraction enjoyed by the individual, the less willing the individual will be to give up some income in order to increase free time fraction further.

The amount of income generated by the labour market may also be formally linked to national average free time fraction by modelling a country's domestic product.

This is done by again using a Cobb-Douglas production function, following Pandey et al (2006) [158]. The output in this instance is the national GDP, denoted as G_C, and the factors of production are the national capital investment, K_C, and the annual supply of labour within the country, L_C:

$$G_C = A_P K_C^{1-\theta} L_C^{\theta} \tag{3.10}$$

where A_P is a productivity constant, that accounts for other factors affecting production, such as technological advancements and education level. The other parameter θ is the fraction of the GDP paid to workers as wages, as will now be shown:

The price of labour, p_L, is the marginal GDP with respect to labour supply, at constant levels of productivity and capital, i.e.:

$$p_L = \frac{dG_C}{dL_C} = \frac{\theta G_C}{L_C} \tag{3.11}$$

so that:

$$\theta = \frac{p_L L_C}{G_C} \tag{3.12}$$

The numerator in equation (3.12), which is the product of the price of labour and the labour supply, is the total wages paid to employees. Thus equation (3.12) shows that θ is the wage share of the GDP.

Furthermore, the supply of labour may be seen to be equal to the total population of a country, N_C, multiplied by the population-averaged work-time fraction:

$$L_C = N_C w = N_C(1-f) \tag{3.13}$$

where equation (3.4) has been used in the last step. Substituting into equation (3.10) gives:

$$G_C = A_P K_C^{1-\theta} N_C^{\theta} (1-f)^{\theta} \tag{3.14}$$

The GDP per person, G, is then:

$$G = \frac{G_C}{N_C} = A_P \left(\frac{K_C}{N_C} \right)^{1-\theta} (1-f)^{\theta} = AK^{1-\theta}(1-f)^{\theta} \tag{3.15}$$

where K is the capital investment per person.

Equation (3.15) shows that average income is related both inversely and non-linearly to the free time fraction. This curve is a constraint that is determined by the collective actions of individuals within a society and links the average individual's income to his free time fraction. It will now be assumed that these collective actions of a society will be such that the life quality is maximised for the average individual, subject to the above constraint. The maximisation occurs when the indifference curve defined by equation (3.8) is tangent to the constraint curve defined by equation (3.15). This situation is demonstrated in Figure 1, which presents data relevant to UK conditions in 2007. This figure shows the downwards curving income constraint, and the convex indifference curves. These three curves represent different levels of the life quality index, Q_f. The highest curve gives the highest quality of life. This curve, however, is unobtainable as it always lies above the constraint line. The lowest curve has parts that lie within the constraint, but any individual on this curve can increase his quality of life within the constraint. Hence the curve that maximises life quality subject to the constraint is tangent to the constraint line. The condition of tangency is met when the derivatives of the two curves are equal. Figure 1 also shows shaded regions where low values of free time fraction or very low income levels may compromise the individual's health, and are therefore excluded. These levels are not precisely defined. It is sufficient for these purposes that the trade-off occurs outside these shaded regions.

If the point of tangency is located at (f_0, G_0), then the derivative of the indifference curve is given by the negative of equation (3.9), evaluated at these points:

$$\left.\frac{dG}{df}\right|_{f_0,G_0} = -MRS = -\frac{G_0}{qf_0} \tag{3.16}$$

The derivative of the constraint line of equation (3.15) is:

$$\left.\frac{dG}{df}\right|_{f_0,G_0} = -\frac{\theta G_0}{1-f_0} \tag{3.17}$$

Matching the derivatives of (3.16) and (3.17) gives:

$$-\frac{G_0}{qf_0} = -\frac{\theta G_0}{1-f_0} \tag{3.18}$$

which can be solved for q, the only unknown parameter. This gives:

$$q = \frac{1}{\theta}\frac{1-f_0}{f_0} = \frac{1}{\theta}\frac{w_0}{1-w_0} \tag{3.19}$$

where, clearly, $f_0 = 1 - w_0$. The meaning of the parameter q may be further explored by rearranging equation (3.9) to give:

$$q = -\frac{G}{f}\frac{df}{dG} = -\eta_f \tag{3.20}$$

which is valid for $dG/df > 0$. The parameter η_f is the income elasticity of free time fraction. Elasticity is a measure of the sensitivity of relative changes in a variable following a relative change in another variable. The parameter q thus emerges as the modulus of this elasticity parameter.

3.3 The Trade-Off between Income and Life Expectancy

The second trade-off investigated is between income and free time fraction. The nature of this trade-off is different from the first trade-off, which was determined by a collective bargaining process made at a societal level. The trade-off between income and life expectancy occurs when health and safety schemes are being considered. Such a health and safety scheme can be expected to improve life expectancy by a certain amount, but at a cost. This cost may be borne by each individual in society, even if the individual does not directly benefit from the health and safety improvement, in line with the compensation notions of Kaldor (1939) [120] and Hicks (1939) [92] (see also Boadway and Bruce (1984) [21] and Johansson (1991) [117]).

The income-life expectancy trade-off is assumed to be independent of the free-time fraction. This means that a new life quality index, Q_X, may be formed, in a similar manner to equation (3.6), by dividing the general life quality index given by equation (3.5) by f, which is now being treated as a constant, rather than as a variable. Hence:

$$Q_X = \frac{Q}{f} = G^q X \tag{3.21}$$

As is the case with the first trade-off, it is possible for an individual to give up some income for additional life expectancy, whilst still retaining his initial level of life quality. It is also clear that excessive spend on life expectancy improvement will reduce the individual's life quality, whilst suitably small spends will increase life quality. Thus the maximum reasonable spend for a health and safety scheme defines the indifference curves for this trade-off. The set of values of G and X that define the indifference curve at a constant level of life quality, denoted as $\overline{Q_X}$, must satisfy:

$$\overline{Q_X} = G^q X \tag{3.22}$$

which can be solved for G, to obtain:

$$G = \frac{\overline{Q_f}^{1/q}}{X^{1/q}} \tag{3.23}$$

Equation (3.23) is analogous to equation (3.8), except the variable *X* is now used in place of the variable *f*. Hence, this equation is also convex in the *X*-*G* plane. This means that the MRS of life expectancy for income is also diminishing with increasing life expectancy, and is given as:

$$MRS = -\frac{dG}{dX} = \frac{\overline{Q_f}^{1/q}}{qX^{(1/q)+1}} = \frac{G}{qX} \tag{3.24}$$

Intuitively, this means that the higher the life expectancy the individual enjoys, the less willing he will be to give up income in order to raise life expectancy further. Equation (3.24) can be rearranged to give:

$$-dG = \frac{G}{q}\frac{dX}{X} \tag{3.25}$$

Here *-dG* is taken as the infinitesimal amount of income which should be exchanged for an infinitesimal increase in life expectancy, *dX*. In practice, these infinitesimal changes are replaced by small changes in income and life expectancy of $-\delta G$ and δX respectively. Thus, equation (3.25) becomes:

$$-\delta G = \frac{G}{q}\frac{\delta X}{X} \tag{3.26}$$

where the value of *q* has been calculated from equation (3.19). Thus, the first trade-off is used to determine the elasticity parameter *q*, which is then used in calculating the maximum reasonable income an individual should give up to achieve a given increase in life expectancy. It may be noted that equation (3.24) can be rearranged to give:

$$q = -\frac{G}{X}\frac{dX}{dG} = -\eta_X \tag{3.27}$$

which is valid for $dG/dX > 0$. Equation (3.27) is analogous to equation (3.20). Here, the parameter η_X is the income elasticity of life expectancy. Comparing equations (3.20) and (3.27), it is obvious that $\eta_X = \eta_f$. The reason why this is may be seen by considering the expected free time from now on:

$$F = fX \tag{3.28}$$

The total differential of (3.28) is:

$$dF = \frac{\partial F}{\partial f} df + \frac{\partial F}{\partial X} dX = Xdf + fdX \tag{3.29}$$

so that:

$$\frac{dF}{F} = \frac{df}{f} + \frac{dX}{X} \tag{3.30}$$

In the first trade-off, it was assumed that X was held constant, so that $dX = 0$. Under this condition the relative change in the free time fraction is equal to the relative change in the expected free time remaining:

$$\frac{df}{f} = \left.\frac{dF}{F}\right|_{X=const} \tag{3.31}$$

while in the second trade-off, the assumption was that f was constant, so that $df = 0$. Here, it is the relative change in life expectancy that is equal to the relative change in the expected free time remaining:

$$\frac{dX}{X} = \left.\frac{dF}{F}\right|_{f=const} \tag{3.32}$$

Thus equation (3.20) may be re-expressed as:

$$q = -\eta_f = -\frac{G}{F}\frac{dF}{dG}\bigg|_X \tag{3.33}$$

while equation (3.27) may be re-written as:

$$q = -\eta_X = -\frac{G}{F}\frac{dF}{dG}\bigg|_f \tag{3.34}$$

Equations (3.33) and (3.34) demonstrate that the income elasticity of expected free time remaining is the same in both instances. This suggests that the two considered trade-offs are specific instances of a more fundamental trade-off between income and expected free time remaining.

3.4 Utility and Discounting in the Life Quality Index

In each of the life quality indices derived above, one constant feature was the G^q term. For $0 < q < 1$, this term has the form of a utility function, known as a "power utility". If utility is denoted $U(G)$, then the utility of income is:

$$U(G) = G^q \qquad 0 \le q \le 1 \tag{3.35}$$

The notion of utility expresses the personal value derived from the consumption of goods. The bounds on the value of q are necessary to preserve the law of diminishing marginal utility. This economic law is based on the observation that individuals value extra gains in commodities more highly when the commodity is scarce than when it is plentiful. This law, when applied to the G^q term, which represents the utility of income, means that the first amount of earnings will give the individual the greatest value, as he will be able to afford such essentials as food and clothing. Subsequent increases in earnings will then be valued at an ever diminishing rate, as the individual will then begin to spend more on life's luxuries. The marginal utility is:

$$\frac{dU}{dG} = qG^{q-1} \qquad 0 \le q \le 1 \tag{3.36}$$

which decreases with increasing income, hence, diminishing marginal utility. An important economic parameter derived from utility theory is the income elasticity of marginal utility, η_{MU}. This is given by:

$$\eta_{MU} = \frac{\left(\frac{dU}{dG}\right)}{G} \div \frac{d\left(\frac{dU}{dG}\right)}{dG} = \frac{d^2U}{dG^2} \div \frac{\left(\frac{dU}{dG}\right)}{G} = q - 1 \tag{3.37}$$

The negative value of this quantity (which is more useful because it is positive) has been studied extensively, and is used by the Treasury to determine how to appropriately discount future effects, see [95]. This negative elasticity has also been shown to be identically equal to a parameter known as the "coefficient of relative risk aversion", or "risk aversion" for short [12], [164]. This parameter describes a person's attitude towards risk. If a person has a risk aversion of zero, then he is described as "risk neutral". Higher values of risk aversion indicate that the individual is willing to pay greater amounts in insurance to protect against risk. If the risk aversion is denoted as ε, then it is given as:

$$\varepsilon = -\eta_{MU} = 1 - q \qquad 0 \leq \varepsilon \leq 1 \tag{3.38}$$

As risk is the central focus of this research, the risk aversion parameter is judged to be a more relevant way of describing and assessing risk, and will replace the elasticity parameter, q. The bounds on the risk aversion and the elasticity parameter are a consequence of the use of the power utility function of equation (3.35). The upper bound on the risk aversion can be removed by instead using a more general utility function first introduced by Atkinson (1970) for the study of income inequality [13]. The Atkinson utility function is defined as:

$$\begin{aligned} U(G) &= \frac{G^{1-\varepsilon} - 1}{1-\varepsilon} \qquad && \varepsilon \geq 0, \varepsilon \neq 1 \\ &= \ln G && \varepsilon = 1 \end{aligned} \tag{3.39}$$

This utility function thus allows for risk aversions greater than unity, and so is a more general function than the power utility. If this utility function were to be used to derive the J-value, it would be necessary to substitute this into the life-quality index, and apply the trade-offs of section 3.2 and 3.3. However, the amount to spend in order to remain on the $\overline{Q_X}$ indifference curve, which is the maximum reasonable amount an individual should be prepared to spend to achieve a given increase in life expectancy, is unaffected by the use of this alternative utility function. In fact, it may be shown that the maximum spend is unaffected by the use of a more general class of utility functions given by:

$$U(G) = \frac{G^{1-\varepsilon} - k_1}{k_2} \tag{3.40}$$

These utility functions are known as "affine transformations" of the power utility function. The proof of the invariance of the maximum spend under affine transformations of the utility function is given in Appendix A. As the maximum reasonable spend is independent of the type of utility function used, the more simple power utility function will be retained in the rest of the development here.

Substituting the risk aversion, ε, as given by equation (3.38) into equation (3.19), which relates the elasticity parameter to measurable and observable quantities, gives:

$$\varepsilon = 1 - \frac{1}{\theta}\frac{w_0}{1-w_0} = \frac{1 - \frac{(\theta+1)}{\theta}w_0}{1-w_0} \tag{3.41}$$

The utility interpretation allows the life quality index to be viewed as the summation of the annual utilities over the whole of the future lifetime of the average individual. This interpretation provides a mechanism for extending the life quality index to include discounting.

It is widely accepted that individuals will prefer commodities that are available for consumption at the present time to commodities which can only be consumed sometime in the future. This concept may be applied to determine the utility of future income, which can be discounted back to the present value using a chosen discount rate.

Let the earnings per year averaged across all individuals of age *a* be $c(a)$ (£/year). If all individuals have the same utility function, so that for each person, the utility for that year's earnings will be:

$$U(c(a)) = (c(a))^{1-\varepsilon} \tag{3.42}$$

If the income is growing at a real, compound rate, r_g, so that the income at a later age, τ, will be given by:

$$c(\tau) = e^{r_g(\tau-a)}c(a) \tag{3.43}$$

and the utility of this income will be:

$$U(c(\tau)) = e^{r_g(\tau-a)(1-\varepsilon)}(c(a))^{1-\varepsilon} \tag{3.44}$$

The utility attained at future age τ may be discounted back to the present age *a* by multiplying by $e^{-r_d(\tau-a)}$, where r_d is the real rate of time preference, which will also be termed the "discount rate". Thus the net present utility to an individual of age *a* of the income he will generate later in the age interval $\tau + d\tau$, is:

$$\begin{aligned} e^{-r_d(\tau-a)}(c(\tau))^{1-\varepsilon}\,d\tau &= e^{-r_d(\tau-a)}e^{r_g(\tau-a)(1-\varepsilon)}(c(a))^{1-\varepsilon}\,d\tau \\ &= e^{-(r_d-r_g(1-\varepsilon))(\tau-a)}(c(a))^{1-\varepsilon}\,d\tau \\ &= e^{-r(\tau-a)}(c(a))^{1-\varepsilon}\,d\tau \end{aligned} \tag{3.45}$$

where *r* is the net discount rate, given by:

$$r = r_d - (1-\varepsilon)r_g \tag{3.46}$$

Clearly, however, the individual will only be able to benefit from a utility τ - a years later if he is still alive at age τ. This aspect may be included by considering survival probabilities. The probability of an individual surviving to age τ given that he has already survived to age a is denoted as $S(\tau|a)$. This is also the probability that the utility given by equation (3.44) will be achieved.

The expected value, $R(a)$, of the future discounted utility for an average individual of age a, is found by multiplying the discounted utility of equation (3.45) by the probability that the utility is achieved, $S(\tau|a)$, and integrating over all possible lengths of life to come:

$$\begin{aligned} R(a) &= \int_{\tau=a}^{\infty} S(\tau|a) e^{-r(\tau-a)} (c(a))^{1-\varepsilon} \, d\tau \\ &= (c(a))^{1-\varepsilon} \int_{\tau=a}^{\infty} S(\tau|a) e^{-r(\tau-a)} d\tau \end{aligned} \tag{3.47}$$

Equation (3.47) may be interpreted in light of the equation for life expectancy, $X(a)$, for an individual of age a, namely:

$$X(a) = \int_{\tau=a}^{\infty} S(\tau|a) d\tau \tag{3.48}$$

which will be derived in more detail in chapter 4. Comparing equation (3.48) with the integral on the right hand side of equation (3.47), it is apparent that the latter integral may be regarded as a "discounted life expectancy", $X_d(a)$:

$$X_d(a) = \int_{\tau=a}^{\infty} S(\tau|a) e^{-r(\tau-a)} d\tau \tag{3.49}$$

Clearly, equation (3.48) and (3.49) are equal when the discount rate is zero. The relationship between life expectancy and discounted life expectancy is shown graphically in Figure 2, which uses mortality data from the ONS [145], and uses a net discount rate of 2.5%. Substituting (3.49) into (3.47), and assuming a constant income i.e.: $c(a) = c$, the expected value of future discounted utility is:

$$R(a) = c^{1-\varepsilon} X_d(a) \tag{3.50}$$

For a group of individuals with varying ages, the average value of discounted utility is found by multiplying $R(a)$ by the probability density for age, $p(a)$, for the individuals within the group, and integrating over the appropriate age range:

$$\int_{a_1}^{a_2} p(a)R(a)da = c^{1-\varepsilon}\int_{a_1}^{a_2} p(a)X_d(a)da = c^{1-\varepsilon} X_d \tag{3.51}$$

where X_d is the average life expectancy for a group of individuals of ages between a_1 and a_2. If the population being considered is the general public, then the integration limits are $a_1 = 0$ and $a_2 = \infty$. If the population under consideration is the workforce, then the limits of integration are $a_1 \sim 18$ and $a_2 \sim 65$. The parameter, c, is now set equal to the national average income, rather than the income of the group. This is done as a result of an ethical decision in order to avoid different treatments of high earning and low earning income groups with regard to safety spend. The national average income is estimated by the GDP per person, and so in setting $c = G$, equation (3.51) can be seen to be a discounted life quality index of the form given by equation (3.6):

$$Q_{f,d} = G^{1-\varepsilon} X_d \tag{3.52}$$

The same procedure as laid out in section 3.3 may be followed to derive the effect of discounting on the income-life expectancy trade-off. The discounted MRS of life expectancy for income is:

$$MRS = -\frac{dG}{dX_d} = \frac{G}{(1-\varepsilon)X_d} \tag{3.53}$$

Following equations (3.25) and (3.26), the maximum amount of income, -δG, that should be given up to achieve in increase in discounted life expectancy, δX_d, is then:

$$-\delta G = \frac{G}{1-\varepsilon}\frac{\delta X_d}{X_d} \tag{3.54}$$

This maximum discounted payment can then be used to derive the maximum amount a group should be willing to pay for a protection system, which is then used to derive the J-value.

3.5 The J-Value

The results of the two trade-offs will now be used to derive the J-value. Equation (3.54) relates the maximum reasonable amount of annual income to give up, δG, in exchange for an increase in discounted life expectancy, δX_d. If the benefits of the risk reduction are experienced by a population of size, N, then the maximum reasonable annual amount the population should be willing to pay, which is denoted as δG_N, is the product of the population size and the individual maximum reasonable payment:

$$\delta G_N = -N\delta G = \frac{NG}{1-\varepsilon}\frac{\delta X_d}{X_d} \tag{3.55}$$

This figure is the maximum annual spend for achieving the given discounted life expectancy improvement. This annual spend can be related to a single lump sum spend, by noting that the average length of time over which the cost is paid is equal to the population's base discounted life expectancy, X_d. Thus the series of annual payments can be discounted back to the present time in a similar manner to equation (3.45), except the period over which the discounting is applied is now equal to X_d. From equation (3.55), the maximum amount that is reasonable to spend on a health and safety measure to protect N people between times t and dt is:

$$\delta G_N dt = \frac{NG}{1-\varepsilon}\frac{\delta X_d}{X_d} dt \tag{3.56}$$

which will have a value discounted back to time, t = 0, of:

$$e^{-r_d t}\delta G_N dt = e^{-r_d t}\frac{NG}{1-\varepsilon}\frac{\delta X_d}{X_d}dt \tag{3.57}$$

The maximum amount of money, δV_N, a group of N people would then be reasonably expected to spend on a protection measure that affords them an improved discounted life expectancy of δX_d, expressed as an up-front lump sum, can be found by integrating equation (3.57) from the time of installation of the measure, which is set to be at time, t = 0, to the life expectancy of the group at the time of installation, namely, $t = X_d$:

$$\begin{aligned}\delta V_N &= \int_{t=0}^{t=X_d} e^{-r_d t}\delta G_N dt = \frac{NG}{1-\varepsilon}\frac{\delta X_d}{X_d}\int_{t=0}^{t=X_d} e^{-r_d t}dt \\ &= \frac{NG\delta X_d}{1-\varepsilon}\frac{\left(1-e^{-r_d X_d}\right)}{r_d X_d}\end{aligned} \tag{3.58}$$

which applies when $r_d > 0$. For the case when $r_d = 0$, it is noted that $e^{-y} \rightarrow 1 - y$ as $y \rightarrow 0$. Hence:

$$\frac{\left(1-e^{-r_d X_d}\right)}{r_d X_d} \rightarrow 1 \tag{3.59}$$

as $r_d \rightarrow 0$. Hence the general expression for the maximum reasonable up-front lump sum spend on the safety system is:

$$\begin{aligned}\delta V_N &= \frac{NG\delta X_d}{1-\varepsilon}\frac{\left(1-e^{-r_d X_d}\right)}{r_d X_d} \qquad \text{for } r_d > 0 \\ &= \frac{NG\delta X_d}{1-\varepsilon} \qquad \text{for } r_d = 0\end{aligned} \tag{3.60}$$

The final step in deriving the J-value is achieved by linking the maximum reasonable spend to the actual cost of any such protection system that improves life expectancy.

If the up-front cost, which will be denoted as $\delta\hat{V}_N$, is known, then the J-value is the ratio of the known cost to the maximum reasonable cost:

$$\begin{aligned} J = \frac{\delta\hat{V}_N}{\delta V_N} &= \frac{(1-\varepsilon)\delta\hat{V}_N}{NG\delta X}\frac{r_d X_d}{\left(1-e^{-r_d X_d}\right)} \qquad \text{for } r_d > 0 \\ &= \frac{(1-\varepsilon)\delta\hat{V}_N}{NG\delta X} \qquad \text{for } r_d = 0 \end{aligned} \tag{3.61}$$

For safety schemes with costs greater than what is the maximum reasonable, $J > 1$, indicating that the scheme offers poor value for money, and will result in a reduction in life quality for the affected population. Schemes that cost less than the maximum reasonable amount will have $J < 1$, which means that the scheme offers good value for money, and will result in an improved life quality for the affected population. Schemes that have a calculated J-value of unity will preserve the initial life quality. This can be represented as an indifference curve in the *X*-*G* plane, as shown in Figure 3. This figure uses data from the Office for National Statistics [145], [149]. The point marked on the graph is the average income and life expectancy (with no discounting) for the population. A move to any other point on the curve would preserve the life quality index, and so has a J-value of unity. A move into the area above the curve would increase the life quality index, either by increasing life expectancy or income, and so such a move would have a J-value of less than unity. Conversely, a move into the area below the curve would have a corresponding J-value of greater than unity.

The J-value is thus a dimensionless indicator of the cost-effectiveness of safety schemes. Aside from the net discount rate, which is usually chosen to be either 0% per annum, or 2.5% per annum, all the input parameters are fully objective and easily measurable from reliable statistics. The following three chapters will describe the techniques and methods needed to estimate these input parameters.

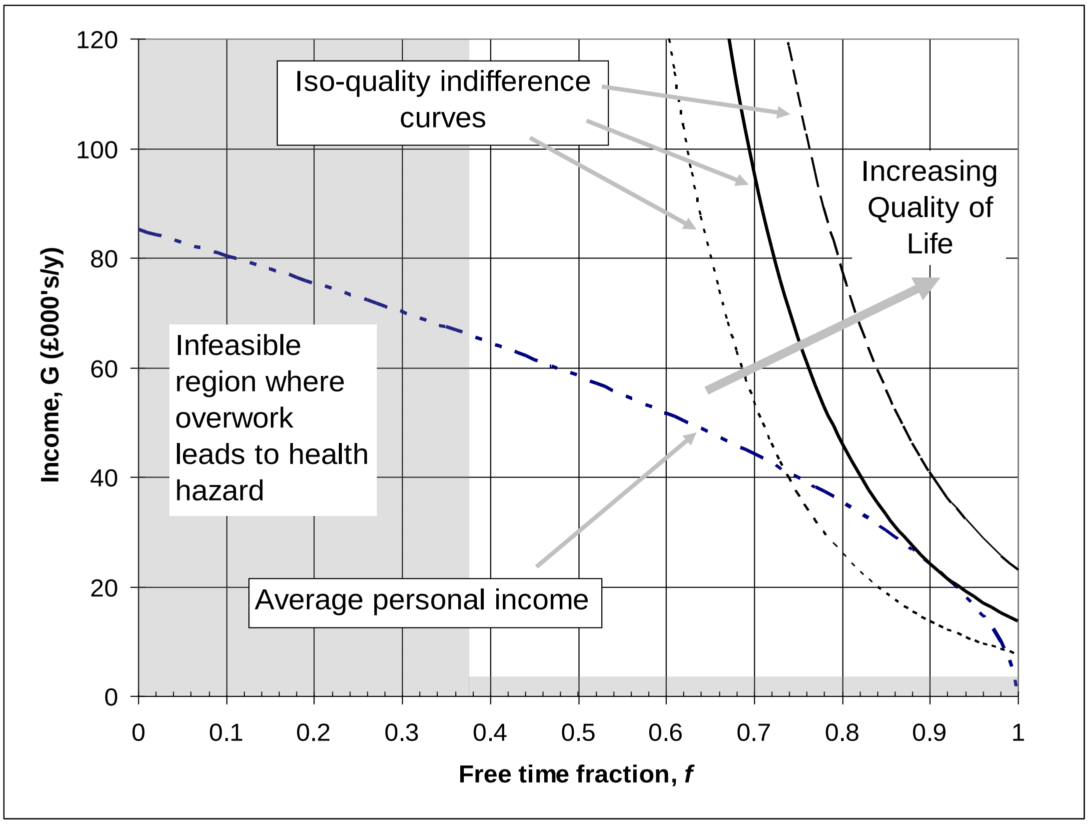

Figure 1 Indifference curves of quality of life against the income constraint.

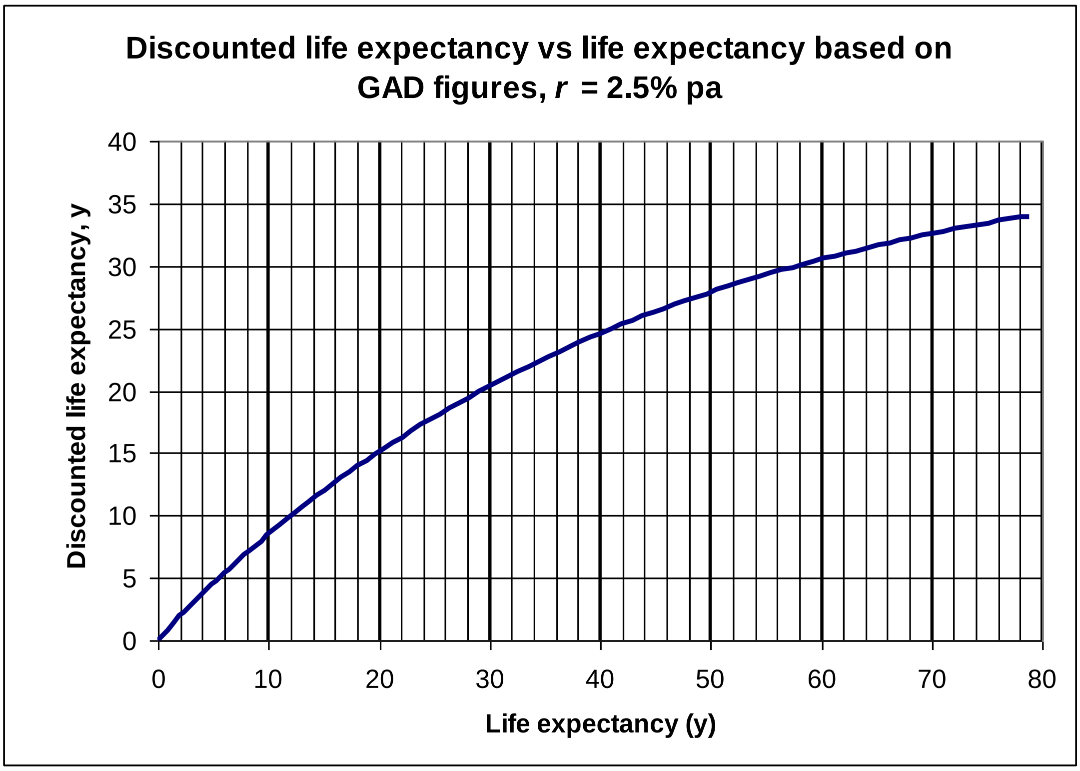

Figure 2 Discounted life expectancy versus life expectancy at r = 2.5% pa, based on ONS figures.

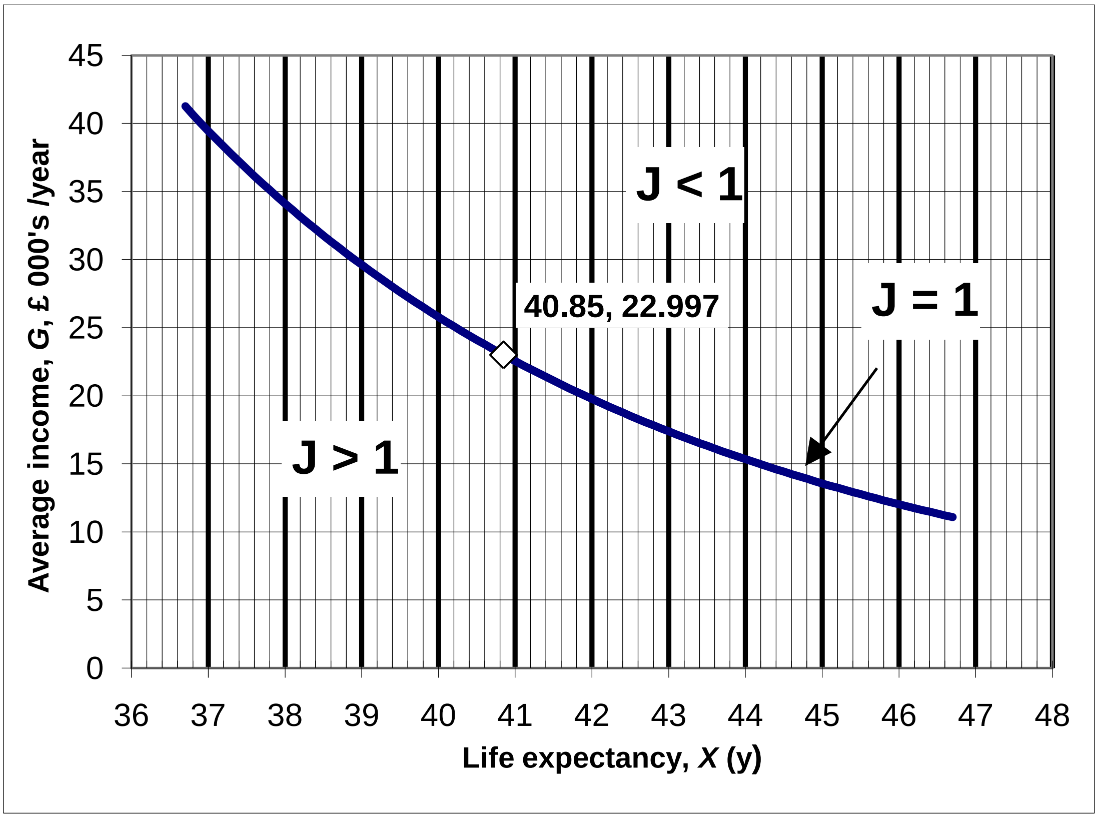

Figure 3 J = 1 indifference curve for income against (undiscounted) life expectancy.

Chapter 4 Fundamental Relationships between Parameters Used in Life-Expectancy Calculations

4.1 Characterising and Modelling the Survival of Populations

In this section the technical details required for the calculation of life expectancy are presented. Life expectancy can be calculated in two ways – the first being through a general probabilistic theory of survival, where the central concepts are the hazard rate and the survival probability, which are dependent upon age. These concepts then allow the age-specific life expectancy to be determined. The second way is through the life table method, in which a theoretical cohort is exposed to rates of mortality experienced by a general population, and followed to extinction. The relationships between these two methods are also described. The theoretical framework of survival models and life tables is now well established (for example, see Chiang (1968) [31]), and this chapter gives an overview of the relevant concepts. These concepts are used extensively in chapter 5, and to a lesser extent in subsequent chapters. One quantity that has received little attention in the literature is the population-averaged life expectancy (although Keyfitz (1985) has briefly discussed this, see [126]). This chapter will thus show how this quantity is calculated, and give some useful approximations. The discounted life expectancy is also described[4].

In order to calculate the average life expectancy, knowledge of the age distribution of the population is required. It is shown that this distribution can be determined from the survival probabilities when it is assumed that the population is in a steady state, such that the number of births each year is always equal to the annual number of deaths. This special population is also known as the stationary population. A different age distribution is required if the average life expectancy is to be determined for a workforce. Here it is assumed that the distribution is uniform between the age of recruitment and the age of retirement, and zero outside these ages.

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[4] The derivation of the average life expectancy and it's discounted equivalent are partly based on the appendices of Thomas et al (2006c) [184], although some new relations are derived here.

4.2 The Hazard Rate and the Survival Probability

Suppose the probability of dying between ages t and $t + dt$ is $f_d(t)dt$. Here age is treated as a continuous variable, so that someone aged 20 and three months has t = 20.25 years. The parameter $f_d(t)$ is then the probability density for the random age of death, T. The cumulative distribution function, $D(t)$, is then the probability of dying at any point from birth to age t, so that $T \le t$, and is the integral of the probability density function from age zero to age t:

$$D(t) = \Pr(T \le t) = \int_0^t f_d(u)du \tag{4.1}$$

The cumulative distribution function is also related to the probability density by:

$$\frac{dD(t)}{dt} = f_d(t) \tag{4.2}$$

For any age, any given individual must have either died or survived. Hence the probability of either dying or surviving from birth to age t must be equal to unity:

$$D(t) + S(t) = 1 \tag{4.3}$$

Where $S(t)$ is the probability of surviving from birth to age t. This is also the probability of dying after age t, which may be related to the probability density of death by:

$$S(t) = \Pr(T > t) = \int_t^\infty f_d(u)du \tag{4.4}$$

Differentiating equation (4.2) gives:

$$\frac{dS(t)}{dt} = -\frac{dD(t)}{dt} \tag{4.5}$$

so that:

$$\frac{dS(t)}{dt} = -f_d(t) \tag{4.6}$$

The immediate hazard faced by an individual of age t is the probability that T will be between t and $t + dt$, given that he has survived so far. The immediate hazard is denoted $h(t)dt$, where $h(t)$ is the hazard rate, and is given formally by:

$$h(t)dt = \Pr(t < T \le t + dt \mid T > t) \tag{4.7}$$

The conditional probability can be written in terms of the joint probability:

$$h(t)dt = \frac{\Pr(t < T \le t + dt \cap T > t)}{\Pr(T > t)} \tag{4.8}$$

Because the event $t < T \le t + dt$ guarantees that the event $T > t$ occurs, the equation is reduced to:

$$h(t)dt = \frac{\Pr(t < T \le t + dt)}{\Pr(T > t)} \tag{4.9}$$

The probability that death occurs between ages t and $t + dt$ is $f_d(t)dt$, and the probability that death occurs after age t is $S(t)$, so that:

$$h(t)dt = \frac{f_d(t)dt}{S(t)} \tag{4.10}$$

and substituting in equation (4.6):

$$h(t)dt = -\frac{dS(t)}{dt}\frac{dt}{S(t)} = -\frac{dS(t)}{S(t)} \tag{4.11}$$

Equation (4.11) can be integrated to give:

$$S(t) = \exp\left(-\int_0^t h(t)dt\right) = \exp(-W(t)) \tag{4.12}$$

where:

$$W(t) = \int_0^t h(t)dt \tag{4.13}$$

is the cumulative hazard rate. The probability that an individual will survive to age *t*, given that he has already survived to age *a*, is denoted *S*(*t*|*a*), and is given formally as:

$$\begin{aligned} S(t \mid a) &= \Pr(T > t \mid T > a) \\ &= \frac{\Pr(T > t \cap T > a)}{\Pr(T > a)} \end{aligned} \tag{4.14}$$

Because surviving to age *t* guarantees that the individual will have survived to age *a*, this equation simplifies to:

$$S(t \mid a) = \frac{\Pr(T > t)}{\Pr(T > a)} = \frac{S(t)}{S(a)} \tag{4.15}$$

This conditional probability of surviving to age *t* given that age *a* has already been reached can be expressed in terms of the hazard rate as:

$$S(t \mid a) = \frac{\exp\left(-\int_0^t h(t)dt\right)}{\exp\left(-\int_0^a h(t)dt\right)} = \exp\left(-\int_a^t h(t)dt\right) \tag{4.16}$$

or:

$$S(t \mid a) = \exp(-W(t \mid a)) \tag{4.17}$$

Where $W(t|a)$ is the conditional cumulative hazard rate.

4.3 The Survival Probability and Life Expectancy

The life expectancy is the expected value of the future life to come, which is a random variable. For an individual of age a the random life to come is denoted as $\chi(a)$. This is related to another random variable, the age of death T, by:

$$\chi(a) = T - a \tag{4.18}$$

The probability density will be the probability of death in the interval t to $t + dt$, given that the individual has survived to age a, and will be denoted $g_d(t|a)$. Following the arguments of equations (4.7) to (4.10), this will be:

$$\begin{aligned} g_d(t\,|\,a) &= \frac{f_d(t)}{S(a)} \qquad \text{for } t \ge a \\ &= 0 \qquad\qquad \text{for } t < a \end{aligned} \tag{4.19}$$

It can be readily verified that the integral of this quantity of all values of t, is equal to unity, as would be expected from a probability density function. The quantity $g_d(t|a)dt$ is therefore the probability that the random variable $\chi(a) = T - a$ will take the value $(t - a)$, for those that have survived to age a. The expected value of the life to come, given that age a has already been attained is the life expectancy, $X(a)$, given as:

$$\begin{aligned} E(T - a \,|\, T \ge a) &= X(a) \\ &= \int_a^\infty (t-a) g_d(t\,|\,a)dt \\ &= \int_a^\infty t g_d(t\,|\,a)dt - a\int_a^\infty g_d(t\,|\,a)dt \\ &= \int_a^\infty t \frac{f_d(t)}{S(a)} dt - a \end{aligned} \tag{4.20}$$

where equation (4.4) has been used in the last step. The integral on the right hand side can be integrated by parts. For the integral:

$$\int_a^\infty t f_d(t)dt \tag{4.21}$$

put:

$$\begin{aligned} &u = t \\ &\frac{du}{dt} = 1 \\ &\frac{dv}{dt} = f_d(t) \\ &v = D(t) = 1 - S(t) \end{aligned} \tag{4.22}$$

using:

$$\int_a^\infty u\frac{dv}{dt}dt = [uv]_a^\infty - \int_a^\infty v\frac{du}{dt}dt \tag{4.23}$$

then:

$$\begin{aligned} \int_a^\infty t f_d(t)dt &= [t - tS(t)]_a^\infty - \int_a^\infty (1 - S(t))dt \\ &= \lim_{t\to\infty}(t - tS(t)) - a + aS(a) - \lim_{t\to\infty}(t) + a + \int_a^\infty S(t)dt \end{aligned} \tag{4.24}$$

because $S(\infty) = 0$, this reduces to:

$$\int_a^\infty t f_d(t)dt = aS(a) + \int_a^\infty S(t)dt \tag{4.25}$$

substituting into (4.20) gives:

$$X(a) = \int_a^\infty \frac{S(t)}{S(a)}dt = \int_a^\infty S(t \mid a)dt \tag{4.26}$$

4.4 Relationship to the Life Table Functions

The life table presents data on mortality rates and length of life for individuals within a population. The life table in its usual form delineates individuals by gender and age. In the J-value model individuals are usually not delineated by gender, which is achieved via a simple averaging process. However, if the problem requires gender to be delineated (for example a particular workforce may be mostly male), then this can be easily achieved. In the UK, the life tables are published by the Office for National Statistics, see [145]. The life table consists of five functions, each of which can be determined from two pieces of information: the mid-year population, n_a, at age a, and the number of people who die, D_a, at age a. The life table functions are discrete variables, which is a consequence of the fact that each individual is grouped according to his present (discrete) age. The relationship between the life table functions and the hazard rate and survival probabilities will now be explored.

