Dependent Risk Modelling and Ruin Probability: Numerical Computation and Applications

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Declaration A

I herewith declare that I have produced this thesis without assistance of any third parties other than the co-authors of the papers. Additionally, without making use of aids other than those specified: notions taken over directly or indirectly from other sources have been identified as such. This thesis have not previously been presented in identical or similar form to any other UK or foreign examination board.

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Shouqi Zhao

Declaration B

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Abstract

In this thesis, we are concerned with the finite-time ruin probabilities in two alternative dependent risk models, the insurance risk model and the dual risk model, including the numerical evaluation of the explicit expressions for these quantities and the application of the probabilistic results obtained. We first investigate the numerical properties of the formulas for the finite-time ruin probability derived by Ignatov and Kaishev (2000, 2004) and Ignatov et al. (2001) for a generalized insurance risk model allowing dependence. Efficient numerical algorithms are proposed for computing the ruin probability with a prescribed accuracy in order to facilitate the following studies. We then propose a new definition of alarm time in the insurance risk model, which generalizes that of Das and Kratz (2012), expressed in terms of the joint distribution of the time to ruin and the deficit at ruin. The alarm time is devised to warn that the future ruin probability within a finite-time window has reached a pre-specified critical level and capital injection is required. Due to our definition, the implementation of the alarm time highly relies on the computation of the finite-time ruin probability, which utilizes the previous results on computing the ruin probability with a prescribed accuracy. The results of the ruin probability and the alarm time are then transferred nicely to a generalized dual risk model, whose name stems from its duality to the insurance risk model, through an enlightening link established between the two risk models. Finally, based on the two alternative risk models, we introduce a framework for analyzing the risk of systems failure based on estimating the failure probability, and illustrate how the probabilistic models and results obtained can be applied as risk analytic tools in various practical risk assessment situations, such as systems reliability, inventory management, flood control via dam

management, infection disease spread and financial insolvency.

Chapter 1

Introduction

This thesis focuses on the ruin probability within a finite time horizon in dependent risk models and the numerical implementation and applications. We start from insurance risk model. The classical insurance risk model assuming independence among claim severities and claim arrivals has been believed unrealistic and cannot meet the needs of practical risk modelling in reality. Research on ruin probability beyond the classical risk model has intensified significantly in recent year. More general ruin probability models assuming dependence between claim amounts and/or claim arrivals and non-linear aggregate premium income have been considered in the actuarial and applied probability literature. Such models are better suited to reflect the dependence in the arrival and severity of losses generated by portfolios of insurance policies. Exploring ruin probability theoretically and numerically, under these more general dependence assumptions, is of utmost importance within the Solvency II framework of internal insolvency-risk model building.

For this purpose, we consider the generalized insurance risk model first considered by Ignatov and Kaishev (2000, 2004) and Ignatov et al. (2001), where the premium income function is assumed an arbitrary non-negative non-decreasing real function of time only, and the claim amounts are assumed arbitrarily distributed with any dependent structures, following a homogeneous Poisson claim arrival process. Under such reasonably general assumptions, we explore the explicit ruin probability formulas and their efficient numerical implementation to facilitate the following studies, and demonstrate that the formulas are useful not only theoretically but also in computing ruin probabilities in the dependent risk model. The latter is important in practical applications. For example, as recently pointed out by Das and Kratz (2012), the need to evaluate the Ignatov-Kaishev ruin probability formulas naturally arises in the context of designing early warning systems against ruin of insurance companies. This need also arises in the context of reserving and risk capital allocation in particular, for operational risk, see Kaishev et al. (2008).

We then propose a new definition of alarm time in the insurance risk model, which generalizes that of Das and Kratz (2012), expressed in terms of the joint distribution of the time to ruin and the deficit at ruin. The key inspiration for this part of research is the idea introduced by Kaishev et al. (2008) that, instead of locking up a significant amount of reserve capital initially, part of it could be invested more profitably and reserved at a later instant, without sacrificing the predetermined overall solvency target. Kaishev et al. (2008) demonstrate that allocating capital in two portions, one initially and one at a later instant, leads to the same (99%) non-ruin probability as in the case of the entire capital being reserved at the start of the period. In order to have a fair comparison, the two strategies assume equal amount of premium (and capital) accumulated at the end of the period, but different premium rates. The approach of Kaishev et al. (2008) has been recently extended by Das and Kratz (2010, 2012) who base their framework on the notion of alarm time. The latter is a future time instant at which short-term ruin probability is alarmingly high and exceeds the predetermined threshold level. The additional portion of capital could then be reserved at the alarm time so that the probability of ruin falls below the threshold level. It should be noted that such a capital allocation strategy is determined at the start of the reserving period and is reflected in the capital and future premium income function which models the allocation of the portions of risk capital and the accumulation of premiums over time.

Due to our newly proposed definition of alarm time, its implementation highly relies on the computation of the finite-time ruin probability, which utilizes the previous results on computing the ruin probability with a prescribed accuracy. The new definition also involves the joint distribution of the time to ruin and the deficit at ruin, motivated by the idea that, even though the company may get ruined, the deficit at ruin may be small, allowing the company to easily borrow and recover. Therefore, incorporating deficit at ruin in the definition of alarm time allows one to emphasize only ruin cases with significant deficit. We therefore derive new expressions for the joint distribution of the time to ruin and the deficit at ruin, under more general assumptions.

The results of the ruin probability and the alarm time in the insurance risk model are then transferred nicely to a generalized dual risk model, whose name stems from its duality to the insurance risk model, through an enlightening link established between the two risk models. As noted by Avanzi et al. (2007), while the insurance risk model is suitable for modelling insurance risks, the dual risk model has a wider application. It describes well the operation of companies specializing in geological exploration of minerals and petroleum, pharmaceutical research, and technological discoveries and inventions, where routine operations generate continuous expenses over time and occasional discoveries or inventions bring stochastic capital gains to the company. It can also be applied in modelling the operation of research and development departments from companies in other industries. Or alternatively, these could be banks, hedge funds or other investment companies, receiving capital gains from their investment and other financial operations, while at the same time experiencing permanently accumulating operational expenses. Thus, the results obtained in the dual risk model are highly applicable for modelling the insolvency of such institutions.

Finally, we turn our attention to the practical application of the dependent risk models introduced and the probabilistic results obtained in this thesis. Based on the two alternative risk models, we introduce a framework for analyzing the risk of systems failure based on estimating the failure probability, and illustrate how the probabilistic models and results obtained can be applied as risk analytic tools in various practical risk assessment situations, such as systems reliability, inventory management, flood control via dam management, infection disease spread and financial insolvency.

1.1 Chapter summaries

This thesis is organized as a series of papers, each of which is presented in a separate chapter. All the four chapters/papers have been submitted to peer reviewed journals. It is worth pointing out that all the papers are based on joint work with my PhD supervisors. In what follows, we summarize the main results of each chapter, and a list of the publications arising from this thesis is provided in Section 1.2.

In Chapter 2, we consider a generalized insurance risk model assuming an arbitrary non-negative, non-decreasing premium income function, possibly dependent claim severities with any dependence structures, following a homogeneous Poisson claim arrivals, and focus on the finite-time ruin probability in this dependent risk model. First, we summarize the explicit ruin probability formulas which appear in the papers by Ignatov and Kaishev (2000, 2004) and Ignatov et al. (2001) and provide their unified treatment in terms of classical Appell polynomials. This is achieved by establishing some enlightening connections between these formulas. Second, we consider their efficient numerical implementation, and demonstrate that the formulas are useful not only theoretically but also in computing ruin probabilities in the generalized insurance risk model. For this purpose, we propose a method of computing survival probability with a prescribed accuracy and introduce and examine a simulation-based method employing order statistics proposed by Dimitrova and Kaishev (2013). Extensive numerical experiments and comparisons are provided, covering cases of both discrete and continuous, dependent and independent claim severities.

Chapter 3 presents a new definition of an alarm time in the insurance risk model, which generalizes that of Das and Kratz (2012). The insurance risk model considered in this chapter is more general than that considered in Chapter 2, as we here further relax the assumption of Poisson claim arrivals. In our new definition of alarm time, we incorporate not only the finite-time ruin probability, but also the deficit at ruin. The motivation for this is that, even though the company may get ruined, the deficit at ruin may be small, allowing the company to easily borrow and recover. Therefore, incorporating deficit at ruin in the definition of alarm time allows one to emphasize only ruin cases with significant deficit. In order to numerically evaluate the alarm time in the more general insurance risk model, we summarize existing ruin formulas and also derive some new closed-form expressions for the joint probability of the ruin time and the deficit at ruin under the relaxed assumptions, in terms of exponential Appell polynomials, introduced by Ignatov and Kaishev (2012). Based on our new definition of alarm times, we formulate an optimal dynamic capital allocation problem. Based on its numerical solution, we demonstrate that reserving capital sequentially in portions at the alarm times, rather than reserving the capital initially as a lump sum, leads to higher finite-time survival probability. We therefore demonstrate such dynamic capital allocation strategies are very appealing from a solvency risk management perspective. Extensive numerical examples are also provided.

In Chapter 4, we turn our attention to the dual risk model, whose name stems from its duality to the insurance risk model. We consider a generalized dual risk model assuming any non-negative non-decreasing cumulative operational cost function and arbitrary capital gain arrival process and focus on the ruin probability over a finite horizon, which has not been considered before. First, by establishing an enlightening connection between the two models, a trajectory hitting an upper bound and a trajectory hitting a lower bound, we link the dual risk model with its corresponding insurance risk model. By revisiting the formulas of survival probability in two reasonably general insurance risk models considered by Ignatov and Kaishev (2004) and Ignatov and Kaishev (2012), we obtain explicit formulas for the finite-time survival probability in our generalized dual risk model for exponential and Erlang capital gains, in terms of classical Appell polynomials and the exponential Appell polynomials, respectively. The results are then generalized to the case where capital gains follow a linear combination of exponential distributions or a hyperexponential distribution. The latter result is then used to obtain the survival probability for arbitrarily distributed capital gains, including heavy-tailed families. We further relax the independence assumptions in our dual risk model and introduce certain dependence structures between capital gains and/or inter-arrival times in order to make the model more realistic. Finally, we address the problem of risk capital allocation in the dual risk model, which to the best of our knowledge, has not been previously considered in the literature. We base our approach on the ideas of Kaishev et al. (2008) of distributing the initial capital over a finite-time horizon without affecting a fixed desired sufficiently high level of survival probability in the insurance risk model. These ideas have been further extended by Das and Kratz (2012), who introduced the concept of alarm time, an early warning system to the problem of risk capital allocation. In our work, we transfer these ideas and concepts to the dual risk model and illustrate them numerically.

In the last chapter, we are concerned with the possible applications of the two risk models considered in this thesis. Chapter 5 develops a framework for analyzing the risk of systems failure based on estimating the failure probability. The latter is defined as the probability that a certain risk process, characterizing the operations of a system, reaches a possibly time-dependent critical risk level within a finite-time interval. Under general assumptions, we utilize the probabilistic results obtained in previous chapters for the failure probability and also the joint probability of the time of the occurrence of failure and the excess of the risk process over the risk level. We illustrate how to interpret the model parameters of the two alternative risk models in order to reflect the specifics of the concrete practical risk assessment problems and how the probabilistic results obtained can be successfully applied in several important areas of risk analysis among which systems reliability, inventory management, flood control via dam management, infection disease spread and financial insolvency. Numerical illustrations are also presented.

1.2 Publications arising from this thesis

Chapter 2: *On the evaluation of finite-time ruin probabilities in a dependent risk model.*
This chapter has been submitted to a peer reviewed journal as:

Dimitrova, D.S., Kaishev, V.K., Zhao, S. 2014. On the evaluation of finite-time ruin probabilities in a dependent risk model.

Chapter 3: *Early warning of bankruptcy and risk capital allocation based on the time to ruin and the deficit at ruin.*
This chapter has been submitted to a peer reviewed journal as:

Dimitrova, D.S., Kaishev, V.K., Zhao, S. 2014. Early warning of bankruptcy and risk capital allocation based on the time to ruin and the deficit at ruin.

Chapter 4: *On finite-time ruin probabilities in a generalized dual risk model with dependence.*
This chapter has been submitted to a peer reviewed journal as:

Dimitrova, D.S., Kaishev, V.K., Zhao, S. 2014. On finite-time ruin probabilities in a generalized dual risk model with dependence.

Chapter 5: *Modelling finite-time failure probabilities in risk analysis applications.*
This chapter has been submitted to a peer reviewed journal as:

Dimitrova, D.S., Kaishev, V.K., Zhao, S. 2014. Modelling finite-time failure probabilities in risk analysis applications.

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Chapter 2

On the evaluation of finite-time ruin probabilities in a dependent risk model

On the evaluation of finite-time ruin probabilities in a dependent risk model

Abstract

This paper establishes some enlightening connections between the explicit formulas of the finite-time ruin probability established by Ignatov and Kaishev (2000, 2004) and Ignatov et al. (2001) for a risk model allowing dependence. The numerical properties of these formulas are investigated and efficient algorithms for computing ruin probability with prescribed accuracy are presented. Extensive numerical comparisons and examples are provided.

Keywords: finite time ruin probability, dependent risk modelling, Appell polynomials, numerical implementation, order statistics

Chapter 3

Early warning of bankruptcy and risk capital allocation based on the time to ruin and the deficit at ruin

Early warning of bankruptcy and risk capital allocation based on the time to ruin and the deficit at ruin

Abstract

We present a new definition of an alarm time, which generalizes that of Das and Kratz (2012), and is expressed in terms of the joint distribution of the ruin time and the deficit at ruin under a general insurance risk model with dependence. In order to evaluate the alarm time, we summarize existing ruin formulas and also derive some new closed-form expressions for the joint probability of the ruin time and the deficit at ruin, in terms of exponential Appell polynomials, introduced by Ignatov and Kaishev (2012b). Based on our new definition of alarm times, we formulate an optimal dynamic capital allocation problem. Based on its numerical solution, we demonstrate that reserving capital sequentially in portions at the alarm times, rather than reserving the capital initially as a lump sum, leads to higher finite-time survival probability. We therefore demonstrate such dynamic capital allocation strategies are very appealing from a solvency risk management perspective. Extensive numerical examples are also provided.

Keywords: alarm time, joint distribution of the ruin time and the deficit at ruin, exponential Appell polynomials, capital allocation, reserving

3.1 Introduction

Problems of allocating and managing risk capital in insurance companies have attracted considerable attention in the actuarial science literature. Allocating capital between different business units or lines of business has been extensively explored (see e.g. Dhaene et al. 2003; Tsanakas 2004, 2009; Dhaene et al. 2012). Under an alternative approach (see Embrechts and Samorodnitsky 2003; Embrechts et al. 2004), risk capital is reserved at the start of a period, so as to ensure that the probability of the company's future ruin is less than a predetermined solvency target level. It can be argued however, that instead of locking up a significant amount of reserve capital initially, part of it could be invested more profitably and reserved at a later instant, without sacrificing the predetermined overall solvency target. This time-distributed (dynamic) capital allocation approach has first been considered by Kaishev et al. (2008), who demonstrate that allocating capital in two portions, one initially and one at a later instant, leads to the same (99%) non-ruin probability as in the case of the entire capital being reserved at the start of the period. In order to have a fair comparison, the two strategies assume equal amount of premium (and capital) accumulated at the end of the period, but different premium rates. The approach of Kaishev et al. (2008) has been recently extended by Das and Kratz (2010, 2012) who base their framework on the notion of alarm time. The latter is a future time instant at which short-term ruin probability is alarmingly high and exceeds the predetermined threshold level. The additional portion of capital could then be reserved at the alarm time so that the probability of ruin falls below the threshold level. It should be noted that such a capital allocation strategy is determined at the start of the reserving period and is reflected in the capital and future premium income function which models the allocation of the portions of risk capital and the accumulation of premiums over time.