The first function of the life table is the central rate of mortality, m_a. This is the average death rate over the interval (a, a + 1), and is defined as:

$$m_a = \frac{D_a}{n_a} \tag{4.27}$$

The second function is q_a, which is the conditional probability that someone aged exactly a will survive to age a + 1. This is the number of people who die at age a, divided by the number of people who have reached age a. Note that the number of people who have reached age a is not the same as the mid-year population because there will be a number of people who will reach age a, but will have died before the population estimate is made. If it is assumed that deaths are distributed uniformly throughout the interval (a, a + 1), then the number of people who will have died before the population estimate is made will be $D_a/2$. Thus the number of people who reach age a is $n_a + D_a/2$, and q_a is given by:

$$q_a = \frac{D_a}{n_a + \frac{D_a}{2}} = \frac{2m_a}{2 + m_a} \tag{4.28}$$

Alternatively, if deaths are distributed exponentially over the interval (a, a + 1), then q_a is related to the central rate by:

$$q_a = 1 - e^{-m_a} \tag{4.29}$$

The next function in the life table is the number of survivors at each age, l_a. The life table uses a hypothetical cohort of individuals which are followed through to extinction as they experience the observed mortality rates. The initial size of the cohort, l_0, is known as the radix, and is usually taken to be 100,000. Thus, l_a is the number of this initial 100,000 who have survived to age a. If l_a is known, then l_{a+1} can be determined from:

$$l_{a+1} = (1 - q_a)l_a \tag{4.30}$$

The l_a's can also be related to the radix by:

$$l_a = \prod_{t=0}^{a-1} (1 - q_t) l_0 \tag{4.31}$$

The fourth function in the life table is the number of deaths in the hypothetical cohort at each age, d_a, given by:

$$d_a = q_a l_a = l_a - l_{a+1} \tag{4.32}$$

The last function in the life table is the life expectancy at age a, which is usually denoted e_a. Note that the life expectancy defined in the previous section, which is denoted $X(a)$, is a continuous function based on general survival probabilities, whilst e_a is a discrete function describing the average length of life for the hypothetical life table cohort. The relationship between $X(a)$ and e_a will be discussed below. The life expectancy e_a is given by:

$$e_a = \frac{1}{l_a} \sum_{t=a}^{\infty} \frac{(l_t + l_{t+1})}{2} \tag{4.33}$$

The correspondences between the life table functions and the probabilistic survival functions may now be explored. The survival probability may be immediately related to the number of survivors. The ratio of the number of survivors to the size of the initial cohort, l_a/l_0, is the probability of surviving from birth to age a. This is the survival probability, $S(a)$. Thus, in the context of the life table functions, the survival probability may be given by:

$$S(a) = \frac{l_a}{l_0} \tag{4.34}$$

It is important to note that $S(a)$ is a general function describing the probability of survival, whilst the l_a's are specific only to the life table cohort. The notation is also slightly awkward, in that $S(a)$ is continuous, whilst the l_a's are discrete functions only defined at specific ages. Nevertheless, this awkwardness can be avoided by using interpolation methods to estimate the life table functions inside the interval, e.g. at $l_{a+1/2}$.

The conditional survival probability, $S(t \mid a)$ is given by:

$$S(t \mid a) = \frac{l_t}{l_a} \tag{4.35}$$

The hazard rate $h(a)$ can be given either by the conditional probability of death, q_a, or by the central rate of mortality, m_a, depending on the assumption made regarding how the deaths are distributed in the interval $(a, a + 1)$. This can be shown by noting that q_a is:

$$q_a = \Pr(a < T \leq a + 1 \mid T > a) \tag{4.36}$$

which can be written as:

$$q_a = \frac{\Pr(a < T \leq a + 1 \cap T > a)}{\Pr(T > a)} = \frac{\Pr(a < T \leq a + 1)}{\Pr(T > a)} \tag{4.37}$$

The probability that death occurs between ages a and $a + 1$ is equal to the difference between the probability of surviving to age a and the probability of surviving to age $a + 1$, i.e.: $S(a) - S(a+1)$. This means that q_a can be written as:

$$q_a = \frac{S(a) - S(a+1)}{S(a)} \tag{4.38}$$

If deaths are distributed uniformly over the interval $(a, a + 1)$, then:

$$f_d(a + \delta a) = f_d(a) \quad \text{for } 0 < \delta a \leq 1 \tag{4.39}$$

The survival probability over the interval is:

$$S(a + \delta a) = S(a) - f_d(a)\delta a \quad \text{for } 0 < \delta a \leq 1 \tag{4.40}$$

So that:

$$S(a+1) = S(a) - f_d(a) \tag{4.41}$$

Substituting (4.41) into (4.38) gives:

$$\begin{aligned} q_a &= \frac{f_d(a)}{S(a)} \\ &= h(a) \end{aligned} \tag{4.42}$$

where equation (4.10) has been used in the last step. Hence, when deaths are uniformly distributed over the interval, then the hazard rate is equal to the conditional probability of dying in the interval. If, however, the deaths are distributed exponentially over the interval $(a, a + 1)$, then the hazard rate is equal to the central rate of mortality, m_a. This can be seen by noting that the central rate is given by:

$$m_a = \frac{S(a) - S(a+1)}{\int_a^{a+1} S(u)du} \tag{4.43}$$

where the denominator is the average survival probability over the interval (*a*, *a* + 1). If deaths are exponentially distributed, then:

$$f_d(u) = \lambda e^{-\lambda u} \qquad \text{for } a < u \le a+1 \tag{4.44}$$

The survival probability is:

$$S(u) = e^{-\lambda u} \qquad \text{for } a < u \le a+1 \tag{4.45}$$

So that the hazard rate, $h(a)$, is equal to λ, a constant over the interval. Substituting (4.45) into (4.43) gives:

$$\begin{aligned} m_a &= \frac{e^{-\lambda a} - e^{-\lambda(a+1)}}{\int_a^{a+1} e^{-\lambda u} du} \\ &= \frac{e^{-\lambda a} - e^{-\lambda(a+1)}}{\frac{-1}{\lambda}\left(e^{-\lambda(a+1)} - e^{-\lambda a}\right)} \\ &= \lambda = h(a) \end{aligned} \tag{4.46}$$

Thus, when deaths are exponentially distributed, the hazard rate is equal to the central rate of mortality.

It is worth noting that, for most populations, the conditional probability of death is generally very small for most ages. This means that the central rate is approximately equal to the conditional probability of death, as can be verified from equations (4.28) and (4.29). This means that:

$$m_a \approx q_a \approx h(a) \quad \text{for } m_a << 1 \tag{4.47}$$

This approximation is not valid at very young or old ages. Approximating the hazard rates by the q_a's is generally more realistic, as a uniform distribution of deaths is a more reasonable assumption than an exponential assumption. However, using the exponential assumption, which means that the hazard rate is constant between years, does enable simpler calculations, and is often preferred.

Finally, the continuous life expectancy, $X(a)$, as given by equation (4.26), is equal to the discrete life expectancy, e_a, of equation (4.33). This can be seen by re-writing (4.33) as:

$$e_a = \frac{1}{l_a}\sum_{t=a}^{\infty}\frac{(l_t + l_{t+1})}{2} = \sum_{t=a}^{\infty}\frac{(S(t\,|\,a) + S(t+1\,|\,a))}{2} \tag{4.48}$$

where equation (3.35) has been used. This summation is equal to the integral of the conditional survival probability from $t = a$ to $t = \infty$, when the trapezium method is used for numerically evaluating the integral. This means that (4.48) can be written as:

$$e_a = \int_{t=a}^{t=\infty} S(t\,|\,a)dt = X(a) \tag{4.49}$$

Thus, the life expectancy based on general survival probabilities should be numerically equal to the life expectancy of the life table cohort, when the trapezium method is used to evaluate the integral of (4.26) or (4.49).

4.5 Calculation of Life Expectancies in the J-Value Model

In the J-value model, the hazard rates are assumed to be equal to the central rates of mortality, m_a, which are obtained from the latest UK life tables, published annually by the ONS [145]. Separate tables are published for males and females, and so the male and female central rates can be averaged by calculating:

$$h(a) = m_a = f_{male} m_a^{male} + (1 - f_{male}) m_a^{female} \tag{4.50}$$

where f_{male} is the proportion of the population that is male, so that $1 - f_{male}$ is the proportion that is female. For public hazards, it is usually assumed that male and female numbers are equal, so that $f_{male} = 0.5$, whereas for industrial occupational hazards, a value of $f_{male} = 1$ may be more suitable.

The hazard rates are then integrated to give the cumulative hazard rate of equation (4.13). As the central rates of mortality are used for the hazard rates, the hazard rates are mid-interval values. This means that the integration can be performed by simply summing the hazard rates:

$$W(a) = \int_0^a h(u)du = \sum_0^a h(u) \tag{4.51}$$

However, there is a problem with the final age interval, since not everybody will be predicted to die by the end of it. This is remedied by adding an open age interval after the last one which approximates the mortality of the remaining cohort. The UK life tables provide data up to the age interval (100, 101), and so the additional age interval is for (101, ∞). This approximation is due to Silcocks (2001) [175], who assumes that the mortality rate of the final interval considered continues indefinitely, and shows that the final hazard rate, $h(101)$, is:

$$h(101) = \frac{2h(100)}{m_{100}} = 2 \tag{4.52}$$

So that the final cumulative hazard rate is:

$$W(101) = W(100) + 2 \tag{4.53}$$

The cumulative hazard rates are then used to calculate the survival probabilities, using equation (4.12). The survival probabilities can then be integrated using the trapezium rule to determine the life expectancy:

$$X(a) = \frac{1}{S(a)} \int_{a}^{\infty} S(t)dt = \frac{1}{S(a)} \sum_{a}^{101} \frac{S(t) + S(t+1)}{2} \tag{4.54}$$

The reason that this method is used to calculate life expectancies, rather than simply taking them from the life tables, is that this method allows the change in life expectancy to be easily calculated following a change in the hazard rate. The life expectancies calculated using this method compare well with the life table values. Exactly how well they correspond is statistically tested in chapter 9, where the sensitivities of the life expectancy calculations to the assumptions regarding the hazard rates and the methods of integration are assessed.

In the J-value model, the population-averaged life expectancy is usually required. The method for averaging over the population will now be described.

4.6 The Steady State Population Distribution

It is assumed that within the general population, the annual number of births is always equal to the annual number of deaths, so that the total population size is always constant. Such a population has a fixed age distribution, and is known as the “steady state”, or “stationary” population [126].

Suppose that the population density at age a is $n(a)$, implying that the number of people between ages a and $a + da$ is $n(a)da$. The number $n(a)$ may also be regarded as the rate at which members of the population are reaching age a. This number will be equal to the birth rate, $n(0)$, multiplied by the probability of surviving to age a, $S(a)$:

$$n(a) = n(0)S(a) \tag{4.55}$$

The total number in the population, N_{Pop}, is the integral of $n(a)$ over all ages:

$$N_{Pop} = \int_{0}^{\infty} n(a)da = n(0)\int_{0}^{\infty} S(a)da \tag{4.56}$$

From equation (4.26), it is noted that:

$$\int_0^\infty S(a)da = X(0) \tag{4.57}$$

is the life expectancy at birth. This means that (4.56) may be rearranged to give the birth rate:

$$n(0) = \frac{N_{Pop}}{X(0)} \tag{4.58}$$

Substituting the birth rate into (4.55):

$$n(a) = \frac{N_{Pop}S(a)}{X(0)} \tag{4.59}$$

and so the population density, $p(a)$, is:

$$p(a) = \frac{n(a)}{N_{Pop}} = \frac{S(a)}{X(0)} \tag{4.60}$$

This is the age structure of the steady state population. It is constant and can be calculated readily. This distribution is shown in Figure 4, which is based on UK data from 2007 to 2009. Also shown in this figure is the actual distribution for the UK population in this time period. There is clearly some difference between the two distributions. However, as is discussed in more detail in chapter 9, the population-averaged parameters needed for J-value calculations are relatively insensitive to the exact distribution used. The steady-state distribution is therefore a simple but powerful distribution which can give sufficiently accurate results.

The death rate between ages, a and $a + da$, is given by the number of people in that age range multiplied by the probability of dying in that interval, given survival to age, a, i.e. the hazard rate, $h(a)$:

$$
\begin{aligned}
n(a)h(a) &= \frac{S(a)}{X(0)} N_{Pop} h(a) \\
&= \frac{N_{Pop}}{X(0)} f_d(a)
\end{aligned}
\tag{4.61}
$$

where equation (4.10) has been used in the last step. The total death rate is found by integrating over all ages:

$$
\begin{aligned}
\int_0^\infty n(a)h(a)da &= \frac{N_{Pop}}{X(0)} \int_0^\infty f_d(a)da \\
&= \frac{N_{Pop}}{X(0)}
\end{aligned}
\tag{4.62}
$$

which is equal to the birth rate, given by equation (4.58), as is expected in a steady state population.

4.7 The Average Life Expectancy

The average life expectancy, *X*, for the general population is given as:

$$
X = \int_0^\infty p(a)X(a)da \tag{4.63}
$$

where the age distribution is given by equation (4.60). Although the average life expectancy can be readily calculated from this equation, it is also possible to gain further insight into the average life expectancy by noting that:

$$
\begin{aligned}
\int_0^\infty p(a)X(a)da &= \int_0^\infty \frac{p(a)}{S(a)} \left(\int_a^\infty S(t)dt \right) da \\
&= \int_0^\infty \frac{1}{X(0)} \left(\int_a^\infty S(t)dt \right) da \\
&= \int_0^\infty \left(\int_a^\infty p(t)dt \right) da
\end{aligned}
\tag{4.64}
$$

where equation (4.26) has been used. The order of integration may be reversed to give:

$$\int_0^\infty \left(\int_a^\infty p(t)dt \right) da = \int_0^\infty p(t) \left(\int_0^t da \right) dt = \int_0^\infty p(t)tdt \qquad (4.65)$$

so that:

$$X = \int_0^\infty p(t)tdt = t_{av} \qquad (4.66)$$

where t_{av} is the mean age in the population. Thus, in the steady state population, the mean life to come is equal to the mean life already experienced.

In the J-value model, it is also necessary to evaluate the average life expectancy for the workforce, as discussed in section 3.5. In this situation it is inappropriate to use the general population age distribution. If data is available regarding the age structure of the workforce under analysis, then this data may be used. However, the age distribution of a general workforce may be approximated by a simple but realistic uniform distribution that does not require any input data. This is given by:

$$\begin{aligned} p(a) &= \frac{1}{a_{ret} - a_{rec}} \quad \text{for } a_{rec} < a \leq a_{ret} \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned} \qquad (4.67)$$

where a_{rec} and a_{ret} are the age of recruitment into the workforce and age at retirement, respectively. The average life expectancy is:

$$X = \frac{1}{a_{ret} - a_{rec}} \int_{a_{rec}}^{a_{ret}} X(a)da \qquad (4.68)$$

For the UK, appropriate recruitment and retirement ages are 20 and 60, respectively. Although employment does occur outside these ages, the proportion of these workers is relatively small, and so can be disregarded for the purposes of the uniform distribution model. The general population average life expectancy is usually close to the working population average life expectancy. For UK data from 2007 to 2009, the corresponding figures were 41.17 years and 41.16 years for populations with an equal gender ratio.

4.8 The Effect of Discounting on Life Expectancy

In section 3.4 it was noted that a discounted life expectancy could be derived as:

$$X_d(a) = \int_a^\infty S(t \mid a) e^{-r(t-a)} dt \tag{4.69}$$

where r is the discount rate. This can be re-written as:

$$\begin{aligned} X_d(a) &= \int_a^\infty \frac{S(t)e^{-rt}}{S(a)e^{-ra}} dt \\ &= \int_a^\infty \frac{S_d(t)}{S_d(a)} dt \end{aligned} \tag{4.70}$$

where $S_d(t)$ is the discounted survival probability:

$$\begin{aligned} S_d(t) &= S(t)e^{-rt} \\ &= e^{-W(t)}e^{-rt} \\ &= e^{-\int_0^t (h(u)+r)du} \\ &= e^{-W_d(t)} \end{aligned} \tag{4.71}$$

where equations (4.12) and (4.13) have been used, and where:

$$W_d(t) = \int_0^t (h(u) + r) du \tag{4.72}$$

is the discounted cumulative hazard rate (although in this case the effect of discounting is to increase the cumulative hazard rate, rather than decrease it, as the term “discounting” may suggest). A discounted hazard rate may also be defined as:

$$h_d(t) = h(t) + r \tag{4.73}$$

so that:

$$W_d(t) = \int_0^t h_d(u)du \tag{4.74}$$

Hence all the variables required for life expectancy calculations can be viewed as having a discounted counterpart.

The discounted average life expectancy is:

$$X_d = \int_0^\infty p(a) X_d(a) da \tag{4.75}$$

This can be developed as:

$$\begin{aligned} X_d &= \int_0^\infty \frac{S(a)}{X(0)} \int_a^\infty \frac{S(t)}{S(a)} e^{-r(t-a)} dtda \\ &= \int_0^\infty \int_a^\infty \frac{S(t)}{X(0)} e^{-r(t-a)} dtda \\ &= \int_0^\infty \int_a^\infty p(t) e^{-r(t-a)} dtda \end{aligned} \tag{4.76}$$

the order of integration can be reversed to give:

$$= \int_0^\infty p(t) \int_0^t e^{-r(t-a)} dadt \tag{4.77}$$

$$= \int_0^\infty p(t) e^{-rt} \int_0^t e^{ra} da dt$$

$$= \int_0^\infty p(t) e^{-rt} \frac{\left(e^{rt} - 1\right)}{r} dt$$

$$= \frac{1}{r} \int_0^\infty p(t) \left(1 - e^{-rt}\right) dt$$

the exponential term can be expanded as:

$$\begin{aligned} 1 - e^{-rt} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} r^n t^n}{n!} \\ &\approx rt - \frac{r^2 t^2}{2} \end{aligned} \tag{4.78}$$

substituting into (4.77) gives:

$$\begin{aligned} X_d &\approx \int_0^\infty p(t) \left(t - \frac{rt^2}{2} \right) dt \\ &= \int_0^\infty p(t) t dt - \frac{r}{2} \int_0^\infty p(t) t^2 dt \\ &= t_{av} - \frac{r}{2} t_{av}^2 \\ &= X - \frac{r}{2} t_{av}^2 \end{aligned} \tag{4.79}$$

where equation (4.66) has been used, and where t^2_{av} is the mean-square age in the population. Equation (4.79) thus linearly relates the discounted life expectancy to the undiscounted life expectancy and the discount rate.

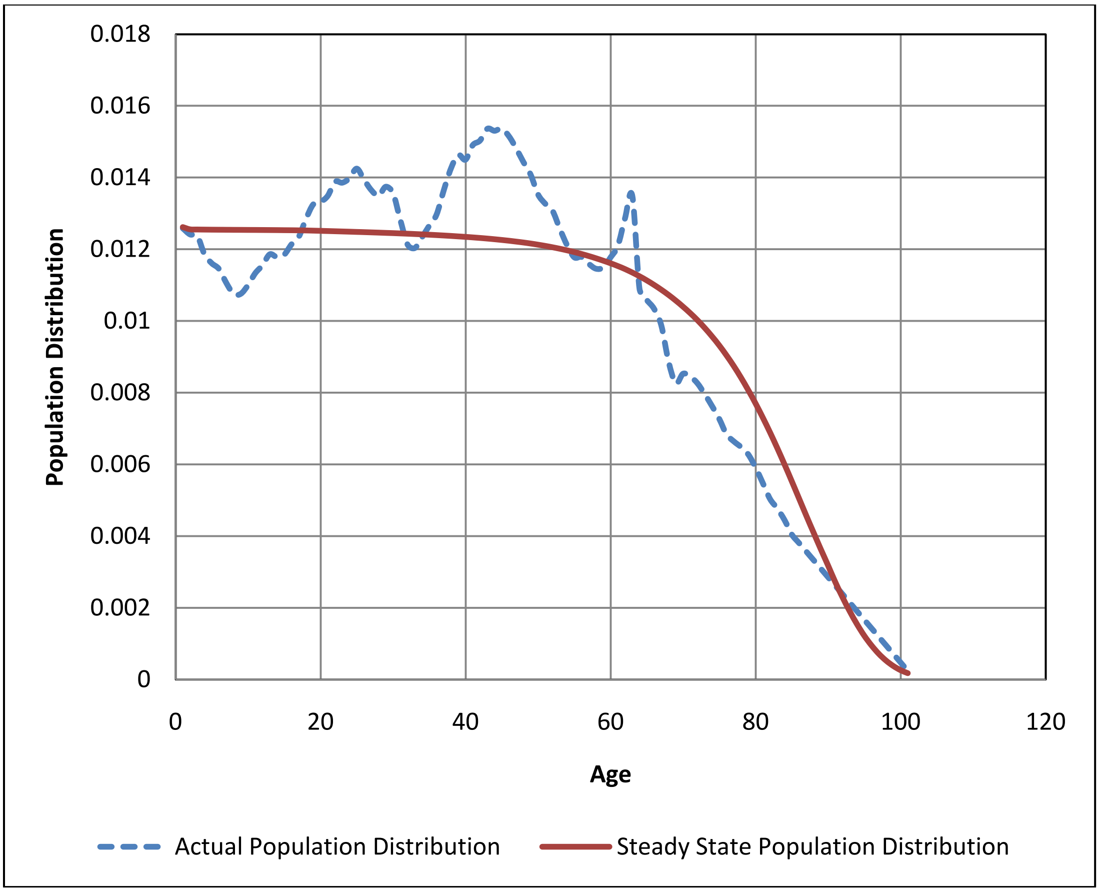

Figure 4 Population distributions calculated from UK data for 2007-2009.

Chapter 5 Calculations for the Change in Life Expectancy Following a Hazard Perturbation

5.1 Modelling Changes in Life Expectancy

Perhaps the most important parameter of the J-value equation is the change in life expectancy caused by exposure to a risk, or resulting from its mitigation. This parameter is especially important when considering the effects of risks that do not become manifest until many years after the initial exposure to the hazard. The calculation of this parameter requires some detailed and technical explanation, which will be given in this section. This section is partly based on Thomas et al (2006c) [184], who derived equations for the change in life expectancy following a prolonged radiation exposure, including the effects on individuals entering or leaving the exposed population. Here a more general model is presented, in which exposures can result in immediate or delayed responses. Exposures that result in absolute or relative hazard perturbations (i.e. perturbations where the magnitude is independent or dependent on the initial hazard rate) are also modelled. Air pollution risks are also modelled explicitly.

The fundamental concepts for understanding the effects of hazards are those of exposure and response. Both of these are characterised by probability density functions. The response of an exposure to a hazard is of particular importance, as it relates the exposure to the resulting increase in probability of death. In many situations, exposure to the hazard is characterised by an immediate increase in mortality rates, which then return to normal when the exposure has stopped. An example of this would be industrial accidents. There is only a risk of death from an accident at the workplace during the time spent at work. After an individual leaves work, he is no longer at risk from this hazard. A hazard with this type of response may be called an “immediate” hazard. This contrasts with exposures to substances such as particulate matter, radiation or other carcinogens, where the resulting increase in mortality occurs some years after the initial exposure. Such types of response may be called “delayed” hazards. Each hazard has its own characteristic response following exposure. The general methodology for modelling the exposures and response, and the consequent change in life expectancy, will now be discussed.

5.2 Exposures

Suppose that the exposure to a hazard begins at time x = 0, and lasts until time x = T_R. Let the rate of exposure felt by an individual be $b(x)$. The units of this quantity are (additional deaths/person-year), although the additional deaths may not occur until many years in the future. In order to clarify what is meant by this term, it will be presented for two types of hazard: immediate risks and delayed radiation risks. For immediate risks, the "exposure" is simply the act of being in a situation where there is an elevated chance of death. For example, this may be working from a height, where there is some chance of experiencing a fatal fall. It may also be travelling in a car or a train, where there is some risk of being in a fatal crash. In these situations, death occurs either during or shortly after the initial exposure period, which is the reason why they are referred to as "immediate" risks. If the additional number of fatalities per year from a given hazard is $\Lambda(x)$ in an exposed population of N (assumed constant), then the individual exposure rate is:

$$\begin{aligned} b(x) &= \frac{\Lambda(x)}{N} \quad \text{for } 0 < x \le T_R \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{5.1}$$

This is shown schematically in Figure 5. For delayed radiation risks, the exposures are in terms of the annual amount of radiation dose received by an individual, $d_r(x)$, measured in Sieverts per year (Sv.year^{-1}). In order to relate the dose to the additional number of deaths, this is multiplied by the total dose-risk coefficient, c_T (Sv^{-1}). The individual exposure rate for radiation is then:

$$\begin{aligned} b(x) &= c_T d_r(x) \quad \text{for } 0 < x \le T_R \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{5.2}$$

The total individual exposure, b_{tot}, is the integral of the exposure rate:

$$b_{tot} = \int_0^\infty b(x)dx \tag{5.3}$$

This is the additional fatalities per person exposed to the hazard, which is also the probability of death resulting from the exposure. The fraction of all fatalities caused by the exposure in the interval x to $x + dx$ will then be $b(x)dx/b_{tot}$, implying that the probability density for causing death from exposure will be $g(x)$, given by:

$$g(x) = \frac{b(x)}{b_{tot}} \tag{5.4}$$

5.3 Responses

As was discussed above, risks can be thought of as having "characteristic" responses. The response is the period of time over which excess mortality is assumed to occur following an exposure, expressed as a probability density function. Suppose that $f_M(y)dy$ is the probability that the excess mortality resulting from the exposure occurs between times y and $y + dy$. The variable, y is the time that has elapsed between the time of induction, x, and the current time, τ, so that $y = \tau - x$. This is shown in Figure 6. The probability that both an exposure occurs between times x to $x + dx$, and an excess mortality is observed between times y and $y + dy$, will be:

$$f_M(y)g(x)dxdy = f_M(\tau - x)g(x)dxd\tau \tag{5.5}$$

But death at time τ could have resulted from exposure over the preceding possible times, x. The total probability density for death at time τ, $f_T(\tau)$, resulting from exposure from any time, is the integral of (5.5) from the start of the exposure to the current time, τ:

$$f_T(\tau) = \int_0^{\tau} f_M(\tau - x)g(x)dx \tag{5.6}$$

5.4 Increase in Hazard Rate – Absolute and Relative Models

An individual who is exposed to some hazard will experience an increased probability of death. This is modelled mathematically by perturbing the hazard rate,

$h(a)$, for an individual of age a. The perturbation can be modelled in two ways: by using an “absolute risk” model; or by using a “relative risk” model. In an absolute risk model, the additional hazard rate is independent of the individual’s existing probability of death, whilst in a relative risk model the additional hazard rate is proportional to the initial hazard rate.

The probability density given in (5.6) is based on the assumption that excess mortality is certain to occur. In an absolute risk model, the probability density that an individual will die at time τ as a direct result of the exposure, is the product of the probability of death from the exposure, b_{tot}, and the probability density for death at time τ, $f_T(\tau)$. This is then the additional hazard rate faced by the individual. If the individual is aged a at the start of the exposure, then after τ years his age will be $t = a + \tau$. The additional hazard rate faced by an individual of age t, given initial exposure at age a, is denoted $\delta h_{abs}(t|a)$, where:

$$\begin{aligned} \delta h_{abs}(t \mid a) &= b_{tot} f_T(\tau) = b_{tot} \int_0^{\tau} f_M(\tau - x) g(x) dx \\ &= \int_0^{t-a} f_M(t - a - x) b(x) dx \end{aligned} \tag{5.7}$$

In a relative risk model, the increase in the hazard rate faced by an individual of age t, given initial exposure at age a, $\delta h_{rel}(t|a)$, is proportional to the hazard rate $h(t)$. Since the hazard rate is the probability density of immediate death, this parameter replaces the excess mortality probability density function, $f_M(y)$. However, it is still necessary to retain some way of modelling the distribution of the excess mortalities. This is done by introducing the function $\phi_0(y)$, which plays a similar role to $f_M(y)$, except that it is not a probability distribution. It is also dimensionless, which is required for consistency. The two functions are related to each other by:

$$f_M(y) = \frac{\phi_0(y)}{\int_0^{\infty} \phi_0(y) dy} \tag{5.8}$$

The integral in the denominator can thus be thought of as the number of effective mortality years experienced following an exposure. The perturbed hazard rate is then:

$$\delta h_{rel}(t \mid a) = h(t)\int_{0}^{t-a} \phi_0(t-a-x)b(x)dx \qquad (5.9)$$

5.5 Increase in Cumulative Hazard Rate

Following a perturbation in the hazard rate, the cumulative hazard rate, $W(t)$ will be increased to:

$$\begin{aligned} W(t) + \delta W(t \mid a) &= \int_{0}^{t} [h(u) + \delta h(u \mid a)]du \\ &= \int_{0}^{t} h(u)du + \int_{a}^{t} \delta h(u \mid a)du \end{aligned} \qquad (5.10)$$

where the lower bound on the second integral has been changed from $u = 0$ to $u = a$, as the change in the hazard rate only occurs at ages equal to or greater than the present age a. This means that:

$$\delta W(t \mid a) = \int_{a}^{t} \delta h(u \mid a)du \qquad (5.11)$$

where $\delta W(t|a)$ is the increase in the cumulative hazard rate at age t, following an exposure at age a, and $\delta h(.)$ refers to either the absolute or relative change in hazard rate, depending on the risk model used.

5.6 Decrease in Life Expectancy

From equations (4.12) and (4.26), the life expectancy can be written as:

$$X(a) = e^{W(a)}\int_{a}^{\infty} e^{-W(t)}dt \qquad (5.12)$$

Following a perturbation in the hazard rate, the life expectancy decreases by an amount:

$$X(a) - \delta X(a) = e^{W(a)} \int_a^\infty e^{-(W(t)+\delta W(t|a))} dt \qquad (5.13)$$

so that:

$$\begin{aligned} \delta X(a) &= e^{W(a)} \int_a^\infty e^{-W(t)} \left(1 - e^{-\delta W(t|a)}\right) dt \\ &= \frac{1}{S(a)} \int_a^\infty S(t) \left(1 - e^{-\delta W(t|a)}\right) dt \end{aligned} \qquad (5.14)$$

For small changes in the cumulative hazard rate, the exponential term can be approximated, using $e^{-x} \approx 1 - x$. The change in life expectancy at age a is then:

$$\delta X(a) \approx \frac{1}{S(a)} \int_a^\infty S(t) \delta W(t \mid a) dt \qquad (5.15)$$

5.7 Decrease in Average Life Expectancy

The change in average life expectancy following a hazard rate perturbation can then be calculated by averaging the change in age-dependent life expectancy over the required population distribution:

$$\delta X = \int_0^\infty p(a) \delta X(a) da \qquad (5.16)$$

where the population age distributions are determined for the general public and the workforce, as described in section 4.7.

Thus, in order to calculate the change in average life expectancy all that is required is knowledge of the distribution of the exposure rate, $b(x)$, and of the mortality response distribution, $f_M(y)$. Some simple, limiting distributions of these functions will now be explored, and the corresponding change in life expectancy will be calculated.

5.8 Limiting Exposure and Response Distributions

Although equations (5.7) to (5.16) allow for the calculation of the change in life expectancy following a hazard perturbation, it is instructive to investigate some of the limiting distributions of the exposure and response functions, and the consequent behaviour of the perturbed hazard rate and associated functions. The limiting distributions are when the exposures and responses are either very short or indefinitely long, and maintained at a constant level throughout. There are therefore four limiting distributions which may be investigated. These are shown in Table 2, which lists the exposure distribution, the excess mortality distribution, the change in hazard rate, and the change in cumulative hazard rate for absolute and relative risk models. One result of note is that the change in hazard rate for a short exposure and long response is the same as for a long exposure with a short response in the relative risk model. The change in cumulative hazard rate and thus change in life expectancy will therefore also be the same. For the absolute risk model, the short exposure/long response hazard perturbation is only different from the long exposure/short response hazard perturbation by a scaling factor, Ω, which is the length of time which the response lasts for following a single exposure.

Once the cumulative hazard rates are calculated for the limiting exposures, the associated change in life expectancy and average life expectancy can be calculated, using equations (5.15) and (5.16). However, some of these limiting distributions may be developed further to give a simple expression for the changes in life expectancy. These will now be shown.

Firstly, the shortest hazard rate perturbation will arise when there is a point exposure at $x = 0$, with an immediate response, with no delayed component. This will occur, for example, following an explosion, which lasts for a short period of time, and will only cause fatalities at that instant. Although in reality any event must have a finite duration, for the purposes of modelling, the exposure can be modelled as only occurring at a single point. The exposure distribution and the response distribution are therefore defined only at a single point, as given in Table 2. These will be repeated below, for clarity:

$$\begin{aligned} b(x) &= b \text{ for } x = 0 \\ &= 0 \text{ otherwise} \end{aligned} \tag{5.17}$$

and:

$$\begin{aligned} f_M(y) &= 1 \text{ for } y = 0 \\ &= 0 \text{ otherwise} \end{aligned} \tag{5.18}$$

so that, for the absolute risk model:

$$\begin{aligned} \delta h_{abs}(t \mid a) &= \int_0^{t-a} f_M(t-a-x)b(x)dx \\ &= bf_M(t-a) \\ &= b \text{ for } t = a \\ &= 0 \text{ otherwise} \end{aligned} \tag{5.19}$$

For the relative risk model, the dimensionless $\phi_0(y)$ function is used instead of $f_M(y)$:

$$\begin{aligned} \delta h_{rel}(t \mid a) &= bh(t)\phi_0(t-a) \\ &= bh(a) \text{ for } t = a \\ &= 0 \text{ otherwise} \end{aligned} \tag{5.20}$$

The change in the cumulative hazard rate is:

$$\delta W(t \mid a) = b \text{ for } t \geq a \tag{5.21}$$

for the absolute risk model, and:

$$\delta W(t \mid a) = bh(a) \text{ for } t \geq a \tag{5.22}$$

for the relative risk model. The change in life expectancy is then:

$$\begin{aligned} \delta X(a) &\approx \frac{1}{S(a)} \int_a^\infty S(t) \delta W(t \mid a) dt \\ &= bX(a) \end{aligned} \tag{5.23}$$

for absolute risks. This means the change in life expectancy is directly proportional to the initial life expectancy, with the constant of proportionality equal to the excess mortality rate. For relative risks, the change in life expectancy is given by:

$$\delta X(a) \approx bh(a) X(a) \tag{5.24}$$

The change in average life expectancy in the absolute risk model is then:

$$\delta X = bX \tag{5.25}$$

For relative risks, the change in average life expectancy is:

$$\delta X = b \int_0^\infty p(a) h(a) X(a) da \tag{5.26}$$

This can be further developed by noting that:

$$\begin{aligned} \int_0^\infty p(a) h(a) X(a) da &= \int_0^\infty \frac{S(a)}{X(0)} h(a) \left(\int_a^\infty \frac{S(t)}{S(a)} dt \right) da \\ &= \int_0^\infty h(a) \left(\int_a^\infty \frac{S(t)}{X(0)} dt \right) da \\ &= \int_0^\infty h(a) \left(\int_a^\infty p(t) dt \right) da \end{aligned} \tag{5.27}$$

the order of integration can then be reversed to give:

$$\int_0^\infty h(a)\left(\int_a^\infty p(t)dt\right)da = \int_0^\infty p(t)\left(\int_0^t h(a)da\right)dt \\ = \int_0^\infty p(t)W(t)dt \tag{5.28}$$

the integral in equation (5.26) thus emerges as the population-averaged cumulative hazard rate. This can be developed still further, by noting that:

$$\int_0^\infty p(t)W(t)dt = \frac{-\int_0^\infty S(t)\ln S(t)dt}{X(0)} \\ = \frac{-\int_0^\infty S(t)\ln S(t)dt}{\int_0^\infty S(t)dt} \\ = H \tag{5.29}$$

where *H* is known as the population entropy, as defined by Keyfitz (1985) [126]. The change in life expectancy is then:

$$\delta X = bH \tag{5.30}$$

Although the developments of equations (5.20) to (5.30) are only strictly true for exposures to the general population, it is also possible to define related measures for exposures to the working population, where instead of having an integral with bounds from zero to infinity; the bounds will be the age at recruitment and the age of retirement. The two measures will be similar, however, and so the above will usually be a satisfactory approximation for the working population as well. Thus, for the simple limiting distribution of a point exposure with immediate response, the change in average life expectancy is given by the simple equations (5.25) and (5.30), although these only apply to small exposure rates, so that the linear approximation used in (5.15) will be valid. For larger exposure rates, the more accurate exponential version in (5.14) should be used. Doing so would not present any difficulties, but would not result in the simple formulas just presented.

In the absolute risk model, the other limiting exposures may also be developed further into simple expressions. As has already been discussed, the hazard rate perturbation following a short exposure with long response will be equal up to a scaling factor to the perturbation following a long exposure with a short response. Table 2 gives the change in cumulative hazard rate as:

$$\delta W(t \mid a) = \frac{b(t-a)}{\Omega} \tag{5.31}$$

which is a generalised version of the two limiting distributions. The parameter Ω is the length of duration of the response following a single exposure. For a short exposure with a prolonged duration, a value of Ω = 100 would be appropriate, whilst for a prolonged exposure with a short response, a value of Ω = 1 should be used. The resulting change in life expectancy is:

$$\begin{aligned} \delta X(a) &\approx \frac{b}{\Omega S(a)} \int_a^\infty S(t)(t-a)dt \\ &= \frac{b}{\Omega S(a)} \int_a^\infty tS(t)dt - \frac{baX(a)}{\Omega} \end{aligned} \tag{5.32}$$

The change in average life expectancy is then:

$$\begin{aligned} \delta X &= \int_0^\infty p(a)\left[\frac{b}{\Omega S(a)}\int_a^\infty tS(t)dt - \frac{baX(a)}{\Omega}\right]da \\ &= \frac{b}{\Omega}\left[\int_0^\infty \frac{1}{X(0)}\int_a^\infty tS(t)dtda - \int_0^\infty p(a)aX(a)da\right] \\ &= \frac{b}{\Omega}\left[\int_0^\infty\int_a^\infty tp(t)dtda - \int_0^\infty \frac{S(a)}{X(0)}a\int_a^\infty \frac{S(t)}{S(a)}dtda\right] \\ &= \frac{b}{\Omega}\left[\int_0^\infty\int_a^\infty tp(t)dtda - \int_0^\infty a\int_a^\infty p(t)dtda\right] \end{aligned} \tag{5.33}$$

Reversing the order of integration gives:

$$
\begin{aligned}
\delta X &= \frac{b}{\Omega}\left[\int_0^\infty tp(t)\int_0^t dadt - \int_0^\infty p(t)\int_0^t adadt\right] \\
&= \frac{b}{\Omega}\left[\int_0^\infty t^2 p(t)dt - \int_0^\infty \frac{t^2}{2} p(t)dt\right] \\
&= \frac{b}{2\Omega}\left[\int_0^\infty t^2 p(t)dt\right] \\
&= \frac{b}{2\Omega} t_{av}^2
\end{aligned}
\tag{5.34}
$$

where t^2_{av} is the mean-square age. Thus, in the limiting case when either there is a short exposure with long response duration, or a long exposure with a short response duration, the change in average life expectancy is directly proportional to the mean-square age of the population, and to the exposure rate.