A system of alarm times at which certain portions of capital should be reserved for the purpose of reducing finite-time ruin risk is referred by Das and Kratz (2012) as alarm system of an insurance company. The authors have also examined the effectiveness of the proposed strategy by numerically comparing the finite- and infinite-time ruin probabilities in the scenarios with and without an alarm system. Their comparison results show that the model employing an alarm system provides a higher probability of survival in a longer time horizon. Analytical bounds for the difference between the ruin probabilities in the two cases are also obtained. It is worth noting that, the alarm time, as defined by Das and Kratz (2012), requires only the evaluation of finite-time ruin (survival) probability. However, instead of considering analytic expressions for the finite-time ruin probability, the authors compute it by simulation, which may introduce significant (simulation) error in the evaluation of the alarm time. The reason for this, as noted by the authors, is that the time unit with which the alarm time is computed is not specified in absolute terms and a unit of time may correspond to a long period. Therefore, a small inaccuracy in the simulated alarm time can make a big difference in absolute terms. It should also be mentioned that Das and Kratz (2012) do not consider the problem of how much capital to allocate at each alarm time, which is in fact the most important question in practical risk capital allocation.

This paper can be viewed as a follow-up from Das and Kratz (2012), and our objective is four-fold. First, we propose a new definition of alarm time which generalizes that of Das and Kratz (2012). In our new definition (see Definition 3.2.2), we incorporate not only the finite-time ruin probability, but also the deficit at ruin. The motivation for this is that, even though the company may get ruined, the deficit at ruin may be small, allowing the company to easily borrow and recover. Therefore, incorporating deficit at ruin in the definition of alarm time allows one to emphasize only ruin cases with significant deficit. Second, as discussed above, since simulation may introduce significant errors, we give explicit expressions for the finite-time survival probability and the joint probability of the time to ruin and the deficit at ruin, which we also use to evaluate the alarm time numerically. Explicit expressions for these two probabilities, under a reasonably general risk model allowing dependence, are summarized in Section 3.2.2. Some new expressions are also derived under further extensions of the model, which is our third contribution in the paper (see Theorem 3.5.1 in the Appendix). Our fourth contribution is related to suggesting an optimal way of determining the amount of capital which needs to be allocated at the alarm times as part of an alarm system for an insurance company (see Section 3.3). We recall that such an optimal capital allocation problem has not been considered by Das and Kratz (2012).

The paper is organized as follows. In Section 3.2.2, we introduce the framework we are concerned with in this paper and the related assumptions and notations. Various expressions for the finite-time survival probability and the joint probability of the time to ruin and the deficit at ruin are provided under different assumptions. We then generalize the definition of alarm time proposed by Das and Kratz (2012) in Section 3.2.1, where we incorporate in the new definition both the ruin probability and the deficit at ruin. We study the impacts of different parameters in the model with extensive numerical examples and parameter settings, covering both discrete and continuous, dependent and independent claim severities. In Section 3.2.4, we devise a system with multiple alarms based on the definition of a single alarm. Section 3.3 is devoted to studying the capital allocation strategies in the alarm system. Optimization problems of capital allocation are formulated and attempted in simple cases with only one and two alarms in Sections 3.3.1 and 3.3.2 respectively. Section 3.4 concludes the paper.

3.2 Alarm times and alarm systems

In this section we introduce the concepts of alarm time and alarm system and demonstrate how the latter can be used to allocate risk capital over time. For the purpose we first introduce the insurance risk model and related notation.

3.2.1 Definition of an alarm time

We start by considering the notion of alarm time, as defined by Das and Kratz (2012), and then we provide its further generalization. Under general insurance risk model, which has first been considered in Ignatov and Kaishev (2000, 2004) and Ignatov et al. (2001), the possibly dependent random variables $W_1, W_2, \ldots$ denote claim severities, and $Y_1, Y_2, \ldots$ denote their partial sums, with joint density $f_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k)$ or joint probability mass function $p_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k)$ in case $Y_1, Y_2, \ldots$ are discrete. Let $\tau_1, \tau_2, \ldots$ denote the claim inter-arrival times assumed independent, identically distributed random variables. We first assume $\tau_1, \tau_2, \ldots$ follow an exponential distribution with mean $1/\lambda$, i.e. $\tau_i \sim$Exp(λ), $i = 1, 2, \ldots$, an assumption which will later be relaxed (see Theorem 3.5.1). Thus, the number of claims up to time t is modelled by the Poisson process $N_t = \max\{i : \tau_1 + \cdots + \tau_i \leq t\}$, $t > 0$. We denote by $T_1, T_2, \ldots$ the arrival times of consecutive claims, i.e. $T_i = \tau_1 + \ldots + \tau_i$, $i = 1, 2, \ldots$. Let u_0 be the initial capital at time 0 and $g(t)$ denote the cumulative premium income function of an insurance company (or a particular line of business), which is assumed a non-negative and non-decreasing continuous real-valued function defined on $\mathbb{R}_+$ with $g(0) = 0$. Thus, $h(t) = u_0 + g(t)$ is the cumulative capital and premium income function. For brevity, we will refer to $h(t)$ as the capital-premium function. It is worth mentioning that the function $h(t)$ may have discontinuities, resulting from e.g. lump-sum capital injections, in which case we define $h^{-1}(z) = \inf\{t : h(t) \geq z\}$. The insurance company's surplus process is expressed as $R_t = h(t) - S_t$, where $S_t = Y_{N_t}$ is the aggregate claim amount process. The instant of ruin T is thus defined as

$$T := \inf\{t : t > 0,\ R_t \leq 0\},$$

or $T = \infty$ if $R_t \geq 0$ for all t, and the deficit at ruin Y is defined as $Y = R_{T_+}$ if $T < \infty$. We consider a finite-time interval $[0, x]$, and denote by $P(T > x)$ and $P(T < x, Y > y)$ the probability of non-ruin in $[0, x]$ and the probability that ruin occurs before time x with a deficit Y exceeding y. It should be noted that we consider this general insurance risk model since it incorporates jumps in the capital-premium function $h(t)$ and therefore allows to implement the definitions of alarm time (and capital allocation strategies) which follow. We note that this has not been possible under the classical insurance risk model, where $h(t)$ is assumed continuous, strictly linearly increasing.

We will address the following capital allocation problem. A total risk reserve capital $u = u_0 + u_1$ of an insurance company (or a line of business) is available at time 0. Instead of reserving the entire capital u, only part of it, u_0, is reserved initially. A second part, u_1, is released (for possible investment) and earns interest with a certain interest rate r. At an appropriate later moment of time, say t_A, the accumulated amount $u_1 e^{rt_A}$ will then be reserved. The following question then arises. Given initial capital u_0 at time 0, is there a future time instant, $t_A > 0$, before which ruin is very unlikely, but becomes very probable shortly after, with ruin probability exceeding an alarmingly high threshold level. If t_A exists, it can be called the alarm time, at which the additional capital, $u_1 e^{rt_A}$, should be added (injected) in the capital-premium function, $h(t)$, so as to assure that the probability of ruin falls below a required threshold level. This approach provides a strategy for a dynamic allocation of a total risk capital amount $u = u_0 + u_1$ in two portions, u_0 initially at time $t = 0$, and u_1 later at time t_A. It should be noted that such a strategy is determined at time 0 by the capital-premium function $h(t) \equiv h_0(t) = u_0 + g(t)$, which will then be accordingly modified and becomes $h_1(t) = u_0 + u_1 e^{rt_A} \mathbb{I}_{\{t \geq t_A\}} + g(t)$. The question of how to determine u_0 and u_1 optimally is addressed in Section 3.3. The alarm time t_A has been given the following formal definition by Das and Kratz (2012).

Definition 3.2.1 (Das and Kratz (2012)) *The alarm time* $t_A = t_A(a, b, d; h_0(t))$ *is defined as*

$$\begin{aligned} t_A &= inf\{t > 0 : P(T \leq t + d | T > t) \geq 1 - a \ and \ P(T > t) \geq 1 - b\} \\ &= inf\left\{t > 0 : P(T > t) \geq max\Big(1 - b, \frac{1}{a}P(T > t + d)\Big)\right\}, \end{aligned} \tag{3.1}$$

where a *and* b *are prescribed probabilities and* d *represents the length of a pre-specified future time interval (window).*

As noted by Das and Kratz (2012), the value of the parameter b should be considerably small to ensure that the probability of ruin before the alarm time (i.e. on $[0, t_A]$) is minimal; the value of the parameter a needs to be moderately small (but not too small), so that the prospect of ruin within $[t_A, t_A + d]$ is realistic and a remedial action (e.g. topping up capital) will be required; the length of the time interval d has to be moderate, neither very small, which leaves little possibility for remedial action to be effected, nor very large, which indicates that ruin may occur in the distant future and there is no strong immediate likelihood of it at time t_A. In fact, a and d could be inter-related.

Clearly, in order to compute the alarm time specified in Definition 3.2.1, one only needs to be able to compute the finite-time survival probability $P(T > t)$. For this purpose, one can use the explicit expressions summarized in Section 3.2.2 under various assumptions for the risk model parameters.

It is reasonable to assume, however, that in practice, when a technical ruin occurs and the deficit at ruin is insignificant, the insurance company would be able to cover it through external funding and remain operational.

Therefore, it is reasonable to incorporate the deficit at ruin in the definition of alarm time and highlight only ruin cases with more significant deficit at ruin. More precisely, under Definition 3.2.2 proposed below, instead of detecting a time window $[t_A, t_A + d]$ within which ruin probability is alarmingly high (i.e. higher than a critical level b), we consider the likelihood that ruin within $[t_A, t_A + d]$ occurs in combination with a large deficit at ruin $Y > y$, where y is a predetermined threshold level. Therefore, this new definition is a refinement of Definition 3.2.1, which dismisses ruin cases when the deficit at ruin is too small to cause serious concerns for the company.

Definition 3.2.2 *The alarm time $t_A = t_A(a, b, d; h_0(t), y)$ is defined as*

$$\begin{aligned} t_A &= inf\{t > 0 : P(T \le t + d, Y > y | T > t) \ge 1 - a \text{ and } P(T > t) \ge 1 - b\} \\ &= inf\{t > 0 : P(T \le t + d, Y > y) - P(T \le t, Y > y) \ge (1 - a)P(T > t) \\ &\qquad \text{and } P(T > t) \ge 1 - b\}, \end{aligned} \tag{3.2}$$

where Y denotes the deficit at ruin and y is a pre-defined threshold level.

Clearly, Definition 3.2.2 is more general and includes Definition 3.2.1 as a special case when $y = 0$. It is worth pointing out that the two conditions in (3.2) may not always be fulfilled simultaneously, i.e. we may not necessarily be able to find such a time point t_A in all cases, because, for all possible values t such that $P(T > t) \ge 1 - b$, the ruin probability (jointly with the deficit at ruin) may not increase significantly within the future time interval $[t, t + d]$ for fixed a, d and y. In Das and Kratz (2012), such cases are referred to as "no alarm". However, under Definition 3.2.2, we define the alarm time in such cases as

$$t_A = \inf\{t > 0 : P(T > t) < 1 - b\}, \tag{3.3}$$

i.e. the alarm time is when the survival probability drops below the prescribed critical level, which makes Definition 3.2.2 more comprehensible, avoiding "no alarm" situations appearing in Tables 2 and 3 of Das and Kratz (2012).

As can be seen from (3.2), computing the alarm time given by Definition 3.2.2 requires computing not only the finite-time survival probability, but also the joint distribution of the time to ruin and the deficit at ruin, $P(T < t, Y > y)$. Explicit expressions for the latter joint distribution under various assumptions for the risk model parameters will also be provided in Section 3.2.2.

3.2.2 Explicit expressions for $P(T > x)$ and $P(T < x, Y > y)$

In this section, we summarize the existing explicit expressions for the finite-time survival probability and the joint distribution of the time to ruin and the deficit at ruin in the insurance risk model described in Section 3.2.1. We also provide some explicit results for $P(T > x)$ and $P(T < x, Y > y)$ under more general model assumptions. We note that in order for these expressions to be applied to compute t_A, x should be replaced by t.

Under the insurance risk model discussed in Section 3.2.1, the following explicit expression for the probability of non-ruin within a finite-time interval $[0, x]$, $P(T > x)$, assuming continuous claim severities, has been derived in Ignatov and Kaishev (2004),

$$P(T > x) = e^{-\lambda x}\Big(1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_{y_1}^{h(x)} \cdots \int_{y_{k-1}}^{h(x)} A_k(x; \nu_1, \ldots, \nu_k) \times f_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k) dy_k \ldots dy_1\Big), \quad (3.4)$$

where $\nu_k = h^{-1}(y_k)$, and $A_k(x; \nu_1, \ldots, \nu_k)$ for $k = 1, 2, \ldots$ are the classical Appell polynomials of degree k with a coefficient in front of x^k equal to $1/k!$, uniquely defined as

$$\begin{aligned} A_0(x) &= 1, \\ A'_k(x; \nu_1, \ldots, \nu_k) &= A_{k-1}(x; \nu_1, \ldots, \nu_{k-1}), \\ A_k(\nu_k; \nu_1, \ldots, \nu_k) &= 0, \; k = 1, 2, \ldots, \end{aligned}$$

where $\nu_1 \leq \ldots \leq \nu_k$, $\nu_i \in \mathbb{R}$. It can directly be seen that formula (3.4) is also valid for discrete claim severities (see Dimitrova et al. 2013a) in which case it takes the form:

$$P(T > x) = e^{-\lambda x}\Big(1 + \sum_{k=1}^{n} \lambda^k \sum_{y_1=1}^{n-(k-1)} \sum_{y_2=y_1+1}^{n-(k-2)} \cdots \sum_{y_k=y_{k-1}+1}^{n} A_k(x; \nu_1, \ldots, \nu_k) \times p_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k)\Big), \quad (3.5)$$

where n is the integer part of $h(x)$, i.e. $n = \lfloor h(x) \rfloor$.

It has been illustrated by Dimitrova et al. (2013a) that formulas (3.4) and (3.5) are computationally appealing. For details on the numerical properties of the two formulas, we refer to Dimitrova et al. (2013a) where extensive numerical experiments are provided.

Explicit expressions for the joint distribution of the time to ruin T and the deficit at ruin Y, $P(T < x, Y > y)$, $x > 0$, $y \geq 0$, are obtained by Ignatov and Kaishev (2012a), in terms of the classical Appell polynomials, for discrete and continuous claim severities respectively given by

$$
\begin{aligned}
&P(T < x, Y > y) = 1 - \sum_{y_1=1}^{m-1} p_{Y_1}(y_1) - \sum_{y_1=m}^{l} p_{Y_1}(y_1) e^{-\lambda h^{-1}(y_1 - y)} \\
&-e^{-\lambda x} \left(1 - \sum_{y_1=1}^{l} p_{Y_1}(y_1) \right) \\
&+\sum_{k=2}^{l} \sum_{C_k} p_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k) \Big\{ B_{k-2} \left(h^{-1}\left(y_{k-1}\right); \nu_1, \ldots, \nu_{k-2} \right) \\
&-B_{k-1} \left(h^{-1}\left(y_k - y\right); \nu_1, \ldots, \nu_{k-1} \right) \Big\} \\
&+\sum_{k=2}^{\infty} \sum_{D_k} p_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k) \Big\{ B_{k-2} \left(h^{-1}\left(y_{k-1}\right); \nu_1, \ldots, \nu_{k-2} \right) \\
&-B_{k-1} \left(x; \nu_1, \ldots, \nu_{k-1} \right) \Big\},
\end{aligned}
\tag{3.6}
$$

where $m = \lfloor y \rfloor + 1$, $n = \lfloor h(x) \rfloor$, $l = \lfloor h(x) + y \rfloor$,

$C_k = \{(y_1, \ldots, y_k) : 1 \leq y_1, 1 + y_{i-1} \leq y_i, i = 2, \ldots, k-1, y_{k-1} + y \leq y_k < h(x) + y\}$,

$D_k = \{(y_1, \ldots, y_k) : 1 \leq y_1, 1 + y_{i-1} \leq y_i, i = 2, \ldots, k-1,\ y_{k-1} \leq h(x) \leq h(x) + y \leq y_k < +\infty\}$ and

$B_k(z; \nu_1, \ldots, \nu_k) = e^{-\lambda z} \left[A_0 + \lambda A_1\left(z; \nu_1\right) + \cdots + \lambda^k A_k\left(z; \nu_1, \ldots, \nu_k\right) \right]$, and

$$
\begin{aligned}
P(T < x, Y > y) &= \int_{y}^{+\infty} f(y_1)dy_1 - \int_{y}^{h(x)+y} e^{-\lambda h^{-1}(y_1-y)} f(y_1)dy_1 \\
&-e^{-\lambda x} \int_{h(x)+y}^{+\infty} f(y_1)dy_1 \\
&+\sum_{k=2}^{\infty} \int \dots \int_{C_k} \Big\{ B_{k-2}\left(h^{-1}\left(y_{k-1}\right); \nu_1, \dots, \nu_{k-2}\right) \\
&-B_{k-1}\left(h^{-1}\left(y_k - y\right); \nu_1, \dots, \nu_{k-1}\right) \Big\} f(y_1, \dots, y_k)dy_k \dots dy_1 \\
&+\sum_{k=2}^{\infty} \int \dots \int_{D_k} \Big\{ B_{k-2}\left(h^{-1}\left(y_{k-1}\right); \nu_1, \dots, \nu_{k-2}\right) \\
&-B_{k-1}\left(x; y_1, \dots, y_{k-1}\right) \Big\} f(y_1, \dots, y_k)dy_k \dots dy_1,
\end{aligned} \tag{3.7}
$$

where $C_k = \{(y_1, \dots, y_k) : 0 < y_1 < \dots < y_{k-1} \leq y_{k-1} + y < y_k < h(x) + y\}$, $D_k = \{(y_1, \dots, y_k) : 0 < y_1 < \dots < y_{k-1} < h(x) \leq h(x) + y \leq y_k < +\infty\}$ and $B_k(z; \nu_1, \dots, \nu_k) = e^{-\lambda z}\left[A_0 + \lambda A_1\left(z; \nu_1\right) + \dots + \lambda^k A_k\left(z; \nu_1, \dots, \nu_k\right)\right]$.

It is not difficult to verify that, when $y = 0$, formulas (3.6) and (3.7) coincide with (3.5) and (3.4) respectively. The proof of that is a simplified version (special case) of the proof of Corollary 3.5.3 (see also Appendix A in Dimitrova et al. 2014).

In what follows, we relax the assumption of Poisson claim arrivals. As a more general case, we assume the inter-arrival times follow an independent non-identical Erlang distribution, i.e. $\tau_i \sim$Erlang(m_i, λ_i), and a Poisson arrival process can be viewed as a special case where $m_i = 1$ and $\lambda_i = \lambda$. The following explicit expression for the finite-time survival probability has been obtained by Ignatov and Kaishev (2012b), assuming continuous claim severities with an arbitrary dependence structure governing $Y_1, Y_2, \ldots$,

$$P(T > x) = e^{-\lambda_1 x} + \sum_{k=1}^{\infty} \int \cdots \int_{0 \le y_1 \le \ldots \le y_{j(k)} \le h(x)} B_k(x) f\left(y_1, \ldots, y_{j(k)}\right) dy_{j(k)} \cdots dy_1, \tag{3.8}$$

where $j(k),\ k = 0, 1, 2, \ldots,$ is an integer-valued function such that

$$m_1 + \ldots + m_{j(k)} \le k < m_1 + \ldots + m_{j(k)} + m_{j(k)+1}, \tag{3.9}$$

and B_k are called exponential Appell polynomials, defined recursively as

$$B_k(x) = \lambda_{j(k-1)+1} e^{-\lambda_{j(k)+1} x} \int_{h^{-1}\left(y_{j(k)}\right)}^{x} e^{\lambda_{j(k)+1} z} B_{k-1}(z) dz,\ k = 1, 2, \ldots,$$

with $B_0(x) = e^{-\lambda_1 x}$. A discrete version of (3.8) can be directly deduced. The corresponding formula for the joint distribution of the time to ruin and the deficit at ruin, $P(T < x, Y > y)$, is derived in the Appendix, see formula (3.23) in Corollary 3.5.4 (see also Appendix B in Dimitrova et al. 2014), and Corollary 3.5.5 states that, when $y = 0$, formula (3.23) coincides with (3.8).

A further generalization can be made by assuming that the inter-arrival times are independent, non-identically distributed as a linear combination of exponential r.v.s, i.e. $\tau_i = \sum_{j=1}^{m_i} \alpha_{ij} \eta_{ij}$, where the constants $\alpha_{ij} > 0$ and $\eta_{ij} \sim$Exp(λ_{ij}). The latter assumption is rather general, and includes both the Erlang and exponential assumptions on the inter-arrival time distribution as special cases. Under this general assumption, the following explicit expression for the non-ruin probability has been derived by Dimitrova et al.

(2013b),

$$P(T > x) = \sum_{k=0}^{\infty} \int_{0 \le y_1 \le \ldots \le y_{j(k)} \le h(x)} \cdots \int B_k(x) f(y_1, \ldots, y_{j(k)}) dy_{j(k)} \cdots dy_1, \quad (3.10)$$

where $j(k)$ is defined as in (3.9),

$$B_k(x) = \theta_k e^{-\theta_{k+1} x} \int_{\nu_k}^{x} e^{\theta_{k+1} z} B_{k-1}(z) dz, k = 1, 2, \ldots,$$

with $B_0(x) = e^{-\theta_1 x}$, $\theta_n = \frac{\lambda_{ij}}{\alpha_{ij}}$, such that $\sum_{s=1}^{i-1} m_s < n \le \sum_{s=1}^{i} m_s$ and $j = n - \sum_{s=1}^{i-1} m_s$, and $0 \le \nu_1 \le \nu_2 \le \ldots$ is a sequence of real numbers denoting

$$\underbrace{h^{-1}(0) \le \ldots \le h^{-1}(0)}_{m_1 - 1} \le \underbrace{h^{-1}(y_1) \le \ldots \le h^{-1}(y_1)}_{m_2} \le \ldots,$$

correspondingly. Furthermore, under these general assumptions, we derive the joint distribution of the time to ruin and the deficit at ruin, $P(T < x, Y > y)$, in Theorem 3.5.1 (see expression (3.13) and its detailed proof in Appendix 3.5), and Corollary 3.5.3 demonstrates that, when $y = 0$, formula (3.13) coincides with (3.10).

It is worth mentioning that, although we only give the expressions for continuous claim severities under the more general assumptions of inter-arrival times, it is straightforward to obtain the form that is valid for discrete claim amounts.

In this section, we have summarized explicit expressions for the finite-time survival probability and the joint probability of the time to ruin and the deficit at ruin under various assumptions to support the numerical computation of alarm times defined in Definitions 3.2.1 and 3.2.2. As mentioned previously, to avoid the inaccuracy introduced by numerical simulations, we choose to use explicit formulas for the these quantities, in contrast to Das and Kratz (2012), who resort on simulation. Therefore, the explicit expressions summarized in this section will be highly important and helpful in the numerical analysis provided next.

3.2.3 Sensitivity analysis

In what follows, we provide some sensitivity analysis of the alarm time (3.2) against various parameters based on the following two examples, which cover cases of discrete and continuous, dependent and independent claim severities.

Example 3.2.3 *Claim amounts follow an i.i.d logarithmic distribution with parameter* α*, i.e.* $W \sim Log(\alpha)$ *with a generic p.m.f.* $P(W = i) = -\alpha^i/(i \ln(1-\alpha))$.

Example 3.2.4 *Claim amounts are inter-dependent with Pareto marginals, i.e.* $W \sim Pareto(\alpha, \beta)$ *with a generic p.d.f.* $f(w) = \alpha^\beta(\alpha + w)^{-\beta-1}\beta$*, and the dependence structure is modelled by a rotated Clayton copula, which models upper tail dependence, with density*

$$c^{RCl}(u_1, \ldots, u_k; \theta) = \theta^k \frac{\Gamma(1/\theta + k)}{\Gamma(1/\theta)} \prod_{i=1}^{k} (1-u_i)^{-\theta-1} \Big(\sum_{i=1}^{k} (1-u_i)^{-\theta} - k + 1 \Big)^{-1/\theta - k},$$

where $\theta \in (0, \infty)$ *denotes the dependence parameter.*

Based on Examples 3.2.3 and 3.2.4 and expression (3.2), we compute the alarm time in Definition 3.2.2 with various sets of parameters and illustrate how it varies against different parameter values. For computational simplicity, we assume Poisson claim arrivals and a linear premium income function $g(t) = ct$, where c denotes a constant premium income rate, i.e. $h_0(t) = u_0 + ct$.

Table 3.1: Alarm time t_A with $b = 0.25$ and $y = 0$, for different a and d, based on Example 3.2.3. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u_0 = 10$ and $c = 1$.

d	a								
	0.3	0.325	0.35	0.375	0.4	0.425	0.45	0.475	0.5
2.5	2.32	2.32	2.32	2.32	2.32	2.24	2.04	1.67	1.38
2.75	2.32	2.32	2.32	2.24	2.04	1.78	1.48	1.18	1.04
3	2.32	2.32	2.15	1.82	1.58	1.33	1.05	0.86	0.71
3.25	2.32	2.06	1.62	1.40	1.18	0.88	0.70	0.56	0.42
3.5	1.88	1.46	1.26	1.07	0.86	0.61	0.41	0.28	0.15
3.75	1.43	1.13	0.88	0.74	0.53	0.30	0.14	0.02	0
4	1.03	0.82	0.63	0.44	0.25	0.03	0	0	0
4.25	0.70	0.52	0.34	0.17	0	0	0	0	0
4.5	0.42	0.25	0.08	0	0	0	0	0	0
4.75	0.15	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0

Table 3.2: Alarm time t_A with $b = 0.25$ and $y = 0.2$, for different a and d, based on Example 3.2.3. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u_0 = 10$ and $c = 1$.

d	a								
	0.3	0.325	0.35	0.375	0.4	0.425	0.45	0.475	0.5
2.5	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.08
2.75	2.32	2.32	2.32	2.32	2.32	2.32	2.32	1.88	1.80
3	2.32	2.32	2.32	2.32	2.32	2.22	1.84	1.56	1.28
3.25	2.32	2.32	2.32	2.32	2.08	1.86	1.37	1.14	0.88
3.5	2.32	2.32	2.28	1.88	1.83	1.22	1.02	0.86	0.62
3.75	2.32	2.32	1.88	1.74	1.16	0.88	0.76	0.54	0.31
4	2.32	1.88	1.61	1.21	0.86	0.64	0.45	0.25	0.03
4.25	1.88	1.51	1.17	0.88	0.53	0.35	0.17	0	0
4.5	1.83	1.12	0.88	0.60	0.25	0.08	0	0	0
4.75	1.12	0.86	0.60	0.28	0	0	0	0	0
5	0.86	0.57	0.30	0.01	0	0	0	0	0

Table 3.3: Alarm time t_A with $b = 0.25$ and $y = 0.5$, for different a and d, based on Example 3.2.3. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u_0 = 10$ and $c = 1$.

d	a								
	0.3	0.325	0.35	0.375	0.4	0.425	0.45	0.475	0.5
2.5	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32
2.75	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32
3	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32
3.25	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32
3.5	2.32	2.32	2.32	2.32	2.32	2.32	2.32	2.32	1.85
3.75	2.32	2.32	2.32	2.32	2.32	2.32	2.32	1.84	1.60
4	2.32	2.32	2.32	2.32	2.32	2.32	1.88	1.61	1.12
4.25	2.32	2.32	2.32	2.32	2.32	2.16	1.75	1.10	0.81
4.5	2.32	2.32	2.32	2.32	2.32	1.82	1.53	0.81	0.55
4.75	2.32	2.32	2.32	2.32	1.88	1.61	0.83	0.56	0.23
5	2.32	2.32	2.32	2.32	1.76	1.04	0.58	0.27	0

In Tables 3.1, 3.2 and 3.3, we compare the alarm times obtained for various choices of a, d and y, based on Example 3.2.3, where the claim severities follow an i.i.d. logarithmic distribution. As noted by Das and Kratz (2012), under the original definition (3.1), the alarm time decreases with the increase of a and d, which is rather intuitive. Here, we observe similar effects under the new definition. Particularly, for smaller values of a and/or d, the first condition in (3.2) cannot be fulfilled prior to the survival probability decreasing below the pre-specified level $1 - b$. Hence, the alarm time is determined solely by definition (3.3) and the level of the parameter b. In fact, b is used in order to cap the alarm time, and its impact on t_A signifies when a and d are small and definition (3.3) is employed. It can also be observed that the alarm time increases along with y. This is reasonable, because for a high level of the deficit threshold y, it takes longer time t for $P(T \leq t + d, Y > y | T > t)$ to become significant enough to reach an alarm time, and so the latter is postponed.

Based on Example 3.2.4, where the claims are assumed dependent, with a joint distribution modelled by a rotated Clayton copula with Pareto marginals, Figure 3.1 illustrates how the alarm time varies for different choices of the initial capital level, u_0, and with the increase of the dependence level, θ. Clearly, the alarm time, t_A, increases with the value of u_0 and also it occurs later when stronger dependence is assumed, and the increase in t_A decelerates with the increase in θ. This is because higher level of dependence among claims, modelled by Rotated Clayton copula, facilitates the occurrence of clusters of either small claims or large claims. Due to the choices of the parameters, it is more likely to have a series of small claims, which raises the survival probability and therefore defers the alarm time, t_A. With further increase in the dependence level θ, its impact gradually vanishes after certain level of θ and the increase in the alarm time also diminishes.

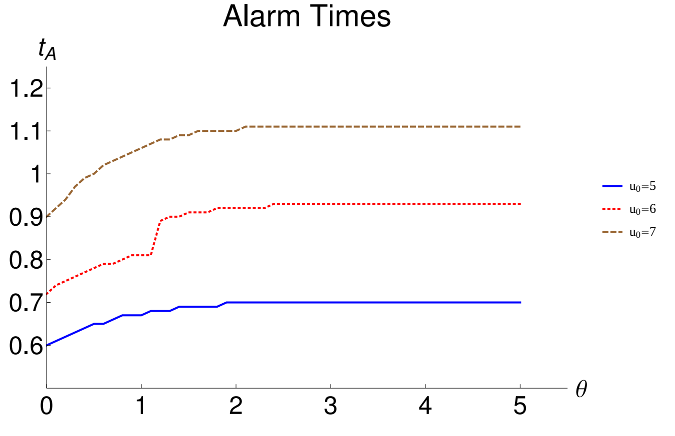

Figure 3.1: Alarm time t_A against dependence level θ, based on Example 3.2.4. Parameter values: $\lambda = 1$, $\alpha = 5$, $\beta = 2$, $c = 10$, $a = 0.4$, $b = 0.1$, $d = 2$ and $y = 0.5$. Blue (solid), red (dotted) and brown (dashed) lines represent $u_0 = 5, 6, 7$ respectively.

So far, we have provided a new definition of a single alarm time, taking into account both the time to ruin and the deficit at ruin, and have numerically illustrated the impacts of the various model parameters. In the next section, we develop an alarm system, i.e. a system with multiple alarm times and capital injections at each alarm time.

3.2.4 A system with multiple alarms

In this section, we introduce the concept of an alarm system, i.e. a system with multiple alarm times and corresponding capital injections at each alarm time. Suppose we have an initial capital $u_0 = h(0)$, which ensures a high solvency level for the insurance company (line of business) until time t_{A_1}, which is the first alarm time determined according to definition (3.2). It is reasonable to suppose that the company will inject capital at the alarm time to maintain the solvency target and keep the company operational. Now, we can generalize this single-alarm time procedure and develop a system with multiple alarms $\{t_{A_i}\}_{i\geq 1}$, topping up capital at each alarm time t_{A_i} with an amount u_i. It is worth noting that all alarm times are determined sequentially with respect to time 0, and so are the corresponding capital amounts injected at each alarm time. Next, we provide the formal definition.