A similar, related expression for the change in life expectancy following a prolonged exposure to a hazard that has a long response duration may also be derived for the absolute risk model. Table 2 gives the increase in the cumulative hazard rate for such an exposure as:

$$
\delta W(t \mid a) = \frac{b(t-a)^2}{2\Omega} = \frac{b}{2\Omega}\left[t^2 + a^2 - 2ta\right] \tag{5.35}
$$

The associated change in life expectancy is:

$$
\begin{aligned}
\delta X(a) &\approx \frac{b}{2\Omega S(a)}\int_a^\infty S(t)\left[t^2 - 2ta + a^2\right]dt \\
&= \frac{b}{\Omega}\left[\frac{1}{2S(a)}\int_a^\infty S(t)t^2 dt - \frac{a}{S(a)}\int_a^\infty S(t)tdt + \frac{a^2 X(a)}{2}\right]
\end{aligned}
\tag{5.36}
$$

The change in average life expectancy is:

$$
\begin{aligned}
\delta X &= \frac{b}{\Omega}\int_0^\infty p(a)\left[\frac{1}{2S(a)}\int_a^\infty S(t)t^2dt - \frac{a}{S(a)}\int_a^\infty S(t)tdt + \frac{a^2X(a)}{2}\right]da \\
&= \frac{b}{\Omega}\int_0^\infty \frac{S(a)}{X(0)}\left[\frac{1}{2S(a)}\int_a^\infty S(t)t^2dt - \frac{a}{S(a)}\int_a^\infty S(t)tdt + \frac{a^2X(a)}{2}\right]da \\
&= \frac{b}{\Omega}\int_0^\infty \left[\frac{1}{2X(0)}\int_a^\infty S(t)t^2dt - \frac{a}{X(0)}\int_a^\infty S(t)tdt + \frac{a^2}{2X(0)}\int_a^\infty S(t)dt\right]da \\
&= \frac{b}{\Omega}\left[\int_0^\infty\int_a^\infty p(t)\frac{t^2}{2}dtda - \int_0^\infty a\int_a^\infty p(t)tdtda + \int_0^\infty \frac{a^2}{2}\int_a^\infty p(t)tdtda\right]
\end{aligned}
\tag{5.37}
$$

Reversing the order of integration gives:

$$
\begin{aligned}
\delta X &= \frac{b}{\Omega}\left[\int_0^\infty p(t)\frac{t^2}{2}\int_0^t dadt - \int_0^\infty p(t)t\int_0^t adadt + \int_0^\infty p(t)t\int_0^t \frac{a^2}{2}dadt\right] \\
&= \frac{b}{\Omega}\left[\int_0^\infty p(t)\frac{t^3}{2}dt - \int_0^\infty p(t)\frac{t^3}{2}dt + \int_0^\infty p(t)\frac{t^3}{6}dt\right] \\
&= \frac{b}{\Omega}\left[\int_0^\infty p(t)\frac{t^3}{6}dt\right] \\
&= \frac{b}{6\Omega}t_{av}^3
\end{aligned}
\tag{5.38}
$$

where t^3_{av} is the mean-cube age. Thus, in the limiting case when there is a prolonged exposure with long response duration, which represents the maximum limiting case, the change in average life expectancy is directly proportional to the mean-cube age of the population, and to the exposure rate.

Although the change in life expectancy can be calculated for the relative risk model from equations (5.15) and (5.16), and from the change in the cumulative hazard rate given in Table 2 for these limiting distributions, there are no such simple expressions for the change in average life expectancy as there are for the absolute risk model. It is also worth noting that the above equations have been developed under the assumption that the exposed population is the general population. When the working population is considered, the equations will not be valid as the integration limits will need modifying. Also, prolonged exposures experienced whilst at work will only be felt until the age of retirement at the latest. This means that the change in cumulative hazard rate would need to be modified accordingly.

5.9 Modelling the Effects of Radiation and Pollution

The general framework for estimating changes in life expectancy laid out above may now be used to model more specific risks, namely, those from exposures to radiation and pollution.

The effects of exposure to radiation are modelled by following the treatment of Lord Marshall et al (1982) [134], and Thomas et al (2006 – 07) [118], [184], [185]. These treatments recognise the fact that, following an exposure to radiation there is a substantial period in which no effects are seen. After this there are stochastic effects for a long duration in which increased mortality will result, although these stochastic effects will eventually die out. This effect can be modelled by assuming that the additional fatalities occur between times ω_1 and ω_2 after exposure, where reasonable values are ω_1 = 10 years and ω_2 = 40 years. It is also assumed that the excess mortality period is uniform between these years. All previous treatments have assumed that the effects of radiation follow the absolute risk framework, and this is also assumed by the International Commission on Radiological Protection (ICRP), who recommends internationally recognised radiation risk values which are used in setting safety levels worldwide. The excess mortality distribution is therefore given by:

$$f_M(y) = \frac{1}{\Omega} = \frac{1}{\omega_2 - \omega_1} \quad \text{for } \omega_1 \le y < \omega_2 \qquad (5.39)$$
$$= 0 \qquad \text{otherwise}$$

where $\Omega = \omega_2 - \omega_1$ is the duration of the latent stochastic effects following a single exposure, which will be taken as 30 years. This distribution may also be modelled more conveniently using step or jump functions, $J_p(x)$, given as:

$$J_p(x) = 1 \quad \text{for } x \ge 0 \qquad (5.40)$$
$$= 0 \quad \text{for } x < 0$$

The mortality distribution is then given by:

$$f_M(y) = \frac{1}{\Omega}\left(J_p(y-\omega_1) - J_p(y-\omega_2)\right) \tag{5.41}$$

It may be observed from equation (5.8) that:

$$\phi_0(y) = J_p(y-\omega_1) - J_p(y-\omega_2) \tag{5.42}$$

as:

$$\int_0^\infty \phi_0(y)dy = \Omega \tag{5.43}$$

Thus the effects of a short exposure to radiation are modelled as having no effect for ten years, before increasing mortality risk for a thirty year period, whence all effects die out. This distribution is shown in Figure 7.

As has been discussed above, exposures to radiation are expressed in terms of the annual amount of radiation dose received by an individual, which is measured in Sieverts per year (Sv.year^{-1}). Radiation doses are related to the additional number of deaths by multiplying the annual dose by the dose-risk coefficient, c_T (Sv^{-1}). The dose-risk coefficient is determined from the 2007 ICRP recommendations [113], who recommend a “detriment adjusted” lifetime cancer risk coefficient of 0.055 Sv^{-1} for the general population, and 0.041 Sv^{-1} for those of working age. These detriment adjusted figures include non-fatal effects of radiation. However, in life expectancy calculations, the required risk coefficient must only refer to fatal effects, and so the above figures are inappropriate. Although the required figures are not given explicitly by the ICRP, they can be calculated from data they present, which is 0.041 Sv^{-1} for the general population, and 0.032 Sv^{-1} for the working population. However, if these figures were applied to the change in life expectancy calculations, they would underestimate the actual loss of life expectancy experienced by individuals in the population. This is because not all individuals would experience the full effect of the delayed risk, as they may die before the effects have occurred. In order to accommodate for those who do not experience the full risk, the ICRP nominal risk figure needs to be adjusted upwards. The method for doing this is given in Thomas and Jones [187], who show that the ICRP risk coefficients need to be multiplied by a compensating factor. Using the latest data, the compensating factor is given as 1.43 for the general population and 1.32 for the working population. The appropriate dose risk coefficient, c_T, is then the product of the ICRP figure and the compensating factor, for both the general and the working population. This is:

$$\begin{aligned} c_T &= 0.058 \text{ for the general population} \\ &= 0.042 \text{ for the working population} \end{aligned} \tag{5.44}$$

It has been assumed in the above discussion that the working population is entirely composed of males. If the workforce is assumed to be composed of equal amounts of men and women, then the compensating factor is decreased to 1.27 and c_T is reduced to 0.041. For a working population entirely composed of females, the compensating factor is 1.23 and the risk coefficient is 0.039. These values are shown in Table 3.

The exposure rate, $b(x)$, is given by:

$$b(x) = c_T d_r(x) \tag{5.45}$$

where $d_r(x)$ is the annual dose received (Sv.year^{-1}). One further issue which needs noting is that the ICRP also recommend that if any individual were to be exposed to particularly high doses or high dose rates, then a “dose and dose rate effectiveness factor” (DDREF) should be applied to the risk estimates. The recommended value for DDREF is 2. It is assumed that this applies to doses greater than 100 mSv. Therefore, the exposure rate is more accurately given as:

$$\begin{aligned} b(x) &= c_T d_r(x) \quad \text{for } d_r \leq 0.1 \\ &= 2c_T d_r(x) \quad \text{for } d_r > 0.1 \end{aligned} \tag{5.46}$$

However, the event of an individual receiving a dose of this magnitude would be exceedingly rare in normal circumstances, and so this effect will not be considered in the rest of this section, but will be considered at a later stage when assessing the impacts of a large nuclear accident.

For a uniform exposure to a radiation dose of d_r Sieverts lasting for T_R years, the exposure rate is given as:

$$b(x) = c_T d_r \left(1 - J_p(x - T_R)\right) \tag{5.47}$$

The hazard rate increase is then equal to:

$$\begin{aligned}\delta h(t \mid a) &= \int_0^{t-a} f_M(t-a-x) b(x) dx \\ &= \frac{c_T d_r}{\Omega} \int_0^{t-a} \left(J_p(t-a-\omega_1-x) - J_p(t-a-\omega_2-x)\right)\left(1 - J_p(x-T_R)\right) dx \\ &= k_{rad} d_r \int_0^{t-a} \phi_0(t-a-x)\left(1 - J_p(x-T_R)\right) dx\end{aligned} \tag{5.48}$$

where $k_{rad} = c_T/\Omega$ is the risk coefficient per year, also known as the distributed risk coefficient. It can be seen that any values of $x \geq T_R$ will not contribute to the integral. This means that (5.48) can be re-written as:

$$\delta h(t \mid a) = k_{rad} d_r \int_0^{T_R} \phi_0(t-a-x) dx \tag{5.49}$$

The variable of integration can now be changed. Put:

$$\begin{aligned}&z = t - a - x \\ &dz = -dx \\ &x = 0 \Rightarrow z = t - a \\ &x = T_R \Rightarrow z = t - a - T_R\end{aligned} \tag{5.50}$$

Hence:

$$
\begin{aligned}
\delta h(t \mid a) &= -k_{rad} d_r \int_{t-a}^{t-a-T_R} \phi_0(z) dz \\
&= k_{rad} d_r \int_{t-a-T_R}^{t-a} \phi_0(z) dz \\
&= k_{rad} d_r \psi_0(t-a)
\end{aligned}
\tag{5.51}
$$

where $\psi_0(t - a)$ is the prolonged hazard perturbation pattern, following the notation of Thomas et al [118], [184], [185]. This can be written out in full as:

$$
\begin{aligned}
&\psi_0(\tau) = (\tau - \omega_1) J_p(\tau - \omega_1) - (\tau - \omega_2) J_p(\tau - \omega_2) \\
&- (\tau - \omega_1 - T_R) J_p(\tau - \omega_1 - T_R) + (\tau - \omega_2 - T_R) J_p(\tau - \omega_2 - T_R)
\end{aligned}
\tag{5.52}
$$

The perturbed hazard rate can then be used to determine the perturbed cumulative hazard rate, and consequently the change in life expectancy and average life expectancy using the equations shown above. The perturbed cumulative hazard rate is:

$$
\begin{aligned}
\delta W(t \mid a) &= \int_a^t \delta h(u \mid a) du \\
&= k_{rad} d_r \int_a^t \psi_0(u-a) du
\end{aligned}
\tag{5.53}
$$

Proceeding again by changing the variable of integration, by putting:

$$
\begin{aligned}
&z = u - a \\
&dz = du \\
&u = a \Rightarrow z = 0 \\
&u = t \Rightarrow z = t - a
\end{aligned}
\tag{5.54}
$$

so that:

$$
\begin{aligned}
\delta W(t \mid a) &= k_{rad} d_r \int_0^{t-a} \psi_0(z) dz \\
&= k_{rad} d_r \psi_1(t-a)
\end{aligned}
\tag{5.55}
$$

where $\psi_1(t-a)$ is the integrated prolonged hazard rate pattern, again following the notation of Thomas et al. This can be written out in full as:

$$\begin{aligned}\psi_1(\tau) = &\frac{1}{2}(\tau-\omega_1)^2 J_p(\tau-\omega_1) - \frac{1}{2}(\tau-\omega_2)^2 J_p(\tau-\omega_2)\\ &-\frac{1}{2}(\tau-\omega_1-T_R)^2 J_p(\tau-\omega_1-T_R)\\ &+\frac{1}{2}(\tau-\omega_2-T_R)^2 J_p(\tau-\omega_2-T_R)\end{aligned} \tag{5.56}$$

The change in life expectancy, by equation (5.15) is then:

$$\begin{aligned}\delta X(a) &= \frac{1}{S(a)}\int_a^\infty S(t)\delta W(t \mid a)dt\\ &= \frac{k_{rad}d_r}{S(a)}\int_a^\infty S(t)\psi_1(t-a)dt\end{aligned} \tag{5.57}$$

and the change in average life expectancy is:

$$\delta X = k_{rad}d_r\int_0^\infty \frac{p(a)}{S(a)}\int_a^\infty S(t)\psi_1(t-a)dtda \tag{5.58}$$

for the general population, this can be developed as:

$$\begin{aligned}\delta X &= k_{rad}d_r\int_0^\infty \frac{S(a)}{X(0)S(a)}\int_a^\infty S(t)\psi_1(t-a)dtda\\ &= k_{rad}d_r\int_0^\infty\int_a^\infty \frac{S(t)}{X(0)}\psi_1(t-a)dtda\\ &= k_{rad}d_r\int_0^\infty\int_a^\infty p(t)\psi_1(t-a)dtda\end{aligned} \tag{5.59}$$

reversing the order of integration gives:

$$\delta X = k_{rad}d_r\int_0^\infty p(t)\int_0^t \psi_1(t-a)dadt \tag{5.60}$$

The variable of integration can be changed, by putting:

$$\begin{aligned} z &= t - a \\ dz &= -da \\ a &= 0 \Rightarrow z = t \\ a &= t \Rightarrow z = 0 \end{aligned} \tag{5.61}$$

Hence:

$$\begin{aligned} \int_0^t \psi_1(t-a)da &= -\int_t^0 \psi_1(z)dz \\ &= \int_0^t \psi_1(z)dz \\ &= \psi_2(t) \end{aligned} \tag{5.62}$$

where $\psi_2(t)$ is the twice-integrated prolonged hazard rate pattern, which can be written out in full as:

$$\begin{aligned} \psi_1(\tau) = &\frac{1}{6}(\tau-\omega_1)^3 J_p(\tau-\omega_1) - \frac{1}{6}(\tau-\omega_2)^3 J_p(\tau-\omega_2) \\ &-\frac{1}{6}(\tau-\omega_1-T_R)^3 J_p(\tau-\omega_1-T_R) \\ &+\frac{1}{6}(\tau-\omega_2-T_R)^3 J_p(\tau-\omega_2-T_R) \end{aligned} \tag{5.63}$$

the average change in life expectancy is thus:

$$\delta X = k_{rad} d_r \int_0^\infty p(t)\psi_2(t)dt \tag{5.64}$$

The integral can be readily evaluated using life table data, and by setting ω_1 = 10 years and ω_2 = 40 years, for any given exposure duration T_R. Although the above equation only applies to the general population, an equivalent calculation could readily be made for the working population from equation (5.58).

The effects of pollution can be modelled in a similar manner to those of radiation. It has long been recognised that inhalation of pollutants can increase mortality. Most of the data used for modelling pollution effects have been based on the 2009 Committee on the Medical Effects of Air Pollutants (COMEAP) recommendations [39]. The main difference between pollution risks and radiological risks are that pollution risks are presented as relative risks, in contrast to the absolute risk model of radiation effects.

The COMEAP report discusses the fact that pollution has been observed to cause immediate effects, and so it is assumed that there is no incubation period. The report did not discuss the duration of time for which these effects are observed, and so to estimate this, data regarding the effects of cigarette smoking (which results in exposures to similar kinds of pollutants) were used, see Kawachi et al (1993) [122], and Kenfield et al (2008) [125]. The studies have found that, upon cessation of smoking, risks begin decreasing immediately, although it can take over twenty years for the risks to return to those that have never smoked. However, the authors note that other studies have found evidence supporting both much shorter and much longer time periods than this. The studies also find that over 75% of the risk decrease occurred before the 15th year of cessation. It was therefore decided to use 15 years as the time taken for stochastic effects of a short exposure to pollution to die out. As for radiation risks, a rectangular excess mortality function will be used to model the distribution. Although such a rectangular function will overestimate the risks as they decrease up to the 15th year, the function will also underestimate the excess risks which still remain after the 15th year. These two features will tend to cancel each other out, so that on average, the rectangular function does not lose too much accuracy. However, a better model would be to fit a parametric curve to the observed data, which would be a linear or exponential decline. These issues remain for further work. As the relative risk framework is being used, the excess mortality distribution is given by the dimensionless $\phi_0(y)$ function:

$$\phi_0(y) = 1 - J_p(y - \omega_2) \tag{5.65}$$

where in this case, ω_2 =15 years (and ω_1 = 0 years, so that $J_p(y - \omega_1) = 1$ for all y). This distribution is shown in
Figure 8.

The COMEAP report also recommended that the best indicator for pollution effects was exposure to PM2.5 particulate matter (particles with diameter less than or equal to 2.5 μm), and that exposure to other larger particulate matter and industrial pollutants such nitrogen dioxide, carbon monoxide and ozone are not associated with significantly increased mortality when the effect of PM2.5 is accounted for. The report finds evidence suggesting that sulphur dioxide does increase mortality, but decides against recommending quantification of direct effects of this pollutant, noting that there were difficulties in separating the effects of particulate matter and sulphur dioxide exposure. Thus, the hazard rate perturbation for pollution is expressed in terms of exposures to increases in the concentration of PM2.5 particulate matter, which is measured in units of micro-grams per cubic metre (μg. m^{-3}). The exposure rate, $b(x)$ is then given by:

$$b(x) = k_{poll}\delta c(x) \tag{5.66}$$

where k_{poll} is the exposure-risk coefficient for pollution (μ$g^{-1}m^3$), and $\delta c(x)$ is the increase in concentration (μg.m^{-3}) associated with pollutant emissions at time x. The COMEAP report's main recommendation is that the relative risk following an increase in PM2.5 concentration of 10 μg.m^{-3} will be 6%. The relative risk is related to the exposure risk coefficient by:

$$RR = e^{k_{poll}\delta c} \tag{5.67}$$

see, for example, [166]. Since a concentration increase of 10 μg.m^{-3} leads to relative risk increase of 6%, then RR = 1.06 when δc = 10 μg.m^{-3}. The exposure risk coefficient can thus be determined as:

$$k_{poll} = \frac{\ln(RR)}{\delta c} = \frac{\ln(1.06)}{10} = 5.8 \times 10^{-3} \ \mu g^{-1} m^3 \tag{5.68}$$

The 95% confidence limits for the relative risk are given in the COMEAP report as 2% to 11%, meaning that the 95% confidence limits for the pollution exposure-risk coefficient are (2.0 – 10.4) $\times 10^{-3}$ $\mu g^{-1} m^{3}$. It is also worth mentioning that this risk coefficient does not need adjusting in the manner described for radiation risks above, as the coefficient does not express the lifetime at risk as the radiation coefficients do.

For a uniform exposure to a pollution concentration of δc ($\mu g.m^{-3}$) lasting for T_R years, the exposure rate is given as:

$$b(x) = k_{poll}\delta c\left(1 - J_p\left(x - T_R\right)\right) \tag{5.69}$$

the hazard rate perturbation is then:

$$\begin{aligned}\delta h_{rel}(t \mid a) &= h(t)\int_0^{t-a}\phi_0(t-a-x)b(x)dx \\ &= h(t)k_{poll}\delta c\int_0^{t-a}\left(1 - J_p(t-a-\omega_2)\right)\left(1 - J_p(x-T_R)\right)dx\end{aligned} \tag{5.70}$$

This can be developed in a similar manner as for radiation exposures above, to give:

$$\delta h_{rel}(t \mid a) = h(t)k_{poll}\delta c\,\psi_0(t-a) \tag{5.71}$$

where $\psi_0(t - a)$ is as given above, except with ω_1 set equal to zero. Writing out in full:

$$\begin{aligned}\psi_0(\tau) = \tau &- (\tau-\omega_2)J_p(\tau-\omega_2) - (\tau - T_R)J_p(\tau - T_R) \\ &+ (\tau - \omega_2 - T_R)J_p(\tau - \omega_2 - T_R)\end{aligned} \tag{5.72}$$

The hazard rate perturbation can then be used to calculate the increase in the cumulative hazard rate, and hence the change in life expectancy and change in average life expectancy. However, because of the presence of the hazard rate $h(t)$ in the calculations, there does not exist any simple solutions involving the integrated hazard rate patterns, $\psi_1(t)$ and $\psi_2(t)$.

Estimating the increase in concentration, δc, presents some difficulties, as this datum is not usually published. However, data will normally be available for the emission rate of the pollutant. In order to determine concentration increase from emission rates, it is necessary to model the dispersion mechanisms of the plume of pollution. The ExternE project has employed some sophisticated models in order to determine concentration increases, and the impacts on the population [77]. It has been noted that it is possible to simplify the calculations considerably with a simple model which nevertheless gives good approximations to the more complex model. This model was developed by Rabl et al (2005) [80], and is known as the "uniform world model". In this model, the collective increase in concentration is related to the emission rate E ($\mu g.s^{-1}$), via the following equation:

$$\delta c = \frac{\rho E}{v} \tag{5.73}$$

In which ρ is the population density of the area over which the pollution is dispersed, and is taken as 80 people.m^{-3}, which is the value for central Europe, including both land and sea [80]. The parameter v is the deposition velocity of pollution and is taken as 0.0027 ms^{-1} for PM2.5 [178].

As equation (5.73) gives the collective increase in concentration experienced by the entire population affected, the resulting calculation will give the collective change in average life expectancy. The collective change in average life expectancy is equal to $N\delta X$ and so, for the purposes of determining J-values, an estimation of the actual number of exposed people is not required. One further point is that strictly speaking, the change in life expectancy calculation should be performed using European mortality rates. However, this has not been done here, as only UK data was used. Using UK mortality rates will, nevertheless, give conservative results, as the UK has lower mortality rates than the rest of Europe taken as a whole [206], so that the life expectancy is higher. Changes in life expectancy are broadly proportional to the initial life expectancy, for example, see equation (5.25). Consequently the calculated change in life expectancy for widely circulated PM2.5 emissions will be an overestimate of the more accurate figure that would be determined if European mortality statistics were used.

5.10 Accounting for those Entering and Leaving the Population during a Prolonged Exposure

The above analysis of the change in life expectancy following a prolonged exposure has, up until now, been assuming that those exposed to the hazard are alive at the start of the exposure. A more accurate calculation would account for members who enter and leave the population during a prolonged exposure. For the general public, only those entering the population by being born in the midst of an exposure need to be accounted for. Members of the public who might not experience the full prolonged exposure because of death are already accounted for in the method laid out above. For the working population, individuals may enter the population through recruitment, and may leave through retirement. There may be other processes by which people enter and leave the exposed population, such as relocation, redundancy or through injury, but these including these processes would require a more sophisticated analysis than is warranted here.

The methods for calculating the effects of exposure to members of the public born during a prolonged exposure, and to members of the workforce who are recruited and who retire during a prolonged exposure, are given by Thomas et al (2009) [186], [185] and Jones et al (2007) [118]. These methods will be briefly outlined below.

Members of the public who are born immediately after the start of the prolonged exposure which lasts for T_R years will be subject to an exposure that lasts for T_R years. If it is assumed that the exposure rate is constant, and if the response is modelled with a step function, as was done for radiation and pollution, then the increase in hazard rate will be proportional to the prolonged hazard perturbation pattern:

$$\delta h(t \mid a) \propto \psi_0(t - a) \tag{5.74}$$

where $\psi_0(z)$ is given by:

$$\psi_0(z) = \int_{z-T_R}^{z} \phi_0(y)dy \tag{5.75}$$

see equation (5.51). The dependence on the exposure time T_R can be made explicit by writing $\psi_0^{(T_R)}(z)$. The member born immediately after the start of the prolonged exposure will have age a = 0. The hazard rate perturbation will then be proportional to:

$$\delta h(t\,|\,0) \propto \psi_0^{(T_R)}(t) \tag{5.76}$$

If the other factors, such as exposure rate and whether the hazard follows the absolute or relative risk model, are known, then the hazard rate perturbation at future age t for an individual of initial age zero can be determined. This can then be used to determine the cumulative hazard rate and hence the change in life expectancy at age zero.

An individual born i years after the exposure will not experience the full prolonged exposure. Instead, he well experience $T_R - i$ years of the exposure. His initial age will still be zero, and so his hazard rate perturbation can be modelled as:

$$\delta h^{(i)}(t\,|\,0) \propto \psi_0^{(T_R-i)}(t) \tag{5.77}$$

where the dependence of the hazard rate perturbation on the number of years since the initial exposure, i, has been made explicit, and where, for clarity:

$$\psi_0^{(T_R-i)}(z) = \int_{z-(T_R-i)}^{z} \phi_0(y)dy = \int_{z-T_R+i}^{z} \phi_0(y)dy \tag{5.78}$$

The hazard rate perturbation will then lead to a change in life expectancy at age zero of $\delta X^{(i)}(0)$, where again, the dependence i has been made explicit. Individuals born in the range $0 \le i < T_R$ will continue to experience the prolonged hazard, but an individual born T_R years or more after the exposure will face no exposure. Under the assumption of a steady state population, the number of individuals being born each year is constant. The average loss of life expectancy for all members still to be born, which will be denoted as δX_{born}, will then be:

$$\delta X_{born} = \frac{\int_0^{T_R} \delta X^{(i)}(0)di}{T_R} \tag{5.79}$$

The total number of individuals, N_2, who will be born in the period of the exposure, T_R, is simply the product of the steady state birth-rate, $n(0)$, given by equation (4.58), and T_R:

$$N_2 = n(0)T_R = N_{Pop}\frac{T_R}{X(0)} \tag{5.80}$$

The total population that experiences the exposure will be the sum of the existing population and those born during the time of exposure:

$$N_{Tot} = N_{Pop} + N_{Pop}\frac{T_R}{X(0)} = N_{Pop}\left(1+\frac{T_R}{X(0)}\right) \tag{5.81}$$

The average loss of life expectancy for this group of people, which will be denoted as δX_{all} will then be the weighted average of the loss of life expectancy of those already alive during the exposure and those who will be born during it:

$$\begin{aligned}\delta X_{all} &= \frac{N_{Pop}\delta X + N_{Pop}\frac{T_R}{X(0)}\delta X_{born}}{N_{Pop}\left(1+\frac{T_R}{X(0)}\right)} = \frac{\delta X + \frac{T_R}{X(0)}\delta X_{born}}{\left(1+\frac{T_R}{X(0)}\right)} \\ &= \frac{X(0)\delta X + T_R\delta X_{born}}{X(0)+T_R}\end{aligned} \tag{5.82}$$

Modelling the recruitment and retirement of a working population can be done in a similar manner. For example, the recruitment process can be seen as being similar to the birth process, but the initial age will be a_{rec}, i.e. about 20, rather than zero for those born, so they will have a hazard rate perturbation of:

$$\delta h^{(i)}(t \mid a_{rec}) \propto \psi_0^{(T_R - i)}(t - a_{rec}) \tag{5.83}$$

which can then be used to calculate the change in life expectancy, $\delta X^{(i)}(a_{rec})$, as long as the exposure rate, and whether the hazard is an absolute or a relative risk-type, are known. The change in average life expectancy is:

$$\delta X_{rec} = \frac{\int_0^{T_R} \delta X^{(i)}(a_{rec})\, di}{T_R} \tag{5.84}$$

The retirement process does pose some additional complications, in that individuals need to be partitioned according to the amount of time they will be exposed to the prolonged hazard, with individuals who are about to retire seeing none of the prolonged hazard, whilst those workers who are below age $a_{ret} - T_R$, where a_{ret} is about 60, will experience the full exposure. Putting a_M as the maximum age an employee can have and still see the full exposure:

$$a_M = a_{ret} - T_R \tag{5.85}$$

The hazard rate perturbation for an individual aged $a_M + i$ at the start of the prolonged exposure will be:

$$\delta h^{(i)}(t \mid a_M + i) \propto \psi_0^{(i)}(t - (a_M + i)) \tag{5.86}$$

which can be used to calculate the change in life expectancy, $\delta X^{(i)}(a_M + i)$, and the average change in life expectancy of those retiring will be:

$$\delta X_{ret} = \frac{\int_0^{T_R} \delta X^{(i)}(a_M + i)di}{T_R} \tag{5.87}$$

Finally, the group of workers in the age range $a_{rec} \le a \le a_M$, who experience the full prolonged exposure will have a hazard rate perturbation of:

$$\delta h^{(i)}(t \mid a_{rec} + i) \propto \psi_0^{(i)}(t - (a_{rec} + i)) \tag{5.88}$$

for $0 \le i \le (a_M - a_{rec})$. This can then be used to calculate the change in life expectancy, $\delta X^{(i)}(a_{rec} + i)$. The average change in life expectancy of this group will be:

$$\begin{aligned} \delta X_{work} &= \frac{\int_0^{a_M - a_{rec}} \delta X^{(i)}(a_{rec} + i)di}{a_M - a_{rec}} \\ &= \frac{\int_0^{a_{ret} - a_{rec} - T_R} \delta X^{(i)}(a_{rec} + i)di}{a_{ret} - a_{rec} - T_R} \end{aligned} \tag{5.89}$$

The average change in life expectancy for the entire workforce who experiences some of the prolonged perturbation will be the weighted average:

$$\delta X_{all-work} = \frac{T_R(\delta X_{rec} + \delta X_{ret}) + (a_{ret} - a_{rec} - T_R)\delta X_{work}}{a_{ret} - a_{rec} + T_R} \tag{5.90}$$

5.11 The Effect of Discounting on the Hazard Rate Perturbations

It was shown in section 4.7 that the effect of discounting was to modify the hazard rate to:

$$h_d(t) = h(t) + r \tag{5.91}$$

where r is the discount rate. This discounted hazard rate then allows the discounted cumulative hazard rate, discounted survival probability, discounted life expectancy, and discounted average life expectancy to be determined from the associated calculations. In the absolute risk model, the perturbed hazard rate is independent of the initial hazard rate, and so discounting has no effect:

$$\delta h_{d.abs}(t \mid a) = \delta h_{abs}(t \mid a) \tag{5.92}$$

The associated change in the cumulative hazard rate will also be unaffected by the discount rate. The change in life expectancy will be:

$$\delta X_d(a) = \frac{1}{S_d(a)} \int_a^{\infty} S_d(t) \delta W(t \mid a) dt \tag{5.93}$$

which is dependent on the discount rate. The discounted average change in life expectancy is then calculated in the usual manner. For the relative risk model, the discounted hazard rate perturbation is:

$$\begin{aligned} \delta h_{d.rel}(t \mid a) &= (h(t) + r) \int_0^{t-a} \phi_0(t - a - x) b(x) dx \\ &= \delta h_{rel}(t \mid a) + r \int_0^{t-a} \phi_0(t - a - x) b(x) dx \end{aligned} \tag{5.94}$$

The associated change in the cumulative hazard rate will also be dependent on the discount rate:

$$\delta W_d(t \mid a) = \int_a^{t} \delta h_{d.rel}(u \mid a) dt \tag{5.95}$$

which can then be used to calculate $\delta X_d(a)$ and δX_d, in the same manner as discussed above.

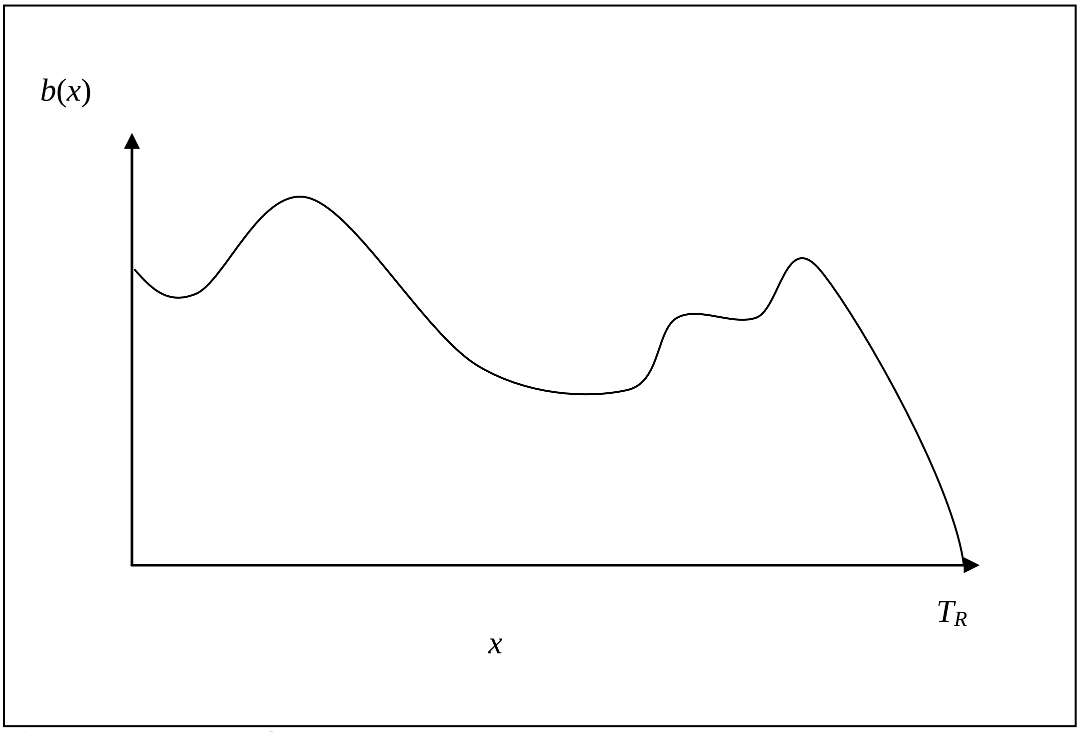

Figure 5 Exposure rate, $b(x)$, over time, x.

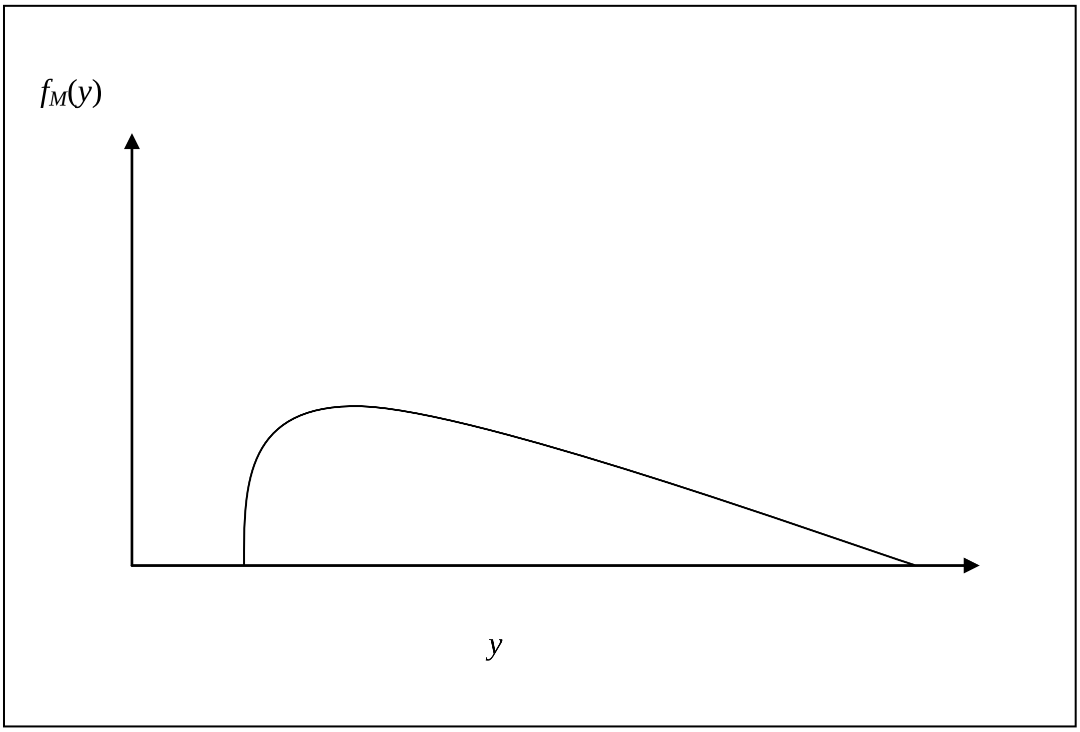

Figure 6 Probability density for the mortality period, y.

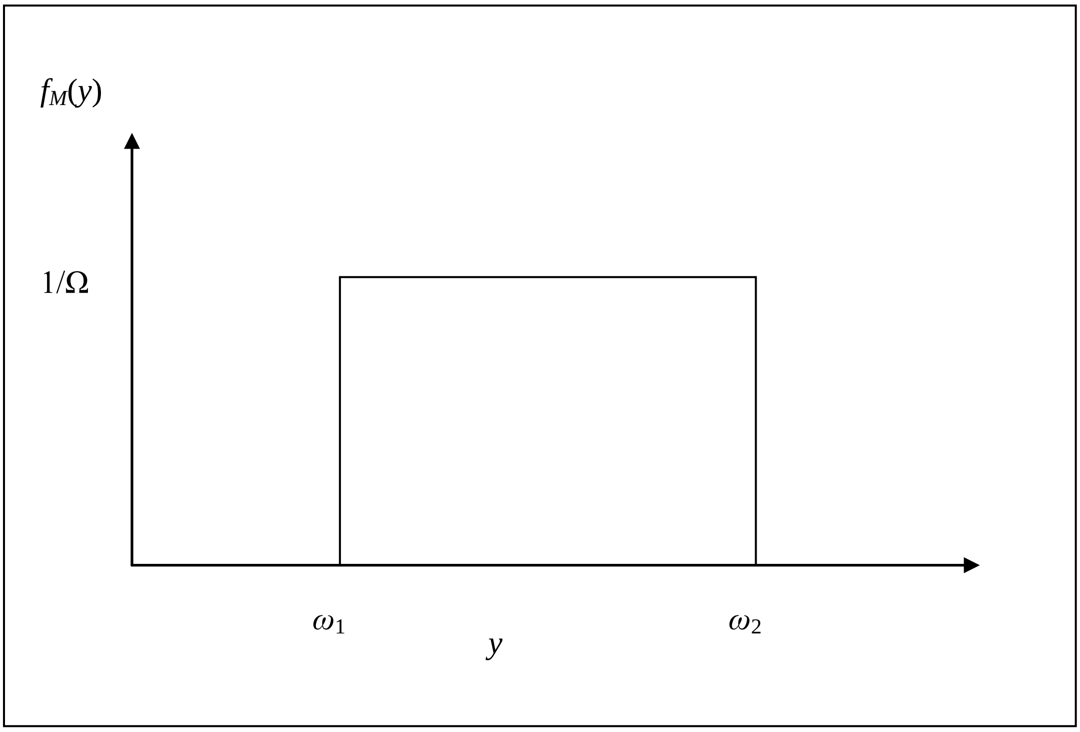

Figure 7 The excess mortality probability distribution for radiation-induced cancer.