Definition 3.2.5 *Given the first i alarm times, $t_{A_1}, \ldots, t_{A_i}$, and the corresponding capital-premium function $h_i(t) = u_0 + \sum_{k=1}^{i} u_i \times \mathbb{I}_{\{t \geq t_{A_i}\}} + g(t)$, the $i+1$-th alarm time, $t_{A_{i+1}} = t_{A_{i+1}}(a_{i+1}, b_{i+1}, d_{i+1}; h_i(t), y_{i+1})$, with prespecified sequences of values $\{a_i\}_{i\geq 1}$, $\{b_i\}_{i\geq 1}$, $\{d_i\}_{i\geq 1}$ and $\{y_i\}_{i\geq 1}$, is defined* *as follows,*

$$\begin{aligned} t_{A_{i+1}} &= inf\Big\{t > t_{A_i} : P(T \le t + d_{i+1}, Y > y_{i+1} | T > t) \ge 1 - a_{i+1} \\ &\quad and\ P(T > t | T > t_{A_i}) \ge 1 - b_{i+1}\Big\} \\ &= inf\{t > t_{A_i} : P(T \le t + d_{i+1}, Y > y_{i+1}) - P(T \le t, Y > y_{i+1}) \\ &\quad \ge (1 - a_{i+1}) P(T > t)\ and\ P(T > t | T > t_{A_i}) \ge 1 - b_{i+1}\}, \end{aligned} \tag{3.11}$$

or

$$t_{A_{i+1}} = inf\{t > t_{A_i} : P(T > t | T > t_{A_i}) < 1 - b_{i+1}\}, \tag{3.12}$$

if the two conditions in (3.11) cannot be satisfied simultaneously.

Note that, theoretically, one can have an infinite number of alarm times and the corresponding sequence can be computed sequentially according to definitions (3.11) and (3.12). However, in practice, it would be highly unrealistic to consider many alarm times due to the increasing complexity of the alarm system and the related solvency concerns. Hence, in our numerical illustrations, we restrict the number of alarm times to a maximum of 3, which, besides practical considerations, also provides computational convenience. In what follows, we revisit Examples 3.2.3 and 3.2.4 to numerically illustrate the concept of multiple alarm times, and in the following Section 3.3, we illustrate complete alarm systems. For simplicity, we use constant values of a, b, d and y when determining the consecutive alarm times, i.e. $a_i = a$, $b_i = b$, $d_i = d$ and $y_i = y$. We also assume a constant premium income rate c, i.e. $g(t) = ct$, and a fixed size of each capital injection, $u_i = \Delta u$.

Example 3.2.3 Revisited. Using the assumptions of Example 3.2.3, i.e. logarithmically distributed claim severities, we compute the alarm times according to definition (3.11) and illustrate these in Table 3.4. The latter are computed for a certain set of parameter values, specified in Table 3.4, with varying threshold level for the deficit at ruin Y, and assuming a constant amount of top up capital, Δu, which should be added at each alarm time to keep the business running healthily. It is not surprising to see that, with the increase of the size Δu of the capital injections, the alarm times occur later and later. However, it can be observed that the incremental increase in t_{A_3} is larger than that in t_{A_2}, which can be explained as a compounded effect.

Table 3.4: First 3 alarm times $t_{A_1}, t_{A_2}, t_{A_3}$ under a system with multiple alarms, based on Example 3.2.3. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u_0 = 10$, $c = 1$, $a = 0.4$, $b = 0.25$ and $d = 4$.

Δu	$y = 0$			$y = 0.1$			$y = 0.2$		
	t_{A_1}	t_{A_2}	t_{A_3}	t_{A_1}	t_{A_2}	t_{A_3}	t_{A_1}	t_{A_2}	t_{A_3}
0.2	0.25	0.34	0.45	0.54	0.58	0.83	0.86	1.03	1.18
0.4	0.25	0.39	0.70	0.54	0.69	1.08	0.86	1.14	1.52
0.6	0.25	0.44	0.98	0.54	0.82	1.35	0.86	1.18	1.86
0.8	0.25	0.59	1.26	0.54	0.88	1.59	0.86	1.31	2.19
1	0.25	0.66	1.54	0.54	0.92	1.81	0.86	1.45	2.52

Example 3.2.4 Revisited. Under the assumption of Example 3.2.4 (distribution of claim severities modelled by rotated Clayton copula with Pareto marginals), in Figure 3.2 the first three alarm times are illustrated against varying dependence level, θ. It is observed that each alarm time is again postponed as dependence intensifies, which has been discussed previously in connection to the results of Figure 3.1. Although the first alarm time is too small to clearly illustrate this phenomenon, it is well expressed in the

graphs of the following two alarm times (see Figure 3.2).

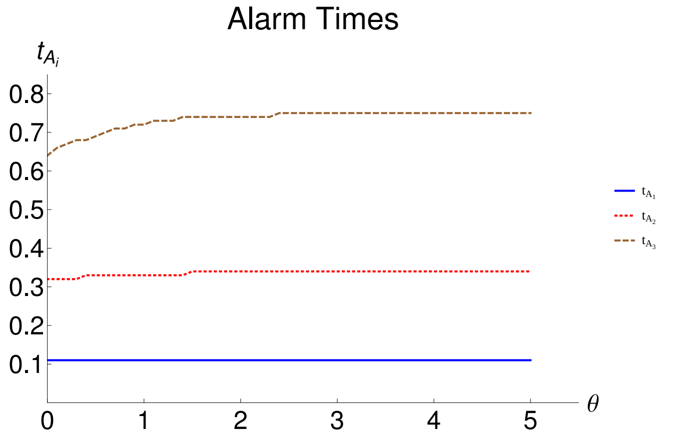

Figure 3.2: First 3 alarm times $t_{A_1}, t_{A_2}, t_{A_3}$ under a system with multiple alarms, based on Example 3.2.4. Parameter values: $\lambda = 1$, $\alpha = 5$, $\beta = 2$, $u_0 = 5$, $c = 10$, $\Delta u = 1$, $a = 0.4$, $b = 0.03$, $d = 2$ and $y = 0.5$. Blue (solid), red (dotted) and brown (dashed) lines represent alarm times 1, 2, 3, respectively.

3.3 Capital allocation strategies

As noted already, a system with multiple alarm times with sequential capital injections is treated as a capital allocation strategy alternative to reserving the entire amount of risk capital initially. The effectiveness of this alternative strategy has been numerically examined by Das and Kratz (2012), where, to draw a fair comparison with a system without alarms, the total capital under two alternatives is equated and the time value of money is considered. The comparison is made under various choices of interest rate, and the proposed system with alarms is believed to outperform the non-alarm system when interest rate is high or the time horizon is long. However, as we have noted, Das and Kratz (2012) have not considered the problem of how to optimize capital allocation at the alarm times. In this section, we study capital allocation strategies and attempt to solve related optimization problems. More precisely, we formulate the problems as follows.

1. *How many alarm times (capital injections) should a company allow of within the time interval considered* $[0, x]$*?*

2. *Given a fixed number of alarm times, what is the optimal amount of each capital injection (which will decide the following alarm time)?*

3. *How does the rate of return, which is assumed constant over the whole period considered, affect the optimal capital allocation strategies?*

As discussed previously, despite mathematical feasibility, in reality, it is highly unlikely for a company to allow of many alarm times due to practical and regulatory concerns. Hence, in this paper, we only explore the cases with a single alarm time and two alarm times within the period considered, and thus, focus on problems 2 and 3. We employ the finite-time survival probability as our criterion when studying problems 2 and 3 and assessing the solutions. As the finite-time survival probability itself is sensitive to the length of the chosen time horizon $[0, x]$, the optimal capital allocation strategy may also vary accordingly when we consider different times. Thus, we are also exploring whether there is a unified solution when we consider time horizons with different lengths.

In the following two sections, we analyze systems with one and two alarm times respectively, formulate the above mentioned (optimization) problems 2 and 3 more precisely and provide some numerical illustrations to their solutions.

3.3.1 Capital allocation strategy in a system with a single alarm time

In this section, we study capital allocation strategies in a single-alarm system and provide numerical solutions to the optimization problems 2 and 3 listed above. First, let us formulate the problems more precisely. Suppose at time 0 an insurance company has capital of size u in total. It can either choose to reserve all the capital at time 0 in order to achieve a high solvency level within a pre-specified horizon $[0, x]$, or, alternatively, opt for smaller initial capital of size u_0 and invest the remaining part to earn a fixed rate of return $r > 0$. The accumulated amount of the investment, $(u - u_0)e^{rt_A}$, will then be added back to the capital at the alarm time t_A, determined by definition (3.2) (or (3.3)). Recall that here we consider a capital allocation strategy when a single alarm time, $0 \leq t_A \leq x$, is allowed. Thus the alarm time $t_A = t_A(u_0)$ is solely determined by the amount of the initial capital u_0, the problem is to find the optimal level of capital, u_0, to set aside initially, given a fixed total amount u, and the optimality is decided by the finite-time survival probability over the period $[0, x]$, $P(T > x)$. Mathematically, the optimization problem can be formulated as follows.

Problem 3.1. Given a fixed total amount of capital u, find an appropriate level of initial capital u_0^* such that

$$u_0^* = \arg\max_{0 \leq u_0 \leq u} P\Big(T > x; h_1(t; u_0, t_A(u_0))\Big),$$

where the alarm time $t_A = t_A(u_0)$ is determined according to (3.2) (or (3.3)) with $h_0(t) = u_0 + g(t)$, and the capital-premium function $h_1(t) =$

$h_1(t; u_0, t_A(u_0)) = u_0 + (u - u_0)e^{rt_A(u_0)} \times \mathbb{I}_{\{t \geq t_A(u_0)\}} + g(t)$, is used in finding u_0^*.

Next, we illustrate the solution to this one-dimensional optimization problem through the following numerical example.

Example 3.2.3 Revisited. We recall the setup in Example 3.2.3, where the claim severities are assumed to follow an i.i.d. logarithmic distribution. We numerically exhaust all possible values for the initial capital u_0, resulting in different alarm times t_A, and evaluate the finite-time survival probability $P\Big(T > x; h_1(t; u_0, t_A(u_0))\Big)$ for each $t_A(u_0)$ in order to find the maximum survival probability, and hence, determine the optimal level of u_0. As mentioned previously, depending on the length of the time horizon $[0, x]$, which the company might consider, the optimal solution to Problem 3.1 may vary. Here, we illustrate the sensitivity of the solution to Problem 3.1 to various choices of x, but with all the choices, we ensure the alarm time (moment for capital injection) falls in the time period considered, i.e. $0 < t_A < x$. Since we are also interested in how the rate of return r will affect the result, two different values of r are selected for comparison. For simplicity, we again assume a constant premium income rate c, i.e. $g(t) = ct$, $h_0(t) = u_0 + ct$ and $h_1(t) = h_1(t; u_0, t_A(u_0)) = u_0 + (u - u_0)e^{rt_A(u_0)} \times \mathbb{I}_{\{t \geq t_A(u_0)\}} + ct$.

Figure 3.3 illustrates the alarm times t_A determined by the corresponding choices of initial capital u_0. It is worth mentioning that for $u_0 \leq 6.1$ the alarm goes off immediately at time 0. The finite-time survival probability $P\Big(T > x; h_1(t; u_0, t_A(u_0))\Big)$ is plotted in Figure 3.4 for different choices of x, and the blue (solid) and red (dotted) lines correspond to two different levels of rate of return, $r = 5\%$ and $r = 10\%$ respectively. Clearly,

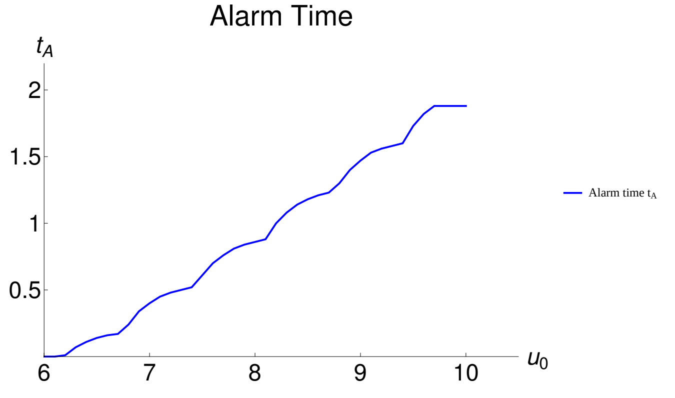

Figure 3.3: Alarm time t_A for varying choices of initial capital u_0. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u = 10$, $c = 1$, $a = 0.4$, $b = 0.25$ and $d = 3$.

a unique optimal solution to Problem 3.1, u_0^*, is found for each choice of $x = 2, 3, 4, 5$, and illustrated in the four panels of Figure 3.4. It is interesting to see that the panel in the left top, where $x = 2$, has a slightly different pattern from the others. This may be because the time interval considered is not long enough; thus, the survival probability significantly depends on the investment returns and the marginal increase in initial capital is not sufficient to compensate for the decrease in the injected portion. As can be seen, the optimal level of initial capital, u_0^*, is deceleratively increasing with the length of the time horizon $[0, x]$. This is reasonable since, when considering a longer time interval $[0, x]$, the optimal time for the capital injection to take place is postponed, which in turn requires more initial capital, and this effect gradually vanishes as x becomes sufficiently large. Figure 3.4 also demonstrates that, for a given time horizon $[0, x]$, different choices of rate of return r lead to the same optimal level of initial capital u_0^*. In other words, the rate of return r has no impact on optimizing the capital allocation in this case.

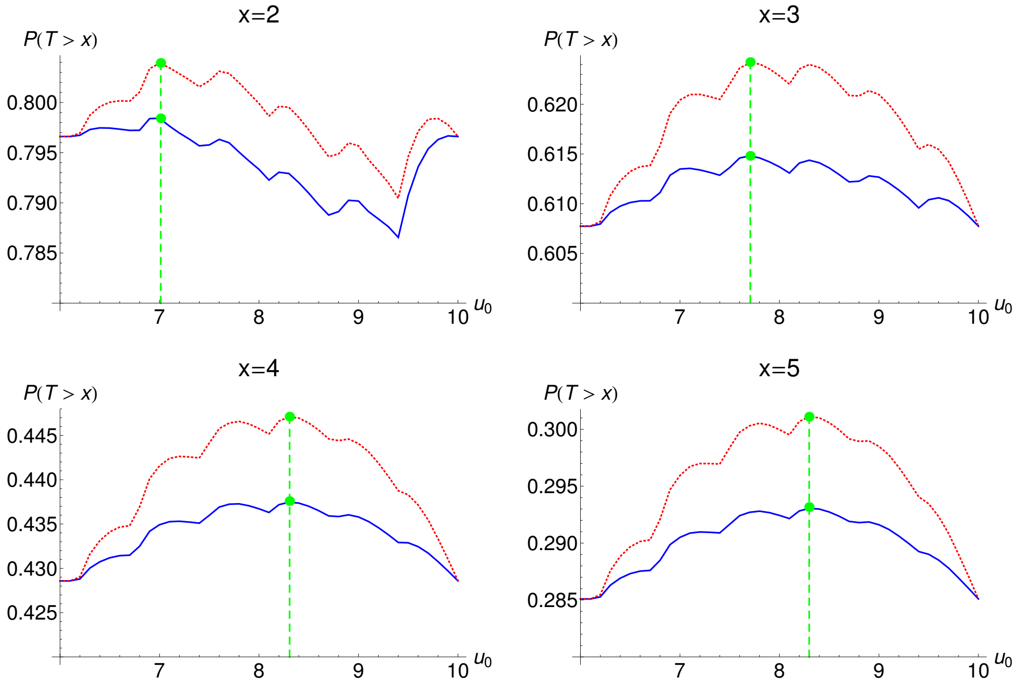

Figure 3.4: Survival probability $P(T > x)$ against different levels of initial capital u_0. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u = 10$, $c = 1$, $a = 0.4$, $b = 0.25$ and $d = 3$. Blue (solid) and red (dotted) lines represent $r = 5\%$ and $r = 10\%$ respectively.

3.3.2 Capital allocation strategy in a system with two alarm times

This section is devoted to studying optimal capital allocation strategies in a system allowing two alarm times. We formulate the problem in a somewhat different way from the case with a single-alarm system, with the consideration that the initial capital u_0, held by an insurance company, may be pre-set at a certain level due to related regulations and solvency requirements. Thus, it may not be possible for the company to choose a (lower) initial capital level in order to pursue higher investment returns. We therefore assume a fixed split between the initial capital and the portion used for investment, respectively denoted by u_0 and u'. As we now allow two alarm times in the system and the first alarm time, t_{A_1}, will be pre-determined by the level of u_0, what remains to be optimized is the portion to be injected at the first alarm time, t_{A_1}, which, according to definitions (3.11) and (3.12), will then determine the second alarm time, t_{A_2}, when all the remaining capital will be injected along with the investment returns. We thus divide u' into two parts, u_1 and $u'-u_1$, so the capital increments to be injected at the two alarm times are $u_1 e^{rt_{A_1}}$ and $(u'-u_1)e^{rt_{A_2}}$ respectively. Hence, the problem of finding optimal capital allocation strategy is now equivalent to finding the appropriate capital portion u_1 which maximizes the finite-time survival probability $P(T > x)$ over the time horizon $[0, x]$ considered by the company. Thus, we formulate the following optimization problem.

Problem 3.2. Given a fixed total amount of capital $u = u_0 + u'$ and a fixed level of u_0, find an appropriate level of the first capital portion u_1

such that

$$u_1^* = \arg\max_{0 \le u_1 \le u'} P\Big(T > x; h_2(t; u_1, t_{A_2}(u_1))\Big),$$

where the alarm time $t_{A_2} = t_{A_2}(u_1)$ is determined according to (3.11) (or (3.12)) with $h_1(t) = u_0 + u_1 e^{rt_{A_1}} \times \mathbb{I}_{\{t \ge t_{A_1}\}} + g(t)$, and the capital-premium function $h_2(t) = h_2(t; u_1, t_{A_2}(u_1)) = u_0 + u_1 e^{rt_{A_1}} \times \mathbb{I}_{\{t \ge t_{A_1}\}} + (u' - u_1)e^{rt_{A_2}(u_1)} \times \mathbb{I}_{\{t \ge t_{A_2}(u_1)\}} + g(t)$ is used in finding u_1^*.

This is again a one-dimensional optimization problem and we illustrate it numerically in the following example.

Example 3.2.3 Revisited. We again consider the setup in Example 3.2.3 with i.i.d. logarithmically distributed claim amounts. Here, since u_0 and t_{A_1} are both pre-determined, we compute the second alarm time, t_{A_2}, for all possible values of $u_1 \in [0, u']$ and then evaluate the finite-time survival probability $P\Big(T > x; h_2(t; u_1, t_{A_2}(u_1))\Big)$ to seek for the optimal value of u_1 and thus find the optimal capital allocation strategy. Similarly to the previous section, here we again consider various time horizons $[0, x]$ and rate of return r, and assume a linear premium income function $g(t) = ct$, so that $h_2(t) = h_2(t; u_1, t_{A_2}(u_1))) = u_0 + u_1 e^{rt_{A_1}} \times \mathbb{I}_{\{t \ge t_{A_1}\}} + (u' - u_1)e^{rt_{A_2}(u_1)} \times \mathbb{I}_{\{t \ge t_{A_2}(u_1)\}} + ct$.

In Figure 3.5, we plot the second alarm time t_{A_2} against varying sizes of u_1 for two different choices of rate of return r, with the blue (solid) and red (dotted) lines representing $r = 5\%$ and $r = 10\%$ respectively. Clearly, for fixed u_1, when a higher rate of return r is applied, higher returns are accrued on u_1 until time t_{A_1} and thus, the second alarm time t_{A_2} is postponed. Figure 3.6 illustrates the optimal solution of Problem 3.2 for four choices of time horizon $[0, x]$. Compared to the case of single-alarm

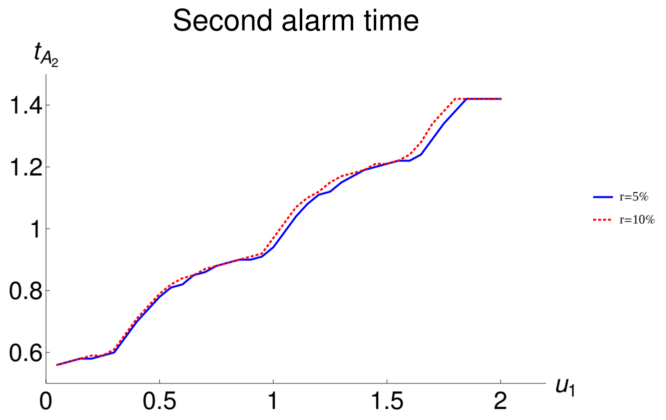

Figure 3.5: The second alarm time t_{A_2} for different values of u_1. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u_0 = 10$, $u' = 2$, $c = 1$, $a = 0.4$, $b = 0.25$ and $d = 4$. Blue (solid) and red (dotted) lines represent $r = 5\%$ and $r = 10\%$ respectively.

system, there is a slightly larger gap between the two curves corresponding to 5% and 10% rate of return. This can again be explained with the higher returns accrued on u_1 and $u' - u_1$. Apart from this, there is not much difference in the pattern of the curves compared with the one-alarm system. It is interesting to see that for $x \geq 3$, the size of u_1 that maximizes the finite-time survival probability remains the same, and the effect of the optimal size of u_1 increasing with the length of the time horizon considered completely vanishes. Thus, it is more conclusive that, when a longer time interval is considered, there tends to be a fixed optimal strategy for capital allocation. However, it is worth pointing out that the rate of return r is still playing a negligible role on optimizing the capital allocation.

3.4 Concluding remarks

We have presented a new definition of an alarm time, t_A, which is expressed in terms of the joint probability of the time to ruin T and the deficit at ruin

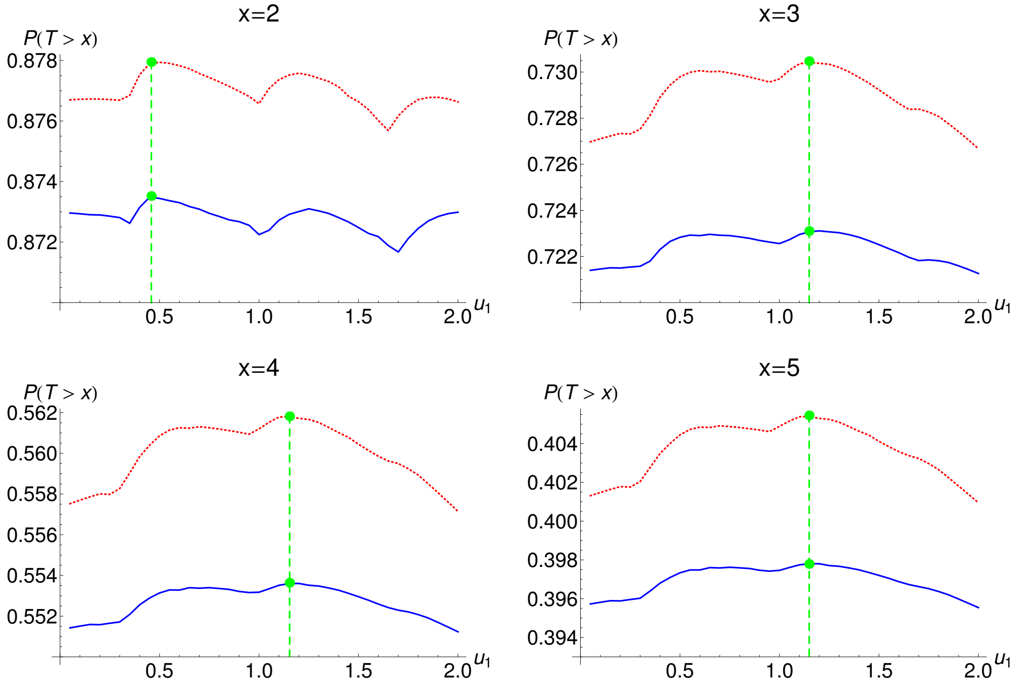

Figure 3.6: Survival probability $P(T > x)$ against different choices of u_1. Parameter values: $\lambda = 2$, $\alpha = 0.7$, $u_0 = 10$, $u' = 2$, $c = 1$, $a = 0.4$, $b = 0.25$ and $d = 4$. Blue (solid) and red (dotted) lines represent $r = 5\%$ and $r = 10\%$ respectively.

Y in a general risk model. It generalizes a definition based solely on ruin probability, recently introduced by Das and Kratz (2012). Alarm times are consecutive time instants, at which portions of risk capital should be reserved sequentially so as to maintain the probability of finite-time ruin below a certain pre-specified target level. We have further summarized some closed-form ruin probability formulas obtained by Ignatov and Kaishev (2012b) and more recently investigated by Dimitrova et al. (2013b) and Dimitrova et al. (2014) and derived some new expressions for the joint distribution of T and Y, which generalizes some of the above mentioned results. We have shown further that, by solving an appropriate optimal capital allocation problem, it is possible to optimally determine the amount of risk capital which should be injected sequentially at the alarm times so as to maximize the finite-time survival probability. We have also undertaken a thorough numerical model sensitivity investigation which has revealed some interesting properties of the alarm time and the alarm system.

We summarize the following key findings from the numerical experiments. It has been shown that the parameters in the definition of alarm time are inter-related, and the values should be selected carefully in order to obtain meaningful results. It has also been noted that, due to the specific settings of our optimal capital allocation problem, there tends to be a fixed optimal allocation strategy as a longer time period is considered, which is independent of the rate of return. However, if we focus on a shorter time horizon, the optimal allocation strategy will involve a smaller portion of initial capital and a larger portion of capital injection at the alarm time.

In this work, the concept of alarm system has been carried out as an alternative capital allocation strategy to reduce the size of initial capital, which is practically highly preferable for the insurance companies. However, how to allocate capital optimally still remains an open question. In this paper, finite-time survival probability has been employed as the key criterion to assess the optimality. However, other risk measures could also be considered. The framework could also be formulated in different ways. It is worth mentioning that the problems formulated in this paper were quite simple, but we hope these could provide some enlightenment for further studies. Due to the nature of the definition of the alarm times, the (computational) complexity embedded may snowball with the increase of the dimension of the problems to be solved.

Another interesting direction to further expand the framework is to move from a static problem to a more dynamic setting. In practice, the operations of the company may change, as well as the solvency requirements. Hence, a more dynamic and adaptive definition of an alarm system could be developed to reflect the information and the data collected over time. An empirical study based on real data from the industry also sounds appealing. These are all in the scope of our future work.

3.5 Appendix

Derivation for the joint probability of the time to ruin and the deficit at ruin in the insurance risk model with independent non-identical inter-arrival times following a linear combination of exponential distributions

Here, we provide a detailed derivation for the joint probability of the time to ruin and the deficit at ruin in the insurance risk model with independent non-identical inter-arrival times following a linear combination of exponential distributions. In what follows, we only consider the case of continuous claim severities. However, we note that, for the case of discrete claim amounts, an analogous derivation can be followed and similar results should be reached.

In the insurance risk model, let τ_i, $i = 1, 2, \ldots$, be a sequence of independent non-identical random variables, denoting the inter-arrival times. We assume the inter-arrival times follow a combination of exponential distributions, i.e. $\tau_i = \sum_{j=1}^{m_i} \alpha_{ij}\eta_{ij}$, where $\eta_{ij} \sim$Exp(λ_{ij}). Let $l_i = m_1 + \cdots + m_i$, $i = 1, 2, \ldots$. Let $\{\tilde{\tau}_n\}_{n\geq 1}$ be a sequence of independent, exponentially distributed random variables with parameters $\theta_1, \theta_2, \ldots$ correspondingly, i.e. $\tilde{\tau}_n \sim$Exp(θ_n), such that $\theta_n = \frac{\lambda_{ij}}{\alpha_{ij}}$, where $l_{i-1} < n \leq l_i$ and $j = n - l_{i-1}$.

Thus, we have

$$(\tilde{\tau}_1 + \cdots + \tilde{\tau}_{l_1}, \tilde{\tau}_{l_1+1} + \cdots + \tilde{\tau}_{l_2}, \ldots) \stackrel{d}{=} (\tau_1, \tau_2, \ldots).$$

Obviously, in this more refined representation of the claim arrivals in terms of sums of exponentials we have that

$$\theta_1, \ldots, \theta_{l_1}, \theta_{l_1+1}, \ldots, \theta_{l_2}, \ldots \equiv \frac{\lambda_{11}}{\alpha_{11}}, \ldots, \frac{\lambda_{1m_1}}{\alpha_{1m_1}}, \frac{\lambda_{21}}{\alpha_{21}}, \ldots, \frac{\lambda_{2m_2}}{\alpha_{2m_2}}, \ldots.$$

In the sequel it will be convenient to use the notation $\tilde{\tau}_1^*, \tilde{\tau}_2^*, \ldots$ for the r.v.s $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$, in the case when $\theta_n = 1$.

Denote by $T_1 = \tau_1, T_2 = \tau_1 + \tau_2, \ldots$, the moments of claim arrivals and introduce the sequence of random variables $\tilde{T}_1 = \tilde{\tau}_1$, $\tilde{T}_2 = \tilde{\tau}_1 + \tilde{\tau}_2, \ldots$. Obviously, we can also write $T_i = \tilde{T}_{l_i}$ $i = 1, 2, \ldots$. Let us also consider the partial sums, Y_i, $i = 1, 2, \ldots$ of the consecutive claim amounts, $Y_1 = W_1, Y_2 = W_1 + W_2, \ldots$ with probability density function

$$f_{Y_1,\ldots,Y_i}(y_1, \ldots, y_i) = \begin{cases} \varphi(y_1, \ldots, y_i), & \text{if } 0 \leq y_1 \leq \ldots \leq y_i \\ 0 & \text{otherwise} \end{cases},$$

where $\varphi(y_1, \ldots, y_i) \geq 0$ for $0 \leq y_1 \leq \ldots \leq y_i$ and

$$\int \cdots \int_{0 \leq y_1 \leq \ldots \leq y_i} \varphi(y_1, \ldots, y_i) dy_1 \ldots dy_i = 1.$$

We will also denote by $F_{Y_1,\ldots,Y_i}(y_1, \ldots, y_i)$, the cdf of $Y_1, \ldots, Y_i$.

We now introduce the non-decreasing sequence of variables $\tilde{Y}_1, \tilde{Y}_2, \ldots$, independent of $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$ and such that $0 = \tilde{Y}_1 = \ldots = \tilde{Y}_{l_1-1} \leq Y_1 = \tilde{Y}_{l_1} =$

$\ldots = \tilde{Y}_{l_2-1} \leq Y_2 = \tilde{Y}_{l_2} = \ldots = \tilde{Y}_{l_3-1} \leq \ldots$.

Then we have the following theorem.

Theorem 3.5.1 *The probability $P(T < x, Y > y)$, $x > 0$, $y \geq 0$, is given by (assuming $l_1 \geq 2$)*

$$P(T < x, Y > y) =$$
$$\sum_{k=1}^{\infty} \int_{0}^{h(x)} \cdots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}+y}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \ldots, \nu_{l_k-2}\right) f_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k)$$
$$dy_k \; dy_{k-1} \ldots dy_1$$
$$-\sum_{k=1}^{\infty} \int_{0}^{h(x)} \cdots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}+y}^{h(x)+y} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k - y); \nu_1, \ldots, \nu_{l_k-1}\right) f_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k)$$
$$dy_k \; dy_{k-1} \ldots dy_1$$
$$-\sum_{k=1}^{\infty} \int_{0}^{h(x)} \cdots \int_{y_{k-2}}^{h(x)} \int_{h(x)+y}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \ldots, \nu_{l_k-1}\right) f_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k) dy_k \; dy_{k-1} \ldots dy_1, \tag{3.13}$$

where

$$\mathfrak{B}_k(z; \nu_1, \ldots, \nu_k) = \sum_{i=0}^{k} B_i(z; \nu_1, \ldots, \nu_i),$$

$B_k(z; \nu_1, \ldots, \nu_k) \equiv B_k(z)$ are the (classical) exponential Appell polynomials defined recurrently by

$$B_k(z) = \theta_k e^{-\theta_{k+1} z} \int_{\nu_k}^{z} e^{\theta_{k+1} w} B_{k-1}(w) dw, k = 1, 2, \ldots$$

with $B_0(z) = \mathfrak{B}_0(z) = e^{-\theta_1 z}$ and $0 \leq \nu_1 \leq \nu_2 \leq \ldots$ is a sequence of real *numbers denoting*

$$\underbrace{h^{-1}(0) \leq \ldots \leq h^{-1}(0)}_{m_1-1} \leq \underbrace{h^{-1}(y_1) \leq \ldots \leq h^{-1}(y_1)}_{m_2} \leq \ldots,$$

correspondingly.