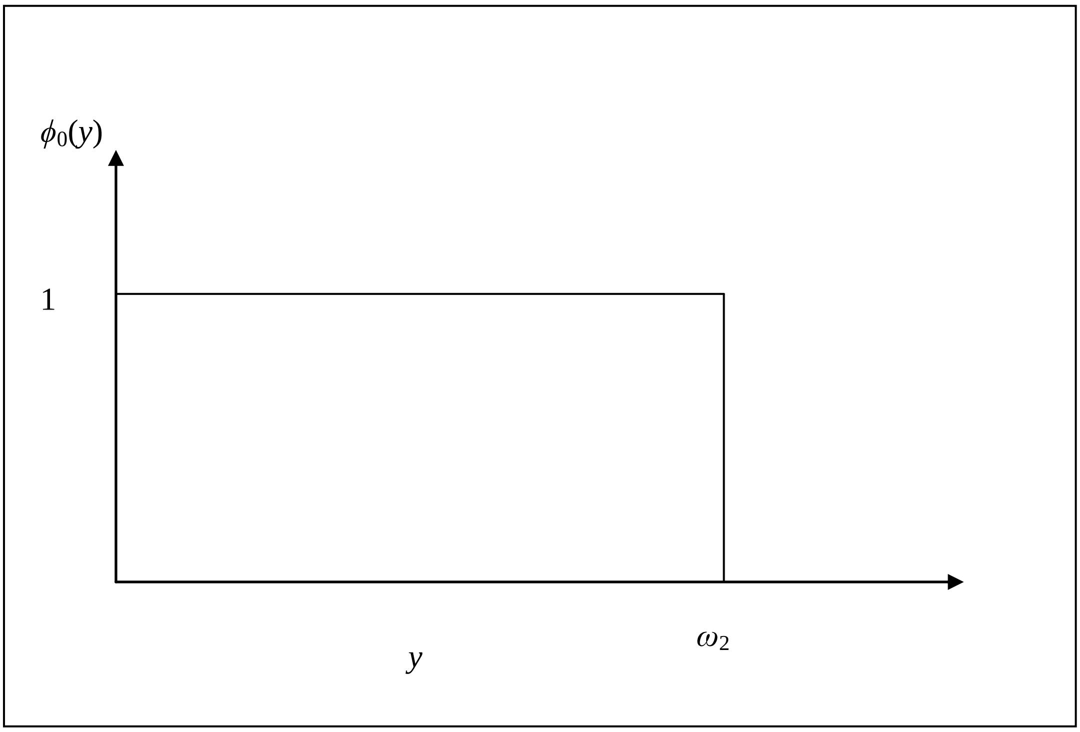

Figure 8 The excess mortality distribution for pollution-induced mortality.

Exposure Type	Response Type	$b(x)$	$f_M(y)/\phi_0(y)$	$\delta h_{abs}(t\|a)$/ $\delta h_{rel}(t\|a)$	$\delta W(t\|a)$ (abs/rel)
Short	Short	= b for $x = 0$ = 0 otherwise	= 1 at $y = 0$ = 0 otherwise	= b/ $bh(t)$ for $t = a$ = 0 otherwise	= b/ $bh(a)$ for $t \geq a$
Short	Long		= Ω^{-1}/1 for $0 \leq y < \Omega$	= $b\Omega^{-1}$/ $bh(t)$ for $t \geq a$	= $b(t-a)\Omega^{-1}$/ $b(W(t)- W(a))$ for $t \geq a$
Long	Short	= b for $x \geq 0$	= 1 at $y = 0$ = 0 otherwise	= b/ $bh(t)$ for $t \geq a$	= $b(t-a)$/ $b(W(t)- W(a))$ for $t \geq a$
Long	Long		= Ω^{-1}/1 for $0 \leq y < \Omega$	= $b\Omega^{-1}(t-a)$/ $bh(t)(t-a)$ for $t \geq a$	= $b(2\Omega)^{-1}(t-a)^2$/ $b(\int h(u)(u-a)du)$ for $t \geq a$

Table 2 Hazard rate perturbations for limiting exposure and response distributions, assumed to be uniform over the specified period. The parameter Ω is the length of time which the prolonged response lasts for. For a long response lasting for the rest of the exposed individual's life, a value of Ω ~ 100 years would be appropriate.

Population Type	Compensating Factor	Dose-Risk Coefficient, c_T (Sv^{-1})
General Population	1.43	0.058
Working Population, 100% Males	1.32	0.042
Working Population, 50:50 Gender Split	1.27	0.041
Working Population, 100% Females	1.23	0.039

Table 3 Values of the compensating factor and dose-risk coefficient for different populations, using latest data.

Chapter 6 Fundamental Relationships for the Calculation of Work-Life Expectancy and the Work-Time Fraction

6.1 Characterising Working Time Behaviour

The preceding two sections described the technical details required for calculating the average length of time remaining for a population, knowledge of which is required in the J-value framework. It is also necessary to calculate the average length of *working* time remaining for a population. This is needed to determine the average work-time fraction, w_0, which is required for the calculation of the risk aversion coefficient in the J-value, as discussed in section 3.2. The average work-time fraction is the average fraction of time the population will spend in work from now on. Related to this parameter is its complement, the average free time fraction, $f_0 = 1 - w_0$. This section describes the methodology for calculating these parameters, which are related to the life expectancy calculations of chapter 4. Indeed, it is shown that the average life expectancy is required to calculate w_0. Also needed is the average work-life expectancy, which is the population averaged length of working time remaining.

6.2 The Work-Time Fraction

Consider an individual of age a in a population with age probability distribution $p(a)$. The individual's life expectancy is $X(a)$. This is the expected value of his life to come from now on. If the individual expects to work for $y_w(a)$ years from now on, which will be termed the work-life expectancy, then his average free time remaining from now on, $F(a)$, will be:

$$F(a) = X(a) - y_w(a) \tag{6.1}$$

Averaging over the entire population gives the average free time remaining, F, in terms of the average life expectancy and the average work-life expectancy, y_w:

$$F = \int_0^\infty p(a)F(a)da = \int_0^\infty p(a)X(a)da - \int_0^\infty p(a)y_w(a)da \\ = X - y_w \tag{6.2}$$

which may also be expressed as:

$$F = \left(1 - \frac{y_w}{X}\right)X \tag{6.3}$$

Comparing this equation with (3.1), it is clear that the average work-time fraction in the population, w, is given by:

$$w = \frac{y_w}{X} \tag{6.4}$$

In section 3.2 it was explained how the work-time fraction relates to the elasticity parameter, q, which is used to describe the trade-offs that are made in maximising the life-quality index. It was assumed that, on average, society's preferences for working will be such that the trade-off between income and free time is optimised for life-quality. This then allowed the optimal work-time fraction, w_0, to be defined, which was assumed to be equal to the average work-time fraction for the population, so that $w_0 = w$.

In order to calculate w_0, the average work-life expectancy needs to be estimated. The method for doing this will now be presented.

6.3 Work-Life Expectancy

It will be assumed that both the population and the job market are in a steady state. The probability, $p_{sw}(t|a)$, of the average individual of age a being in work at a future age t, is the probability that he will have survived to that age, $S(t|a)$, multiplied by the probability that a person of age, t, is in work, $p_w(t)$:

$$p_{sw}(t|a) = p_w(t)S(t|a) = p_w(t)\frac{S(t)}{S(a)} \tag{6.5}$$

If the average person of age, t, works for a fraction of the time, $g_w(t)$, when in work, then the fraction of time, $z_w(t|a)$, someone of age a can expect to be working at future age t, is:

$$z_w(t|a) = g_w(t)p_w(t)\frac{S(t)}{S(a)} \tag{6.6}$$

Thus the amount of time that such a person can expect to work between ages t and $t + dt$ will be $z_w(t|a)\delta t$, and the total time that someone of age, a, can expect to work from now on, $y_w(a)$, may be found by integrating from the current age over all possible future ages to the end of life:

$$y_w(a) = \int_{t=a}^{\infty} z_w(t|a)dt = \frac{1}{S(a)}\int_a^{\infty} g_w(t)p_w(t)S(t)dt \tag{6.7}$$

In the simplest case, the probability that a person of age t is in work, $p_w(t)$, and the fraction of the time the average person of that age spends in work, $g_w(t)$, may be regarded as uniform over the working age, and zero outside it:

$$\begin{aligned} p_w(t) &= 0 && \text{for } t < t_{rec} \\ &= p_w && \text{for } t_{rec} \le t < t_{ret} \\ &= 0 && \text{for } t \ge t_{rec} \end{aligned} \tag{6.8}$$

and:

$$\begin{aligned} g_w(t) &= 0 && \text{for } t < t_{rec} \\ &= g_w && \text{for } t_{rec} \le t < t_{ret} \\ &= 0 && \text{for } t \ge t_{ret} \end{aligned} \tag{6.9}$$

where t_{rec} is the starting age for work, while t_{ret} is the retirement age, so that

$$
\begin{aligned}
g_w(t)\,p_w(t) &= 0 && \text{for } t < t_{rec} \\
&= g_w p_w && \text{for } t_{rec} \le t < t_{ret} \\
&= 0 && \text{for } t \ge t_{ret}
\end{aligned}
\tag{6.10}
$$

Substituting from equation (6.10) into equation (6.7) gives:

$$
\begin{aligned}
y_w(a) &= \frac{g_w p_w}{S(a)} \int_{t_{srec}}^{t_{ret}} S(t)dt && \text{for } a < t_{rec} \\
&= \frac{g_w p_w}{S(a)} \int_{a}^{t_{ret}} S(t)dt && \text{for } t_{rec} \le a < t_{ret} \\
&= 0 && \text{for } a \ge t_{ret}
\end{aligned}
\tag{6.11}
$$

The assumption of a uniformly distribution for the employment probability, $p_w(t)$, and hours of work, $g_w(t)$, is somewhat simplistic. The sensitivity of the work-time parameters to the type of distribution is assessed in chapter 9, where the uniform distribution is compared to observed data for the UK, which appears more normally distributed.

When using the more general equation (6.7), the average work-life expectancy is then given by:

$$
\begin{aligned}
y_w &= \int_0^{\infty} p(a) y_w(a) da \\
&= \int_0^{\infty} \frac{p(a)}{S(a)} \int_a^{\infty} g_w(t) p_w(t) S(t) dt da
\end{aligned}
\tag{6.12}
$$

This expression can be simplified first by noting that, from equation (4.60):

$$
\begin{aligned}
y_w &= \int_0^{\infty} \frac{1}{X(0)} \int_a^{\infty} g_w(t) p_w(t) S(t) dt da \\
&= \int_0^{\infty} \int_a^{\infty} p(t) g_w(t) p_w(t) dt da
\end{aligned}
\tag{6.13}
$$

The employment rate, $p_w(t)$, can be written as:

$$p_w(t) = \frac{n_w(t)}{n(t)} \tag{6.14}$$

where $n_w(t)$ is the number of people working at age t, and $n(t)$ is the number alive at age t. The fraction of time spent working is:

$$g_w(t) = \frac{h_w(t)}{168} \tag{6.15}$$

where $h_w(t)$ is the weekly hours worked at age t, and 168 is the number of hours in a week. The average work-life expectancy is then:

$$\begin{aligned} y_w &= \frac{1}{168}\int_0^\infty\int_a^\infty \frac{p(t)h_w(t)n_w(t)}{n(t)}dtda \\ &= \frac{1}{168N_{Pop}}\int_0^\infty\int_a^\infty H_w(t)dtda \end{aligned} \tag{6.16}$$

where equation (4.60) has again been used to substitute $p(t)/n(t)$ for $1/N_{Pop}$, where N_{Pop} is the total population size, and where $H_w(t) = n_w(t)h_w(t)$ is the total person-hours worked per week at age t. The order of integration can be reversed to give:

$$\begin{aligned} y_w &= \frac{1}{168N_{Pop}}\int_0^\infty H_w(t)\int_0^t dadt \\ &= \frac{1}{168N_{Pop}}\int_0^\infty tH_w(t)dt \end{aligned} \tag{6.17}$$

If the simple case of uniformly distributed working hours between the age of recruitment, t_{rec} and the age of retirement, t_{ret}, is used, then:

$$\begin{aligned} H_w(t) &= \frac{H_T}{t_{ret} - t_{rec}} && \text{for } t_{rec} \le t < t_{ret} \\ &= 0 && \text{otherwise} \end{aligned} \tag{6.18}$$

where H_T is the total person hours worked per week in the population, a figure which can be readily obtained from national statistics, as will be described in more detail in chapter 8. The average work-life expectancy is then:

$$
\begin{aligned}
y_w &= \frac{H_T}{168 N_{Pop}(t_{ret} - t_{rec})} \int_{t_{rec}}^{t_{ret}} t\,dt \\
&= \frac{H_T}{168 N_{Pop}(t_{ret} - t_{rec})} \frac{(t_{ret}^2 - t_{rec}^2)}{2} \\
&= \frac{H_T}{168 N_{Pop}(t_{ret} - t_{rec})} \frac{(t_{ret} + t_{rec})(t_{ret} - t_{rec})}{2} \\
&= \frac{(t_{ret} + t_{rec})}{2} \frac{H_T}{168 N_{Pop}} \\
&= t_{w.av} \frac{H_T}{168 N_{Pop}}
\end{aligned}
\tag{6.19}
$$

where the ratio (t_{ret} + t_{rec}) / 2 is the average working age in the population, $t_{w.av}$, under the uniform distribution assumption.

6.4 Approximations for the Work-Time Fraction

The average work-time fraction can then be estimated, from equation (6.4), as:

$$
\begin{aligned}
w &= \frac{y_w}{X} = \frac{t_{w.av}}{X} \frac{H_T}{168 N_{Pop}} \\
&= \frac{t_{w.av}}{t_{av}} \frac{H_T}{168 N_{Pop}}
\end{aligned}
\tag{6.20}
$$

where equation (4.66) has been used. It may be noted that the average working age is generally very close to the average population age. It was mentioned in section 4.7 that the average age using 2007 – 2009 UK data was around 41.2 years. The average age working age of a uniformly distributed working population is 40 years. Their ratio is thus close to unity. This means that the average work-time fraction may be approximated as:

$$w \approx \frac{H_T}{168 N_{Pop}} \tag{6.21}$$

The two quantities can be estimated readily from national population and labour market statistics, as will be described in chapter 8. In practice, the more accurate (6.20) is used in the estimation. Although the above equations are suitable for measuring *w*, more insight can be gained into this parameter by noting that the ratio $H_T/168$ is the total hours worked in the average week divided by the number of hours in a week. This quantity is therefore the number of person-weeks worked per week. This can be scaled up by multiplying the numerator and denominator by the number of weeks in a year. The scaled up quantity is then the annual person-years worked, N_{py}, and so:

$$w \approx \frac{N_{py}}{N_{Pop}} \tag{6.22}$$

The average work-time fraction thus emerges as the annual per capita person-years worked within the population. This is effectively the procedure advocated by Pandey et al (2006) [158].

Chapter 7 The Value of Life and Life-Years

7.1 The Value of Delaying a Fatality

The methods for using the J-value framework to derive more commonly used valuations of human life will now be presented. The starting point is deriving the value of delaying an immediate fatality by some nominal amount. The maximum reasonable value to spend on increasing life expectancy is given by equation (3.60). This would be such that the J-value was unity. This can be generalised to other situations in which $J \neq 1$ by multiplying by J:

$$\begin{aligned} \delta V_N &= J\,\frac{NG\delta X_d}{1-\varepsilon}\,\frac{\left(1-e^{-r_d X_d}\right)}{r_d X_d} && \text{for } r_d > 0 \\ &= J\,\frac{NG\delta X_d}{1-\varepsilon} && \text{for } r_d = 0 \end{aligned} \tag{7.1}$$

The value, $V_D^{(N)}(x_d)$, of delaying an imminent threat of death by x_d discounted years is found by integrating (7.1) from $X_d = 0^+$ to $X_d = x_d$, where 0^+ indicates the fact that death is imminent but has not actually happened:

$$\begin{aligned} V_D^{(N)}\left(x_d\right) &= JN\,\frac{G}{1-\varepsilon}\int_{0^+}^{x_d}\frac{1-e^{-r_d X_d}}{r_d X_d}\,dX_d && \text{for } r_d > 0 \\ &= JN\,\frac{G}{1-\varepsilon}\int_{0^+}^{x_d} dX_d = JN\,\frac{G}{1-\varepsilon}\,x_d && \text{for } r_d = 0 \end{aligned} \tag{7.2}$$

If only one individual is concerned, then N = 1 in equation (7.2), and using the notation $V_D(x_d) = V_D^{(1)}(x_d)$, then:

$$\begin{aligned} V_D\left(x_d\right) &= J\,\frac{G}{1-\varepsilon}\int_{0}^{x_d}\frac{1-e^{-r_d X_d}}{r_d X_d}\,dX_d && \text{for } r_d > 0 \\ &= J\,\frac{G}{1-\varepsilon}\,x_d && \text{for } r_d = 0 \end{aligned} \tag{7.3}$$

The integral in equation (7.3) has no closed form solution, but can be evaluated numerically. In order to retain accuracy, the integral is expressed as a series sum, which is much easier to evaluate. This is done by first making the replacement:

$$\begin{aligned} &z = r_d X_d \\ &dz = r_d dX_d \\ &X_d = 0 \Rightarrow z = 0 \\ &X_d = x_d \Rightarrow z = r_d x_d = Z \end{aligned} \tag{7.4}$$

so that:

$$\int_0^{x_d} \frac{1 - e^{-r_d X_d}}{r_d X_d} dX_d = \frac{1}{r_d} \int_0^Z \frac{1 - e^{-z}}{z} dz \tag{7.5}$$

The Taylor series expansion for e^{-z} is:

$$e^{-z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \tag{7.6}$$

so that:

$$1 - e^{-z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n!} \tag{7.7}$$

and:

$$\frac{1 - e^{-z}}{z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{n-1}}{n!} \tag{7.8}$$

The integral therefore becomes:

$$\int_0^Z \frac{1 - e^{-z}}{z} dz = \int_0^Z \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{n-1}}{n!} dz \tag{7.9}$$

As the integral of a sum is equal to the sum of an integral, equation (7.9) can be written as:

$$\begin{aligned}\int_0^Z \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{n-1}}{n!} dz &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_0^Z z^{n-1} dz \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} Z^n}{nn!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (r_d x_d)^n}{nn!}\end{aligned} \tag{7.10}$$

substituting into (7.5):

$$\int_0^{x_d} \frac{1-e^{-r_d X_d}}{r_d X_d} dX_d = \frac{1}{r_d} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (r_d x_d)^n}{nn!} \tag{7.11}$$

which can be readily evaluated numerically. The sum converges to the correct solution very rapidly. After two terms, the error is about 2%, and after three terms, the error is about 0.3%, for typical values of r_d and x_d. Even for high values of r_d and x_d, the error is still less than 1% after three terms.

7.2 The Value of Temporarily Preventing a Fatality, VTPF

The above analysis of the value of delaying a fatality by x_d years may be extended to the case where the immediate threat to life is completely eliminated, returning the individual back to his initial state. The more common term for this value is the VPF – the value of preventing a fatality. However, this phrase is a circumlocution, as it is impossible to prevent a fatality – all individuals will eventually die. Hence the phrase adopted here is the "value of temporarily preventing a fatality", or VTPF, which acknowledges this problem.

The maximum number of years an individual can gain from having an immediate threat to his life removed is his initial discounted life expectancy in the absence of the threat. If the age of the individual is known, then this maximum value is thus

$X_d(a)$. If the age of the individual is unknown, then the average discounted life expectancy, X_d, will be the best estimate of the number of years gained from temporarily preventing the fatality. Thus the VTPF may be written as $V_P(X(a))$, or more simply, $V_P(a)$, for when age is known, where:

$$\begin{aligned} V_P(a) &= J\frac{G}{1-\varepsilon}\int_0^{X_d(a)}\frac{1-e^{-r_d X_d}}{r_d X_d}dX_d && \text{for } r_d > 0 \\ &= J\frac{G}{1-\varepsilon}X_d(a) && \text{for } r_d = 0 \end{aligned} \tag{7.12}$$

or when the age is unknown:

$$\begin{aligned} V_P &= J\frac{G}{1-\varepsilon}\int_0^{X_d}\frac{1-e^{-r_d X_d}}{r_d X_d}dX_d && \text{for } r_d > 0 \\ &= J\frac{G}{1-\varepsilon}X_d && \text{for } r_d = 0 \end{aligned} \tag{7.13}$$

The VTPF when age is unknown may be used as an indicator of the population averaged VTPF. Another way of averaging would be to integrate the age-dependent VTPF over the population distribution:

$$V_{P.av} = \int_0^{\infty} p(a)V_P(a)da \tag{7.14}$$

when the discount rate is zero, these two methods of averaging are identical. For $r_d > 0$, the values are still close, with the age-independent VTPF being slightly higher.

7.3 The Value of a Discounted Life-Year, VODLY

The value of a discounted life year, $v_d(x_d)$, is the amount that should be paid to extend life by one year. This is equal to the difference in the value of a delayed fatality between a delay of $x_d + 1$ years and x_d years:

$$v_d(x_d) = V_D(x_d + 1) - V_D(x_d) \tag{7.15}$$

so that:

$$v_d\left(x_d\right) = J\,\frac{G}{1-\varepsilon}\int_{x_d}^{x_d+1}\frac{1-e^{-r_d X_d}}{r_d X_d}\,dX_d \qquad \text{for } r_d > 0$$
$$= J\,\frac{G}{1-\varepsilon} \qquad \text{for } r_d = 0 \tag{7.16}$$

The integral may be evaluated by noting that it can be developed as a sum, following the same method as was shown in section 7.1, except with the limits of integration changed. This means the sum will be altered to:

$$\begin{aligned}
\int_{x_d}^{x_d+1}\frac{1-e^{-r_d X_d}}{r_d X_d}\,dX_d &= \frac{1}{r_d}\int_{Z}^{Z+r_d}\frac{1-e^{-z}}{z}\,dz \\
&= \frac{1}{r_d}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\int_{Z}^{Z+r_d} z^{n-1}dz \\
&= \frac{1}{r_d}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{nn!}\left((Z+r_d)^n-(Z)^n\right) \\
&= \frac{1}{r_d}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{nn!}Z^n\left(\left(1+\frac{r_d}{Z}\right)^n-1\right)
\end{aligned} \tag{7.17}$$

Since $r_d/Z = 1/x_d$ will typically be small, the bracketed term may be approximated as:

$$\begin{aligned}
\left(1+\frac{1}{x_d}\right)^n - 1 &\approx 1+\frac{n}{x_d}-1 \\
&= \frac{n}{x_d}
\end{aligned} \tag{7.18}$$

substituting back into (7.17) gives:

$$\frac{1}{r_d x_d}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}Z^n = \frac{1}{r_d x_d}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\left(r_d x_d\right)^n \tag{7.19}$$

Comparing (7.19) with (7.7), it is apparent that:

$$\frac{1}{r_d x_d}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}(r_d x_d)^n = \frac{1-e^{-r_d x_d}}{r_d x_d} \tag{7.20}$$

and so the VODLY is:

$$\begin{aligned} v_d(x_d) &= J\frac{G}{1-\varepsilon}\frac{1-e^{-r_d x_d}}{r_d x_d} && \text{for } r_d > 0 \\ &= J\frac{G}{1-\varepsilon} && \text{for } r_d = 0 \end{aligned} \tag{7.21}$$

The VODLY is thus dependent upon the length of the achieved delay, but only at high discount rates and large delays. For low discount rates or delays, the VODLY is approximately constant.

7.4 An Alternative Model of the VODLY, the VODLYA

An alternative characterisation of the VODLY, which will be called the VODLYA, would be the average value of a discounted life-year, v_{av}, achieved by returning the individual to his or her normal life expectancy. This is simply equal to the ratio of the VTPF to the initial life expectancy of an individual, $X(a)$:

$$v_{ave}(a) = \frac{V_P(a)}{X(a)} \tag{7.22}$$

or, when age is not known:

$$v_{ave} = \frac{V_P}{X} \tag{7.23}$$

By comparing the above equation with equations (7.12) and (7.13), it can be seen that the VODLYA is equal to the VODLY when the discount rate is zero. They are also close for non-zero discount rates.

7.5 The Hazard Elimination Premium, HEP

The VTPF, VODLY and VODLYA all provide a valuation for the extension of life by a certain amount of time. It is also possible to define a value for a given level of risk reduction, and it is natural to first consider the value of completely eliminating a given risk. In such a situation, an individual or a population would be exposed to some detrimental hazard that is causing a reduction in life expectancy. Upon elimination of the hazard, the life expectancy is returned to the average value for the general public. Such a measure thus provides a maximum reasonable amount to be spent on completely eliminating a given risk, and is termed the "Hazard Elimination Premium", or HEP.

This measure has useful applications in the field of comparative risk analysis, in which different risk-exposing systems will produce costs on an exposed population, and the best system is the one which minimises this cost for a given output. The HEP calculates the total improvement in life expectancy in absence of the risk, and monetises it to produce a common measure of this cost. The HEP is given by equation (7.1), repeated below:

$$\begin{aligned}\delta V_N &= J\,\frac{NG\delta X_d}{1-\varepsilon}\,\frac{\left(1-e^{-r_d X_d}\right)}{r_d X_d} && \text{for } r_d > 0 \\ &= J\,\frac{NG\delta X_d}{1-\varepsilon} && \text{for } r_d = 0\end{aligned} \tag{7.1}$$

where here the change in discounted life expectancy, δX_d is the life expectancy gained from complete elimination of the hazard. The maximum reasonable HEP occurs when J = 1. For a comparative risk analysis to be consistent, then the same value of J should be used for each system studied. However, there may be various practical constraints whereby using different values of J would be warranted. For example, safety regulations may require a disproportion factor to be incorporated into cost considerations for certain systems. The factor of J could then be used for this disproportion factor.

The HEP is a novel concept introduced here for use in the second part of this thesis, in which a comparative risk analysis of UK electricity generating systems is performed. Here the systems under scrutiny are the entire fuel chains involved with various methods of electricity generation, from fuel extraction to waste disposal. These produce costs to the public and workers in terms of extra mortality from pollution and radiation exposure, as well from accidents. Using the tools presented in the preceding chapters i.e. those of the life quality index and J-value, which incorporate models of survival and mortality, and models of working time behaviour, the risks involved with the electricity generation systems under comparison can be objectively measured. These can then be combined using equation (7.1) to produce a set of HEPs for each electricity generating system, in terms of the maximum reasonable amount to spend on risk elimination per unit of electricity generated, which can then be used to compare the different aspects of risk posed by each system.

Chapter 8 Measurement of the Parameters Required for J-Value Analysis and their Tolerances

8.1 Quantifying Parameters and their Uncertainty

The preceding sections have laid out the methods and procedures necessary for the calculation of the parameters required in the J-value model. In this section the estimates of each of these parameters is presented. The methods for estimating the uncertainty of the parameters is also discussed and where possible, the 95% tolerance limits are shown. Some of the work contained in this chapter has been previously published by the author, see Kearns (2010) [123]. However, the majority is either new or is a further extension of the previous work.

The J-value, as given by equation (3.61), is comprised of seven parameters. These are also dependent upon further parameters. Other parameters extraneous to the J-value, such as the VTPF and VODLY may also be calculated from these quantities. Five of the seven J-value parameters can be objectively measured from reliable statistics, a defining feature of the J-value. The only parameters which are not objectively measured are the discount rate, r_d, and the net discount rate, r. The former parameter is usually fixed so that the latter parameter is equal to either 0% per annum or 2.5% per annum, but can also be varied to assess sensitivities, as will be described later. The remaining parameters can be classed as either "context-dependent" parameters or "context-independent" parameters. The context-dependent parameters are those which depend on the specific nature of the protection system, and so cannot be determined *a priori*. These parameters are: the change in discounted life expectancy, δX_d; the number of individuals affected by the protection system, N; and the actual cost of the protection system, $\delta \hat{V}_N$. The context-independent parameters are those which are constant for each protection system, and can be evaluated without knowledge of the protection system. These are: the GDP per person, G; the risk aversion coefficient, ε; the average life expectancy, X; and the growth rate, r_g. These parameters, in turn, are dependent upon other parameters, such as the age distribution, $p(a)$, the survival probability, $S(a)$, the work-time fraction, w_0, etc. Each parameter will now be discussed in turn, and the estimate will be presented.

It is also important that some attempt is made to quantify the uncertainty associated with these measurements. The uncertainty is presented in terms of the tolerance limits. The methods for doing this will also be discussed. Although many of the parameters can be assessed for uncertainty, it is not possible to do this for each one. In particular, those that are not used directly in the J-value equation will not have their uncertainty quantified. Important to the uncertainty analysis is the consideration of the propagation of uncertainty conditions, which relates the uncertainty on a particular variable to the uncertainty of some function of that variable. These considerations allow the tolerance limits on the J-value to be estimated. The propagation of uncertainty is determined by a weighted sum of squares method. If a function, f, is dependent upon k variables, denoted as x_i, for i = 1, 2, ..., k, so that:

$$f = f(x_1, x_2, x_3, ..., x_k) \tag{8.1}$$

and if the variance of each of the x_i's, denoted as $\sigma_{x_i}^2$, are known, then the variance of f, σ_f^2, is given by:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + ... + \left(\frac{\partial f}{\partial x_k}\right)^2 \sigma_{x_k}^2 + \text{corr} \tag{8.2}$$

and the standard deviation is the square root of the variance. The "corr" term represents the contribution to the uncertainty when two or more variables are correlated with each other. For example, if the variables x_1 and x_2 are correlated with correlation coefficient $\rho_{x_1 x_2}$, but all other variables are independent of each other, then the correlations term would be equal to:

$$\text{corr} = 2 \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \sigma_{x_1} \sigma_{x_2} \rho_{x_1, x_2} \tag{8.3}$$

Once the standard deviation has been obtained, the last remaining piece of information required for knowledge of the tolerance limits is the distribution. As will be discussed, many of the parameters have normal distributions. The 95% tolerance limits for such distributions then lie at approximately ± two standard deviations from the mean.

8.2 Gross Domestic Product per Person, *G*

The Gross Domestic Product (GDP) of a country is a measure of economic activity. It is the value of all goods and services produced within the country over the year. The GDP per person is the GDP divided by the total population of the country:

$$G = \frac{GDP}{N_{Pop}} \tag{8.4}$$

In the UK, these figures are published annually by the Office for National Statistics (ONS), in a publication entitled "United Kingdom National Accounts: The Blue Book" [149]. The latest value of *G*, as taken from the Blue Book 2010, is £22,538.

In order to assess the uncertainty on *G*, it is first necessary to estimate the standard deviation of the estimates of the GDP and the population. These uncertainties will then be related to the standard deviation on *G* by:

$$\begin{aligned}\sigma_G^2 &= \left(\frac{\partial G}{\partial GDP}\right)^2 \sigma_{GDP}^2 + \left(\frac{\partial G}{\partial N_{Pop}}\right)^2 \sigma_{N_{Pop}}^2 \\ &\quad + 2\frac{\partial G}{\partial GDP}\frac{\partial G}{\partial N_{Pop}} \sigma_{GDP}\sigma_{N_{Pop}}\rho_{GDP,N_{Pop}} \\ &= G^2\left(\frac{\sigma_{GDP}}{GDP}\right)^2 + G^2\left(\frac{\sigma_{N_{Pop}}}{N_{Pop}}\right)^2 - 2G^2\frac{\sigma_{GDP}}{GDP}\frac{\sigma_{N_{Pop}}}{N_{Pop}}\rho_{GDP,N_{Pop}}\end{aligned} \tag{8.5}$$

so that:

$$\frac{\sigma_G}{G} = \sqrt{\left(\frac{\sigma_{GDP}}{GDP}\right)^2 + \left(\frac{\sigma_{N_{Pop}}}{N_{Pop}}\right)^2 - 2\frac{\sigma_{GDP}}{GDP}\frac{\sigma_{N_{Pop}}}{N_{Pop}}\rho_{GDP,N_{Pop}}} \tag{8.6}$$

where $\rho_{GDP,N_{Pop}}$ is the correlation coefficient between the population size and the GDP. The values of the GDP and N_{Pop} are also given in the Blue Book. For 2009, the GDP was £1.39 trillion and N_{Pop} was 61.8 million.

The uncertainty on the GDP measurement is estimated from [144], which gives data on the subsequent revisions in the estimates of the GDP in a previous publication of the Blue Book. It is assumed that the most up to date value of the GDP will be subject to similar revisions, and that this is the major source of uncertainty on the GDP estimate. The total revisions after the initial Blue Book publication give the relative standard deviation, or coefficient of variation, on the GDP as 0.1%.

The uncertainty on the population can also be estimated from data published by the ONS. An analysis performed by the ONS of 2001 Census data showed that the 95% confidence interval for the 2001 population estimate for England and Wales was ±0.2% of the mean estimate [143]. The relative standard deviation is then 0.2%/1.96 = 0.1%. Had data for the whole of the UK been pooled, rather than just for England and Wales, the error would have been smaller. Although this estimate was for the 2001 population, it will be assumed that the uncertainty is also applicable to the present day population estimate.

The final estimate required to calculate equation (8.6) is the correlation coefficient between the GDP and the population. This can be estimated from ONS time series data [153], which provides the historical values of the GDP and the national population from 1948 to 2008. It is then possible determine how the two vary together, and hence obtain ρ. Performing this calculation gives $\rho = 0.94$: The time series data is shown in Figure 10.

Using the above values in equation (8.6) gives the relative standard error on the GDP per person, σ_G/G as 0.03%. The estimates of the GDP and the population are made by summing a large number of independent records, and so, by the central limit theorem, the uncertainty on each of the estimates will be normally distributed. Thus, G is the ratio of two normally distributed and correlated random variables with different means and standard deviations. The uncertainty on G then follows the ratio distribution, see [94]. This distribution, which cannot be expressed simply, is shown in Figure 9. The distribution is not normal – it is much more sharply peaked. The associated distribution if the uncertainties were normally distributed is also shown in this figure for reference. It is not known what the 95% tolerance limits are for such a distribution, but they will be closer to the mean than for the normal distribution (where the 95% limits are at around $\pm 2\sigma$), which also means that the tolerance interval will be smaller.

8.3 Net Discount Rate, r , Discount Rate, r_d and Growth Rate, r_g

In order to discount the life expectancy and the change in life expectancy, it is necessary to evaluate the net discount rate, r. The net discount rate is a linear combination of the discount rate (or real rate of time preference), r_d, and the annual growth rate, r_g, as given in equation (3.46). The growth rate can be evaluated from the Treasury Green Book [95], who use r_g = 2% per annum. The discount rate can then be chosen to set the net discount rate to be either 0% or 2.5%, which are the two discount rates usually used in J-value analysis, although higher discount rates may also be used. In order to get r = 0%, then it is necessary to set $r_d = (1- \varepsilon) \times r_g$ = 0.3% per annum. To get r = 2.5% it is necessary to put r_d = 2.8% per annum. Different values of the discount rate can also be used to assess the sensitivity of the life expectancy and the J-value to discounting. As the net discount rate is not a directly measured quantity, it will be assumed that there is no uncertainty on this parameter.

8.4 Discounted Average Life Expectancy, X_d, and Other Related Actuarial Parameters

The method for calculating the life expectancy and the other related variables is presented in chapter 4. The fundamental variable in these calculations is the age-dependent hazard rate, $h(a)$. All other actuarial parameters can be calculated once these are known. As was discussed in section 4.4, the way the hazard rate is determined is dependent upon whether deaths are assumed to be uniformly or exponentially distributed over the interval (a, a+1). Section 4.5 discussed the current assumption used in J-value calculations, which is to assume that deaths are exponentially distributed, so that the central rate of mortality is used for the hazard rate. The sensitivity of the results to this assumption is assessed in chapter 9.

The central rates of mortality for the UK population are available in the Office for National Statistics' Interim Life Tables [145]. These are presented in terms of male and female mortality rates, which can then be combined using equation (4.50). Section 4.5 also describes the end correction used to account for the mortality of the final age group of the population.

Section 4.6 details the method used to calculate the population distribution, $p(a)$, under the assumption the population is in a steady state. Again, all that is required to calculate this distribution are the hazard rates. This steady state population distribution is shown in Figure 4, along with actually observed UK population distribution. The effect of using the simplified distribution on the results is assessed in Chapter 9. The probability distribution is also used in calculating other parameters, such as the moments of the distribution. The first moment – the mean age, was shown in section 4.7 to be equal to the average (undiscounted) life expectancy. The value of this parameter is discussed in the next paragraph. In sections 4.8 and 5.8, it was shown how the second moment can be used in approximating the effect of the discount rate on the average life expectancy, and also the value of the change in life expectancy for prolonged exposures and short responses and *vice versa*. The third moment was also found to be useful for calculating the change in life expectancy for prolonged exposures and prolonged responses. As the population is assumed to be in a steady state, these moments are constant over time for the population. The second moment of the distribution, which is the mean-square age, is equal to 2,304 years^2. The third moment, the mean-cube age, is equal to 147,311 years^3. One other parameter which can be calculated from the distribution is the population entropy, H, derived in equation (5.29) as a key parameter in the change in life expectancy resulting from a short relative risk exposure. For most populations, the population entropy lies between zero and unity. Populations that have constant mortality rates over all ages, so that the distribution declines exponentially, will have a population entropy of unity, whilst populations in which the majority of deaths occur within a narrow age range will have a low entropy near zero, for example, see the discussion by Goldman and Lord (1986) [89].

The trend is thus for populations to reduce their entropy as they become more developed over time. For the UK for 2007-2009, the population entropy was 0.13.

The discounted life expectancy, $X_d(a)$, and discounted average life expectancy, X_d, are shown in Figure 11 for discount rates of 0% and 2.5%. Life expectancy at birth, $X_d(0)$, is 79.6 and 34.0 years respectively. The average life expectancy is 41.2 and 22.9 years respectively. These numbers are for the general population, and assume that there is a 50% male/female split at all ages. For a working population distributed uniformly between ages 20 and 60, and which is composed entirely of males, average life expectancies are 39.5 and 23.9 years for discount rates of 0% and 2.5% respectively.

As the discounted average life expectancy is an important parameter in the J-value equation, as given by (3.61), the tolerance limits will be analysed for this parameter. This is done using the following method:

Suppose and individual is selected at random from the population as a whole. The individual will be of random age, A^*. If we know the value of this random age, such that $A^* = a$ (which is taken to mean that the age is between a and a+1), then we may categorise the individual into an age category. The selected individual will have a random life to come, $\chi(a)$, but that life to come, even though random, will be conditioned by the fact the individual has age, a. The relationship is defined formally by:

$$\chi(a) = \{\chi | a < A^* \leq a + 1\} \tag{8.7}$$

where χ is the unconditioned random life to come. The expected value, $X(a)$, of the life to come of an individual of age, a, is the average value of the expected life to come for all $n(a)$ individuals of age a in the population:

$$X(a) = E[\chi(a)] = \frac{\sum_{k=1}^{n(a)} \chi^{(k)}(a)}{n(a)} \tag{8.8}$$

However, if we do not know the age of the randomly selected individual, our best estimate of his life to come will be the weighted, average value, *X*, over all ages:

$$\begin{aligned} X &= E[\chi] \\ &= \underset{a}{E}[E[\chi(a)]] \\ &= \underset{a}{E}[X(a)] \end{aligned} \tag{8.9}$$

The arguments advanced for treating random life to come, χ, transfer one-to-one to the case of the random square of life to come, χ^2. Hence the random square of life to come, given that the individual's age is *a*, is:

$$\chi^2(a) = \{\chi^2 | a < A^* \leq a+1\} \tag{8.10}$$

while the expected value of the square of life to come of an individual of age, *a*, is given formally by

$$E[\chi^2(a)] = \frac{\sum_{k=1}^{n(a)} \left(\chi^2(a)\right)^{(k)}}{n(a)} \tag{8.11}$$

Then, if we do not know the age of the randomly selected individual, our best estimate of the square of his life to come will be the weighted, average value over all ages:

$$\begin{aligned} E[\chi^2] &= \underset{a}{E}[E[\chi^2(a)]] \\ &= \int_0^\infty p(a)E[\chi^2(a)]da \end{aligned} \tag{8.12}$$

The variance of random life to come for individuals selected at random in the population will be $\text{var}[\chi]$, given by:

$$\mathrm{var}[\chi] = E[\chi^2] - (E[\chi])^2 \tag{8.13}$$

which may be expanded using equations (8.9) and (8.12):

$$\mathrm{var}[\chi] = \underset{a}{E}[E[\chi^2(a)]] - X^2 \tag{8.14}$$

But formally, the variance of random life to come, given that the person is aged, *a*, is given by:

$$\begin{aligned}\mathrm{var}[\chi(a)] &= E[\chi^2(a)] - (E[\chi(a)])^2 \\ &= E[\chi^2(a)] - X^2(a)\end{aligned} \tag{8.15}$$

Moreover, it is known, by equation (D.18) of Thomas, Jones and Kearns (2010) [189], that the variance of random life to come for an individual of age, *a*, is:

$$\mathrm{var}[\chi(a)] = 2X(a)(t_{a+.ave} - a) - X^2(a) \tag{8.16}$$

where, from equation (D.15) of *op. cit.*, $t_{a+.ave}$, is the average age of those above age, *a*:

$$t_{a+.ave} = \frac{1}{p(a)X(a)}\int_a^\infty tp(t)dt \tag{8.17}$$

Comparing equations (8.15) and (8.16) shows that:

$$E[\chi^2(a)] = 2X(a)(t_{a+.ave} - a) \tag{8.18}$$

Substituting from equation (8.18) into equation (8.14) gives:

$$
\begin{aligned}
\operatorname{var}[\chi] &= \underset{a}{E}\left[2X(a)\left(t_{a+.ave}-a\right)\right]-X^2 \\
&= 2\int_0^\infty p(a)X(a)\left(t_{a+.ave}-a\right)da - X^2
\end{aligned} \tag{8.19}
$$

This may be developed further, by noting that:

$$
\begin{aligned}
&\int_0^\infty p(a)X(a)\left(t_{a+.ave}-a\right)da \\
&= \int_0^\infty p(a)X(a)t_{a+.ave}da - \int_0^\infty p(a)X(a)ada \\
&= \int_0^\infty\int_a^\infty tp(t)dtda - \int_0^\infty ap(a)\int_a^\infty \frac{S(t)}{S(a)}dtda \\
&= \int_0^\infty\int_a^\infty tp(t)dtda - \int_0^\infty a\int_a^\infty p(t)dtda
\end{aligned} \tag{8.20}
$$

where equations (8.17), (4.26) and (4.60) have been used in the development. The order of integration can be reversed to give:

$$
\begin{aligned}
&\int_0^\infty\int_a^\infty tp(t)dtda - \int_0^\infty a\int_a^\infty p(t)dtda \\
&= \int_0^\infty tp(t)\int_0^t dadt - \int_0^\infty p(t)\int_0^t adadt \\
&= \frac{1}{2}\int_0^\infty p(t)t^2dt \\
&= \frac{t_{av}^2}{2}
\end{aligned} \tag{8.21}
$$

where t^2_{av} is the mean-square age of the population, as discussed above. This means that:

$$
\underset{a}{E}\left[E\left[\chi^2(a)\right]\right] = \underset{a}{E}\left[2X(a)\left(t_{a+.ave}-a\right)\right] = t_{av}^2 \tag{8.22}
$$

The square of the random life to come averaged over all ages of death and over the population is therefore equal to the mean-square age of the population. It has also been established, via equation (4.66), that the average life expectancy, which is the random life to come averaged over all ages of death and over the population (see equation (4.20)), is equal to the mean age in the population. Thus, both the first and second moments of the distribution of the life to come averaged over all ages of death and all ages are equal to the first and second moments of the age distribution. In fact, this may be shown to be true for all moments, a proof of which is given in Appendix B. Thus the general result is that, under steady state conditions, the moments of life to come are equal to the moments of life lived.

Substituting into equation (8.19):

$$\begin{aligned}\mathrm{var}[\chi] &= t_{av}^2 - X^2 = E\left[t^2\right] - X^2 \\ &= E\left[t^2\right] - E[t]^2 \\ &= \mathrm{var}[t]\end{aligned} \tag{8.23}$$

where equation (4.66) has been used, and where the expectation operator $E[.]$ has been introduced to avoid confusion. Thus the variance of the life to come averaged over all ages is therefore equal to the variance of the age distribution.

Using latest UK data, the standard deviation for an individual picked at random, without knowledge of the individual's age, is about 24 years.

In order to derive the variance of the average life expectancy for a whole population of size N_{Pop}, it is assumed that the age distribution of the population is unknown. Each individual can then be treated as having a random life to come of value χ which has mean value X and variance, σ^2, as given by equation (8.23). By the Central Limit Theorem, for large N_{Pop}, the average of the N_{Pop} random variables will be approximately normally distributed with mean X and variance $\mathrm{var}[\chi]/N_{Pop}$. Hence the variance of average life expectancy for a whole population is:

$$\mathrm{var}[X] = \frac{\mathrm{var}[\chi]}{N_{Pop}} = \frac{\mathrm{var}[t]}{N_{Pop}} \tag{8.24}$$

For the UK, var[χ] = 609 years^2. Dividing by the population size of 61.8 million, the variance of the average life expectancy is approximately $1\text{x}10^{-5}$ years^2, and the standard deviation is 0.003 years. For the normal distribution, the 95% tolerance limits lie at ±1.96σ from the mean. The 95% tolerance interval for the average life expectancy is therefore 41.166 – 41.177 years.

8.5 Share of Wages in the GDP, θ

The wage share of the GDP, θ, which was introduced in section 3.2, needs to be estimated in order to estimate the risk aversion coefficient, ε, as given by equation (3.41). The wage share may be determined from national statistics. In the UK, the ONS publish this datum in many publications. Here, data from the monthly "Economic & Labour Market Review" [150] will be used. When estimating θ, there exists a problem of defining exactly what constitutes wages. Most national accounts use the term "compensation of employees" to refer to wages paid by employers to employees. The ONS define "compensation of employees" as the "*Total remuneration payable to employees in cash or in kind. Includes the value of social contributions payable by the employer*" [149]. The main drawback of this definition is that it neglects the income of the self employed, which in some countries can represent a large fraction of the GDP.

It will be recalled that the wage share was defined in section 3.2, equation (3.10) in a "production function", a function that relates two inputs, or "factors of production" to the output produced. In this case, the factors of production were labour and capital, and the output was the Gross Domestic Product. The production function defined in equation (3.10) was of a special type, known as a "Cobb-Douglas" production function, in which the two factors of production are exponentially weighted and formed into a product. A consequence of the Cobb-Douglas production for GDP is that the share of wages should remain constant over time and across countries. This is because if wage rates were to rise relative to capital income, then industries would employ fewer people in order to minimise the loss of profit. If wages were to fall relative to capital, industries could employ more people for the same profit. Thus the wage rate and the employment rate are always engaged in a trade-off, and this trade-off renders θ approximately constant. For further details of this process, see Wolfson (1978) [205]. Using the definition of wages as being equal to only the compensation of employees, θ does not appear to be constant, either over long periods of time or across countries, as shown by Gollin (2002) [88]. Gollin attributes these discrepancies to the practice of neglecting the income of the self employed in the definition of wages. Changing the definition of wages to include the self employed as well as compensation of employees gives new estimates of the wage share that are remarkably consistent with the predictions of the Cobb-Douglas theory. It is for this reason that the income of the self employed is included with the compensation of employees in calculating θ for use in J-value analysis.

The income of the self-employed can be very difficult to measure in some countries. In the UK, the ONS provide estimates of self employed income under the term “mixed income”. The ONS define this as: “*The balancing item on the generation of income account for unincorporated businesses owned by households. The owner or members of the same household often provide unpaid labour inputs to the business. The surplus is therefore a mixture of remuneration for such labour and return to the owner as entrepreneur*” [149]. The last sentence of this quote highlights the difficulty with using mixed income for the self-employed contribution to the GDP. This is that the UK national accounts do not determine how much of the self-employed income is taken as a wage, and how much is fed back into the unincorporated business, which would count as capital formation. This problem has been noted by the ONS, see [193], who solve the problem by assuming the share of mixed income taken as profit is equal to the share of the GDP paid as compensation of employees. For example, if compensation of employees is 60% of the total GDP, then one should assume that 60% of the mixed income is taken by the self-employed as wages, with the rest going as capital formation. Hence, θ is estimated from the national accounts as:

$$\theta = \frac{COE}{GDP} + \frac{COE}{GDP}\frac{MI}{GDP} \tag{8.25}$$

where COE stands for “compensation of employees” and MI stands for “mixed income”, both of which are published in the Economic & Labour Market Review [150]. This publication also gives historical data.

Figure 12 shows θ for the UK from 1955. The average value over this time period is 0.603, and the standard deviation is 0.032. However, as can be clearly seen, there is a large peak at 1975, which began in the early 70's and returned to normal levels during the 80's. This period corresponds to a period of great industrial unrest in the UK. The period from 1984 to present is more stable, and is judged to be a better indicator of the future than the period 1955 to present. Consequently it will be this time series that will be used to calculate θ. The average value for this period is 0.573 and the standard deviation is 0.012, as shown in Figure 13. The coefficient of variation, or relative standard deviation, for θ is thus 2%.

8.6 Work-Life Parameters and Risk Aversion, ε

Chapter 6 discussed the methods for determining the average work-life expectancy, y_w, and the work time fraction, w_0. The only required parameters for estimating y_w were the total hours worked per week in the population, and the size of the population. The total time worked per week can be estimated from the ONS publication "Labour Market Statistics", [146], which is published regularly. Data for 2009 indicate that there were 913 million hours worked per week, on average. The size of the population has already been discussed as being 61.8 million. Using equation (6.19), the average work-life expectancy is 3.5 years. The work time fraction is then this number divided by the average life expectancy. However, rather than using a present value, the work-time fraction is time averaged over the same period as for θ. This is because this parameter has remained remarkably constant over recent decades. Historical data from Labour Market Statistics and the Interim Life Tables can be used to estimate the past values. Life expectancy has increased linearly over this period, whilst the average work-life expectancy has fluctuated between 3.4 to 3.8 years. The average value for the work-time fraction for the period from 1984 to present is 0.091, and the standard deviation is 0.002, so that the coefficient of variation is about 2%. The time series is shown in Figure 13.

The risk aversion coefficient, ε, can then be calculated from equation (3.41). As time series data for both w_0 and θ have been determined, the corresponding risk aversion figures can also be determined over this period. These values are shown in Figure 13.

The risk aversion appears to be quite stable, with a mean value of 0.825, and a standard deviation of 0.005. The coefficient of variation is therefore 0.6%. As the risk aversion is used in the J-value equation, the tolerance limits will be analysed for this parameter. As the standard deviation is known, all that is required in order to place these limits is the distribution. A null hypothesis is formed that the data is distributed normally. This hypothesis is then tested using a normal-quantile plot.

A normal quantile plot compares the observed dataset against the data that would be seen if it were normally distributed. The observed data is first sorted by rank order, and the cumulative proportion is then calculated. The cumulative proportion is denoted *p*. This is then plotted against the quantile function, z_p, defined as:

$$z_p = \Phi^{-1}(p) \tag{8.26}$$

Where $\Phi^{-1}(p)$ is the inverse cumulative distribution function of the normal distribution, and is the value that would be observed at the *p*th quantile for a normally distributed random variable with mean of zero and standard deviation of unity. $\Phi(p)$ is hence defined as:

$$\Phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{p} e^{-\frac{x^2}{2}} dx \tag{8.27}$$

To test whether the null hypothesis can be rejected, a relevant test statistic is computed, which can then be compared to a critical value at a given level of significance. If the test statistic is less than the critical value, the null hypothesis may be confidently rejected. The relevant test statistic in this case is the correlation coefficient, which measures how closely the data and the z_p value change together. If the correlation coefficient were unity, the distribution would be perfectly normal. Table 4 shows the results obtained with the observed dataset for the risk aversion from 1984 to present. The correlation coefficient is 0.976. The significance level for this test is 5%, and the critical value at this level is 0.957, meaning that correlation coefficients below this value would be sufficient to reject the null hypothesis of normally distributed data. Hence, as the correlation coefficient was found to be greater than the critical value, so the null hypothesis may not be rejected. Therefore, it may be inferred that the risk aversion is distributed normally, with a probability of less than 5% that the distribution occurred by chance. Table 5 presents these results. The normal-quantile plot is shown in Figure 14.

For the normal distribution, the 95% tolerance limits lie at ±1.96σ from the mean. The 95% tolerance interval for the risk aversion is then 0.814-0.835.

8.7 Change in Discounted Life Expectancy, δX_d

As discussed in section 8.1, the J-value parameters can be classed as either "context-dependent" or "context-independent", with the former referring to parameters that cannot be determined without prior knowledge of the specifics of the safety system, and the latter referring to those that can. Up until now, this section has been concerned with the estimates of the context-independent parameters. The change in discounted life expectancy, however, is an example of a context-dependent parameter. Chapter 5 details the methods that can be used in order to estimate this parameter. The unknown variables for these calculations are the exposure rate at time x, $b(x)$, the length of time which the exposure lasts for, T_R, and the probability density of the response of the exposure y years after the exposure, $f_T(y)$. Also required is knowledge of whether the risk causes an absolute or relative increase in the initial hazard rate. In section 5.8, some "limiting distributions" were introduced in order to provide some simplified calculations. The limiting distributions used were when the exposure and response functions were short, and when they were long and uniform. The shortest change in life expectancy, which follows from a short exposure with a short response, was found to be:

$$\begin{aligned} \delta X &= bX \quad \text{for absolute risks} \\ &= bH \quad \text{for relative risks} \end{aligned} \tag{8.28}$$

These may also be discounted following the procedure laid out in section 5.11. For similar values of b, the relative risk equation will be smaller than for absolute risks, as $H < X$ (see section 8.4). However, smaller change in life expectancies may also be achieved when the response is delayed, for example with radiation risks, where the response does not become active until ten years after the initial exposure. Upper limits of the change in life expectancy would correspond to long exposures and long responses. It was shown in section 5.8 that, for such situations, the change in life expectancy is proportional to the second and third moment of the population distribution. However, an upper limit for the change in life expectancy may more easily be defined as the initial life expectancy itself, i.e.:

$$\delta X = X \tag{8.29}$$

This is because, in the worst case situation, when instant death occurs, the group of individuals will lose all their life expectancy they had remaining. In such situations, it may be inappropriate to use the equations of section 5.8, because it was assumed that the exposure rate, $b(x)$, was small enough so that the additional survival probability could be approximated with a linear expansion. In situations where there is large loss of life, this assumption will no longer be appropriate, and so the original equations must be used. The loss of accuracy in the life expectancy calculations from using in the linear expansion, for different exposure rates, is investigated in chapter 9.

Thus, although it is not possible to give exact calculations of the change in life expectancy following a hazard exposure without the specific details of the risk, it is possible to give indicative ranges of what the change in life expectancy may be. A lower bound of δX for situations in which there is an immediate one-off exposure with an immediate short response (which may correspond to being in the vicinity of some large explosion, for example), is given by equation (8.28). However, if the risk will result in a response with some delay, such as is the case with radiation exposures, then the change in life expectancy may be lower than this bound. If the delay is sufficiently long enough, there will be no change in life expectancy at all, so that the lower bound for delayed risks is zero. The upper bound for the change in life expectancy is simply the initial life expectancy, X. Introducing discounting can be done as described in section 5.11, but does not pose any additional complications. For example, the upper bound is reduced from X to X_d.

It is also not possible to determine the tolerance limits exactly for the change in life expectancy, unless information of the specific risk is available. Nevertheless, it is also possible to determine a "limiting uncertainty" for this parameter, by making a few assumptions. The assumptions are conservative, so that the uncertainty will tend to be overestimated, rather than underestimated. The method for determining this "limiting uncertainty" will now be described.

Let the frequency of the accident be λ per year. The Poisson distribution gives the probability, $p_\lambda^{(y)}$, of *y* such accidents occurring in the time-interval of length, *T*, as:

$$p_\lambda^{(y)} = \Pr(Y = y) = \frac{e^{-k} k^y}{y!} \quad \text{for } y = 0, 1, 2 \ldots \tag{8.30}$$

where *Y* is the random number of accidents, and *k* is the expected number of accidents in the interval:

$$k = E(Y) = \lambda T \tag{8.31}$$

From (8.30) and (8.31), the probability of no accidents in the interval (so that *Y* = *y* = 0) is:

$$\Pr(Y = 0) = p_\lambda^{(0)} = e^{-\lambda T} \tag{8.32}$$

Hence the probability of one or more accidents in the interval is given by Pr(*Y* ≥ 1), where:

$$\Pr(Y \geq 1) = p_\lambda^{(1+)} = 1 - p_\lambda^{(0)} = 1 - e^{-\lambda T} \tag{8.33}$$

Let us assume that the probability of experiencing an early death as a result of the accident among the exposed group is p_d. Very often $p_d << 1$, especially when the group is large. For an individual in the exposed group, therefore, the probability of early death as a result of the accident is $p_\lambda^{(1+)} p_d$ because the probabilities are independent. This combined probability may be called the probability of being affected, p_{aff} :

$$p_{aff} = p_{\lambda}^{(1+)} p_d \tag{8.34}$$

For simplicity, consider a protection system that eliminates completely the chance of the accident. Let the improvement in lifetime for an individual of age *a*, brought about by the protection system be $\delta\chi(a)$. Clearly, $\delta\chi(a)$ will depend on many random hazards the individual faces apart from the specific accident being prevented, and so will be a random number. It may not be a small quantity: its value could be 80 years or more when an infant is being protected.

Let us consider an accident where death, if it is to occur, is immediate, coincident with the accident. This could apply to an explosion on a petrochemical plant, for example. This risk would be described by a point response function with an instant response, as was discussed in section 5.8 and previously in this section. In such a case, the installation of the protection system will have the effect of restoring the life to come amongst those who would otherwise experience immediate death to its value in the absence of the accident. In this first group of potentially affected people, an individual of age *a*, will experience a change in life to come:

$$\delta\chi(a)\big|_1 = \{\chi | a < A^* \leq a + 1\} = \chi(a) \tag{8.35}$$

where the notation follows that used in section 8.4, i.e. where $\delta\chi(a)$, $\chi(a)$ and A^* are random numbers.

The second group of unaffected people will contain some members who have the same age, *a*, and who would have survived the accident unscathed. For them, there is no change in life to come, and so:

$$\delta\chi(a)\big|_2 = 0 \tag{8.36}$$

The expected value of the first group's change in life to come is:

$$E\left[\delta\chi(a)\big|_1\right] = E\left[\chi \middle| a < A^* \le a+1\right] = E\left[\chi(a)\right] \tag{8.37}$$

while the expected value of the second group's change in life to come is:

$$E\left[\delta\chi(a)\big|_2\right] = E[0] = 0 \tag{8.38}$$

Any given individual in the potentially exposed cohort of people (for example those living near a factory producing toxic chemicals) will have a probability, p_{aff}, of being in the first group and a probability, $1 - p_{aff}$, of being in the second group. This probability is also equal to the ratio of number of eventual deaths from the accident, Λ, to the total number of people exposed to the accident, N, i.e.: Λ/N. This quantity may also be seen to be the integrated exposure rate, b_{tot}, of equation (5.3), which is the probability of death following an exposure. In this situation, where the exposure occurs at a single point, the integrated exposure rate is equal to the single exposure rate, b. Therefore the expected value, $\delta X(a)$, of the life to come of an individual of age a, is given by:

$$\begin{aligned} \delta X(a) &= E\left[\delta\chi(a)\right] \\ &= p_{aff} E\left[\delta\chi(a)\big|_1\right] + \left(1 - p_{aff}\right) E\left[\delta\chi(a)\big|_2\right] \\ &= p_{aff} E\left[\chi(a)\right] + \left(1 - p_{aff}\right) \times 0 \\ &= p_{aff} E\left[\chi(a)\right] \\ &= p_{aff} X(a) \\ &= bX(a) \end{aligned} \tag{8.39}$$

which is the same as the change in life expectancy found in the limiting case of a point exposure and short response found in section 5.8, equation (5.23). However, if we do not know the age of the randomly selected individual, our best estimate of his change in life to come, $\delta\chi$, will be the weighted, average value, δX, over all ages:

$$
\begin{aligned}
\delta X &= E[\delta\chi] = \underset{a}{E}[\delta X_a] = \underset{a}{E}[bX(a)] \\
&= \int_0^\infty p(a) bX(a) da \\
&= bX
\end{aligned}
\tag{8.40}
$$

which confirms equation (5.25).

The same arguments apply to the square of change in life to come. Individuals in the first group of potentially people who have age *a*, will experience a squared change in life to come:

$$
\delta\chi^2(a)\big|_1 = \left\{\chi^2 \middle| a < A^* \le a+1\right\} = \chi^2(a) \tag{8.41}
$$

Individuals of the same age in the second group of unaffected people, who would have survived the accident unscathed, experience no change in life to come. Hence, for those of age *a*, the change in life to come and its square will both be zero:

$$
\delta\chi^2(a)\big|_2 = 0 \tag{8.42}
$$

The expected value of the first group's squared change in life to come is:

$$
E\left[\delta\chi^2(a)\big|_1\right] = E\left[\chi^2 \middle| a < A \le a+1\right] = E\left[\chi^2(a)\right] \tag{8.43}
$$

while the expected value of the second group's squared change in life to come is, of course zero:

$$
E\left[\delta\chi^2(a)\big|_2\right] = E[0] = 0 \tag{8.44}
$$

The expected value of the square of life to come of an individual of age *a*, is given by:

$$\begin{aligned}E\left[\delta\chi^2(a)\right] &= p_{aff}E\left[\delta\chi^2(a)\big|_1\right]+\left(1-p_{aff}\right)E\left[\delta\chi^2(a)\big|_2\right]\\ &= p_{aff}E\left[\chi^2(a)\right]+\left(1-p_{aff}\right)\times 0\\ &= p_{aff}E\left[\chi^2(a)\right]\\ &= bE\left[\chi^2(a)\right]\end{aligned} \tag{8.45}$$

If we do not know the age of the randomly selected individual, our best estimate of the square of his change life to come, $E\left[\delta\chi^2\right]$, will be the weighted, average value over all ages:

$$\begin{aligned}E\left[\delta\chi^2\right] &= \underset{a}{E}\left[\delta\chi^2(a)\right]=\underset{a}{E}\left[bE\left[\chi^2(a)\right]\right]\\ &= \int_0^{\infty} p(a)bE\left[\chi^2(a)\right]da\\ &= b\int_0^{\infty} p(a)E\left[\chi^2(a)\right]da\\ &= bE\left[\chi^2\right]\end{aligned} \tag{8.46}$$

where equation (8.12) has been used. The variance of random change in life to come for individuals selected at random in the population will be $\mathrm{var}[\delta\chi]$, given by:

$$\begin{aligned}\mathrm{var}[\delta\chi] &= E\left[\delta\chi^2\right]-\left(E[\delta\chi]\right)^2\\ &= E\left[\delta\chi^2\right]-\left(\delta X\right)^2\end{aligned} \tag{8.47}$$

Using equations (8.40) and (8.46), we may write:

$$\begin{aligned}\mathrm{var}[\delta\chi] &= bE\left[\chi^2\right]-b^2X^2\\ &= b\left(E\left[\chi^2\right]-bX^2\right)\end{aligned} \tag{8.48}$$

By equation (8.13):

$$E\left[\chi^2\right]=\mathrm{var}(\chi)+X^2 \tag{8.49}$$

Hence:

$$\mathrm{var}[\delta\chi] = b\left(\mathrm{var}(\chi) + X^2(1-b)\right) \tag{8.50}$$

In many cases, $b << 1$, and so:

$$\begin{aligned}\mathrm{var}[\delta\chi] &\approx b\left(\mathrm{var}(\chi) + X^2\right)\\ &= bt_{av}^2\end{aligned} \tag{8.51}$$

where equation (8.23) has been used. The fact that b will be non-negative means that for all possible values of $b : 0 \leq b \leq 1$, $\mathrm{var}[\delta\chi]$ will be bounded above by:

$$\mathrm{var}[\delta\chi] \leq t_{av}^2 \tag{8.52}$$

In the case where the protection system acts to avert a reduction in life to come rather than averting immediate death, once again there will be an affected group, Group 1, whose life to come would have been reduced in the absence of the protection system, and an unaffected group, Group 2, whose life to come would not have been affected whether or not the protection system was in place. The probability of being in Group 1 is p_{aff} and the probability of being in Group 2 is $1 - p_{aff}$. If the risk being averted is still a point exposure, then the exposure rate, b is still equal to p_{aff}, but the exposure now refers to some delayed risk, for example, radiation, in which case, $b = c_T d_r$, where c_T is the risk coefficient, and d_r is the dose received, see equation (5.45).

Consider those of age, a, in Group 1. The installation of the protection system will avert their loss of part of their life to come, so that:

$$\delta\chi(a)\big|_1 = \left\{(R_r\chi) \middle| a < A^* \leq a+1\right\} \tag{8.53}$$

where R_r may be termed the restoration requirement, and will be a random number bounded in (0,1):

$$0 \le R_r \le 1 \tag{8.54}$$

The life to come will be conditioned by the age *a*, and so, in the most general case, will the restoration requirement. For example the same dose of toxin might reduce the life to come of people of different ages by the same absolute amount, leading to a different fractional reduction in life to come. The restoration requirement has the same numerical value as that fractional reduction, and so would be different for people of different ages in this case. However, once age *a*, is specified the two parameters may reasonably be regarded as independent of each other. In the case considered, it is asserted that sensitivity to the same toxin amongst individuals of the same age would not be related generally to how long those individuals will live, which will be conditioned by a very large range of independent factors: occupation, marital status, hobbies, consumption of alcohol etc. Hence:

$$\begin{aligned}\{(R_r\,\chi)|a < A^* \le a+1\} &= \{R_r|a < A^* \le a+1\} \times \{\chi|a < A^* \le a+1\} \\ &= R_r(a)\chi(a)\end{aligned} \tag{8.55}$$

where $R_r(a): 0 \le R_r(a) \le 1$ is the restoration requirement appropriate for age *a*. Hence:

$$\delta\chi(a)|_1 = R_r(a)\chi(a) \tag{8.56}$$

The expected value of change in life to come for those of age *a* in the first group is:

$$E[\delta\chi(a)|_1] = E[R_r(a)\chi(a)] = E[R_r(a)]E[\chi(a)] \tag{8.57}$$

while the expected value of the second group's change in life to come is:

$$E[\delta\chi(a)|_2] = E[0] = 0 \tag{8.58}$$

The expected value, $\delta X(a)$, of the life to come of an individual of age, *a*, is given by:

$$
\begin{aligned}
\delta X(a) &= E[\delta\chi(a)] \\
&= p_{aff} E\left[\delta\chi(a)\big|_1\right] + \left(1 - p_{aff}\right) E\left[\delta\chi(a)\big|_2\right] \\
&= p_{aff} E[R_r(a)] E[\chi(a)] + \left(1 - p_{aff}\right) \times 0 \\
&= p_{aff} E[R_r(a)] E[\chi(a)] \\
&= bE[R_r(a)] E[\chi(a)]
\end{aligned}
\tag{8.59}
$$

However, if we do not know the age of the randomly selected individual, our best estimate of his change life to come, $\delta\chi$, will be the weighted, average value, δX, over all ages:

$$
\begin{aligned}
\delta X &= E[\delta\chi] = \underset{a}{E}[\delta X(a)] = \underset{a}{E}\left[bE[R_r(a)]E[\chi(a)]\right] \\
&= \int_0^\infty p(a) bE[R_r(a)] X(a) da \\
&= b\int_0^\infty p(a) E[R_r(a)] X(a) da
\end{aligned}
\tag{8.60}
$$

For the square of the change in life to come, individuals in the first group of people who have age *a*, will experience a squared change in life to come given by:

$$
\delta\chi^2(a)\Big|_1 = \left\{(R_r\chi)^2 \middle| a < A^* \le a+1\right\} = R_r^2(a)\chi^2(a) \tag{8.61}
$$

since the squares of independent random variables will also be independent.

Meanwhile, those of the same age in the second group of unaffected people will experience no change in life to come. Hence, the square of change in life to come for them is zero, whatever their age:

$$
\delta\chi^2(a)\Big|_2 = 0 \tag{8.62}
$$

The expected value of the first group's squared change in life to come may be written:

$$E\left[\delta\chi^2(a)\big|_1\right] = E\left[R_r^2(a)\chi^2(a)\right] = E\left[R_r^2(a)\right]E\left[\chi^2(a)\right] \tag{8.63}$$

while the expected value of the second group's squared change in life to come is, of course zero:

$$E\left[\delta\chi^2(a)\big|_2\right] = E[0] = 0 \tag{8.64}$$

As the probability of being in the affected group is p_{aff}, the expected value of the square of life to come of an individual of age *a*, is given by:

$$\begin{aligned}
E\left[\delta\chi^2(a)\right] &= p_{aff}E\left[\delta\chi^2(a)\big|_1\right] + \left(1 - p_{aff}\right)E\left[\delta\chi^2(a)\big|_2\right] \\
&= p_{aff}E\left[R_r^2(a)\right]E\left[\chi^2(a)\right] + \left(1 - p_{aff}\right)\times 0 \\
&= p_{aff}E\left[R_r^2(a)\right]E\left[\chi^2(a)\right] \\
&= bE\left[R_r^2(a)\right]E\left[\chi^2(a)\right]
\end{aligned} \tag{8.65}$$

If we do not know the age of the randomly selected individual, our best estimate of the square of his change life to come, $E\left[\delta\chi^2\right]$, will be the weighted, average value over all ages:

$$\begin{aligned}
E\left[\delta\chi^2\right] &= \underset{a}{E}\left[\delta\chi^2(a)\right] = \underset{a}{E}\left[bE\left[R^2(a)\right]E\left[\chi^2(a)\right]\right] \\
&= b\int_0^\infty p(a)E\left[R^2(a)\right]E\left[\chi^2(a)\right]da
\end{aligned} \tag{8.66}$$

The variance of random change in life to come, $\delta\chi$, for individuals selected at random in the population will be $\mathrm{var}[\delta\chi]$, given by equation (8.47).

Using equations (8.60) and (8.66), we may write:

$$\mathrm{var}[\delta\chi] = b\int_0^\infty p(a)E\left[R_r^2(a)\right]E\left[\chi^2(a)\right]da - b^2\left(\int_0^\infty p(a)E[R_r(a)]X(a)da\right)^2 \tag{8.67}$$

Now the variance, $\mathrm{var}[R_r(a)]$, is given by:

$$\mathrm{var}[R_r(a)] = E\left[R_r^2(a)\right] - \{E[R_r(a)]\}^2 \tag{8.68}$$

so that:

$$E\left[R_r^2(a)\right] = \mathrm{var}[R_r(a)] + \{E[R_r(a)]\}^2 \tag{8.69}$$

An analogous route leads to:

$$E\left[\chi^2(a)\right] = \mathrm{var}[\chi(a)] + \{E[\chi(a)]\}^2 \tag{8.70}$$

Substituting from equations (8.67) and (8.69) into equation (8.70) gives:

$$\mathrm{var}[\delta\chi] = b\int_0^\infty p(a)\left(\mathrm{var}[R_r(a)] + \{E[R_r(a)]\}^2\right)\left(\mathrm{var}[\chi(a)] + \{E[\chi(a)]\}^2\right)da - b^2\left(\int_0^\infty p(a)X(a)E[R_r(a)]da\right)^2 \tag{8.71}$$

For the case where the protection system averts immediate death for those in the affected, first group, the restoration requirement is equal to unity, since all life to come is restored:

$$R_r(a) = 1 \text{ for all } a \tag{8.72}$$

This is deterministic, with:

$$
\begin{aligned}
&E[R_r(a)] = 1 \\
&E[R_r^2(a)] = 1 \\
&\mathrm{var}[R_r(a)] = 0
\end{aligned}
\tag{8.73}
$$

In this case, equation (8.67) defaults to equation (8.50), as we would expect. Since the last term in equation (8.71) must be positive, we may conclude that:

$$
\mathrm{var}[\delta\chi] \le b\int_0^\infty p(a)\left(\mathrm{var}[R_r(a)] + \{E[R_r(a)]\}^2\right)\left(\mathrm{var}[\chi(a)] + \{E[\chi(a)]\}^2\right)da \tag{8.74}
$$

Because $R_r(a)$ is bounded on (0,1), it follows that the absolute maximum value of $\mathrm{var}[R_r(a)]$ is $1/4$, see Jacobsen (1969) [116]. This is based on the distribution being bimodal, and concentrated at the extreme values. The same paper demonstrates that the maximum variance of a unimodal distribution on (0,1) is $1/9$. Meanwhile, it is immediately clear that the maximum value of $E[R_r(a)]$ and hence $\{E[R_r(a)]\}^2$ is 1.0.

Using these figures makes it clear that $\mathrm{var}[\delta X]$ is bounded above for all possible probability distributions for restoration requirement, $R_r(a)$, for all values of age, *a*, by:

$$
\begin{aligned}
\mathrm{var}[\delta\chi] &\le \frac{5}{4}b\int_0^\infty p(a)\left(\mathrm{var}[\chi(a)] + \{E[\chi(a)]\}^2\right)da \\
&= \frac{5}{4}b\int_0^\infty p(a)E[\chi^2(a)]da \\
&= \frac{5}{4}bE[\chi^2] \\
&= \frac{5}{4}b\left(\mathrm{var}[\chi] + X^2\right) \\
&= \frac{5}{4}bt_{av}^2
\end{aligned}
\tag{8.75}
$$

where equation (8.23) has been used in the last step. If the probability distribution for the restoration requirement is unimodal, then the upper bound condition is replaced by a slightly smaller value:

$$\mathrm{var}[\delta\chi] \le \frac{10}{9} b t_{av}^2 \tag{8.76}$$

The conditions (8.75) and (8.76) bear a strong similarity to condition (8.52) on the upper bound for $\mathrm{var}[\delta X]$ when the protection system is preventing an accident that would cause only immediate deaths if it occurred.

Because of the small increase that condition (8.75) brings over either of the other possible conditions, (8.76), it will be sufficient for most purposes to use the most conservative estimate of the limiting upper bound implied by condition (8.75), for which we shall use the terminology, "$\lim \mathrm{var}[\delta\chi]$":

$$\lim \mathrm{var}[\delta\chi] = \frac{5}{4} b t_{av}^2 \tag{8.77}$$

Thus using values calculated in section 8.4, i.e., t^2_{av} = 2,304 years^2, the limiting variance on the change in random life to come is 2,880b years^2. This may be compared with 2,560b years^2 if a unimodal distribution is used. Moreover, if immediate-death equation (8.51) is used then the variance on the change in random life to come is 2,304b years^2. Clearly the three figures are similar. Health and safety regulations state that, in the workplace, the probability of being killed in an accident must be no larger than 10^{-3} per year, but the figures are usually of the range 10^{-6} to 10^{-4} per year. Using these figures, the limiting variance on the change in life to come ranges from 0.003 to 3 years^2. The variance on the change in average life expectancy is then this variance divided by the number of people affected by the hazard. A typical workforce will number between 100 and 1,000. The variance in the change in life expectancy, var[δX], then ranges from $3\text{x}10^{-6}$ to 0.003 years^2. The standard deviation then ranges from 0.002 to 0.2 years. Compared to the initial change in life expectancy calculated from such hazard rates, these numbers are large. The coefficient of variation is around 400 to 4,000%. The distribution will also be normal, as the figures are determined from summing together the change in life to come of a number of people. However, because the numbers presented here are only illustrative, no tolerance limits will be placed on the change in average life expectancy parameter. It is sufficient to note that, unless there are a very large number of people affected by the hazard (in excess of 100,000), the tolerance interval will be relatively wide, when compared to the central change in life expectancy. However, in absolute terms, the interval will usually be fairly small.

8.8 Other Context-Dependent Parameters

In addition to the change in average life expectancy, there are two other parameters which are dependent upon the specific nature of the safety system. These are the number of people benefitting from the system, N, and the cost of the protection system, $\delta\hat{V}_N$.

In J-value analysis it is often the case that the number of people affected by the safety system does not need estimation. This is because the change in life expectancy is proportional to the hazard rate, which itself is inversely proportional to the number of people affected, as shown, for example, in equations (5.1) and (5.25). Thus the product of the number of people affected and the change in life expectancy is approximately independent of N. This parameter therefore will usually not contribute any significant uncertainty to the J-value.

The cost of the safety system is assumed to be provided in the details of the safety system itself. An alternative formulation, however, may be to investigate the range of acceptable costs that would still give J-values less than or equal to unity. Little can be said about the uncertainty of the cost of the safety system, except that it is unbounded, being potentially very large. It is therefore important when conducting J-value analyses that some kind of indication of how variation in the cost would affect the results is given. Alternatively, an indication can be given for the permitted variation in the cost estimate that would still maintain a reasonable J-value.

8.9 The J-Value

The J-value is given by equation (3.61), repeated below:

$$J = \frac{\delta \hat{V}_N}{\delta V_N} = \frac{(1-\varepsilon)\delta \hat{V}_N}{NG\delta X} \frac{r_d X_d}{\left(1-e^{-r_d X_d}\right)} \quad \text{for } r_d > 0$$
$$= \frac{(1-\varepsilon)\delta \hat{V}_N}{NG\delta X} \quad \text{for } r_d = 0 \tag{3.60}$$

This can be simplified by noting that, for small r_d:

$$\frac{r_d X_d}{1-e^{-r_d X_d}} \approx 1 + \frac{r_d X_d}{2} \tag{8.78}$$

putting:

$$D_f = 1 + \frac{r_d X_d}{2} \tag{8.79}$$

allows the J-value to be re-written as:

$$J = \frac{(1-\varepsilon)\delta \hat{V}_N}{NG\delta X_d}\left(1 + \frac{r_d X_d}{2}\right)$$
$$= \frac{(1-\varepsilon)\delta \hat{V}_N}{NG\delta X_d} D_f \tag{8.80}$$

Which is valid for all r_d, and D_f is termed the “linearised discount factor”. The methods and results of measuring each of the parameters in the above equation have been laid out in the preceding sections. The uncertainties, which result from either the measurement process itself, or from the natural variation of the parameters, have also been quantified as far as is possible. These individual uncertainties will then propagate through the J-value calculation to give an uncertainty on the J-value itself. As has been discussed, it is not possible to determine the uncertainty from the context-dependent parameters – the change in life expectancy, the number of people affected, and the cost of the safety system – although an indication of the magnitude of the uncertainty on the change in life expectancy was given in section 8.7. A full analysis of the uncertainty of the J-value therefore cannot be given without details of the protection system. However, it is possible to provide an analysis of the "intrinsic uncertainty" of the J-value. This is the uncertainty resulting from the context-independent parameters. This is then a minimum level of uncertainty that will always be present in any J-value estimate, which will increase once knowledge of the uncertainties of the context-dependent parameters is achieved. Intrinsic uncertainty on the J-value will result from uncertainty on the estimate of the GDP per person, G, the risk aversion, ε, and the discount factor, D_f, which itself results from uncertainty on the discounted average life expectancy, X_d. The standard deviation on the J-value is then given by the weighted sum-of-squares method:

$$\sigma_J = \sqrt{\left(\frac{\partial J}{\partial \varepsilon}\right)^2 \sigma_\varepsilon^2 + \left(\frac{\partial J}{\partial G}\right)^2 \sigma_G^2 + \left(\frac{\partial J}{\partial D_f}\right)^2 \sigma_{D_f}^2} \tag{8.81}$$

which can be written as:

$$\frac{\sigma_J}{J} = \sqrt{\left(\frac{\sigma_\varepsilon}{1-\varepsilon}\right)^2 + \left(\frac{\sigma_G}{G}\right)^2 + \left(\frac{\sigma_{D_f}}{D_f}\right)^2} \tag{8.82}$$

note the presence of the $1 - \varepsilon$ term in the denominator of the first term on the right hand side of the equation. This equation therefore gives the coefficient of variation, or the relative standard deviation of the J-value. In order to place tolerance limits, it is necessary to determine the distribution of the J-value. However, this has not been possible, as the uncertainty results from the product of three variables, two of which are taken as having a normal distribution, and the third of which is taken as having the ratio distribution. The variables all have different means and standard deviations. The distribution of such a random product does not appear to have been studied before. It would be possible to infer a distribution via simulation, but this has not been attempted, and remains for further work. Instead, it will be assumed that 95% coverage of the distribution can be achieved with ±2 standard deviations about the mean, i.e. assuming that the distribution approximates the normal distribution.

The uncertainty on the discount factor can be expressed as:

$$\begin{aligned}\sigma_{D_f} &= \left(\frac{\partial D_f}{\partial X_d}\right)\sigma_{X_d} \\ &= \frac{r_d}{2}\sigma_{X_d}\end{aligned} \tag{8.83}$$

It was shown in section 8.4 that the standard deviation on the life expectancy was 0.003 years. The discount rate is not assumed to contribute any uncertainty, and so this value will also be true of the discounted life expectancy. For a value of r_d of 5%, which represents a maximum discount rate that would be used, the standard deviation on D_f is then $8x10^{-5}$. The associated coefficient of variation is 0.004%, which is clearly small. The minimum value is when r_d is zero, in which case the discount factor is also zero, and there is no uncertainty.