Proof: By construction, the event $\{T < x, Y > y\}$ can be expressed as

$$\begin{aligned}
\{T < x, Y > y\} &= \bigcup_{k=1}^{\infty} \left[\bigcap_{i=1}^{k-1} \{Y_i < h(T_i)\} \cap \{Y_k > h(T_k) + y\} \cap \{T_k < x\} \right] \\
&= \bigcup_{k=1}^{\infty} \left[\bigcap_{i=1}^{k-1} \left\{ \tilde{Y}_{l_i} < h\left(\tilde{T}_{l_i}\right) \right\} \cap \left\{ \tilde{Y}_{l_k} > h\left(\tilde{T}_{l_k}\right) + y \right\} \cap \left\{ \tilde{T}_{l_k} < x \right\} \right].
\end{aligned}$$

Clearly,

$$\left\{ \tilde{Y}_{l_i} < h\left(\tilde{T}_{l_i}\right) \right\} \subseteq \left\{ \tilde{Y}_{l_i} < h\left(\tilde{T}_{l_i+w}\right) \right\}$$

for $w = 0, 1, \ldots, m_{i+1} - 1$, which is equivalent to

$$\left\{ \tilde{Y}_{l_i} < h\left(\tilde{T}_{l_i}\right) \right\} \subseteq \left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\}$$

for $l_i \leq r < l_{i+1}$. Therefore, for any $i = 1, 2, \ldots,$

$$\left\{ \tilde{Y}_{l_i} < h\left(\tilde{T}_{l_i}\right) \right\} \subseteq \bigcap_{r=l_i}^{l_{i+1}-1} \left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\}.$$

In addition, for $1 \leq r < l_1$, we also have

$$\left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\} = \left\{ 0 < h\left(\tilde{T}_r\right) \right\} = \Omega,$$

and hence

$$\bigcap_{r=1}^{l_1-1} \left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\} = \Omega,$$

where Ω is the sure event. Thus, we obtain

$$\{T < x, Y > y\} = \bigcup_{k=1}^{\infty} \left[\bigcap_{r=1}^{l_k - 1} \left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\} \cap \left\{ \tilde{Y}_{l_k} > h\left(\tilde{T}_{l_k}\right) + y \right\} \cap \left\{ \tilde{T}_{l_k} < x \right\} \right].$$

We note that the events in the square brackets are mutually exclusive. Hence, we have

$$\begin{aligned}
& P(T < x, Y > y) \\
= \; & P\left(\bigcup_{k=1}^{\infty} \left[\bigcap_{r=1}^{l_k - 1} \left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\} \cap \left\{ \tilde{Y}_{l_k} > h\left(\tilde{T}_{l_k}\right) + y \right\} \cap \left\{ \tilde{T}_{l_k} < x \right\} \right] \right) \\
= \; & \sum_{k=1}^{\infty} P\left(\bigcap_{r=1}^{l_k - 1} \left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\} \cap \left\{ \tilde{Y}_{l_k} > h\left(\tilde{T}_{l_k}\right) + y \right\} \cap \left\{ \tilde{T}_{l_k} < x \right\} \right) \\
= \; & \sum_{k=1}^{\infty} \int_0^x \cdots \int_{t_{l_k - 1}}^x f_{\tilde{T}_1, \ldots, \tilde{T}_{l_k}}(t_1, \ldots, t_{l_k}) dt_{l_k} \ldots dt_1 \int_0^{h(t_1)} \cdots \int_{\tilde{y}_{l_k - 2}}^{h(t_{l_k - 1})} \int_{h(t_{l_k}) + y}^{\infty} \\
& dF_{\tilde{Y}_1, \ldots, \tilde{Y}_{l_k}}(\tilde{y}_1, \ldots, \tilde{y}_{l_k}), \qquad (3.14)
\end{aligned}$$

where $F_{\tilde{Y}_1, \ldots, \tilde{Y}_{l_k}}(\tilde{y}_1, \ldots, \tilde{y}_{l_k})$ is the joint distribution of $\tilde{Y}_1, \ldots, \tilde{Y}_{l_k}$ and $f_{\tilde{T}_1, \ldots, \tilde{T}_{l_k}}$ $(t_1, \ldots, t_{l_k})$ is the joint density of $\tilde{T}_1, \ldots, \tilde{T}_{l_k}$. It can easily be seen that the random vector $\tilde{\boldsymbol{T}} = \left(\tilde{T}_1, \ldots, \tilde{T}_{l_k}\right)'$ coincides in distribution with the random vector $\boldsymbol{B}_{l_k} \tilde{\boldsymbol{\tau}}^*$, i.e., $\boldsymbol{B}_{l_k} \tilde{\boldsymbol{\tau}}^* \overset{d}{=} \tilde{\boldsymbol{T}}$, where $\tilde{\boldsymbol{\tau}}^* = \left(\tilde{\tau}_1^*, \ldots, \tilde{\tau}_{l_k}^*\right)'$, and $\boldsymbol{B}_{l_k} = (b_{ij})_{l_k \times l_k}$ is a $l_k \times l_k$ dimensional matrix, where $b_{ij} = \frac{1}{\theta_j}$, if $i \geq j$, otherwise $b_{ij} = 0$. Then, it is not difficult to see that

$$f_{\tilde{T}_1, \ldots, \tilde{T}_{l_k}}(t_1, \ldots, t_{l_k}) = \begin{cases} e^{-\boldsymbol{1} \cdot \boldsymbol{B}_{l_k}^{-1} \cdot \boldsymbol{t}} \left| \det \boldsymbol{B}_{l_k}^{-1} \right| & \text{if } 0 \leq t_1 \leq t_2 \leq \ldots \leq t_{l_k} \\ 0 & \text{otherwise} \end{cases},$$

where, $\boldsymbol{1} = \Bigg(\underbrace{1, \ldots, 1}_{l_k} \Bigg)$, $\boldsymbol{t} = (t_1, \ldots, t_{l_k})'$, $()'$ stands for transposition, and

$\det \boldsymbol{B}_{l_k}^{-1}$ denotes the determinant of the inverse of $\boldsymbol{B}_{l_k}$. It can also be directly verified that the inverse matrix, $\boldsymbol{B}_{l_k}^{-1} = \left(\tilde{b}_{ij}\right)$, is an incomplete, lower triangular matrix, with non-zero elements only at the main and next lower diagonals, given by $\tilde{b}_{ij} = \theta_i$, if $i = j$, $\tilde{b}_{ij} = -\theta_i$, if $i = j+1$, otherwise $\tilde{b}_{ij} = 0$. Then, $P(T < x, Y > y)$ becomes

$$
\begin{aligned}
& P(T < x, Y > y) \\
= & \sum_{k=1}^{\infty} \int_0^x \cdots \int_{t_{l_k-1}}^x \theta_1 \ldots \theta_{l_k} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_{l_k}(t_{l_k} - t_{l_k-1})\}] \\
& dt_{l_k} \ldots dt_1 \int_0^{h(t_1)} \cdots \int_{\tilde{y}_{l_k-2}}^{h(t_{l_k-1})} \int_{h(t_{l_k})+y}^{\infty} dF_{\tilde{Y}_1,\ldots,\tilde{Y}_{l_k}}(\tilde{y}_1, \ldots, \tilde{y}_{l_k}) \\
= & \sum_{k=1}^{\infty} \int_0^x \cdots \int_{t_{l_k-1}}^x \theta_1 \ldots \theta_{l_k} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_{l_k}(t_{l_k} - t_{l_k-1})\}] \\
& dt_{l_k} \ldots dt_1 \int_0^{h(t_1)} \cdots \int_{\tilde{y}_{l_k-2}}^{h(t_{l_k-1})} \left(\int_{\tilde{y}_{l_k-1}+y}^{\infty} - \int_{\tilde{y}_{l_k-1}+y}^{h(t_{l_k})+y} \right) dF_{\tilde{Y}_1,\ldots,\tilde{Y}_{l_k}}(\tilde{y}_1, \ldots, \tilde{y}_{l_k}) \\
= & \sum_{k=1}^{\infty} \int_0^x \cdots \int_{t_{l_k-2}}^x \theta_1 \ldots \theta_{l_k-1} \Big(\exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_{l_k-1}(t_{l_k-1} - t_{l_k-2})\}] \\
& - \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_{l_k}(x - t_{l_k-1})\}] \Big) dt_{l_k-1} \ldots dt_1 \\
& \int_0^{h(t_1)} \cdots \int_{\tilde{y}_{l_k-2}}^{h(t_{l_k-1})} \int_{\tilde{y}_{l_k-1}+y}^{\infty} dF_{\tilde{Y}_1,\ldots,\tilde{Y}_{l_k}}(\tilde{y}_1, \ldots, \tilde{y}_{l_k}) \\
& - \sum_{k=1}^{\infty} \int_0^x \cdots \int_{t_{l_k-1}}^x \theta_1 \ldots \theta_{l_k} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_{l_k}(t_{l_k} - t_{l_k-1})\}] \\
& dt_{l_k} \ldots dt_1 \int_0^{h(t_1)} \cdots \int_{\tilde{y}_{l_k-2}}^{h(t_{l_k-1})} \int_{\tilde{y}_{l_k-1}+y}^{h(t_{l_k})+y} dF_{\tilde{Y}_1,\ldots,\tilde{Y}_{l_k}}(\tilde{y}_1, \ldots, \tilde{y}_{l_k}).
\end{aligned}
$$

Permuting the two multiple integrals, we obtain

$$
\begin{aligned}
& P(T < x, Y > y) \\
= & \sum_{k=1}^{\infty} \underset{\substack{0<\tilde{y}_1<\ldots<\tilde{y}_{l_k-1}<h(x) \\ \tilde{y}_{l_k-1}+y<\tilde{y}_{l_k}<\infty}}{\int \ldots \int} dF_{\tilde{Y}_1,\ldots,\tilde{Y}_{l_k}}(\tilde{y}_1,\ldots,\tilde{y}_{l_k}) \int_{h^{-1}(\tilde{y}_1)}^{t_2} \ldots \int_{h^{-1}(\tilde{y}_{l_k-2})}^{t_{l_k-1}} \int_{h^{-1}(\tilde{y}_{l_k-1})}^{x} \\
& \theta_1 \ldots \theta_{l_k-1} \Big(\exp[-\{\theta_1 t_1 + \theta_2(t_2-t_1) + \cdots + \theta_{l_k-1}(t_{l_k-1} - t_{l_k-2})\}] - \\
& \exp[-\{\theta_1 t_1 + \theta_2(t_2-t_1) + \cdots + \theta_{l_k}(x - t_{l_k-1})\}] \Big) dt_{l_k-1} \ldots dt_1 \\
& - \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \underset{\substack{0<\tilde{y}_1<\ldots<\tilde{y}_{l_k-1}<h(x) \\ \tilde{y}_{l_k-1}+y<\tilde{y}_{l_k}<h(x)+y}}{\int \ldots \int} dF_{\tilde{Y}_1,\ldots,\tilde{Y}_{l_k}}(\tilde{y}_1,\ldots,\tilde{y}_{l_k}) \int_{h^{-1}(\tilde{y}_1)}^{t_2} \ldots \int_{h^{-1}(\tilde{y}_{l_k-1})}^{t_{l_k}} \int_{h^{-1}(\tilde{y}_{l_k}-y)}^{x} \\
& \theta_1 \ldots \theta_{l_k} \exp[-\{\theta_1 t_1 + \theta_2(t_2-t_1) + \cdots + \theta_{l_k}(t_{l_k} - t_{l_k-1})\}] dt_{l_k} \ldots dt_1 \\
= & \sum_{k=1}^{\infty} \underset{\substack{0<y_1<\ldots<y_{k-1}<h(x) \\ y_{k-1}+y<y_k<\infty}}{\int \ldots \int} dF_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k) \int_{\nu_1}^{t_2} \ldots \int_{\nu_{l_k-2}}^{t_{l_k-1}} \int_{h^{-1}(y_{k-1})}^{x} \theta_1 \ldots \theta_{l_k-1} \\
& \Big(\exp[-\{\theta_1 t_1 + \theta_2(t_2-t_1) + \cdots + \theta_{l_k-1}(t_{l_k-1} - t_{l_k-2})\}] - \\
& \exp[-\{\theta_1 t_1 + \theta_2(t_2-t_1) + \cdots + \theta_{l_k}(x - t_{l_k-1})\}] \Big) dt_{l_k-1} \ldots dt_1 \\
& - \sum_{k=1}^{\infty} \underset{\substack{0<y_1<\ldots<y_{k-1}<h(x) \\ y_{k-1}+y<y_k<h(x)+y}}{\int \ldots \int} dF_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k) \int_{\nu_1}^{t_2} \ldots \int_{\nu_{l_k-1}}^{t_{l_k}} \int_{h^{-1}(y_k-y)}^{x} \theta_1 \ldots \theta_{l_k} \\
& \exp[-\{\theta_1 t_1 + \theta_2(t_2-t_1) + \cdots + \theta_{l_k}(t_{l_k} - t_{l_k-1})\}] dt_{l_k} \ldots dt_1,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&P(T<x,Y>y)\\
=\;& \sum_{k=1}^{\infty}\int_{0}^{h(x)}\cdots\int_{y_{k-2}}^{h(x)}\int_{y_{k-1}+y}^{\infty} f_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k)dy_k\ dy_{k-1}\ldots dy_1\int_{\nu_1}^{t_2}\cdots\int_{\nu_{l_k-2}}^{t_{l_k-1}}\int_{h^{-1}(y_{k-1})}^{x}\\
&\theta_1\ldots\theta_{l_k-1}\Big(\exp[-\{\theta_1t_1+\theta_2(t_2-t_1)+\cdots+\theta_{l_k-1}(t_{l_k-1}-t_{l_k-2})\}]-\\
&\exp[-\{\theta_1t_1+\theta_2(t_2-t_1)+\cdots+\theta_{l_k}(x-t_{l_k-1})\}]\Big)dt_{l_k-1}\ldots dt_1\\
&-\sum_{k=1}^{\infty}\int_{0}^{h(x)}\cdots\int_{y_{k-2}}^{h(x)}\int_{y_{k-1}+y}^{h(x)+y} f_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k)dy_k\ dy_{k-1}\ldots dy_1\int_{\nu_1}^{t_2}\cdots\int_{\nu_{l_k-1}}^{t_{l_k}}\int_{h^{-1}(y_k-y)}^{x}\\
&\theta_1\ldots\theta_{l_k}\exp[-\{\theta_1t_1+\theta_2(t_2-t_1)+\cdots+\theta_{l_k}(t_{l_k}-t_{l_k-1})\}]dt_{l_k}\ldots dt_1,
\end{aligned}
\tag{3.15}
$$

where $0\le\nu_1\le\nu_2\le\ldots$ is a sequence of real numbers denoting

$$
\underbrace{h^{-1}(0)\le\ldots\le h^{-1}(0)}_{m_1-1}\le\underbrace{h^{-1}(y_1)\le\ldots\le h^{-1}(y_1)}_{m_2}\le\ldots,
$$

correspondingly.

Let $\mathfrak{B}_0(z)=e^{-\theta_1 z}$ and for $k=1,2,\ldots,$

$$
\begin{aligned}
&\mathfrak{B}_k(z;\nu_1,\ldots,\nu_k)\equiv\mathfrak{B}_k(z)\\
=\;&\int_{\nu_1}^{t_2}\cdots\int_{\nu_k}^{t_{k+1}}\int_{z}^{\infty}\theta_1\ldots\theta_{k+1}\exp[-\{\theta_1t_1+\theta_2(t_2-t_1)+\cdots+\theta_{k+1}(t_{k+1}-t_k)\}]\\
&dt_{k+1}\ dt_k\ldots dt_1.
\end{aligned}
$$

Permuting the two innermost integrals, we have

$$
\begin{aligned}
\mathfrak{B}_k(z) &= \int_{\nu_1}^{t_2} \dots \int_{\nu_{k-1}}^{t_k} \left(\int_{\nu_k}^{t_{k+1}} \int_{z}^{\infty} \theta_1 \dots \theta_k \; \theta_{k+1} \; \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \right. \\
&\quad \left. + \theta_k(t_k - t_{k-1}) + \theta_{k+1}(t_{k+1} - t_k)\}] dt_{k+1} \; dt_k \right) dt_{k-1} \dots dt_1 \\
&= \int_{\nu_1}^{t_2} \dots \int_{\nu_{k-1}}^{t_k} \left(\int_{\nu_k}^{\infty} \int_{\max\{t_k, z\}}^{\infty} \theta_1 \dots \theta_k \; \theta_{k+1} \; \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \right. \\
&\quad \left. + \theta_k(t_k - t_{k-1}) - \theta_{k+1} t_k\}] \times e^{-\theta_{k+1} t_{k+1}} dt_{k+1} \; dt_k \right) dt_{k-1} \dots dt_1 \\
&= \int_{\nu_1}^{t_2} \dots \int_{\nu_{k-1}}^{t_k} \left(\int_{\nu_k}^{\infty} \theta_1 \dots \theta_k \; \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_k(t_k - t_{k-1}) \right. \\
&\quad \left. - \theta_{k+1} t_k\}] \times e^{-\theta_{k+1} \max\{t_k, z\}} dt_k \right) dt_{k-1} \dots dt_1 \\
&= \int_{\nu_1}^{t_2} \dots \int_{\nu_{k-1}}^{t_k} \int_{\nu_k}^{z} \theta_1 \dots \theta_k \; \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_k(t_k - t_{k-1}) \\
&\quad - \theta_{k+1} t_k\}] e^{-\theta_{k+1} z} dt_k \dots dt_1 \\
&\quad + \int_{\nu_1}^{t_2} \dots \int_{\nu_{k-1}}^{t_k} \int_{z}^{\infty} \theta_1 \dots \theta_k \; \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_k(t_k - t_{k-1}) \\
&\quad - \theta_{k+1} t_k\}] e^{-\theta_{k+1} t_k} dt_k \dots dt_1 \\
&= B_k(z; \nu_1, \dots, \nu_k) + \mathfrak{B}_{k-1}(z),
\end{aligned}
$$

where

$$
\begin{aligned}
& B_k(z; \nu_1, \dots, \nu_k) \equiv B_k(z) \\
= \; & e^{-\theta_{k+1} z} \int_{\nu_1}^{t_2} \dots \int_{\nu_{k-1}}^{t_k} \int_{\nu_k}^{z} \theta_1 \dots \theta_k \; \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_k(t_k - t_{k-1}) \\
& - \theta_{k+1} t_k\}] dt_k \dots dt_1. \qquad (3.16)
\end{aligned}
$$

It can be easily seen that $\mathfrak{B}_k(z;\nu_1,\ldots,\nu_k) = \sum_{i=0}^{k} B_i(z;\nu_1,\ldots,\nu_i)$ with $B_0(z) = \mathfrak{B}_0(z) = e^{-\theta_1 z}$. Thus, we have

$$\begin{aligned}
& \int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-2}}^{t_{l_k-1}} \int_{h^{-1}(y_{k-1})}^{x} \theta_1 \ldots \theta_{l_k-1} \Big(\exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
& + \theta_{l_k-1}(t_{l_k-1} - t_{l_k-2})\}] \\
& - \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots + \theta_{l_k}(x - t_{l_k-1})\}] \Big) dt_{l_k-1} \ldots dt_1 \\
= & \int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-2}}^{t_{l_k-1}} \int_{h^{-1}(y_{k-1})}^{x} \theta_1 \ldots \theta_{l_k-1} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
& + \theta_{l_k-1}(t_{l_k-1} - t_{l_k-2})\}] dt_{l_k-1} \ldots dt_1 \\
& - \int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-2}}^{t_{l_k-1}} \int_{\nu_{l_k-1}}^{x} \theta_1 \ldots \theta_{l_k-1} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
& + \theta_{l_k}(x - t_{l_k-1})\}] dt_{l_k-1} \ldots dt_1 \\
= & \int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-2}}^{t_{l_k-1}} \int_{h^{-1}(y_{k-1})}^{\infty} \theta_1 \ldots \theta_{l_k-1} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
& + \theta_{l_k-1}(t_{l_k-1} - t_{l_k-2})\}] dt_{l_k-1} \ldots dt_1 \\
& - \int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-2}}^{t_{l_k-1}} \int_{x}^{\infty} \theta_1 \ldots \theta_{l_k-1} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
& + \theta_{l_k-1}(t_{l_k-1} - t_{l_k-2})\}] dt_{l_k-1} \ldots dt_1 \\
& - \int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-2}}^{t_{l_k-1}} \int_{\nu_{l_k-1}}^{x} \theta_1 \ldots \theta_{l_k-1} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
& + \theta_{l_k}(x - t_{l_k-1})\}] dt_{l_k-1} \ldots dt_1 \\
= & \ \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \ldots, \nu_{l_k-2}\right) - \mathfrak{B}_{l_k-2}\left(x; \nu_1, \ldots, \nu_{l_k-2}\right) \\
& - B_{l_k-1}\left(x; \nu_1, \ldots, \nu_{l_k-1}\right) \\
= & \ \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \ldots, \nu_{l_k-2}\right) - \mathfrak{B}_{l_k-1}\left(x; \nu_1, \ldots, \nu_{l_k-1}\right), \qquad (3.17)
\end{aligned}$$

noting $h^{-1}(y_{k-1}) = \nu_{l_k-1}$, and

$$\begin{aligned}
&\int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-1}}^{t_{l_k}} \int_{h^{-1}(y_{l_k}-y)}^{x} \theta_1 \dots \theta_{l_k} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
&+\theta_{l_k}(t_{l_k} - t_{l_k-1})\}] dt_{l_k} \dots dt_1 \\
= &\int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-1}}^{t_{l_k}} \int_{h^{-1}(y_{l_k}-y)}^{\infty} \theta_1 \dots \theta_{l_k} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
&+\theta_{l_k}(t_{l_k} - t_{l_k-1})\}] dt_{l_k} \dots dt_1 \\
&-\int_{\nu_1}^{t_2} \cdots \int_{\nu_{l_k-1}}^{t_{l_k}} \int_{x}^{\infty} \theta_1 \dots \theta_{l_k} \exp[-\{\theta_1 t_1 + \theta_2(t_2 - t_1) + \cdots \\
&+\theta_{l_k}(t_{l_k} - t_{l_k-1})\}] dt_{l_k} \dots dt_1 \\
= &\mathfrak{B}_{l_k-1}\left(h^{-1}(y_k - y); \nu_1, \dots, \nu_{l_k-1}\right) - \mathfrak{B}_{l_k-1}\left(x; \nu_1, \dots, \nu_{l_k-1}\right).
\end{aligned} \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.15), formula (3.13) directly follows. Hence, the asserted result holds true.

In what follows, we provide a solution for $B_k(z; \nu_1, \dots, \nu_k)$ to complete the proof. Denoting the multiple integral on the right-hand side in (3.16) by $I_k(z)$, we have

$$B_k(z) = e^{-\theta_{k+1} z} I_k(z).$$

One sees that the derivative of $I_k(z)$ is given by

$$
\begin{aligned}
\frac{dI_k(z)}{dz} &= \int\limits_{\nu_1}^{t_2} \dots \int\limits_{\nu_{k-1}}^{t_k} \int\limits_{\nu_k}^{z} \theta_1 \dots \theta_k \exp[-\{\theta_1 t_1 + \theta_2 (t_2 - t_1) + \dots \\
&\quad + \theta_k (z - t_{k-1}) - \theta_{k+1} z\}] dt_{k-1} \dots dt_1 \\
&= \theta_k e^{(\theta_{k+1} - \theta_k) z} \int\limits_{\nu_1}^{t_2} \dots \int\limits_{\nu_{k-1}}^{t_k} \int\limits_{\nu_k}^{z} \theta_1 \dots \theta_{k-1} \exp[-\{\theta_1 t_1 + \theta_2 (t_2 - t_1) + \dots \\
&\quad + \theta_{k-1} (t_{k-1} - t_{k-2}) - \theta_k t_{k-1}\}] dt_{k-1} \dots dt_1 \\
&= \theta_k e^{(\theta_{k+1} - \theta_k) z} I_{k-1}(z).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{dB_k(z)}{dz} &= -\theta_{k+1} e^{-\theta_{k+1} z} I_k(z) + \theta_k e^{-\theta_{k+1} z} e^{\theta_{k+1} z - \theta_k z} I_{k-1}(z) \\
&= -\theta_{k+1} B_k(z) + \theta_k B_{k-1}(z).
\end{aligned}
$$

It is not difficult to verify that the system of linear differential equations,

$$
\begin{aligned}
B_0'(z) &= -\theta_1 e^{-\theta_1 z} \\
B_k'(z) &= -\theta_{k+1} B_k(z) + \theta_k B_{k-1}(z).
\end{aligned}
$$

for $k = 1, 2, \dots$ with initial conditions

$$
B_0(0) = 1,\ B_k(\nu_k) = 0,\ k = 1, 2, \dots
$$

has a unique solution, given by the following sequence of functions

$$
B_k(z) = \theta_k e^{-\theta_{k+1} z} \int_{\nu_k}^{z} e^{\theta_{k+1} w} B_{k-1}(w) dw\ ,\ k = 1, 2, \dots
$$

with $B_0(z) = e^{-\theta_1 z}$. Actually, $B_k(z)$ are called (classical) exponential Appell polynomials as defined by Ignatov and Kaishev (2012b). Thus, we complete the proof. $\square$

Corollary 3.5.2 *When $l_1 = 1$, the probability $P(T < x, Y > y)$, $x > 0$, $y \geq 0$, is given by*

$$P(T < x, Y > y) = \int_y^\infty f_{Y_1}(y_1) dy_1$$
$$+ \sum_{k=2}^{\infty} \int_0^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}+y}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k)$$
$$dy_k\ dy_{k-1} \dots dy_1$$
$$- \sum_{k=1}^{\infty} \int_0^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}+y}^{h(x)+y} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k - y); \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k)$$
$$dy_k\ dy_{k-1} \dots dy_1$$
$$- \sum_{k=1}^{\infty} \int_0^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{h(x)+y}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k\ dy_{k-1} \dots dy_1. \tag{3.19}$$

Proof: From (3.14), we have

$$\begin{aligned}
& P(T < x, Y > y) \\
= & \sum_{k=1}^{\infty} P\left(\bigcap_{r=1}^{l_k-1}\left\{\tilde{Y}_r < h\left(\tilde{T}_r\right)\right\} \cap \left\{\tilde{Y}_{l_k} > h\left(\tilde{T}_{l_k}\right) + y\right\} \cap \left\{\tilde{T}_{l_k} < x\right\}\right) \\
= & P\left(\{Y_1 > h(T_1) + y\} \cap \{T_1 < x\}\right) + \\
& \sum_{k=2}^{\infty} P\left(\bigcap_{r=1}^{l_k-1}\left\{\tilde{Y}_r < h\left(\tilde{T}_r\right)\right\} \cap \left\{\tilde{Y}_{l_k} > h\left(\tilde{T}_{l_k}\right) + y\right\} \cap \left\{\tilde{T}_{l_k} < x\right\}\right).
\end{aligned} \tag{3.20}$$

For the first term in the right-hand side in (3.20), we have

$$
\begin{aligned}
& P\left(\{Y_1 > h(T_1)+y\} \cap \{T_1 < x\}\right) \\
= \quad & \int_0^x f_{T_1}(t_1) \int_{h(t_1)+y}^{\infty} f_{Y_1}(y_1) dy_1 dt_1 \\
= \quad & \int_0^x \lambda e^{-\lambda t_1} \int_y^{\infty} f_{Y_1}(y_1) dy_1 dt_1 - \int_0^x \lambda e^{-\lambda t_1} \int_y^{h(t_1)+y} f_{Y_1}(y_1) dy_1 dt_1 \\
= \quad & \left(1-e^{-\lambda x}\right) \int_y^{\infty} f_{Y_1}(y_1) dy_1 - \int_y^{h(x)+y} \left(\int_{h^{-1}(y_1-y)}^{x} \lambda e^{-\lambda t_1} dt_1 \right) f_{Y_1}(y_1) dy_1 \\
= \quad & \left(1-e^{-\lambda x}\right) \int_y^{\infty} f_{Y_1}(y_1) dy_1 - \int_y^{h(x)+y} \left(e^{-\lambda h^{-1}(y_1-y)} - e^{\lambda x} \right) f_{Y_1}(y_1) dy_1 \\
= \quad & \int_y^{\infty} f_{Y_1}(y_1) dy_1 - e^{-\lambda x} \int_{h(x)+y}^{\infty} f_{Y_1}(y_1) dy_1 - \int_y^{h(x)+y} e^{-\lambda h^{-1}(y_1-y)} f_{Y_1}(y_1) dy_1.
\end{aligned}
\tag{3.21}
$$

For the second term in the right-hand side in (3.20), it is not difficult to follow the proof of Theorem 3.5.1 and obtain

$$
\begin{aligned}
& \sum_{k=2}^{\infty} P\left(\bigcap_{r=1}^{l_k-1} \left\{ \tilde{Y}_r < h\left(\tilde{T}_r\right) \right\} \cap \left\{ \tilde{Y}_{l_k} > h\left(\tilde{T}_{l_k}\right) + y \right\} \cap \left\{ \tilde{T}_{l_k} < x \right\} \right) \\
= \quad & \sum_{k=2}^{\infty} \int_0^{h(x)} \cdots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}+y}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \ldots, \nu_{l_k-2}\right) f_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k) \\
& dy_k \; dy_{k-1} \ldots dy_1 \\
& -\sum_{k=2}^{\infty} \int_0^{h(x)} \cdots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}+y}^{h(x)+y} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k - y); \nu_1, \ldots, \nu_{l_k-1}\right) f_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k) \\
& dy_k \; dy_{k-1} \ldots dy_1 \\
& -\sum_{k=2}^{\infty} \int_0^{h(x)} \cdots \int_{y_{k-2}}^{h(x)} \int_{h(x)+y}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \ldots, \nu_{l_k-1}\right) f_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k) dy_k \; dy_{k-1} \ldots dy_1.
\end{aligned}
\tag{3.22}
$$

Substituting the results of (3.21) and (3.22) into (3.20), formula (3.19) directly follows. Hence, the asserted result holds true. □

Corollary 3.5.3 *When $y = 0$, the finite-time ruin probability given by Theorem 3.5.1 coincides with the results of Lemma A.1 in Dimitrova et al. (2013b).*

Proof: When $y = 0$, (3.13) can be simplified as

$$
\begin{aligned}
& P(T < x, Y > 0) \equiv P(T < x) \\
= \; & \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) \\
& dy_k \; dy_{k-1} \dots dy_1 \\
& - \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}}^{h(x)} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k); \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) \\
& dy_k \; dy_{k-1} \dots dy_1 \\
& - \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{h(x)}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k \; dy_{k-1} \dots dy_1 \\
= \; & 1 + \sum_{k=2}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) \\
& dy_k \; dy_{k-1} \dots dy_1 \\
& - \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}}^{h(x)} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k); \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) \\
& dy_k \; dy_{k-1} \dots dy_1 \\
& - \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{h(x)}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k \; dy_{k-1} \dots dy_1,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& P(T>x) \\
= \; & -\sum_{k=2}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) f_{Y_1,\dots,Y_k}(y_1,\dots,y_k) \\
& dy_k \; dy_{k-1} \dots dy_1 \\
& + \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}}^{h(x)} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k); \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1,\dots,y_k) \\
& dy_k \; dy_{k-1} \dots dy_1 \\
& + \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{h(x)}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1,\dots,y_k) dy_k \; dy_{k-1} \dots dy_1 \\
= \; & -① + ② + ③.
\end{aligned}
$$

We then have

$$
\begin{aligned}
① \; = \; & \sum_{k=2}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{y_{k-1}}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) f_{Y_1,\dots,Y_k}(y_1,\dots,y_k) \\
& dy_k \; dy_{k-1} \dots dy_1 \\
= \; & \sum_{k=2}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) \left(\int_{y_{k-1}}^{\infty} f_{Y_1,\dots,Y_k}(y_1,\dots,y_k) dy_k \right) \\
& dy_{k-1} \dots dy_1 \\
= \; & \sum_{k=2}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) f_{Y_1,\dots,Y_{k-1}}(y_1,\dots,y_{k-1}) \\
& dy_{k-1} \dots dy_1 \\
= \; & \sum_{k=1}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{k-1}}^{h(x)} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k); \nu_1, \dots, \nu_{l_k-1}\right) f_{Y_1,\dots,Y_k}(y_1,\dots,y_k) dy_k \dots dy_1 \\
= \; & ②,
\end{aligned}
$$

indicating that

$$\begin{aligned}
P(T > x) &= -① + ② + ③ = ③ \\
&= \sum_{k=1}^{\infty} \int_0^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{h(x)}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \dots, \nu_{l_k-1}\right) \\
& \quad f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k \; dy_{k-1} \dots dy_1 \\
&= \sum_{k=1}^{\infty} \int_0^{h(x)} \dots \int_{y_{k-2}}^{h(x)} \int_{h(x)}^{\infty} \sum_{i=0}^{l_k-1} B_i\left(x; \nu_1, \dots, \nu_i\right) \\
& \quad f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k \; dy_{k-1} \dots dy_1 \\
&= \sum_{k=0}^{\infty} \int_0^{h(x)} \dots \int_{y_{k-1}}^{h(x)} \int_{h(x)}^{\infty} \sum_{i=0}^{l_{k+1}-1} B_i\left(x; \nu_1, \dots, \nu_i\right) \\
& \quad f_{Y_1,\dots,Y_{k+1}}(y_1, \dots, y_{k+1}) dy_{k+1} \; dy_k \dots dy_1.
\end{aligned}$$

Permuting the two sums, we obtain

$$\begin{aligned}
P(T > x) &= \sum_{i=0}^{\infty} \sum_{k=j(i)}^{\infty} \int_0^{h(x)} \dots \int_{y_{k-1}}^{h(x)} \int_{h(x)}^{\infty} B_i\left(x; \nu_1, \dots, \nu_i\right) \\
& \quad f_{Y_1,\dots,Y_{k+1}}(y_1, \dots, y_{k+1}) dy_{k+1} \; dy_k \dots dy_1,
\end{aligned}$$

where $j(k)$, $k = 0, 1, 2, \dots$, is an integer-valued function such that

$$l_{j(k)} \leq k < l_{j(k+1)}$$

so that

k	0	1	...	l_1-1	l_1	...	l_2-1	l_2	...	l_3-1	l_3	...
$j(k)$	0	0	...	0	1	...	1	2	...	2	3	...