The above results can then be used to determine the uncertainty on the J-value. Because of the fact that uncertainties are combined in a sum-of-squares manner, the sum is dominated by the largest value, which in this case is the risk aversion term. The GDP per person and the discount factor both produce uncertainties that are negligible, and so can be disregarded from the calculation. The uncertainty on the J-value is then:

$$\begin{aligned}\frac{\sigma_J}{J} &= \frac{\sigma_\varepsilon}{1-\varepsilon} = \frac{0.005}{1-0.825} \\ &= 2.86\%\end{aligned} \tag{8.84}$$

The "internal accuracy" of the J-value has thus been found to be 2.86%. The 95% tolerance interval, which is taken as two standard deviations, is ±5.7%. However, the other case dependent input parameters may also contribute to this uncertainty. If it is possible to assess the uncertainty of the change in life expectancy, then the correlation between this parameter and the initial life expectancy (which will be present in the J-value equation for non-zero discount rates), also needs to be accounted for. The method for accounting for correlations has already been discussed in section 8.1. As the change in life expectancy is approximately linearly dependent upon the initial life expectancy (c.f. equation (5.25)), the correlation coefficient between these two parameters is unity.

8.10 The VTPF, VODLY and VODLYA

Chapter 7 showed how the J-value framework could be used to derive valuations of human lifespan. This was done by first deriving the value of delaying a fatality by some arbitrary number of years. The value of temporarily preventing a fatality (VTPF) is then a specific instance of this, when the delay is set equal to the life expectancy of the individual concerned. This then corresponds to a situation in which a hazard that will cause immediate death to an individual is permanently eliminated, so that the individual regains his or her initial life expectancy. The VTPF, which is therefore age-dependent, is denoted as $V_P(a)$, and is given by equation (7.12). It will be assumed in this section that $J = 1$ is used in the valuations. It was also shown that two average values of the VTPF may be derived, one evaluated at $X_d(a) = X_d$, which may be the case when age is not known, and another one in which the $V_P(a)$ values are averaged over the population, as given by equations (7.13) and (7.14). These two averages were shown to be equal at a 0% discount rate. Using the numbers presented throughout this section, the average VTPF at a 0% discount rate is calculated as about £5.30M. At a 2.5% discount rate, the average VTPF when age is not known is £2.54M, and the population-averaged VTPF is £2.49M. These two average measures are therefore close. Figure 15 shows the average values of the VTPF, and the age dependencies at these two discount rates.

Also derived was the value of a discounted life-year (VODLY), and a related measure, the VODLYA, which is the average value of a discounted life-year over an individual's remaining life. For zero discount rate, both the VODLY and VODLYA are equal and constant, valued simply as $G/(1 - \varepsilon)$, which is about £129,000. For non-zero discount rates, the VODLY depends on which year of an individual's life is being saved. For example, if it is the next year of life that will be saved, then the value is simply equal to the undiscounted VODLY. However, if the year of life that will be saved is some time in the future, then the value will be discounted, and so will be slightly less than the first-year value. The VODLYA is also age-dependent if the discount rate is non-zero. At age zero, the VODLYA has the smallest value, as there are the maximum possible number of years over which to discount. The VODLYA returns to the undiscounted value by the maximum age, when there are no more life-years to discount over. These values are shown in Figure 16.

As the VTPF, VODLY and VODLYA are not inputs to the J-value, no analysis of the associated tolerance limits has been performed. However, the largest contribution to the uncertainty will come from the risk aversion coefficient, ε, as it did with the J-value, with the other parameters contributing a negligible uncertainty. Hence, the coefficient of variation for each of the three valuations of life described above will be 2.87%. As with the J-value, the distribution is not known, and so the tolerance interval cannot be set.

The values of all the parameters described above are summarised in Table 6.

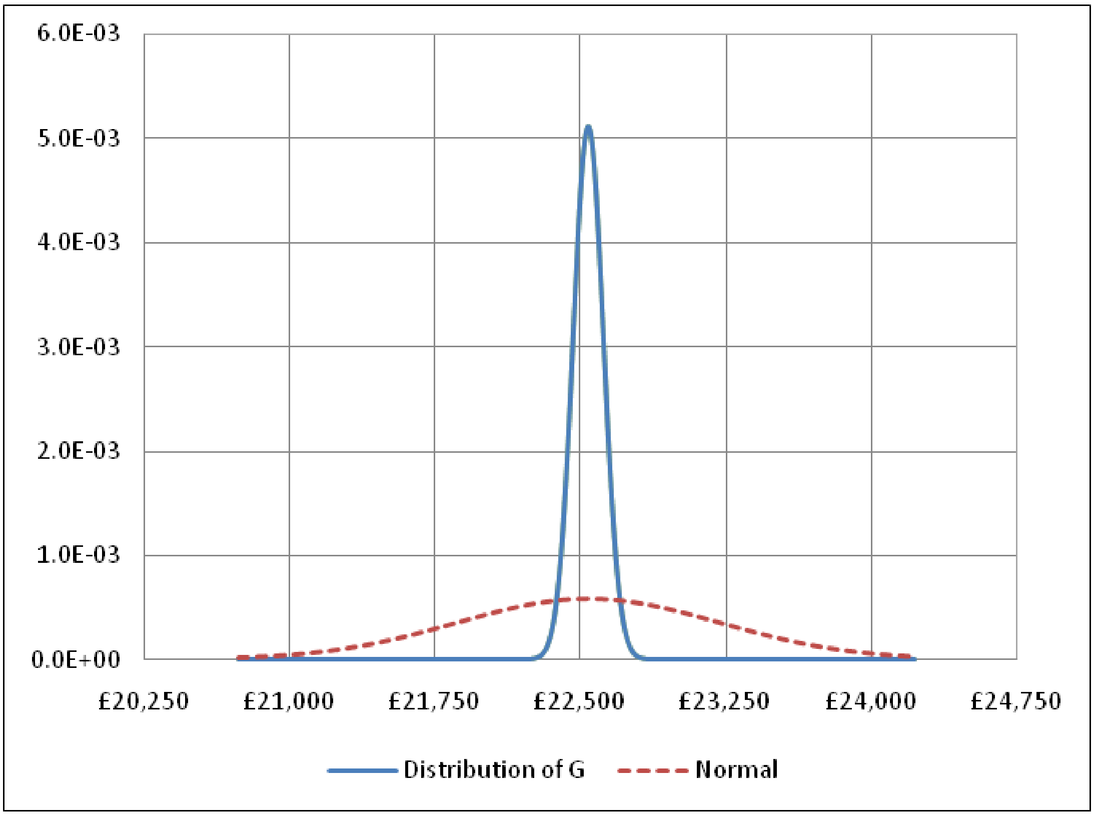

Figure 9 Probability distribution of the GDP per person estimate. Also shown is what the distribution would look like if it were normal.

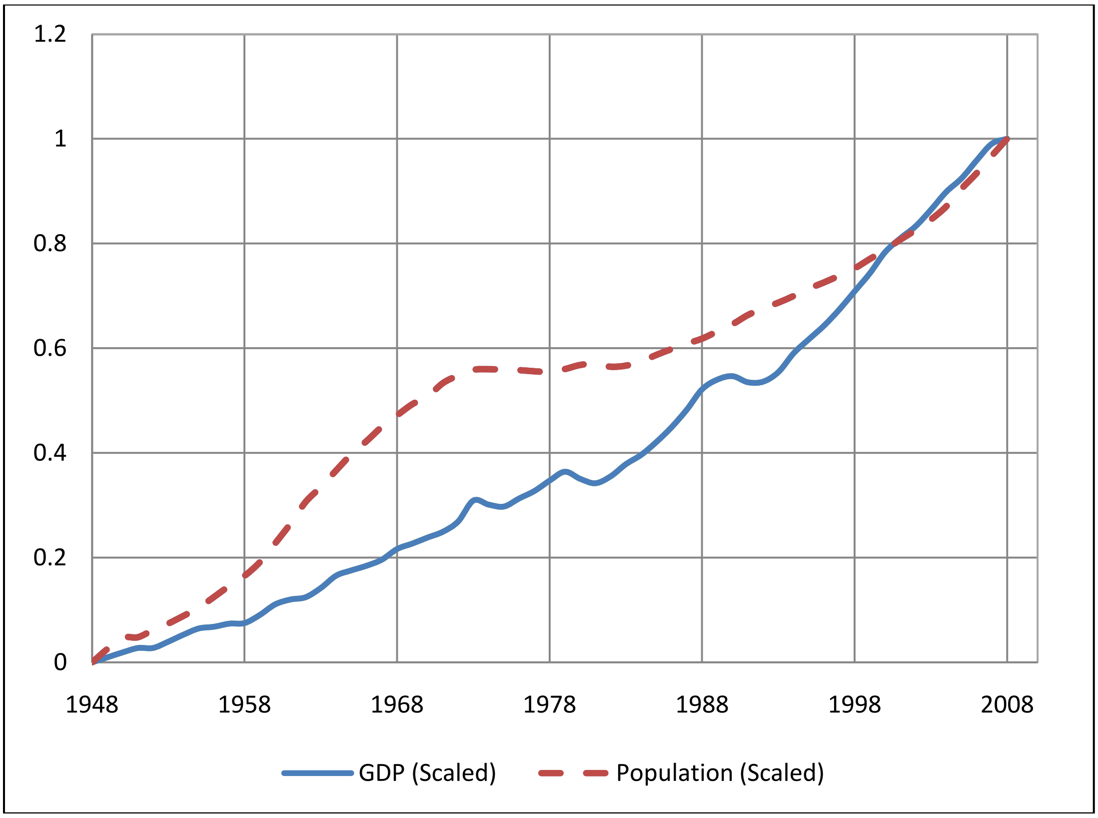

Figure 10 Historical data showing how the UK GDP and population size are correlated. Both are scaled to lie between 0 and 1. At zero, the GDP is about £0.3 billion and the population is about 50 million. At unity, the GDP is about 1.3 billion, and the population is about 62 million.

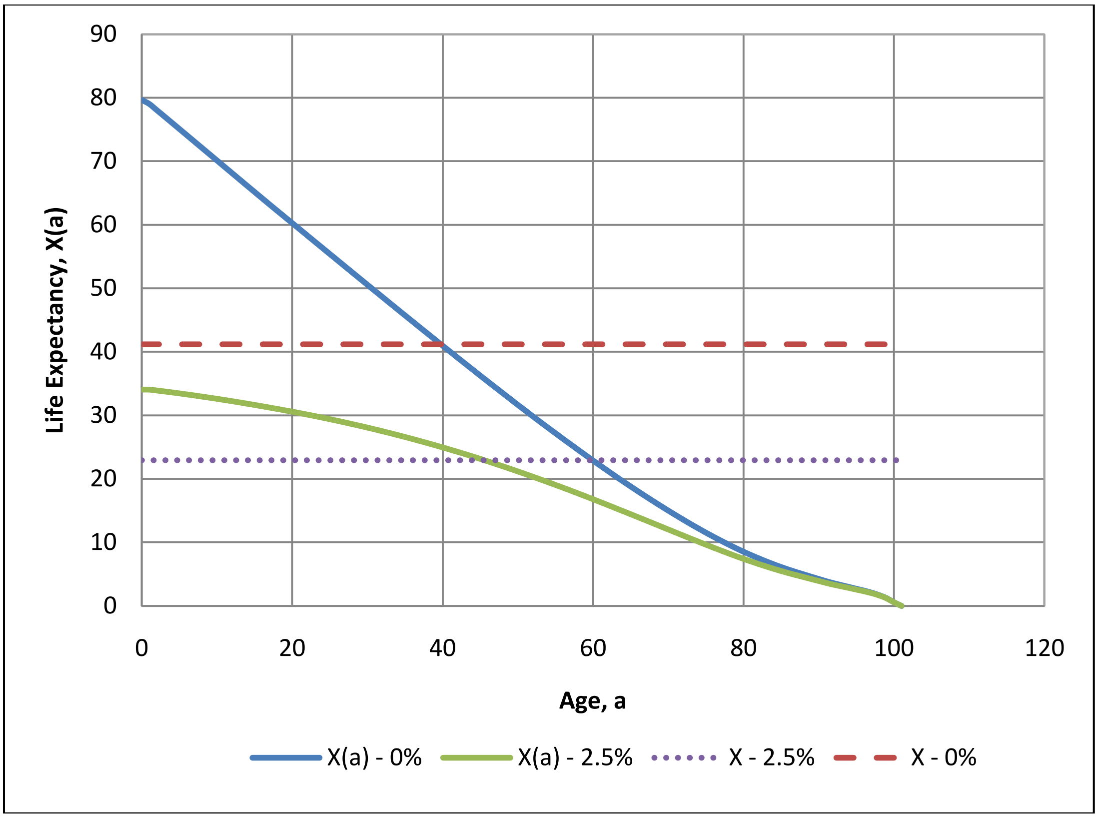

Figure 11 Life expectancy, X_d(a), and average life expectancy, X_d, for discount rates of 0% and 2.5%. Assumed 50% male female split at all ages. Average life expectancies are 41.2 and 22.9 years at 0% and 2.5% discount rate respectively.

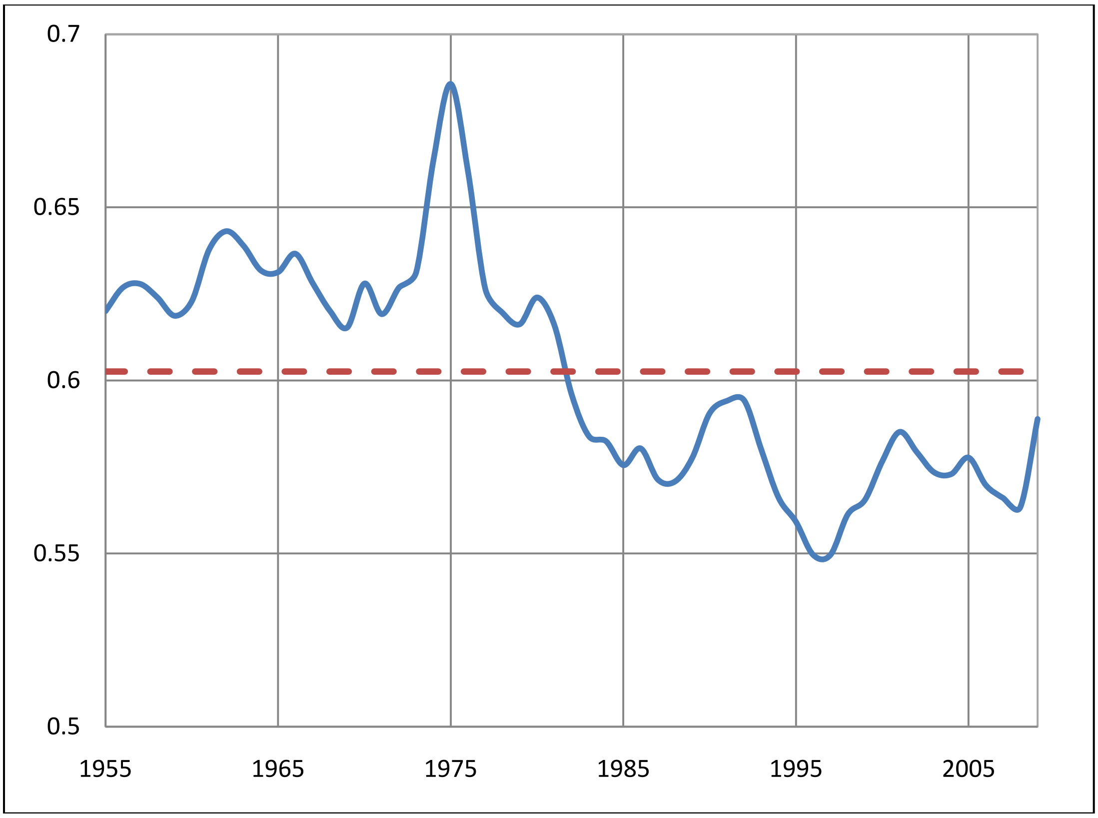

Figure 12 Historical data showing the variation in the wage share of the GDP, θ, for the UK from 1955. Note the large peak at 1975, during a period of considerable industrial unrest. During this period the mean wage share was 0.603, or about 60%, and the standard deviation was 0.032, so that the coefficient of variation is about 5%.

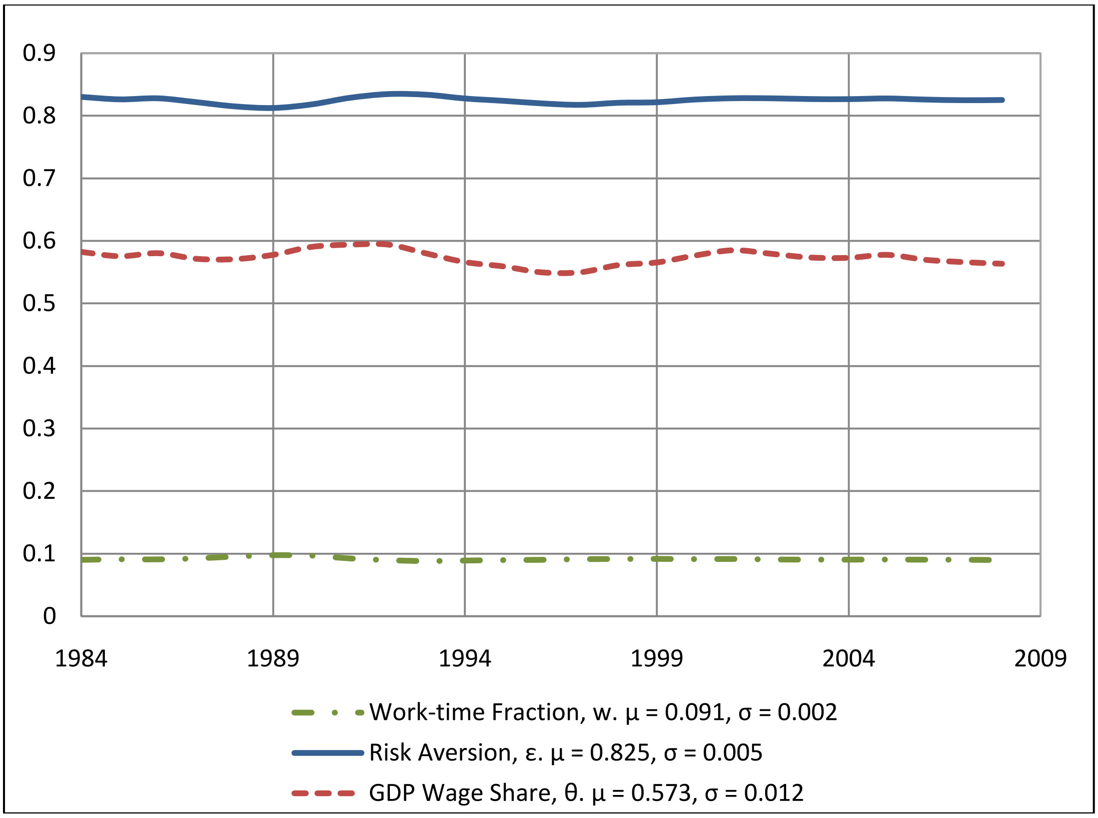

Figure 13 Time series data from the work time fraction, w_0, the wage share of the GDP, θ, and the risk aversion, ε, for available data from 1984 to 2008.

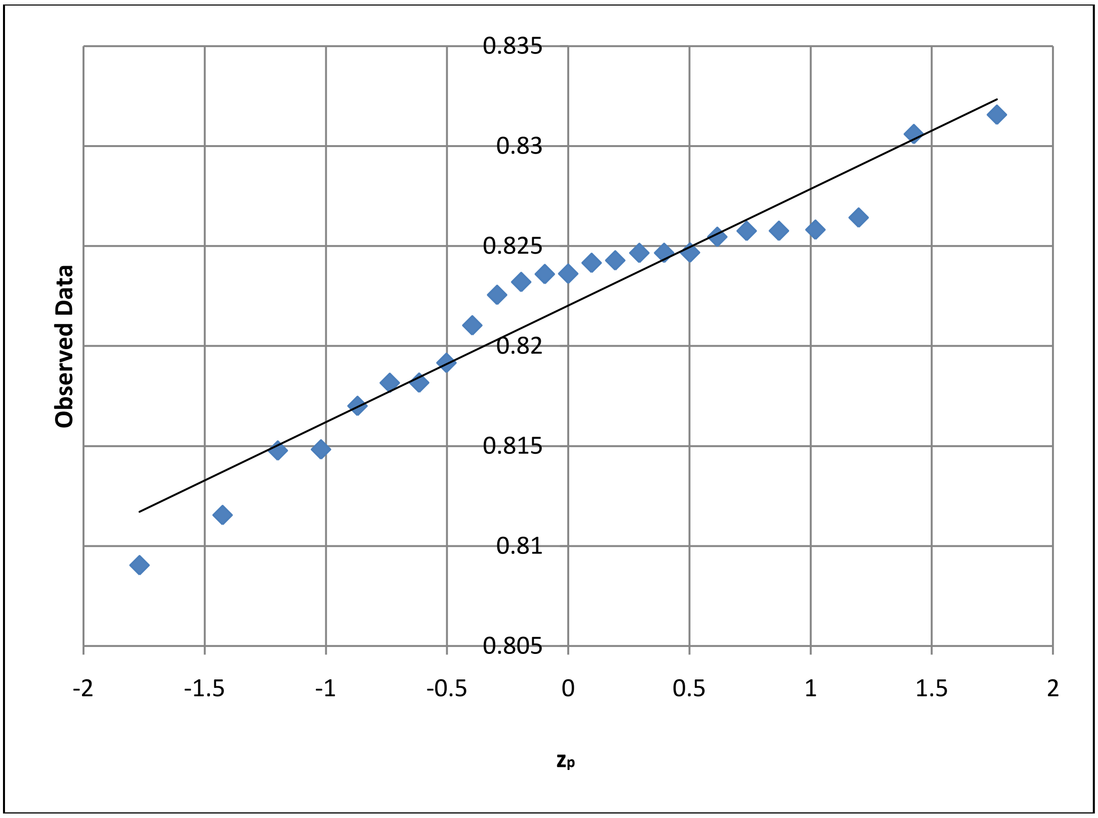

Figure 14 Normal-quantile plot for risk aversion normality test.

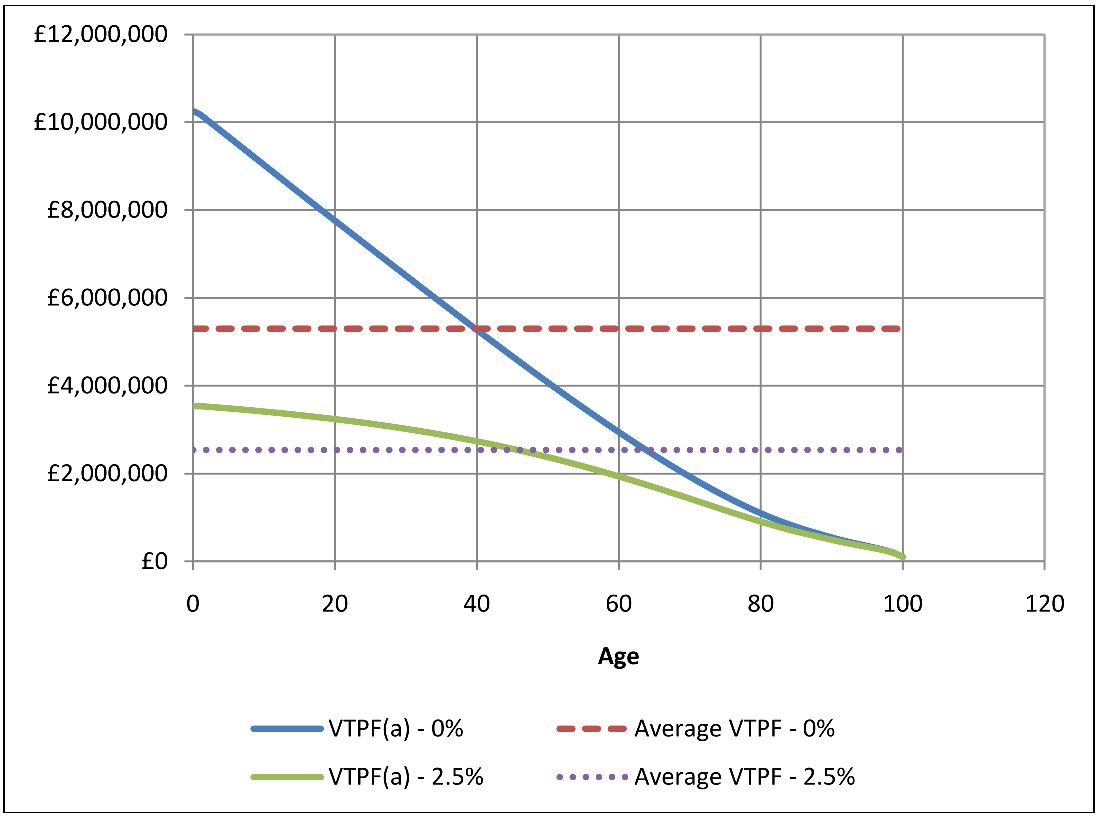

Figure 15 Values of the age dependent VTPF, and the age-averaged VTPF, for discount rates 0% and 2.5%. The average values of the VTPF are £5.3 million and £2.5 million, respectively. These are evaluated at $J = 1$.

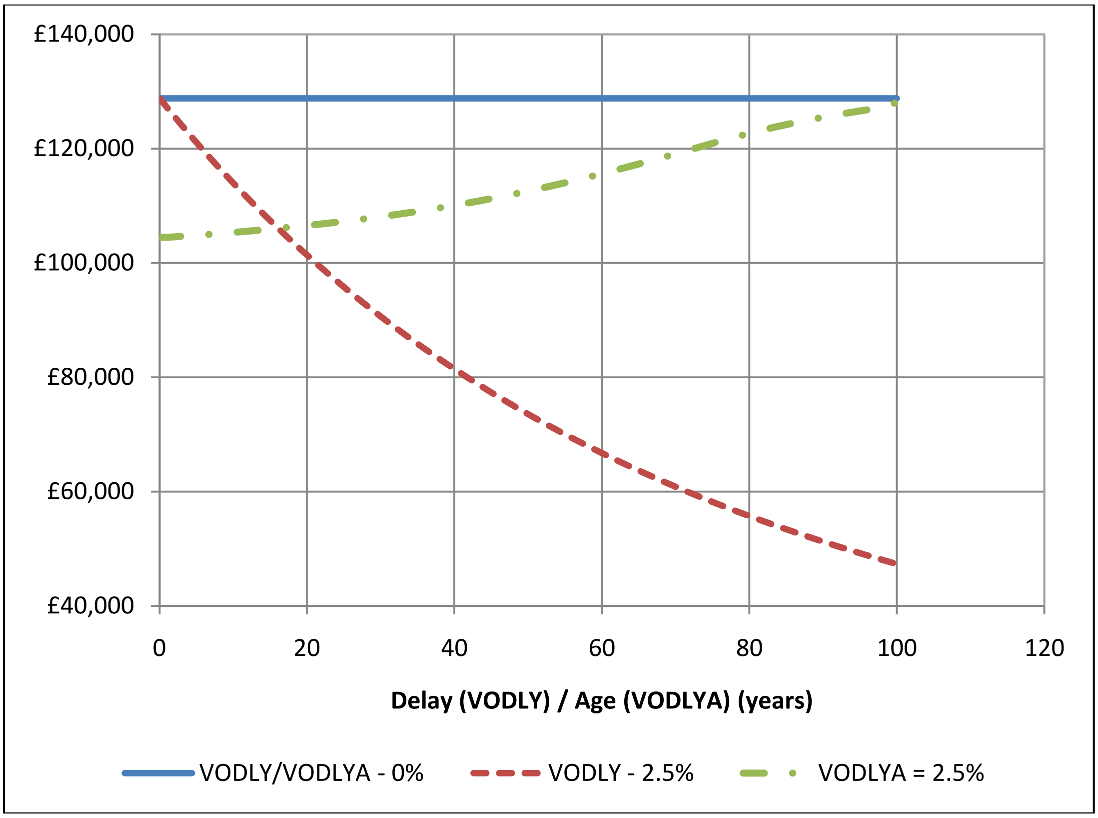

Figure 16 Values of the VODLY and VODLYA, for discount rates of 0% and 2.5%. At 0% discount rate, the VODLY and VODLYA are equal and constant, at about £129,000. The abscissa is for the VODLY is the delay before the life-year is saved, whilst for the VODLYA, the abscissa is the age of the individual.

Sorted Data	Cumulative Proportion, *p*	z_p
0.8127	0.0385	-1.7688
0.8152	0.0769	-1.4261
0.8176	0.1154	-1.1984
0.8183	0.1538	-1.0201
0.8199	0.1923	-0.8694
0.8208	0.2308	-0.7363
0.8217	0.2692	-0.6151
0.8218	0.3077	-0.5024
0.8240	0.3462	-0.3957
0.8249	0.3846	-0.2934
0.8252	0.4231	-0.1940
0.8259	0.4615	-0.0966
0.8260	0.5000	0.0000
0.8262	0.5385	0.0966
0.8266	0.5769	0.1940
0.8267	0.6154	0.2934
0.8276	0.6538	0.3957
0.8277	0.6923	0.5024
0.8279	0.7308	0.6151
0.8279	0.7692	0.7363
0.8280	0.8077	0.8694
0.8287	0.8462	1.0201
0.8301	0.8846	1.1984
0.8335	0.9231	1.4261
0.8346	0.9615	1.7688

Table 4 Data for the normal-quantile plot to test the risk aversion for normality.

Correlation Coefficient	Critical Value, $\alpha = 0.05$
0.98	0.957
The null hypothesis may not be rejected at this level of significance.	

Table 5 Results of the normal-quantile plot.

Parameter	Value (95% Tolerance Limit)
GDP per Person, G (£/y)	22,538 (22,531 – 22,545)
Discount Rate, r_d (/y)	0.3% / 2.8%
Growth Rate, r_g (/y)	2.0%
Net Discount Rate, r (/y)	0% / 2.5%
Life Expectancy, X (years) (general population distribution, 50% male/female ratio, 0% discount rate)	41.17 (41.166 – 41.177)
Mean square age , t^2_{av}, (years^2) (general population distribution, 50% male/female ratio, 0% discount rate)	2,304
Mean cube age , t^3_{av}, (years^3) (general population distribution, 50% male/female ratio, 0% discount rate)	147,311
Population entropy, H	0.13
Theta, θ	0.573
Work-Time Fraction, w_0	0.091
Risk Aversion, ε	0.825 (0.814 – 0.835)
The J-value (J)	N/A (±5.7%)
VTPF, V_P (£), (general population distribution, 50% male/female ratio, 0% discount rate)	5,300,000
VODLY/VODLYA (£), (0% discount rate)	129,000

Table 6 Values of parameters

Chapter 9 Sensitivity Analysis of the J-Value Framework

9.1 The Purpose of Sensitivity Analysis

The sensitivity of the J-value framework to the inherent variability of the input parameters and to the numerous explicit and implicit assumptions necessarily used in developing the model may now be analysed. Such analyses give indications of areas in which the assumptions may need to be used carefully. They may also indicate areas where perhaps less care may be required than had previously been suspected. A sensitivity analysis can also be used to add strength to conclusions, or highlight areas that require further development.

A benefit of the J-value framework is that there is only one key output, the J-value itself. This is dependent upon a number of input parameters. Furthermore, these input parameters can be objectively determined. These factors mean that assessing the J-value framework for sensitivities can be done in a fairly straightforward manner, as will now be described.

9.2 The Sensitivity Coefficients of the J-Value

The initial step in assessing sensitivities is to calculate the sensitivity coefficients of the J-value. Although not yet apparent, this has already been partially done in section 8.9. The sensitivity coefficients of an output with a number of inputs are simply the partial derivatives of the output with respect to each of the inputs. Equation (8.81) relates the uncertainty of the J-value to the uncertainty of the context-independent parameters. This can be expanded further by including all the J-value input parameters:

$$\sigma_J = \sqrt{\begin{aligned}&\left(\frac{\partial J}{\partial \varepsilon}\right)^2 \sigma_\varepsilon^2 + \left(\frac{\partial J}{\partial G}\right)^2 \sigma_G^2 + \left(\frac{\partial J}{\partial N}\right)^2 \sigma_N^2 \\ &+ \left(\frac{\partial J}{\partial \delta X_d}\right)^2 \sigma_{\delta X_d}^2 + \left(\frac{\partial J}{\partial \delta \hat{V}_N}\right)^2 \sigma_{\delta \hat{V}_N}^2 + \left(\frac{\partial J}{\partial D_f}\right)^2 \sigma_{D_f}^2\end{aligned}} \tag{9.1}$$

The sensitivity coefficients are then these partial derivatives. The derivatives can be evaluated readily. As the J-value is a product of factors, all the partial derivatives will be proportional to J, and this can be divided out of the equation to give the coefficient of variation on the J-value in terms of the new sensitivity coefficients, and the uncertainties of the input parameters: it will also be assumed, for simplicity, that there is no correlation between the change in life expectancy and the discount factor, D_f.

$$\frac{\sigma_J}{J} = \sqrt{\begin{array}{l}\left(\frac{1}{1-\varepsilon}\right)^2 \sigma_\varepsilon^2 + \left(\frac{1}{G}\right)^2 \sigma_G^2 + \left(\frac{1}{N}\right)^2 \sigma_N^2 \\ + \left(\frac{1}{\delta X_d}\right)^2 \sigma_{\delta X_d}^2 + \left(\frac{1}{\delta \hat{V}_N}\right)^2 \sigma_{\delta \hat{V}_N}^2 + \left(\frac{1}{D_f}\right)^2 \sigma_{D_f}^2\end{array}} \tag{9.2}$$

These sensitivity coefficients then weight the variances of the input parameters. As each coefficient is the reciprocal of the input parameter, it follows that the smaller the input parameter, the greater the sensitivity coefficient. The uncertainty on the number of people affected by the risk reduction, N, does not contribute much uncertainty, as the J-value is approximately independent of this parameter. Therefore this term and its coefficient may be disregarded from the equation. The GDP per person has been shown to have a relatively small coefficient of variation. Its sensitivity coefficient will also be small, as the GDP per person is a large term in the J-value. This will also usually apply to the cost of the safety system, which usually is at least of the order of £10,000, and can be many orders of magnitude larger than this. Thus, although the uncertainty over this figure may be considerable, the sensitivity coefficient will usually mean that this uncertainty carries little weighting onto the uncertainty of the J-value. However, the possibility that the uncertainty on the cost of the safety system is sufficiently large to dominate the J-value can never be ruled out.

The sensitivity coefficient for the discount factor is only defined for non-zero discount rates, as the uncertainty on D_f is zero for a 0% discount factor. For a discount rate of 2.5%, the discount factor is about 1.5, so that the sensitivity coefficient is 0.67. While this is larger than the coefficients of the GDP per person and the cost of the safety system, it is still relatively small when compared to the remaining coefficients of the risk aversion and the change in life expectancy. The sensitivity coefficient of the risk aversion is different from the other parameters in that it is the reciprocal of the complement of the risk aversion, $1 - \varepsilon$ that appears in the equation. As $\varepsilon = 0.825$, the complement is equal to 0.175, and the reciprocal is 5.7. The final factor is the change in life expectancy. Although this parameter is context-dependent, and as such cannot be determined *a priori*, an indication can be given of its magnitude. Although the maximum possible average loss of life expectancy is the initial life expectancy, $X = 41.2$ years, situations where the protection system offers this kind of benefit are rare. Typical values of the change in life expectancy are from 10^{-5} to 10^{-2} years. The sensitivity coefficient can then be large compared to the others.

Thus, an analysis of the sensitivity coefficients of the J-value indicates that the J-value is most sensitive to the uncertainties and assumptions regarding the risk aversion and the change in life expectancy. Therefore the assumptions made in calculating these parameters will be analysed and tested to see how the calculations compare when more realistic data is used. As the change in life expectancy is closely related to the initial life expectancy, (e.g. see equation (5.25)), the assumptions made in calculating this parameter will also be analysed.

9.3 Sensitivity Analysis of the Life Expectancy Calculations

Calculating the change in life expectancy requires determination of many of the same parameters as the calculation of the initial life expectancy. Indeed, the calculation is actually performed by first calculating the initial life expectancy, and then perturbing the hazard rates. Therefore, analysing the sensitivity of the change in life expectancy parameter will require an analysis of the sensitivity of the initial life expectancy. In this section, such a sensitivity analysis is presented.

Chapter 4 has already presented the methods required to calculate the life expectancy. The method can be broken down into a series of steps:

1. Calculate the hazard rates, $h(a)$,
2. Calculate the cumulative hazard rates, $W(a)$,
3. Calculate the survival probabilities, $S(a)$,

4. Calculate the life expectancies, $X(a)$,
5. Calculate the probability densities, $p(a)$,
6. Calculate the average life expectancy, X.

The effect of discounting does not need to be considered here, and so it will be assumed throughout that the discount rate is zero. In these steps, there are a number of assumptions that need to be made in order to perform the calculation. These assumptions can be varied, and consequently different life expectancies will be produced. The question therefore arises as to which life expectancy is the most accurate. This question can be answered by assuming that the "correct" life expectancies are the ones given by the ONS in their life tables. The method that best approximates the ONS life tables is therefore judged to be the most accurate life expectancies. The discrepancy between the model's calculation and the ONS calculation can be tested statistically. The test can answer whether the difference is statistically significant or not. A null hypothesis is therefore formed that the ONS life table data are distributed according to the model's method. The test performed is Pearson's Chi-Square Test. The test statistic, χ^2_{k-1}, is determined from the summed squared difference between the number of deaths associated in a cohort facing the calculated survival probabilities, denoted as $\hat{E}_a$, and the number of deaths from the life table function, d_a, from equation (4.32):

$$\chi^2_{k-1} = \sum_{a=0}^{k-1} \frac{\left(\hat{E}_a - d_a\right)^2}{\hat{E}_a} \tag{9.3}$$

where:

$$\hat{E}_a = n\left(\hat{S}_a - \hat{S}_{a+1}\right) \tag{9.4}$$

and where k - 1 is the number of degrees of freedom, see, for example, London (1997) [132]. As there are 101 ages in the life table (from age 0 to 100), then k = 101 and the number of degrees of freedom is 100. The parameter n is the sample size, which is the assumed initial size of the cohort that is subject to the hazard rates. This is also equal to the radix, l_0, of the standard life table, and is taken as 100,000. The

$\hat{S}_a$ are the survival probabilities estimated from the model. If the χ^2_{k-1} test statistic is greater than some critical value, then the null hypothesis may be rejected. An upper one-sided test is performed at the 5% significance level. The critical value of the upper tail χ^2_{k-1} statistic at this level is the value at which the complement of the cumulative distribution function of the chi-square distribution with 100 degrees of freedom is equal to 5%. This can be computed from tables, and is approximately equal to 124. If the value of the test statistic is greater than this value, then the null hypothesis is rejected in favour of the alternative hypothesis, namely, that the ONS data is not distributed according to the model under test. The lower the value of the test statistic, the closer the ONS data is to the model. The model that produces the lowest value will be accepted as representing the most accurate life expectancy calculations.

There are a number of assumptions which can be tested. The first is the assumption about the correct value to use for the hazard rate, $h(a)$. In chapter 4 it was argued that either of two functions could be used to approximate the hazard rate. These were the central rate of mortality, m_a, which was shown to be correct if deaths are distributed exponentially throughout the interval $(a, a + 1)$, and the probability of death, q_a, which was shown to be correct if deaths are distributed uniformly over the interval $(a, a + 1)$. These two approximations can then be tested. In addition to these, two other approximations to the hazard rate are also tested. These are:

$$h(a) = -\ln(1 - q_a) \tag{9.5}$$

which also assumes that deaths are distributed exponentially throughout the interval, and should therefore give similar results to the approximation when $h(a)$ is approximated by m_a. Another approximation is given by a quintic polynomial representation of the hazard rate, see Haberman (1994) [90] and McCutcheon (1983) [135]. In this approximation, the hazard rate is given by:

$$h(a)=\frac{1}{12}\begin{Bmatrix}25-48(1-q_a)+36(1-q_a)(1-q_{a+1})\\-16(1-q_a)(1-q_{a+1})(1-q_{a+2})\\+3(1-q_a)(1-q_{a+1})(1-q_{a+2})(1-q_{a+3})\end{Bmatrix}\quad \text{for a}\leq 2$$
$$=\frac{1}{12}\begin{Bmatrix}-(1-q_{a-2})^{-1}(1-q_{a-1})^{-1}+8(1-q_{a-1})^{-1}\\-8(1-q_{a+1})+3(1-q_{a+1})(1-q_{a+2})\end{Bmatrix}\quad \text{for a}>2 \tag{9.6}$$

with:

$$q_1=m_1\left\{\frac{1+0.5m_2}{1+1/12(7m_1+5m_2)+1/3(m_1m_2)}\right\} \tag{9.7}$$

and:

$$q_a=m_a\left\{\frac{1-0.5m_{a-1}}{1+5/12(m_a-m_{a-1})-1/6(m_am_{a-1})}\right\}\quad \text{for a}>1 \tag{9.8}$$

and q_0 is as given in the life table. Although the quintic polynomial approximation to the hazard rate is complex and cumbersome, it will also be tested against the life-table data.