It is not difficult to follow that

$$
\begin{aligned}
& P(T > x) \\
= \; & \sum_{i=0}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{j(i)-1}}^{h(x)} B_i\left(x; \nu_1, \dots, \nu_i\right) \times \\
& \left(\sum_{k=j(i)}^{\infty} \int_{y_{j(i)}}^{h(x)} \dots \int_{h(x)}^{\infty} f_{Y_1,\dots,Y_{k+1}}(y_1, \dots, y_{k+1}) dy_{k+1} \dots dy_{j(i)+1} \right) dy_{j(i)} \dots dy_1 \\
= \; & \sum_{i=0}^{\infty} \int_{0}^{h(x)} \dots \int_{y_{j(i)-1}}^{h(x)} B_i\left(x; \nu_1, \dots, \nu_i\right) \\
& \times \Big(\int_{h(x)}^{\infty} f_{Y_1,\dots,Y_{j(i)+1}}(y_1, \dots, y_{j(i)+1}) dy_{j(i)+1} \\
& + \int_{y_{j(i)}}^{h(x)} \int_{h(x)}^{\infty} f_{Y_1,\dots,Y_{j(i)+2}}(y_1, \dots, y_{j(i)+2}) dy_{j(i)+2} dy_{j(i)+1} \\
& + \int_{y_{j(i)}}^{h(x)} \int_{y_{j(i)+1}}^{h(x)} \int_{h(x)}^{\infty} f_{Y_1,\dots,Y_{j(i)+3}}(y_1, \dots, y_{j(i)+3}) dy_{j(i)+3} dy_{j(i)+2} dy_{j(i)+1} \\
& + \cdots \Big) dy_{j(i)} \dots dy_1 .
\end{aligned}
$$

Noting that the term in the brackets is identically equal to $f_{Y_1,\dots,Y_{j(i)}}(y_1, \dots, y_{j(i)})$, we have

$$
\begin{aligned}
P(T > x) &= \sum_{i=0}^{\infty} \int_{0}^{h(x)} \cdots \int_{y_{j(i)-1}}^{h(x)} B_i\left(x; \nu_1, \ldots, \nu_i\right) f_{Y_1,\ldots,Y_{j(i)}}(y_1, \ldots, y_{j(i)}) dy_{j(i)} \ldots dy_1 \\
&= \sum_{i=0}^{\infty} \int_{0 \le y_1 \le \ldots \le y_{j(i)} \le h(x)} \!\!\!\!\cdots\!\!\int B_i\left(x; \nu_1, \ldots, \nu_i\right) f_{Y_1,\ldots,Y_{j(i)}}(y_1, \ldots, y_{j(i)}) dy_{j(i)} \ldots dy_1,
\end{aligned}
$$

which exactly coincides with the results of Lemma A.1 in Dimitrova et al. (2013b). Thus, we complete the proof. □

Now, we consider the following special case. We assume the inter-arrival times, τ_i, follow an independent non-identical Erlang distribution with shape parameter $m_i > 0$ and rate parameter $\lambda_i > 0$, i.e. $\tau_i \sim$Erlang(m_i, λ_i), with density

$$
f_{\tau_i}(t) = \frac{\lambda_i^{m_i} t^{m_i - 1} e^{-\lambda_i t}}{\Gamma(m_i)},
$$

where m_i's are arbitrary positive integers and λ_i's are positive real numbers. Clearly, $\tau_i \stackrel{d}{=} \sum_{j=1}^{m_i} \tilde{\tau}_i$, where $\tilde{\tau}_i \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(\lambda_i)$. Thus, this can be viewed as a special case of inter-arrival times following a combination of independent non-identical exponential distributions, where $\lambda_{ij} = \lambda_i$, $j = 1, \ldots, m_i$. Hence, without providing further proof, we give the joint probability of the time to ruin and the deficit at ruin in the insurance risk model with independent non-identical Erlang distributed inter-arrival times in Corollary 3.5.4, and Corollary 3.5.5 naturally follows.

Corollary 3.5.4 *The probability $P(T < x, Y > y)$, $x > 0$, $y \geq 0$, is given by (assuming $l_1 \geq 2$)*

$$\begin{aligned}
P(T < x, Y > y) \;\; = \;\; & \sum_{k=1}^{\infty} \int\limits_{0}^{h(x)} \dots \int\limits_{y_{k-2}}^{h(x)} \int\limits_{y_{k-1}+y}^{\infty} \mathfrak{B}_{l_k-2}\left(h^{-1}(y_{k-1}); \nu_1, \dots, \nu_{l_k-2}\right) \\
& f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k \; dy_{k-1} \dots dy_1 \\
& - \sum_{k=1}^{\infty} \int\limits_{0}^{h(x)} \dots \int\limits_{y_{k-2}}^{h(x)} \int\limits_{y_{k-1}+y}^{h(x)+y} \mathfrak{B}_{l_k-1}\left(h^{-1}(y_k - y); \nu_1, \dots, \nu_{l_k-1}\right) \\
& f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k \; dy_{k-1} \dots dy_1 \\
& - \sum_{k=1}^{\infty} \int\limits_{0}^{h(x)} \dots \int\limits_{y_{k-2}}^{h(x)} \int\limits_{h(x)+y}^{\infty} \mathfrak{B}_{l_k-1}\left(x; \nu_1, \dots, \nu_{l_k-1}\right) \\
& f_{Y_1,\dots,Y_k}(y_1, \dots, y_k) dy_k \; dy_{k-1} \dots dy_1, \qquad (3.23)
\end{aligned}$$

where

$$\mathfrak{B}_k(z; \nu_1, \dots, \nu_k) = \sum_{i=0}^{k} B_k(z; \nu_1, \dots, \nu_k),$$

$B_k(z; \nu_1, \dots, \nu_k) \equiv B_k(z)$ *are the (classical) exponential Appell polynomials defined recurrently by*

$$B_k(z) = \lambda_{j(k-1)+1} e^{-\lambda_{j(k)+1} z} \int_{\nu_{j(k)}}^{z} e^{\lambda_{j(k)+1} w} B_{k-1}(w) dw, k = 1, 2, \dots$$

with $B_0(z) = \mathfrak{B}_0(z) = e^{-\lambda_1 z}$, $j(k)$, $k = 0, 1, 2, \dots$, *is an integer-valued function such that*

$$m_1 + \dots + m_{j(k)} \leq k < m_1 + \dots + m_{j(k)} + m_{j(k)+1},$$

and $0 \leq \nu_1 \leq \nu_2 \leq \ldots$ *is a sequence of real numbers denoting*

$$\underbrace{h^{-1}(0) \leq \ldots \leq h^{-1}(0)}_{m_1-1} \leq \underbrace{h^{-1}(y_1) \leq \ldots \leq h^{-1}(y_1)}_{m_2} \leq \ldots,$$

correspondingly.

Corollary 3.5.5 *When* $y = 0$*, the finite-time ruin probability given by Corollary 3.5.4 coincides with the results of Theorem 2.1 in Ignatov and Kaishev (2012b).*

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Chapter 4

On finite-time ruin probabilities in a generalized dual risk model with dependence

On finite-time ruin probabilities in a generalized dual risk model with dependence

Abstract

In this paper, we study the finite-time ruin probability in a reasonably generalized dual risk model, where we assume any non-negative non-decreasing cumulative operational cost function and arbitrary capital gain arrival process. Establishing an enlightening link between this dual risk model and its corresponding insurance risk model, explicit expressions for finite-time survival probability in the dual risk model are obtained under various general assumptions for the distribution of the capital gains. In order to make the model more realistic and general, different dependence structures among capital gains and inter-arrival times and between both are also introduced and corresponding ruin probability expressions are also given. The concept of alarm time, due to Das and Kratz (2012), is applied for the dual risk model within the context of risk capital allocation. Extensive numerical illustrations are provided.

Keywords: dual risk model, finite-time ruin probability, dependent risk modelling, capital allocation, alarm time, (exponential) classical Appell polynomials

Chapter 5

Modelling finite-time failure probabilities in risk analysis applications

Modelling finite-time failure probabilities in risk analysis applications

Abstract

In this paper, we introduce a framework for analyzing the risk of systems failure based on estimating the failure probability. The latter is defined as the probability that a certain risk process, characterizing the operations of a system, reaches a possibly time-dependent critical risk level within a finite-time interval. Under general assumptions, we define two dually connected models for the risk process and derive explicit expressions for the failure probability and also the joint probability of the time of the occurrence of failure and the excess of the risk process over the risk level. We illustrate how these probabilistic models and results can be successfully applied in several important areas of risk analysis among which systems reliability, inventory management, flood control via dam management, infection disease spread and financial insolvency. Numerical illustrations are also presented.

Keywords: finite-time failure probability, dependent risk modelling, alarm time, (exponential) classical Appell polynomials

Chapter 6

Conclusions

6.1 Summary

In this thesis, we have considered two alternative dependent risk models, the insurance risk model and the dual risk model. We have derived explicit expressions for the finite-time ruin probabilities and the joint probabilities of the time to ruin and the deficit at ruin under various general assumptions and proposed efficient algorithms to evaluate the ruin probability with a prescribed accuracy. We have also introduced a new definition of alarm time to detect the instant when the ruin probability in a pre-specified future time interval becomes alarmingly high. It is worth noting that the new definition highly relies on the computation of the finite-time ruin probability and the joint probability of the time to ruin and the deficit at ruin, which have utilized the results obtained previously. Based on the alarm time devised, optimization problem of capital allocation has been studied. We have finally introduced a framework for analyzing the risk of systems failure based on estimating the failure probability and illustrated how these probabilistic models and results can be successfully applied as risk analytic tools in several important areas of risk management. Extensive numerical experiments have been provided.

In Chapter 2, we have shown that the survival probability formulas derived by Ignatov and Kaishev (2000, 2004) and Ignatov et al. (2001) can be derived from one another both in their discrete and continuous versions and thus, can all be expressed in terms of the classical Appell polynomials. Various recurrence expressions for computing these polynomials have been presented, and their numerical properties have been investigated. Furthermore, the numerical efficiency of formulas for the finite-time survival probability have been investigated and conclusions have been drawn about the differences in their summation structure. We have selected the computationally more efficient expressions to evaluate the survival probability. A method of computing the survival probability with a prescribed accuracy has been introduced, which is also applicable for the case of discrete claim amounts whereby the number of summands in the already finite summation can be further reduced. We also studied the order statistics simulation-based method, proposed by Dimitrova and Kaishev (2013), for evaluating the survival probability formulas and provided several numerical examples to demonstrate its performance in dealing with different cases of discrete or continuous, dependent or independent claim severities.

In Chapter 3, we have presented a new definition of an alarm time, t_A, which is expressed in terms of the joint probability of the time to ruin T and the deficit at ruin Y in a general insurance risk model. It generalizes a definition based solely on ruin probability, recently introduced by Das and Kratz (2012). Alarm times are consecutive time instants, at which portions of risk capital should be reserved sequentially so as to maintain the probability of finite-time ruin below a certain pre-specified target level. We have further summarized some closed-form ruin probability formulas obtained by Ignatov and Kaishev (2012) and more recently investigated by Dimitrova et al. (2014a) and Dimitrova et al. (2014b) and derived some new expressions for the joint distribution of T and Y, which generalizes some of the above mentioned results. We have shown further that, by solving an appropriate optimal capital allocation problem, it is possible to optimally determine the amount of risk capital which should be injected sequentially at the alarm times so as to maximize the finite-time survival probability. We have also undertaken a thorough numerical model sensitivity investigation which has revealed some interesting properties of the alarm time and the alarm system.

In Chapter 4, we have considered the problem of finding the probability of ruin in a finite time in a reasonably generalized dual risk model, where we assume any non-negative non-decreasing cumulative operational cost function and arbitrary capital gain arrival process. Establishing an enlightening link between this dual risk model and its corresponding insurance risk model, we obtain explicit expressions for finite-time survival probability in the dual risk model for various reasonably general assumptions for capital gains distribution. Dependence structures among capital gains and inter-arrival times or between both have also been incorporated to make the model more realistic and general and corresponding ruin probability expressions have been obtained. A risk capital allocation approach based on the concept of alarm time, due to Das and Kratz (2012), has been proposed for the dual risk model. A corresponding procedure for the computation of alarm times, where additional capital needs to be injected in order to maintain chance of survival above a certain level, has been developed and implemented numerically. It has to be highlighted that the ruin probabilistic results obtained here and the elegant duality lemma are remarkable since there are very few papers in the literature devoted to the ruin probability in the dual risk model and, to the best of our knowledge, there are no closed-form results. As illustrated, the dual risk model has the potential for much wider applications than the insurance risk model.

In Chapter 5, we have introduced two dually connected stochastic models A and B, representing the generalized insurance risk model and dual risk model respectively, for quantifying the risk of failure and have interpreted the related model parameters and variables to fit specific applications in risk analysis. We have demonstrated that the theoretical results obtained within the modelling frameworks of models A and B are quite general and thus, applicable in providing solutions to many risk analytic problems appearing in systems reliability, inventory management, dam management, infection disease spread and financial insolvency.

6.2 Directions for future research

In Chapter 2, we have studied the order statistics simulation-based method, proposed by Dimitrova and Kaishev (2013), for evaluating the survival probability formulas. In our numerical study, we have not taken advantage of the possibilities for parallelizing the simulations and thus achieving substantial reduction in computation time but it is clear that this is easily achievable. We also note that further improvements of the accuracy of the proposed method could also be achieved borrowing ideas from the importance sampling simulation area of research, which is the subject of ongoing work.

In Chapter 3, the concept of alarm system has been carried out as an alternative capital allocation strategy to reduce the size of initial capital, which is practically highly preferable for the insurance companies. However, how to allocate capital optimally still remains an open question. In this paper, finite-time survival probability has been employed as the key criterion to assess the optimality. However, other risk measures could also be considered. The framework could also be formulated in different ways. It is worth mentioning that the problems formulated in Chapter 3 were quite simple, but we hope these could provide some enlightenment for further studies. Due to the nature of the definition of the alarm times, the (computational) complexity embedded may snowball with the increase of the dimension of the problems to be solved.

Another interesting direction to further expand the framework is to move to a more dynamic setting. In practice, the operations of the company may change, as well as the solvency requirements. Hence, a more dynamic and adaptive definition of an alarm system could be developed to reflect the information and the data collected over time. An empirical study based on real data from the industry also sounds appealing. These are all in the scope of our future work.

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