Another assumption that can be tested is the integration method for the cumulative hazard rate function, $W(a)$. As was discussed in section 4.5, when the central rates of mortality are used as the hazard rate, the cumulative hazard rate can be can be calculated by summing up the hazard rates. However, in more general circumstances this assumption may not be applicable. Therefore, different methods of integration are also tested against the empirical data. These other methods are the trapezium method of integration, with the step length taken as one year. This is equal to the sum, but with the endpoints only contributing half the weight of the other points, i.e.:

$$\int_a^b f(x)dx\approx\sum_{i=a}^{b}f_i-\frac{(f_b+f_a)}{2} \tag{9.9}$$

The cumulative hazard rate can be estimated in an iterative manner through:

$$
\begin{aligned}
W(a+1) &= \int_0^{a+1} h(u)du = \int_0^{a} h(u)du + \int_a^{a+1} h(u)du \\
&= W(a) + \int_a^{a+1} h(u)du \\
&= W(a) + \left(\frac{h(a)+h(a+1)}{2}\right)
\end{aligned}
\tag{9.10}
$$

and where $W(0) = 0$. Another method of integration is Simpson's method, which approximates the integral as a quadratic polynomial:

$$
\int_a^b f(x)dx \approx \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \tag{9.11}
$$

The cumulative hazard rate is then estimated by:

$$
W(a+1) = W(a) + \frac{1}{6}\left(h(a) + 4h\left(a+\frac{1}{2}\right) + h(a+1)\right) \tag{9.12}
$$

Clearly, Simpson's method requires that the hazard rate is evaluated at age a + ½. Elandt-Johnson (1980) [70] gives a general approximation as:

$$
h\left(a+\frac{1}{2}\right) \approx q_a \tag{9.13}
$$

which can then be used to evaluate the integral.

One final assumption that is tested against empirical data is the use of the final age band, as an "end correction" to account for the mortality experience of those older than 101. This correction was discussed further in section 4.5. Here the effect of including such a correction will be tested.

The tests then include four hazard rate approximations, three numerical integral approximations, and two approximations that do and do not include the end correction. There are therefore 24 separate tests. For each of these, the χ^2_{k-1} statistic can be calculated and tested against the critical value. These 24 tests are shown in Figure 17. The most immediate result is the importance of the use of the end correction. All the tests performed without the end correction had χ^2_{k-1} values in excess of the critical value, and therefore had their null hypothesis rejected in favour of the alternative hypothesis – that the life table data did not match up with the model. Another result is that the trapezium rule is generally a poor fit for the data, with three of the four hazard rate tests with the end correction being rejected. This compares with two tests for the summation method, and no tests for Simpson's method. Although all of the hazard rates tested with Simpson's method were less than the critical value, they were not the tests that were closest to the empirical data. The most accurate tests were those that used the summation method and the hazard rates equal to m_a and -ln(1-q_a). The use of q_a for the hazard rate was not found to be accurate. Surprisingly, the quintic polynomial approximation also performed poorly, except in the case when the Trapezium method was used. The overall conclusion of these tests is that the end correction should be used, and that the hazard rate of -ln(1-q_a) and the trapezium method of numerical integration for the cumulative hazard rate should be used for most accuracy. However, using the central rate of mortality, m_a for the hazard rate does not degrade this accuracy very much, and is easier to calculate, as it is given directly in the life tables. Therefore this variable is recommended for use as the hazard rate. These tests thus validate the assumptions used in section 4.5, where the procedures used in calculating the J-value were explained.

Another feature of the change in life expectancy calculations that can be tested is the validity of the linear approximation used in approximating the effect of a hazard rate perturbation on the life expectancy, as used between equations (5.14) and (5.15). The linear approximation is unbounded in the additional hazard rate, whilst the true value is bounded, so that the change in life expectancy is never greater than the initial life expectancy. Figure 18 shows the difference between the two methods. They are very close for low additional hazard rates, but begin to diverge at an additional hazard rate of 0.1 year^{-1}. At this hazard rate the percentage difference is 5% and the change in life expectancy is about 4 years. This is judged to be the upper limit of practicability for the linear approximation. The calculations rapidly diverge after this. At an additional hazard rate of 0.5 year^{-1}, the difference is about 30%. These calculations apply to the situation where there is a single exposure resulting in a risk of immediate death. Prolonged risks will result in higher changes in life expectancy, and hence greater divergences between the linear and true calculations at lower additional hazard rates. Therefore, any calculations of an individual change in life expectancy of about 4 years or greater based on the linear model should instead be done using the true calculations.

9.4 Sensitivity Analysis of the Risk Aversion Calculations

Section 9.2 discussed that the two variables with the highest sensitivity coefficients were the change in life expectancy and the risk aversion coefficient. The previous section has investigated a number of the assumptions which were made in the calculations of the life expectancy and the subsequent change in life expectancy following a perturbation of the hazard rate. Here the assumptions underlying the risk aversion calculations will be investigated.

The risk aversion is dependent upon the share of wages in the GDP, θ, and the optimal work time fraction, w_0, see equation (3.41). The value of θ was taken directly from observed data, and so there were few assumptions made in the calculation. The calculation of w_0, however, requires that a number of simplifying assumptions be made, as was described in chapter 6. It was shown that the work time fraction is equal to the ratio of the work-life expectancy to the life expectancy, as given by equation (6.4). In calculating these two parameters, it was assumed that a), the population is in a steady state, and b) that time spent working is distributed uniformly between recruitment and retirement ages. These two assumptions may now be examined in further depth.

Throughout most of the development so far, it has been assumed that the population is in a steady state, so that the number of people born each year is equal to the number of people dying each year. This assumption produces a certain population distribution that can be readily calculated from the survival probabilities. This distribution is described in more detail in section 4.6. However, actual populations are rarely in a steady state, as they are affected by varying fertility rates, immigration, emigration and health care improvements which reduce mortality. It therefore is pertinent to compare the results of the calculations of population-averaged values that are based on the steady state assumption with the values obtained when actual population figures are used. Data for the actual population size at each age is available for the UK from the ONS [148], from which the probability distribution can be readily estimated.

The other assumption was made in deriving the work-life expectancy, where it was assumed that the time spent working was uniformly distributed over working lifetime, which was taken to start at age 20 and end at age 60. This can be compared against empirical data on time spent working at each age and employment rates, which again is available from the ONS, see [146] and [147]. These then allow the parameters $g_w(t)$ (the fraction of time a worker spends in work at current age t) and $p_w(t)$ (the probability of being employed at age t) to be determined, which can then be used to calculate $y_w(a)$ and y_w, from equations (6.7) and (6.12). The distribution of $g_w(t)$, $p_w(t)$ and their product, $g_w(t)p_w(t)$, are shown in Figure 19 and Figure 20 for the uniform assumption and the actual data. As can be seen, the actual data appears more bell-shaped, with people beginning work before age 20, and retiring after age 60. This data allows a comparison of the calculations of y_w obtained under each circumstance. Because y_w is also a population averaged parameter, it will also depend on the assumption used for the population distribution. There are then four values of y_w that will result from the different assumptions.

The parameters tested for sensitivity to these assumptions are the average life expectancy, X, the work-life expectancy, y_w, the work-time fraction, w_0, and the risk aversion, ε. Four values are determined for the two population distributions and two working time distributions (although the life expectancy is not affected by the working time distribution). The results are shown from Table 7 to Table 10. Note that the data used was from 2008, so that the steady state and uniform working time assumption will not be the same as those presented earlier in chapter 8, as more recent data was used in estimating those figures.

The tables show that the effect of using the actual population distribution increases the life expectancy by about 2%. For the other parameters, the largest difference from the simple steady state population and uniform working distribution assumptions is when both actual distributions are used. The actual distributions increase the work-life expectancy by about 5%, while the work-time fraction increases by about 3%. The effect on the risk aversion is that it is reduced by less than 1%. Thus, the use of actual observed distributions does not affect the risk aversion by much. Furthermore, the simpler distributions lead to a greater risk aversion estimate. In the context of the J-value, this will mean that slightly higher spending on safety will be allowed. The simple distributions are therefore more conservative than the actual distributions.

The risk aversion is thus insensitive to changes in the underlying assumptions about the population and working time. Using the simpler distributions is computationally easier and more efficient, and produces slightly more conservative results. The sensitivity analysis therefore validates the use of the simplifying distributions.

The conclusion of the sensitivity analyses is that the uncertainty on the J-value is most sensitive to the uncertainty on the life expectancy and the risk aversion, as these parameters were found to have the greatest sensitivity coefficients. The change in life expectancy was assessed for sensitivity by testing the underlying life expectancy calculations against ONS life table data. This allowed the assumptions to be picked in order to minimise the difference in the calculations between the model output and the ONS data, thus optimising the accuracy of the life expectancy calculations in the model. The linear approximation used in perturbing the hazard rate for the calculation of the change in life expectancy was also assessed. It was found that for changes life expectancies less than around 4 years, the difference between the linear approximation and the true value was less than 5%, which was judged to be acceptable. However, if the linear model produced a change in life expectancy greater than this, then it would be necessary to recalculate without the linear approximation in order to retain accuracy. Testing the underlying assumptions of the risk aversion showed that use of the simplified population and working time distributions was justified, as they did not affect the risk aversion by much, and also produced more conservative results, in addition to being simpler to calculate. Thus, it is concluded that the J-value is reasonably robust to the use of such simplifying assumptions.

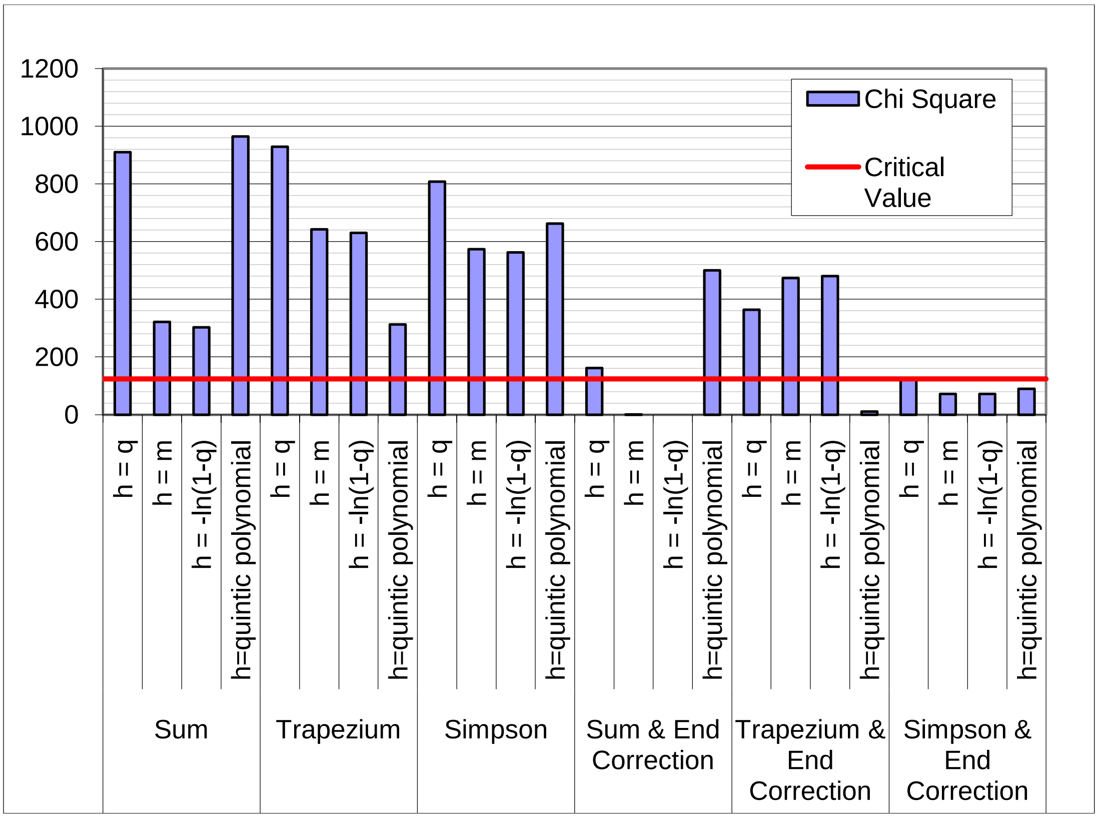

Figure 17 Result of Pearson's chi-square test for 24 tests of: 1. three methods of integrating the cumulative hazard rate (sum, trapezium and Simpson). 2. four different approximations to the hazard rate (*q*, *m*, -ln(1-*q*) and a quintic polynomial), and 3. the effect of using the end correction for the final age band. The lower the chi-square value, the closer the empirical data is to the model. The tests that are greater than the critical value (red line) can be rejected.

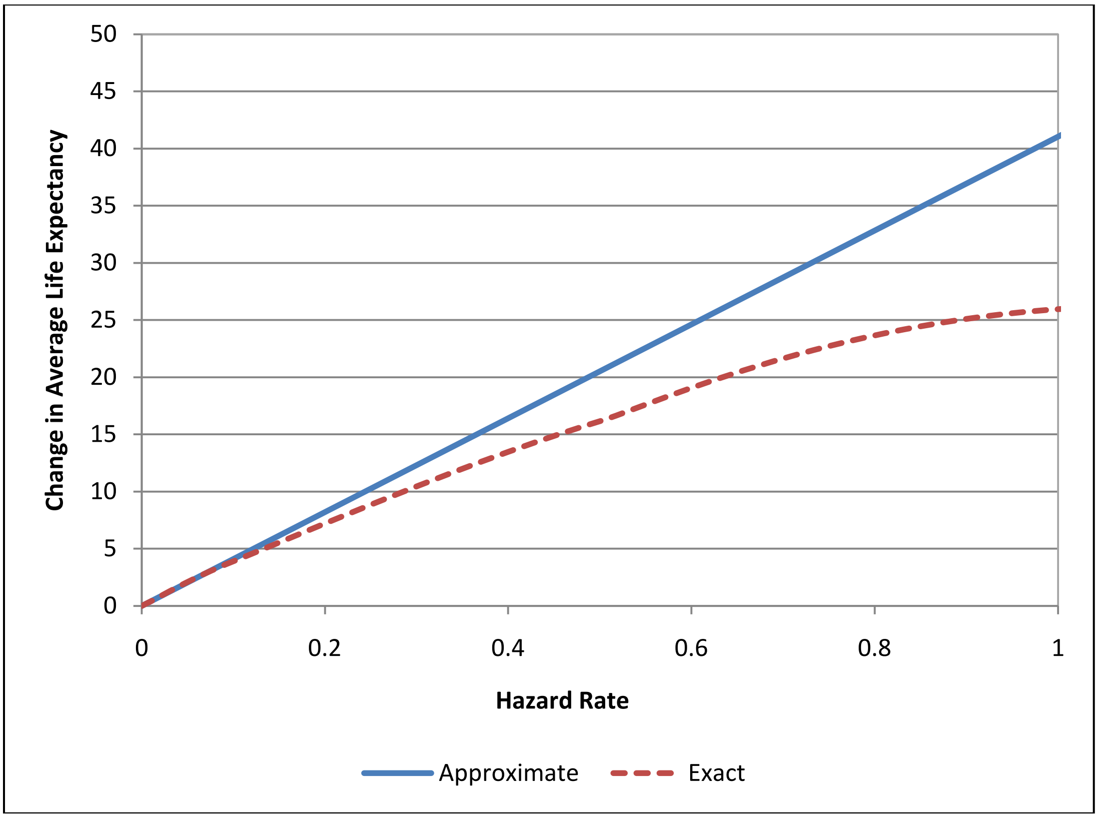

Figure 18 Difference between the linear approximation and the exact calculation of the change in life expectancy, as a function of the hazard rate. The difference between the two is around 5% at a hazard rate of 0.1 year^{-1}, and is nearly 30% at a hazard rate of 0.5 year^{-1}.

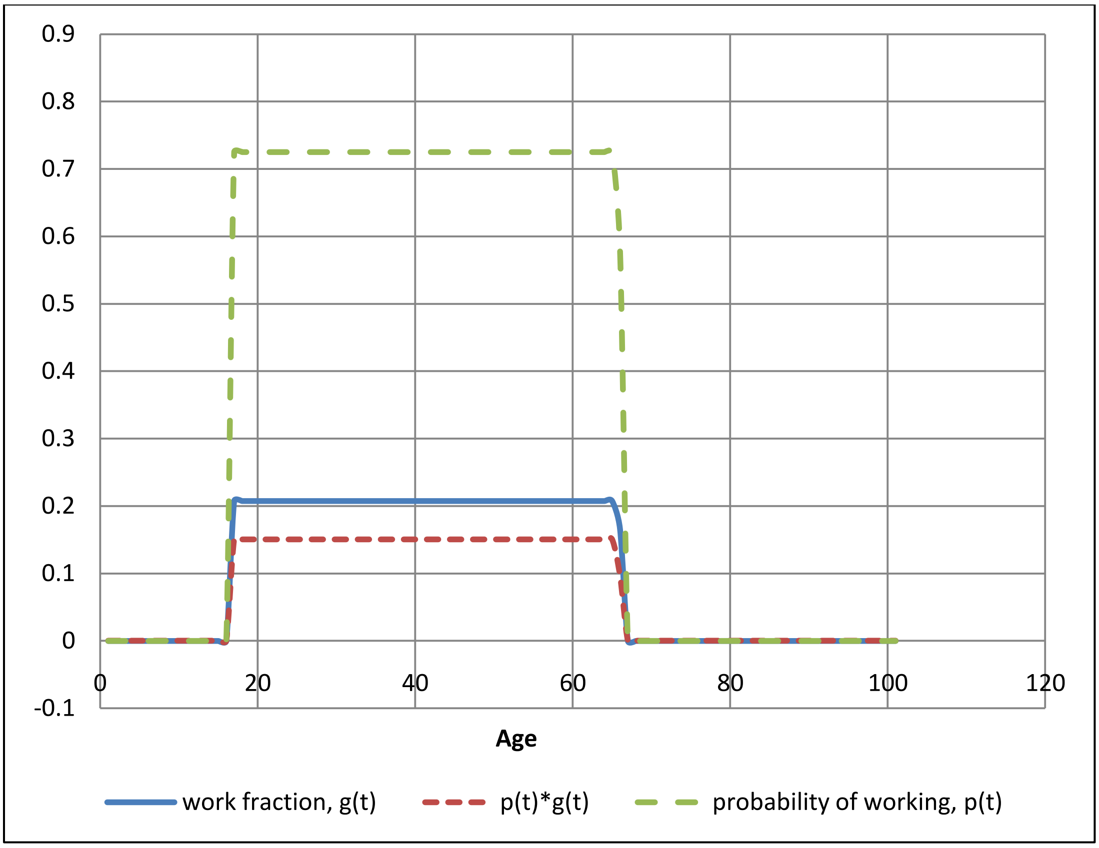

Figure 19 Rectangular distributions for $g_w(t)$, $p_w(t)$ and $p_w(t)g_w(t)$

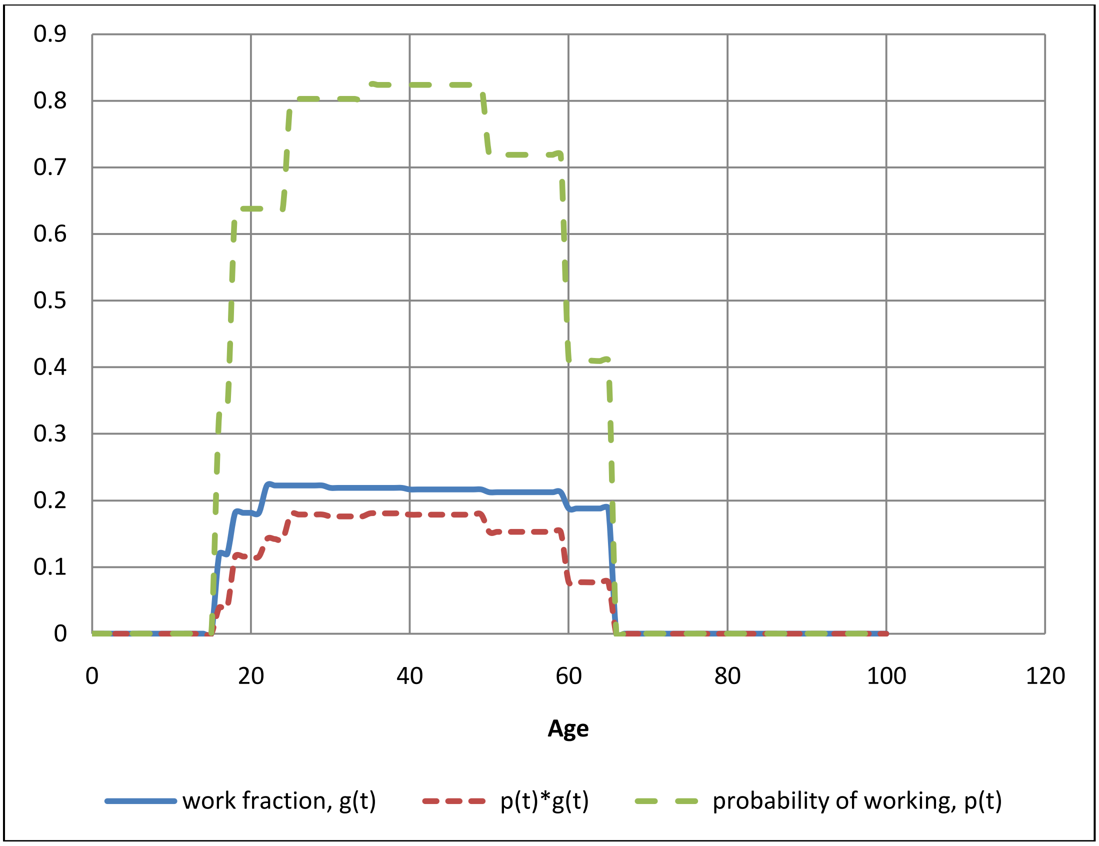

Figure 20 Actual distributions calculated from UK data for 2009 for $g_w(t)$, $p_w(t)$ and $p_w(t)g_w(t)$

Life Expectancy, *X* (years)	**Steady State Population**	**Actual Population**
	41.04	41.82

Table 7 Life expectancy under different population distributions.

Work-Life Expectancy, y_w (years)	**Steady State Population**	**Actual Population**
Uniform Working Time	3.43	3.53
Actual Working Time	3.48	3.59

Table 8 Work-life expectancy under different population and working time distributions.

Work-Time Fraction, w_0	**Steady State Population**	**Actual Population**
Uniform Working Time	0.083	0.084
Actual Working Time	0.085	0.086

Table 9 Work-time fraction under different population and working time distributions.

Risk Aversion, ε	**Steady State Population**	**Actual Population**
Uniform Working Time	0.838	0.836
Actual Working Time	0.835	0.833

Table 10 Risk aversion under different population and working time distributions. The wage share θ is taken as 0.563, which was calculated for 2008 data.

Chapter 10 Extending the J-Value Framework to Include Mitigation of Financial Risks

10.1 The J_2 and J_T-Values

So far, the focus of this thesis has been on introducing and developing the concepts underpinning the valuation of health and safety using the J-value framework. The risks concerned have been physical risks – those that affect human life. Recently, however, the J-value framework has been extended by Thomas et al (2010) [190], [191], [192], to include valuation of financial risks. These are risks to either an individual or an organisation's assets that can be somehow mitigated. A method has been developed that enables the maximum amount that should be spent on mitigating a given risk to be determined. If the amount that the individual or organisation has actually allocated to spend on mitigation is known, then the ratio of the actual spend to the maximum theoretical spend can be calculated. This ratio of financial risks is then the J_2-value. It is then straightforward to generalise to the case where both physical and financial risks are mitigated. If a scheme is being considered that will reduce both risks to assets and risks to life, then the maximum amount that should be spent on the scheme is equal to the sum of the maximum amount that should be spent on reducing physical risk and the maximum amount that should be spent on reducing risks to assets. The ratio of the actual amount spent on the scheme to this theoretical amount is the J_T-value, or "total judgement value". In this section the methods for determining the maximum spend shall be briefly laid out. Full details of the methods are described in the above references.

10.2 The Baseline, Risk Neutral Spend on Risk Reduction

In order to introduce some of the concepts, a simple case will be presented where the organisation is assumed to be risk neutral. If the probability and cost of the accident are known, then the amount that should be spent on reducing the risk can be determined easily. This risk neutral cost is then the baseline cost. In the following sections, it will be shown how the effect of risk-averse decision making increases the cost above this baseline value. Risk aversion is represented in the form of a utility function. In chapter 3, the utility of income, $U(G)$, was introduced. It was also discussed that there are various types of utility functions that can be used, but that two particularly important ones are the power utility function and the Atkinson utility function, which allows $\varepsilon \geq 1$ to be used. These are given by equations (3.35) and (3.39) respectively. In chapter 3, the simpler power utility function was favoured. However, in this section, the Atkinson utility will be used instead. Another change is that the utility of assets, A will used, rather than utility of income. The utility of assets is then given by:

$$\begin{aligned} U(A) &= \frac{A^{1-\varepsilon}-1}{1-\varepsilon} \qquad && \varepsilon \geq 0, \varepsilon \neq 1 \\ &= \ln A && \varepsilon = 1 \end{aligned} \tag{10.1}$$

Risk neutrality corresponds to $\varepsilon = 0$, in which case the utility is:

$$U(A) = A - 1 \tag{10.2}$$

which is thus the difference between current assets and one unit of the asset. In most cases, $A >> 1$, and $U(A) \approx A$, so that the utility of assets is just the assets itself. In this situation, the amount to spend on reducing a risk to the assets can be easily determined. If there is a probability, π_1, that the original assets, A, will be reduced by an amount, C, so that the final assets are $A - C$, then the expected value of the assets will be:

$$\pi_1(A-C) + (1-\pi_1)A = A - \pi_1 C \tag{10.3}$$

and the expected loss is $\pi_1 C$. If there is a scheme that can completely eliminate the risk, but will cost an amount, B, to implement, so that total assets would be $A - B$, then it would only be reasonable to implement the scheme if doing so increased or at least maintained the expected value of the assets in absence of the scheme. Thus, it must satisfy:

$$A - B \geq A - \pi_1 C \tag{10.4}$$

Therefore, the maximum amount that should be spent on the scheme, B_0, is:

$$B_0 = \pi_1 C \qquad (10.5)$$

The maximum value to spend on mitigation is therefore equal to the expected loss resulting from the risk. If the scheme does not completely eliminate the risk altogether, but instead reduces the probability from π_1 to π_2, then the maximum amount to spend is instead:

$$B_0 = (\pi_1 - \pi_2)C \qquad (10.6)$$

which again is the expected value of the loss. Thus, in the risk neutral case, the decisions are made based on expected monetary losses. However, if preferences for risk are considered, then spends must be based on expected loss of utility, rather than loss of assets. This (usually) entails an additional premium, which can be expressed in terms of a "maximum risk multiplier" of the baseline, expected monetary loss, $m_{r.max}$. If the maximum reasonable spend on mitigating risks is denoted δZ_R, then it is given by:

$$\delta Z_R = m_{r\max} B_0 \qquad (10.7)$$

The method for calculating the maximum risk multiplier will be shown in the following section.

10.3 Accounting for Risk Aversion Using the ABCD Model[7]

The ABCD model draws together four important aspects of decision making when regarding risk, three of which were introduced in the previous section. The organisation (or individual) is assumed to have assets, *A* (for the UK measured in £), and faces accident costs, *C* (£) with probability, $\pi_1 = 1 - p_1$ (where p_1 is the probability of no accident occurring). The affected party is considering spending an amount *B* (£) on an environmental protection system that will reduce the probability of incurring those accident costs from π_1 to $\pi_2 = 1 - p_2$, for the common case where

[7] This section largely follows [199].

π_1 is already small (the choice of the letter "B" to denote the cost of the protection system may be regarded as a "balancing" expenditure in certain circumstances). The expected utilities before, $E(u_1)$, and after, $E(u_2)$, the risk-mitigating system is introduced are calculated using the Atkinson utility function (3.39). The final element is the difference in expected utility, D:

$$D(u_1, u_2 \mid \varepsilon) = E(u_1) - E(u_2) \tag{10.8}$$

where dependence on the risk-aversion has been made explicit, and where:

$$E(u_1) = p_1 U(A) + (1 - p_1)U(A - C) \tag{10.9}$$

and:

$$E(u_2) = p_2 U(A - B) + (1 - p_2)U(A - B - C) \tag{10.10}$$

The protection system should be installed only if D is negative or, in the limiting case, $D = 0$.

It is convenient to define another variable, the "reluctance to invest" in the safety system, R_{120A}, as the change in the organisation's utility, D, normalised to the utility of the starting assets, $u_0(\varepsilon) = U(A)$:

$$\begin{aligned} R_{120A} &= \frac{D(u_1, u_2 \mid \varepsilon)}{u_0(\varepsilon)} \\ &= \frac{A^q}{A^q - 1} \left\{ \begin{array}{l} [1 - c_a]^q - [1 - b_a - c_a]^q \\ + p_1 \left(1 - [1 - c_a]^q\right) - p_2 \left([1 - b_a]^q - [1 - b_a - c_{\cdot} \right. \end{array} \right. \end{aligned} \tag{10.11}$$

where $q = 1 - \varepsilon$, and the lower-case letters b_a and c_a indicate normalised costs: $b_a = B/A$ is the cost of the safety system normalised to the assets, $c_a = C/A$ is the accident cost normalised to the assets.

A value of $R_{120A} = 1$ corresponds to a 100% reluctance to invest – the case where the cost of the safety system reduces to zero the expected utility of the organisation. A positive reluctance to invest ($0 < R_{120A} < 1$) indicates that the system is poor value for money, whereas a negative reluctance ($R_{120A} < 0$) corresponds to a desire to invest in the system. It has been shown [190] that as risk-aversion increases, the absolute value of the reluctance decreases towards zero. A scheme that is good value at $\varepsilon = 0$ and a second scheme that would be rejected outright at $\varepsilon = 0$, because of its poor value, both converge towards $R_{120A} = 0$ at large values of risk-aversion. Hence the risk-averse decision maker is unable to discriminate between the merits or demerits of the two schemes at large ε. This is the "point of indiscriminate decision" and occurs where $|R_{120A}| = \delta_{dis}$, with $\delta_{dis} \sim 10^{-6}$ being the discrimination limit. This gives an upper limit to the value of the risk-aversion, which is denoted as ε_{max}.

As was shown in the previous section, when the risk aversion is zero, then decisions are made in purely financial terms, and the maximum that should be spent on the protection system is equal to the reduction in the expected cost of an accident:

$$\begin{aligned} B_0 &= (p_2 - p_1)C \\ &= (\pi_1 - \pi_2)C \end{aligned} \tag{10.12}$$

or equivalently:

$$b_0 = (\pi_1 - \pi_2)c_a \tag{10.13}$$

The risk multiplier, m_r, is defined as the ratio of the actual (normalised) cost of the protection scheme, b_a, to the expected monetary savings it will produce: $m_r = b_a/b_0 \geq 0$.

Thomas et al (2010b)[191] have also shown that for a given protection scheme, the reluctance to invest exhibits a minimum value, and this minimum occurs at a risk-aversion of $\varepsilon = \varepsilon_{pp}$, called the "permission point". This corresponds to the point of maximum desire to invest in the protection scheme. To calculate the permission point a lower bound is set at $\varepsilon_{pp} = 0$, since only risk-averse decisions are considered and not risk-seeking behaviour. There is an upper bound at $\varepsilon_{pp} = \varepsilon_{max}$ where ε_{max} is the risk-aversion at the point of indiscriminate decision. Within these bounds, the minimum of R_{120A} follows three distinct patterns, illustrated in Figure 21. Pattern (1): there is a positive reluctance to invest at zero risk-aversion which decreases monotonically with increasing risk-aversion until the permission point meets the point of indiscriminate decision at $\varepsilon_{pp} = \varepsilon_{max}$. Pattern (2): the reluctance to invest is a (negative) minimum at $\varepsilon_{pp} = 0$, corresponding to the case when the safety system is justified on purely financial grounds, and R_{120A} increases monotonically with risk-aversion until the point of indiscriminate decision. Pattern (3): if the reluctance to invest is close to (positive or negative) zero at zero risk-aversion, then there is a minimum in the R_{120A} function at $0 < \varepsilon_{pp} < \varepsilon_{max}$. These three different patterns are important to keep in mind when evaluating the optimum risk-aversion below.

Calculating the optimum risk-aversion requires the numerical computation of the risk-aversion and the normalised safety spend at the permission point (ε_{pp} and b_{pp} respectively), together with their maximum values which occur at the point of indiscriminate decision (ε_{max} and b_{max}). The latter can also be expressed in terms of the "maximum risk multiplier", m_{rmax}, given by $m_{rmax} = b_{max}/b_0$, with b_0 defined above.

The risk-aversion at the permission point, ε_{pp}, is defined at the minimum of R_{120A}. Differentiating R_{120A} with respect to q yields the objective function $g(b_a, \varepsilon)$. Recalling that $q = 1 - \varepsilon$, the objective function is given as:

$$g(b_a, \varepsilon) = \frac{dR_{120A}}{dq} = \frac{dR_{120P}}{dq} - \ln\left(A^{1-q}\right)R_{120P} = 0 \tag{10.14}$$

where R_{120P} is the reluctance to invest in the safety scheme assuming a power utility function:

$$R_{120P} = p_1 + (1 - p_1)(1 - c_a)^q - (1 - b_a)^q \tag{10.15}$$

and its derivative is:

$$\frac{dR_{120P}}{dq} = (1 - p_1)(1 - c_a)^q \ln(1 - c_a) - (1 - b_a)^q \log(1 - b_a) \tag{10.16}$$

The roots of equation (10.14) yield the desired risk-aversion, ε_{pp}.

Graphical analysis of the variation of $g(b_a, \varepsilon)$ with b_a for fixed ε, shows that the function has two different regimes when $\varepsilon < 1$ and when $\varepsilon > 1$. For $\varepsilon < 1$, the objective function has two roots on the positive and negative going slopes of the function as shown in Figure 22 and Figure 23. As discussed in more detail later, the first of these roots are sought out. For $\varepsilon > 1$, there is only one root, near to $b_a = 1$ (Figure 24 and Figure 25). Finding the roots is made difficult at high values of c by the rapid change in slope as shown in Figure 23 and Figure 25.

Equation (10.14), cannot be solved analytically, and so must be solved numerically. Two distinct approaches to these computations have been taken which were developed independently so that results from the two methods could be compared and used to increase confidence in their accuracy. The first approach was to use the secant method. This naturally follows on from the referred derivative method used in [191], but it uses a finite difference approximation for the derivative of R_{120A} rather than an analytical expression. The permission point, ε_{pp} is incremented as the independent variable towards $\varepsilon_{pp} = \varepsilon_{max}$, yielding values of b_{pp} and b_{max}. The second approach was a technique which was named the "Golden Bisection Method". The minimum in the R_{120A} function is found using a Golden Section Search, without recourse to an analytical derivative. The independent variable is taken as b_a rather than ε, incrementing towards $b_{pp} = b_{max}$. The point of indiscriminate decision is evaluated using the Bisection Method, yielding values for ε_{max} and b_{max}. The very different nature of this algorithm promotes useful diversity in the calculations.

Equation (10.14) can be solved for the objective function using the method of referred derivatives (see Thomas (1997) [181] and (1999) [1]), which was used in Thomas et al (2010b) [191], and which lends itself to computation in a spreadsheet format. The computation can also be extended to more accurate and robust software based algorithms. The initial approach to solving equation (10.14) for the objective function that will be presented here consisted of applying the secant method – a modification of the Newton–Raphson iterative method that uses a finite difference approximation (see e.g. Press (1992) [165]). In the iteration the roots of the objective function are solved holding q constant, and solving for the value of $b_a = b_{pp}$ at the permission point for a given value of ε. The iterative procedure for this is given by:

$$b_{i+1} = \frac{2\delta b_i}{g(\varepsilon, b_i + \delta b) - g(\varepsilon, b_i - \delta b)} g(\varepsilon, b_i) \tag{10.17}$$

with the iteration continuing until $g(\varepsilon,b_{i+1}) < 10^{-6}$ and where $\delta b = 10^{-5}$ is a small increment in b_i. Each solution of equation (10.17), for increasing values of ε, will give the permission pair, $(b_{pp}, \varepsilon_{pp})$.

The procedure progresses by first finding a value for $b_{pp}(\varepsilon = 0)$. Here we use b_0 as a seed value in the iteration. The corresponding value of the risk multiplier, m_r, is denoted by $m_{rlow} = b_{pp}(0)/b_0$. This then proceeds to higher values of $b_a = b_{pp}(\varepsilon + \delta\varepsilon)$ by adding fixed increments, $\delta\varepsilon$, up to $\varepsilon = \varepsilon_{max}$, where, at some point the desire to invest, $-R_{120A}$, will become smaller than δ_{dis}, and the procedure will stop with $\varepsilon = \varepsilon_{pp} = \varepsilon_{max}$, $b_a = b_{pp} = b_{max}$ and $m_r = m_{rmax}$.

The above analysis caters for normalised costs for the protection system in the range $b_{pp}(0) \leq b_a \leq b_{pp}(\varepsilon_{max})$, with the corresponding risk multipliers in the range $m_{rlow} \leq m_r \leq m_{rmax}$. It is assumed that a normalised cost less than $b_{pp}(0)$, is not possible for risk-averse decision makers, although modifying this assumption to include risk seeking decision makers would be a topic for further research.

The Golden Section Search method (see Press (1992) [165]) for determining the permission point pairs $(\varepsilon_{pp}, b_{pp})$ finds the minimum in the R_{120A} function without requiring derivatives of the function. The algorithm first looks for an approximate value of ε_{pp} by evaluating R_{120A} at discrete values of ε with a step size of $\delta\varepsilon = 0.1$, over the range of ε up to the point where the absolute value of R_{120A} is less than the value, δ_{dis}, at the point of indiscriminate decision. If a local minimum is identified then a more accurate estimate of ε_{pp} is obtained by applying a golden section search in the region of the minimum, which ensures that the minimum is found. If there is not a local minimum in the approximate solution – for example, if the minimum is too close to $\varepsilon = 0$ (i.e. $\varepsilon < 2\delta\varepsilon$) – then an iterative approach is taken by decreasing the step size and recalculating ε_{pp} in the region of the minimum, repeating the procedure until the required accuracy is achieved.

An approximate value of the risk-aversion at the point of indiscriminate decision, ε_{max}, was found as above, by evaluating R_{120A} at discrete intervals of ε. This value was refined by applying the Bisection method [165] to evaluate the roots of $|R_{120A}| - \delta_{dis} = 0$ about the approximate solution.

Thus, a brief overview of the methods for calculating the maximum risk multiplier $m_{r.max}$, have been laid out. This parameter then allows the maximum reasonable spend on mitigating financial to be determined, as will be described below. No analytical solution for the maximum risk multiplier can be determined. Indeed, the value is dependent upon the probability of occurrence and the consequence of the risk faced, as well as the initial assets of the organisation (or individual). For further details of the computational methods used in calculating the limits to risk aversion, see Waddington et al (forthcoming) [199].

10.4 The Maximum Reasonable Spend and the New J-Values

Once the maximum risk multiplier has been determined through numerical methods, the maximum reasonable spend can be computed, from equation (10.7), repeated below:

$$\delta Z_R = m_{r\max} B_0 \tag{10.7}$$

The value B_0 is the expected monetary loss resulting from the risk. However, the expected monetary loss may be complicated by factors such as the possibility of the accident occurring multiple times, and the growth of the organisation. These issues are more fully addressed in Thomas and Jones (2010) [192]. Nevertheless, treating B_0 as being equal to the expected monetary loss will be a good approximation in the case of low accident probability and low growth rates.

The J_2-value (or second judgement value) is then the ratio of the actual amount spent on mitigating the risk, denoted as $\delta\hat{Z}$, to the maximum reasonable spend:

$$J_2 = \frac{\delta\hat{Z}}{\delta Z_R} \tag{10.18}$$

If a system protects against both risks to human life as well as to assets, and will cost $\delta\hat{W}$ to implement, then it is also possible to calculate a "total judgement value", J_T:

$$J_T = \frac{\delta\hat{W}}{\delta Z_R + \delta V_N} \tag{10.19}$$

where δV_N is the maximum reasonable spend on protecting human life, as given by equation (3.60). The J_T-value may be interpreted in a similar manner to the J-value, in that J_T-values in the range from zero to unity will be deemed as cost-beneficial, while J_T-values in excess of unity indicate that the scheme offers poor value for money, and should not be implemented. Thus the J_T-value provides a new and full criterion for the adoption or otherwise of a protection scheme to guard against both financial and human costs.

This concludes the exposition and development of the theory and methods required by the J-value framework for the valuation of health and safety, as well as the more recent addition of financial risks. This framework provides original and objective techniques for decision making that encompass a wide variety of types of risk yet still retains an output that is transparent and simple to interpret, and more importantly, provides consistency to a field in which decisions regarding sensible levels of expenditure on a given benefit can vary by eleven orders of magnitude (see Tengs et al (1995) [180]).

The final chapter of part 1 will provide some example calculations in order to illustrate to broad applicability of the techniques.

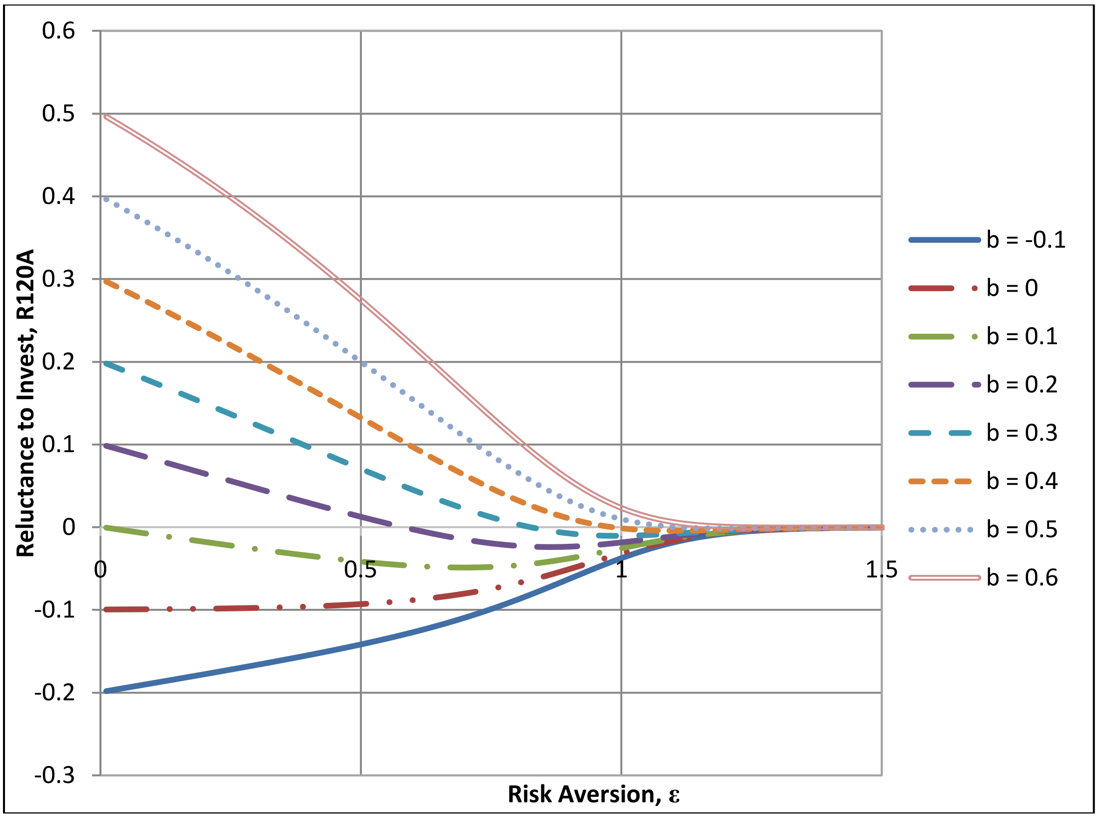

Figure 21 Response of the reluctance to invest (R_{120A}) with increasing risk aversion (ε), for different normalised costs of the safety system ($-0.1 < b < 0.6$). Assets (A) are £180,000, normalised accident cost (c) is 0.995, and the probabilities of no accident with and without the safety system are $p_2 = 1$ and $p_1 = 0.9$ respectively.

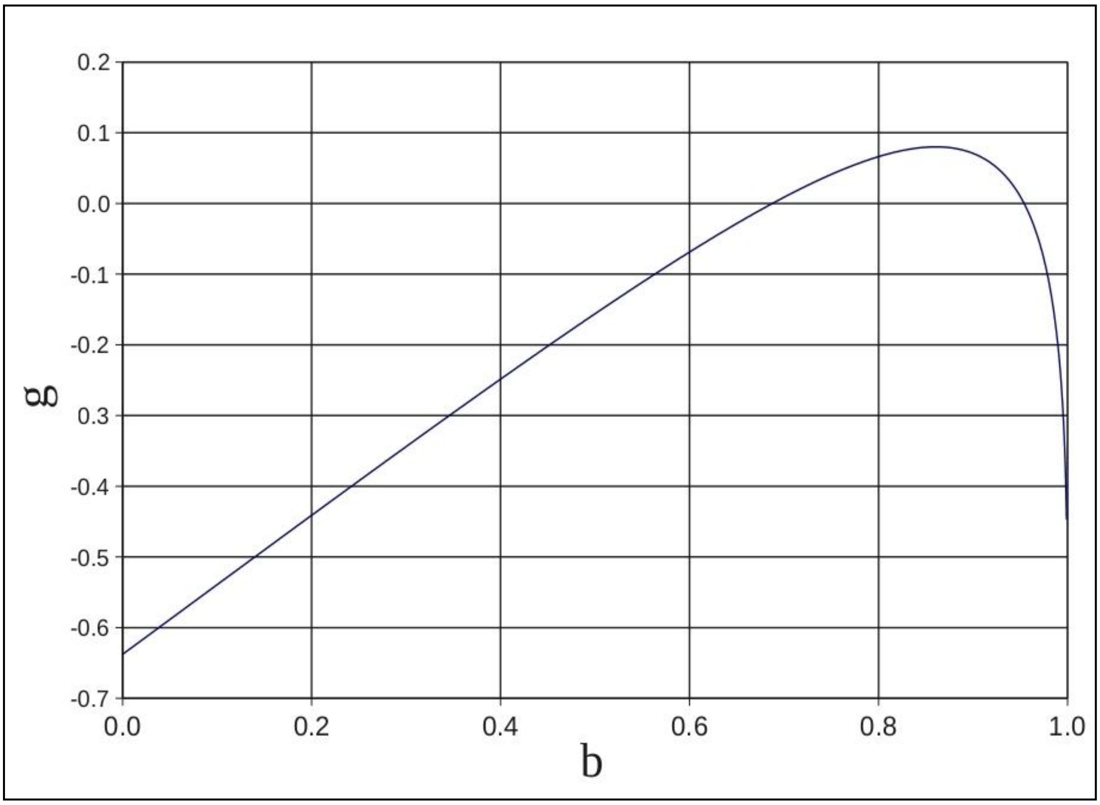

Figure 22 The derivative of the reluctance to invest when $\varepsilon = 0.5$ and $c = 0.9$, illustrating the two roots of the objective function $g(\varepsilon, b) = 0$. The assets are A = £180,000 and all accident probabilities are considered.

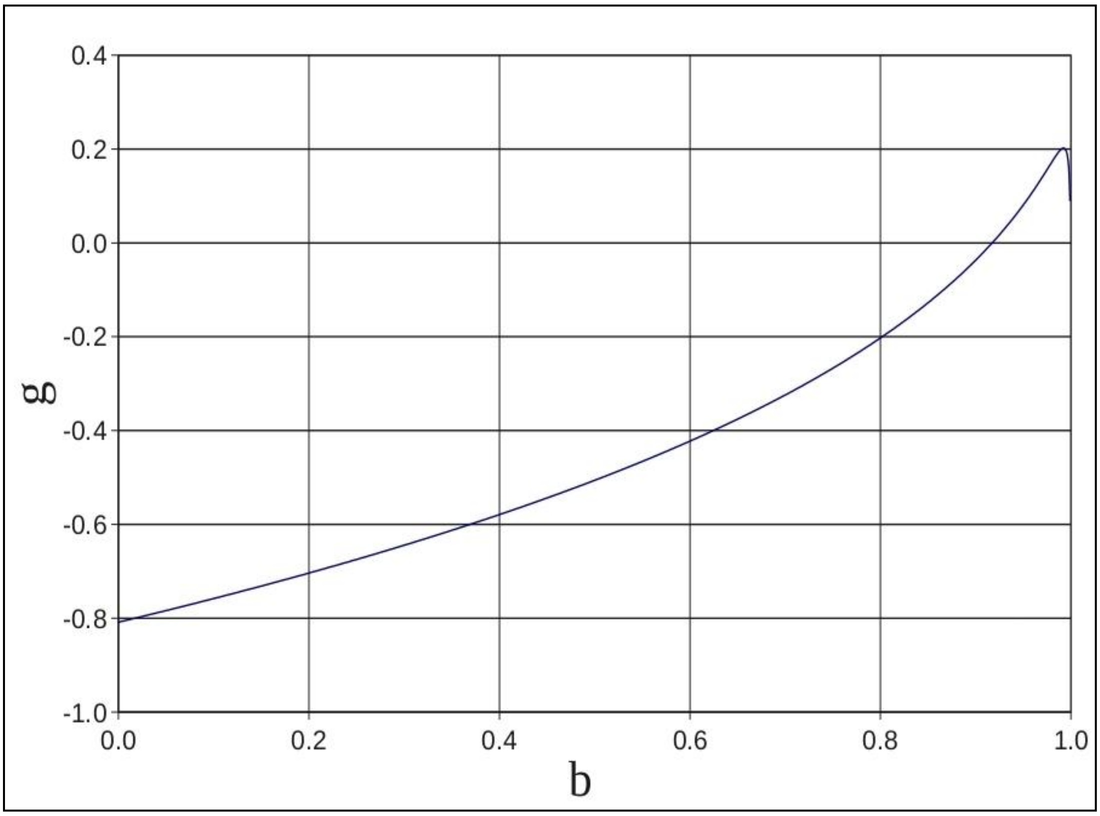

Figure 23 The derivative of the reluctance to invest when $\varepsilon = 0.9$ and $c = 0.999$, illustrating the two roots of the objective function $g(\varepsilon, b) = 0$. Other parameters are the same as Figure 22. Note the steep gradient in the region around the second root.

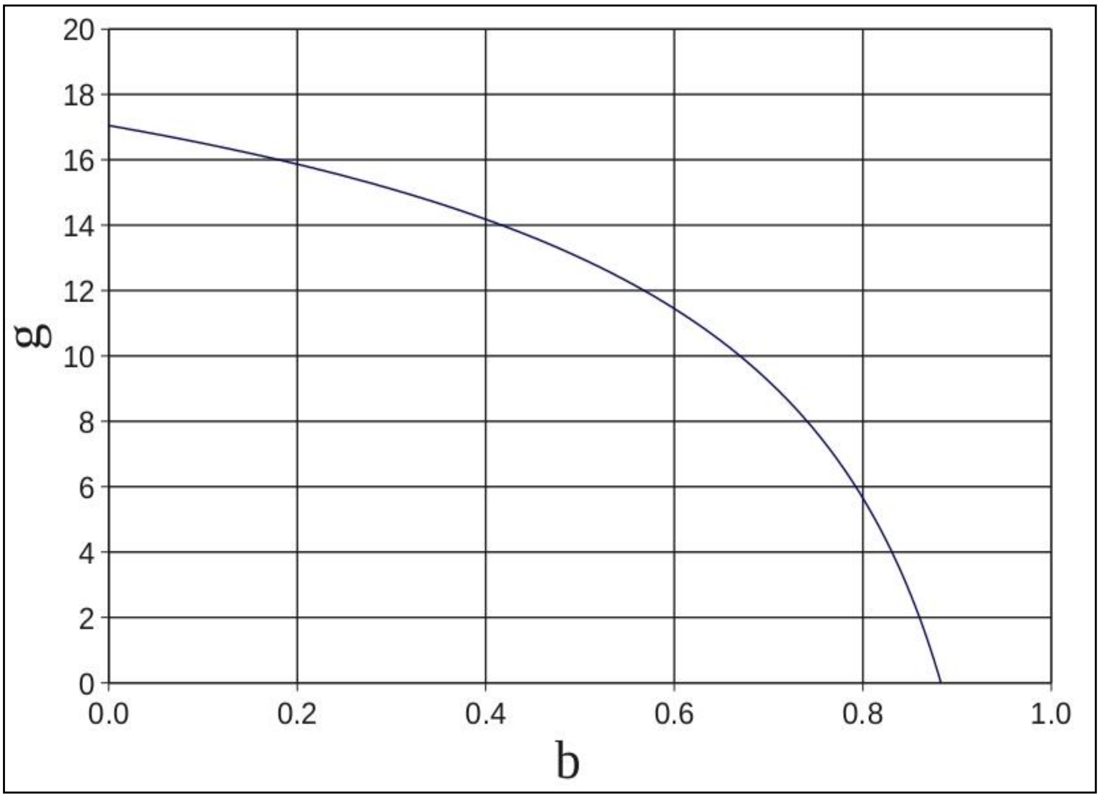

Figure 24 The derivative of the reluctance to invest when ε = 1.5 and c = 0.9, illustrating the single root of the objective function $g(\varepsilon, b) = 0$. Other parameters are the same as Figure 22

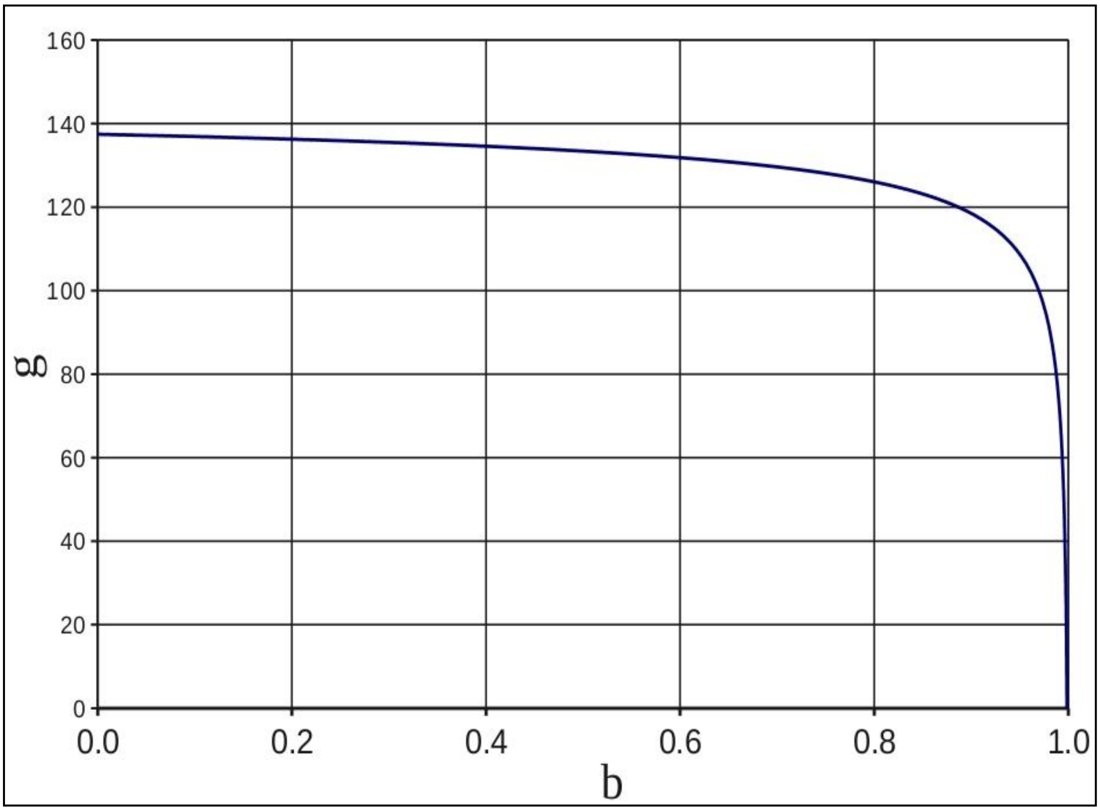

Figure 25 The derivative of the reluctance to invest when ε = 1.5 and c = 0.999, illustrating the single root of the objective function $g(\varepsilon, b) = 0$. Other parameters are the same as Figure 22. Note the steep gradient in the region around the root.

Chapter 11 Example Calculations

11.1 Example Calculations for the J-Value

In this section some example calculations will be shown in order to demonstrate the broad applicability of the J-value. The next three sections will provide calculations for the J-value by considering impact assessments for various health and safety schemes. Following this will be a calculation of the J_2 and J_T value of a protection scheme to mitigate the risk of a large nuclear accident. Finally, a J-value analysis of the ancient VTPF will be provided.

11.2 HSE's Impact Assessment of Various Policies to Limit Occupational Exposures to Respirable Crystalline Silica

A review by the HSE of occupational exposure to respirable crystalline silica (RCS) found that workers were exposed to unacceptable risks. They produced a regulatory impact assessment of four proposed exposure limits, see HSE (2005) [101]. These limits were: i) 0.3 $mg.m^{-3}$, which then was the current limit, but would have been more strictly enforced, as it was suspected that a substantial number of workers were exposed to concentrations in excess of these limits; ii) 0.1 $mg.m^{-3}$; iii) 0.05 $mg.m^{-3}$, and iv) 0.01 $mg.m^{-3}$.

The benefits of these limits were calculated in the document as reduced numbers of deaths from silicosis and lung cancer. Although there were also other benefits assessed in the document, such as prevented disabilities, medical costs and lost output, these are not included here, as only mortality effects are relevant to J-value analysis. It is estimated that policy i) would result in 36 less lung cancer deaths. Policy ii) would reduce lung cancer deaths by 185 while iii) reduced them by 300, and iv) reduced deaths by 455. The number of reduced deaths from silicosis was taken to be the same as for lung cancer. In order to convert these figures into a loss of life expectancy, it was necessary to use national mortality statistics [151] which give data on the age of death from those diseases, from which the average loss of life expectancy per death can be determined. The standard deviation of the loss of life expectancy can also be calculated from the data. These statistics show that lung cancer deaths cause, on average, 13.8 years of lost life per death, whilst silicosis deaths results in 7.3 years of lost life. These numbers can then be multiplied by the number of avoided deaths to arrive at the total improvement in life-expectancy afforded by the regulation, which is equal to $N\delta X$. These are listed in Table 11. The HSE document also lists costs associated with each option. Maximum and minimum cost estimates are given, and these can be averaged to determine a mean cost. J-values can then be determined with the values of the parameters as given in Table 6. The costs of the scheme and the J-values are shown in Table 12, along with the 95% confidence limits. In calculating the tolerance limits, it was assumed that the low and high estimates of the cost of the scheme represented 95% confidence limits, which then allows the standard deviation to be determined. No discounting will be presented here.

As can be seen, the only scheme which has a J-value less than unity is option i), that is, the option to more strictly enforce current limits. However, it is worth noting that there will likely be additional uncertainties associated with the number of deaths avoided by the regulations, as cancer and silicosis involve latent effects, making it difficult to assess the effects of exposures with much accuracy. Given that there will likely be further uncertainties, it seems reasonable to view option ii), which has a J-value slightly greater than the $J = 1$ threshold, as an acceptable figure. Also, when other factors, such as disability costs etc. are considered alongside the J-values, option ii) would be viewed with further favour. To summarise, option i) gives the best value for money, but option ii) may also be considered acceptable given the uncertainty.

The conclusions of the HSE document agreed to some extent with the J-value analysis. It was found that only option i) offered value for money. However, the HSE considered the occupational risks with this option as unacceptable, and so rejected this option, instead favouring option ii).

11.3 Department of Health's Proposal to Reduce the Number of Unnecessary CT Scans

The Department of Health (DH) has recently published a regulatory impact assessment that investigated the use of Computed Tomography (CT) scans in asymptomatic individuals, see Department of Health (2011) [63]. These scans can help in detecting conditions, but expose patients to ionising radiation, which carries health risks, and as such, needs to be justified. The Committee on Medical Aspects of Radiation in the Environment (COMARE) has provided some recommendations which would reduce the risks if implemented. DH's impact assessment reviews the costs and benefits of enforcing COMARE's recommendations.

The report assumes that there are approximately 3,000 individuals who have scans every five years between the age of 40 and 70. Each scan is taken as delivering to the individual a dose of 10 mSv, so that a 40 year old will receive an additional dose of 70 mSv from the extra scans over his or her lifetime. This information alone is sufficient to calculate the loss of life expectancy resulting from these scans. The exposure can be modelled as a series of short exposures, as is indicated in Figure 26. The effect of a single radiation exposure on the additional risk is discussed in section 5.9, which assumes that no response will be observed for the first 10 years, due to the latency of cancer development. There will then be a step change which lasts for 30 years, before the risk response returns to zero. When a series of these responses, which are delayed by five years each, are added together, the overall response is a pyramid shape, shown in Figure 27. The averaging is performed over the population that is at least age 40. The average life expectancy of this cohort is 22.3 years.

The cost of implementing the recommendations is given in the assessment as £45,000 per annum. This is based on 3,000 scans each costing £300, total cost £0.9m, and assuming 5% of this is taken as surplus (presumably after deducting for the costs of operating the scanner and staff costs). The undiscounted present value over the remaining lifetime of the individuals is then £45,000×22.3 = £1,003,500. The J-value for this scheme is 0.31, meaning that implementing COMARE's recommendations will give good value for money. This was also the conclusion of DH's impact assessment. The data is shown in Table 13. No uncertainty estimates are available for the cost of the scheme or the number of people. The tolerance limit is therefore only calculated from the parameters in which the uncertainty is already known. The tolerance limits are therefore small in this case. Again, discounting is not considered.

11.4 Department of Health's Proposal to Reduce the Number of MRSA Infections

Another regulatory impact assessment by the DH, which was published in 2009, reviewed proposals to reduce the number of MRSA infections and deaths in NHS hospitals, see Deparment of Health (2009) [62]. Although the number of MRSA infections had decreased by 74% since 2003, it was felt that there was still substantial variation across hospitals, and the DH believed that there was scope for further reductions. In the impact assessment, two options for reduction were considered. Option i) involved setting targets based around the median. Hospitals with infection rates above the median were required to reduce either to the median or by 20%, whichever was greater. Hospitals below the median were required to reduce by either 20% or to the lower quartile, whichever was least. Option ii) was for all hospitals with rates above the lower quartile to reduce to the lower quartile.

The report assumes that i) would lead to a reduction in MRSA deaths of 86.3 per year, whilst option ii) would reduce MRSA deaths by 109.3 per year. The ONS report that death rates for MRSA are highest amongst the over 85's [152], although MRSA can affect people of all ages. It will be assumed that the average age of death for MRSA is then 85 years. The life expectancy of an 85 year old is about 5.6 years. Thus is will be assumed that each MRSA death causes a loss of life expectancy of 5.6 years.

The assessment assumes that option i) would result in extra staff costs of £7.5 million whilst option ii) would result in extra staff costs of £19.08 million. It was also noted that these costs should be multiplied by 2.4 to account for lost opportunity costs associated with not being able to spend this money in other areas. There would also be some reduction in costs associated with avoided treatments of those who would otherwise have been infected. For option i) these benefits were £1.95 million per annum, whilst for option ii) these benefits were £2.47 million per annum. The total cost of i) was then £16.05 million, whilst for ii) the total cost was £43.32 million per annum. These details are then sufficient to calculate the J-value of the two options. The data is presented in Table 14. In assessing the tolerances, no attempt has been made to account for uncertainty on the cost of the scheme, as the data was not available. Equation (8.77) was used to estimate the standard deviation on the change in life expectancy. This calculation requires knowledge of the probability of being affected by MRSA, *b*. This is given in [62] as 6.3×10^{-5}, resulting from 3,211 MRSA cases in 2008. The standard deviation on the total change in life expectancy, $N\delta X$, can then be calculated as 0.65 years for option i), and 0.82 years for option ii).

As can be seen from Table 14, both options have J-values less than unity, and so offer good value for money. However, option i) has the lower J-value and so would be the preferred option. This was the same conclusion as in the impact assessment.

11.5 Example Calculations for the J_2 and J_T-Value: Mitigating Large Nuclear Accidents

This example uses notional, but realistic figures for a protection system that mitigates the risk of a large nuclear accident. The example is taken from [192]. Suppose an organisation with assets of £10 billion owns a nuclear power plant that has a lifetime of 50 years. It is considering installing a protection system that will reduce the frequency of large accidents from 2×10^{-5} per year to 5×10^{-8} per year. The new protection system would last the life of the plant and would cost $\delta\hat{W} = £4.5\text{M}$, a sum that would include all finance and maintenance costs. A risk analysis has shown that if an accident were to occur, then 5 workers would be killed immediately, while 40 would be exposed to a one-off dose of 300 mSv. Moreover, 500 members of the general public living in a small town close to the plant would receive a one-off dose of 200 mSv, while the remaining 5000 inhabitants of the same town would receive a single dose of 150 mSv. In addition, there would be environmental costs of £5 bn, covering evacuation, relocation, business disruption, decontamination and clean up, amongst others. Should the protection system be installed?

First, it is necessary to determine the average loss of life expectancy resulting from the accident. The dose to the members of each group, and their respective loss of life expectancy, is given in Table 15, where it is shown that the average loss of life expectancy for all those exposed is 0.4 years. These calculations assume a 0% discount rate. The collective loss of life expectancy is then 2,218 years. It was shown in Jones and Thomas (2009) [119] that the average change in life expectancy following a reduction in accident frequency over the lifetime at risk is approximately equal to the product of the average loss of life expectancy following a single accident, the lifetime and the change in frequency. Performing this calculation, the average change in life expectancy over the life of the plant with the given accident reduction is then 3.99×10^{-4} years, and the collective change in life expectancy is 2.2 years. The maximum reasonable spend on protection is then δV_N = £284,939.

The justifiable spend at risk neutrality can be determined from equation (10.6), as: B_0 = £3,165,746. It was shown in Thomas et al (2010a) [191] and Thomas and Jones (2010) [192] that the maximum risk multiplier in this situation is $m_{r\ \max} = 1.34$. Hence from equation (10.7), δZ_R = £4,242,100.

If it is assumed that the cost of the protection system can be partitioned into human costs and environmental costs, then it is possible to calculate the J_2-value. Suppose that, of the total amount $\delta\hat{W}$, an amount $3\times\delta V_N$ has been apportioned to human protection, where the factor of three may arise because of considerations of societal risk or gross disproportion. The J_2-value is then:

$$J_2 = \frac{\delta\hat{W} - 3\delta V_N}{\delta Z_R} = \frac{£4{,}500{,}00 - £854{,}817}{£4{,}242{,}100} = 0.86 \tag{11.1}$$

and thus, based on financial considerations alone, the scheme would represent good value for money. However, for a J_T-value analysis, it is necessary to consider all costs. The J_T-value is then:

$$J_T = \frac{\delta\hat{W}}{\delta Z_R + \delta V_N} = \frac{£4{,}500{,}00}{£4{,}242{,}100 + £284{,}939} = 0.99 \tag{11.2}$$

Thus, $J_T < 1$ and installation of the protection system would be justified.

11.6 J-Value Analysis of the Ancient VTPF

In chapter 2, it was noted that civilisations have been valuing life for millennia. The earliest known valuations of life date back to ca. 1700 BCE, with the Babylonian Code of Hammurabi, and 1400 BCE, with the Book of Leviticus. It was found, using extremely crude calculations, that these Ancient VTPF's were around £100-£400, in current prices. It is possible to perform a rudimentary J-value analysis of these valuations to determine the cost-effectiveness of the health and safety policies of ancient civilisation. Of course, the analysis will not have a high degree of accuracy, but it may nevertheless prove to be informative.

The J-value of the VTPF is given by a rearranged version of either equation (7.13) or (7.14). Discounting will not be included, and so these equations will be identical. Therefore:

$$J = \frac{(1-\varepsilon)V_P}{GX} \tag{11.3}$$

where V_P will be taken to lie in the range £100-£400. Estimates of the world GDP per person have been made for times stretching back to 1 Million BCE [47]. For 1600 BCE (the closest date to the VTPF estimates), the global GDP per person is given as $121, the units of currency are 1990 international dollars. International dollars are dollars that have been adjusted for purchasing power parity (PPP). This can be converted into 1990 UK pounds by multiplying by the ratio of current UK GDP to UK GDP measured in international dollars, which is given by the IMF (2011) [114]. This ratio is about 0.645. The figure can then be adjusted for inflation using ONS time series data on the GDP [153], which amounts to multiplying by 2.26, to give the world GDP per person in 1600 BCE in 2010 UK pounds. This value is £177. It will be assumed that the world GDP per person in 1600 BCE is a sufficiently good estimator of the GDP per person in the Mesopotamian and Eastern Mediterranean region around this time.

In order to estimate the average life expectancy, life table data from ancient Rome was obtained [194]. It is assumed the mortality experience in ancient Rome was similar to that of the civilisations being assessed. The data gives both life expectancy and the population distribution, from which the average life expectancy can be calculated. This was found to be 29 years, although the figure is strongly affected by infant mortality.

The final parameter that needs estimating is the ancient risk aversion, ε. To estimate this, it is necessary to first estimate the ancient work-time fraction, w, and the ancient wage share of the GDP, θ. In section 8.5, it was noted that the wage share is predicted to be constant over time and across countries. It was also noted that this has been experimentally verified. It will be assumed, then, that this constant wage share can be extrapolated back to ancient civilisations. As the UK wage share was found to be about 58%, it will be assumed that the ancient wage share is similar to this. A rounded figure of 60% will therefore be used. The work time fraction is estimated by assuming that individuals would spend the majority of their life working, and so would have little free time. If it is assumed that an individual will commence work at age 8, and will work for the rest of his life, until age 50, and that he will work for one hundred hours a week, then his work-time fraction will be 0.5. Similar figures would also apply to most individuals in the society, so that this figure would be appropriate as an average work time fraction. This then enables the risk aversion to be calculated. However, this raises an immediate problem.

With the figures given above, the risk aversion is about -0.7, i.e. it is negative, indicating risk seeking behaviour. So far, it has been assumed that the fraction of time spent working will be low enough to give risk averse behaviour, which in turn is required if the law of diminishing marginal utility is to be satisfied. This law, that successive amounts of a commodity will be valued at a diminishing rate, is one of the most well established laws in utility theory. However, in the situation where considerable proportions of an individual's life would be spent working, then risk aversion is negative, and the marginal utility increases with the amount of commodity. Thus, in order to proceed with this analysis this law must be given up here. However, the effect of long working hours being associated with risk seeking behaviour and increasing marginal utility is an interesting result which may be considered further in the future.

Thus, the J-value of the ancient VTPF may now be calculated:

$$J = \frac{(1-\varepsilon)V_P}{GX} = \frac{1.7 \times V_P}{£177 \times 29} \approx \frac{V_P}{£3{,}000} \tag{11.4}$$

thus, for values of V_P in the range £100 - £400, the J-value of the VTPF is in the range 0.03 – 0.13. If the work-time fraction is varied up to a high value of 0.8, then the J-value is still considerably less than unity, at 0.52. Thus, this fairly rudimentary analysis indicates that the ancient VTPF's were cost-beneficial.

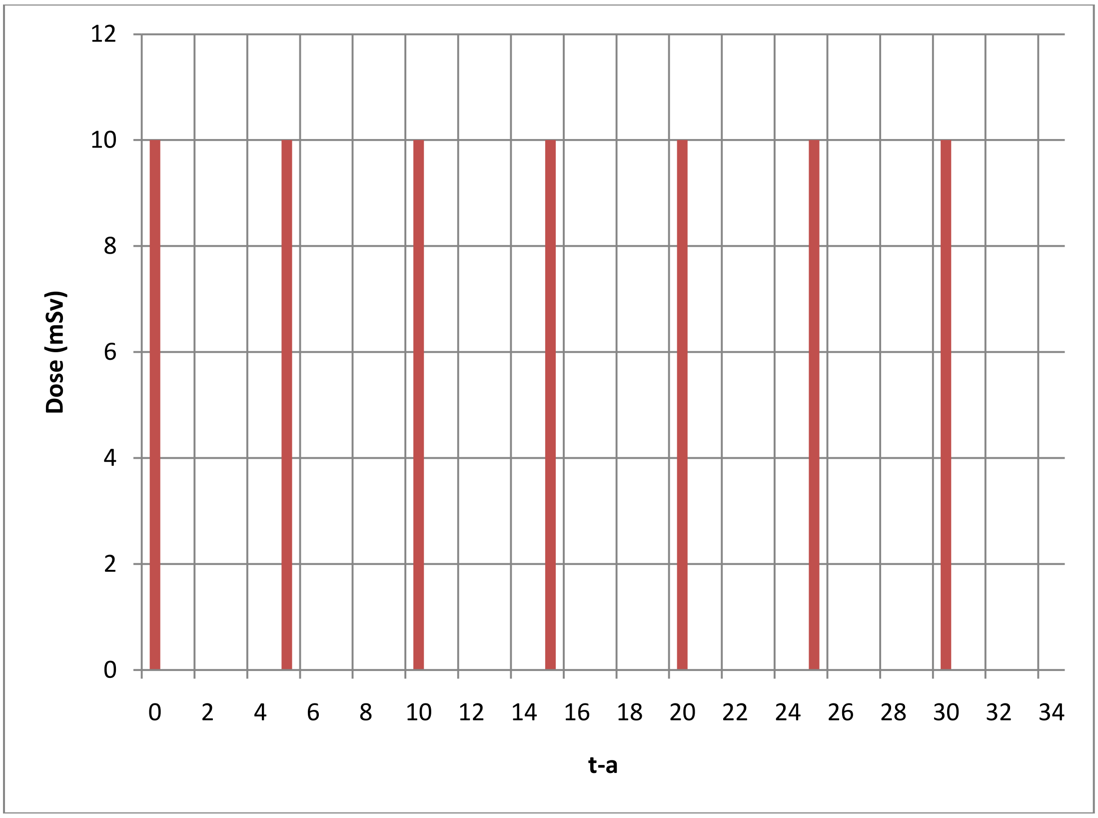

Figure 26 Dose received by individual of age *a* who is undergoing scans at future age *t*.

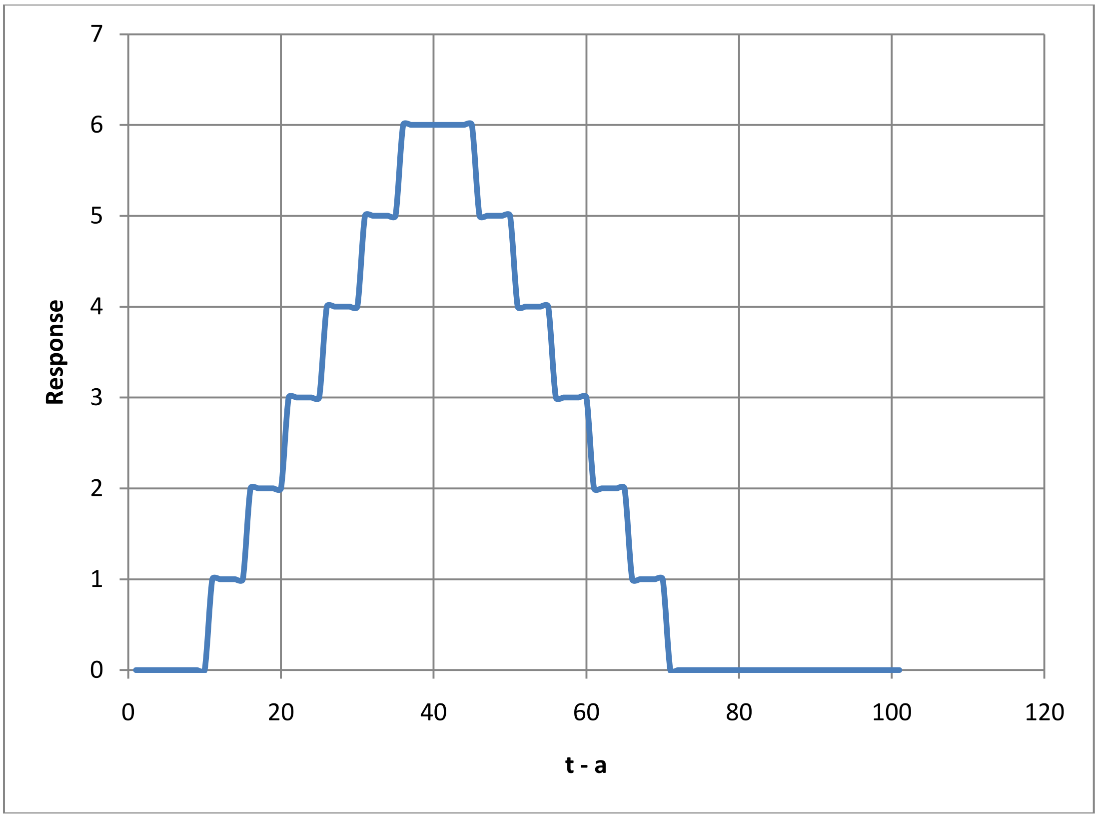

Figure 27 The response of the additional risk faced by an individual of current age *a* at future age *t*, following an exposure type given in Figure 26.

Regulatory Exposure Limit (mg. m^{-3})	Lung Cancer Deaths Avoided	Silicosis Deaths Avoided	Lung Cancer Life-Years Gained	Silicosis Life-Years Gained	Total Life Years Gained, $N\delta X$ (±1 S.D)
i) 0.3	36	36	497	262	759 (±29.2)
ii) 0.1	185	185	2,553	1,348	3,900 (±150)
iii) 0.05	300	300	4,139	2,186	6,325 (±243)
iv) 0.01	455	455	6,278	3,315	9,593 (±369)

Table 11 Deaths avoided and life-years gained for the four exposure limits from HSE's assessment of methods to reduce occupational exposures to respirable crystalline silica.

Regulatory Exposure Limit (mg.m^{-3})	Average Cost of Scheme (£M) (±1 S.D)	J-value (95% Tolerance Limit - ±2 σ_J/J)
i) 0.3	5.2 (±0.05)	0.050 (0.048-0.058)
ii) 0.1	644.0 (±3.06)	1.3 (1.2-1.4)
iii) 0.05	3,528.0 (±38.3)	4.3 (3.9-4.8)
iv) 0.01	13,343.5 (±673.2)	11 (9.3-12)

Table 12 Cost of scheme and J-values using Table 11 data.

Proposal to Implement COMARE's Recommendations	Individual change in Average Life Expectancy, δX (years)	Initial Life Expectancy, X (years)	Cost, = 45,000*X (£)	J-Value (N = 3,000) (95% Tolerance Limit - ±2 σ_J/J)
	8.3×10^{-3}	22.3	1,003,500	0.31 (0.30-0.33)

Table 13 Data for DH's proposal to implement COMARE's recommendations.

Proposal to Reduce Number of MRSA Deaths	Annual Cost (£)	Annual Life Years Gained, $N\delta X$ (years)	J-Value (95% Tolerance Limit - ±2 σ_J/J)
i)	16,050,000	479.0	0.26 (0.25-0.28)
ii)	43,320,000	606.6	0.55 (0.52-0.59)

Table 14 Data for DH's proposal to reduce the number of MRSA deaths.

Group	Group Size	Dose (Sv)	Loss of Life Expectancy per Person (year)
Public	5000	0.15	0.354
Public	500	0.2	0.472
Plant Operators	5	Killed immediately	38.795
Plant Operators	40	0.3	0.401
Average loss of life expectancy per person, δX (years)			0.400
Collective loss of life expectancy, $N\delta X$ (years)			2,218

Table 15 Loss of life expectancy to public and workers following a notional large nuclear accident.