The Two Sector Endogenous Growth Model: An Atlas

Michael Ben-Gad

Department of Economics

City University London

Northampton Square

London EC1V 0HB, UK*

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Abstract

In this paper we investigate the underlying structure of the Lucas (1988) endogenous growth model. We derive analytically, the restrictions on the parameter space that are necessary and sufficient for the existence of balanced growth paths and saddle-path stable local dynamics. We demonstrate that in contrast to the original model, with the addition of an external effect and depreciation in the human capital sector, the Lucas model can be made consistent with the high degrees of intertemporal elasticities of substitution increasingly estimated in the empirical literature—even if there is a significant degree of increasing returns to scale in the physical production sector of the economy. Finally we demonstrate that for a given baseline rate of steady state growth, with the inclusion of modest degrees of depreciation and external effects to the human capital production process, the model can accommodate the widest possible range of economies—including those characterized by low discount factors, high elasticities of intertemporal substitution, increasing returns in the final goods sector, and also both the high rates of population growth and steady state per-capita output growth we observe in many parts of the world today.

JEL classification: O41, E20

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1 Introduction

This paper has two aims. First, to provide an atlas of sorts for the Lucas (1988) two-sector endogenous growth model with human capital model—one extended to include sector-specific external effects and depreciation in both sectors—by mapping out analytically, the precise restrictions on the parameter space necessary and sufficient for the existence of balanced growth paths and for the existence of saddle-path stable equilibria in their vicinity. Second, the Lucas model in its original form has trouble accommodating values for the intertemporal elasticity of substitution that are significantly higher than one, a problem that becomes more acute for high rates of population growth or high rates of per-capita output growth. Generalizing the process of human capital accumulation provides the model with greater flexibility to accommodate the disparate growth experience we observe around the world.

Lucas (1988) introduced the first growth model where endogenously generated human capital generate both labour augmenting technology and total factor productivity, the latter by non-rival spillovers. The two-sector model has featured in increasing numbers of applications in macroeconomics. However, its most important feature remains its ability to capture the empirical regularities in economic growth data that the neo-classical models have difficulty explaining—particularly the apparent lack of long-run convergence, either absolute or conditional we observe in cross-country datasets. Indeed mounting evidence suggests that a satisfactory model must be sufficiently versatile to explain both evidence assembled by Pritchett (1997), Easterly and Levine (2001) and many others that point to divergence, but also the existence of convergence clubs, as in Quah (1997), Canova (2004) and Huang (2005).

Attfield and Temple (2010) point out how sensitive the relationships between steady state ratios and growth are to the intertemporal elasticity of substitution. With this in mind, along with the growing body of empirical evidence pointing towards higher values for the intertemporal elasticity of substitution, this paper demonstrates that by adding a degree of sector-specific external effects and/or depreciation to the human capital sector, the two-sector model can be made consistent with high rates of intertemporal elasticity of substitution, as the well as high rates of population growth and high rates of output growth we commonly observe in many countries in Africa, Asia, and Central and South America.

In Section 2, we present the two-sector endogenous growth model with both depreciation and sector-specific external effects in each sector of the economy, and derive the laws of motion that characterize the model's dynamic behavior. Caballé and Santos (1993) analyze the two-sector model with depreciation in both sectors but only establish some general conditions for the existence of balanced growth paths. Xie (1994) includes external effects in the production sector, while abstracting from depreciation and external effects in the production of human capital. Unlike Caballé and Santos, Xie presents explicit bounds on the parameter space necessary and sufficient to guarantee balanced growth, however only by setting the intertemporal elasticity of substitution strictly equal to the reciprocal of the share of physical capital in the production sector. In this paper we do not impose this restriction.

In Section 3, we derive the steady state values for capital, consumption, and hours of market work. Following Benhabib and Perli (1994) and Ben-Gad (2003), we then use these values to analytically define the restrictions on the parameter space in terms of bounds on the subjective discount rate necessary and sufficient to ensure the existence of interior solutions to the representative agent's optimization program which support unique balanced growth paths. In Section 4, we further restrict the parameter space, by ruling out balanced growth paths characterized by unstable local dynamics.

In Section 5, we begin by considering what country level growth and demographic data imply about the value of the intertemporal elasticity of substitution within the context of the model. We then demonstrate the implications of our analytical results using numerical examples that focus on the behavior of the model in the usually problematic region where the intertemporal elasticity of substitution is greater than one—the region our empirical results imply may be best for parameterizing the model for at least many of the countries considered. Varying the magnitude of both external effects and the intertemporal elasticity of substitution, while fixing the other parameters of the model, we demonstrate that with the inclusion of external effects and depreciation, the two-sector model is able to accommodate the high values for the intertemporal elasticity of substitution as estimated by Hansen and Singleton (1982), Amano and Wirjanto (1997), Mulligan (2002) and Gruber (2006) for the United States, or Hamori (1996) and Fuse (2004) for Japan.

Finally in Section 6, we restrict our attention to those portions of the parameter space that correspond to a wide range of the most empirically relevant rates of growth. Fixing the baseline rate of steady state growth, we demonstrate that including modest degrees of depreciation and external effects to the human capital production process, enables us to calibrate the model for the widest possible range of economies—including those characterized by low discount factors, high elasticities of intertemporal substitution, increasing returns in the production sector, as well as the high rates of population growth found in much of Africa, Asia and Latin America.

2 The Model

The economy is composed of a large number of households whose behavior can be represented by the intertemporal maximization of an infinite-lived representative agent. This agent maximizes utility over time t, by choosing the dynamic path of per-capita consumption, c, and $u \in (0,1)$, the fraction of time as well as per-capita human capital h devoted to working in the production sector:

$$\max_{(c,u,k,h)_{t\geq 0}} \int_0^\infty e^{(n-\rho)t} \frac{\sigma}{\sigma - 1} c^{1-1/\sigma} dt, \tag{1}$$

subject to the constraints:

$$\dot{k} = wuh + (r - \delta - n) k - c, \tag{2}$$

$$\dot{h} = \nu \left[(1-u) h\right]^{1-\gamma} \left[(1-\bar{u}) \bar{h}\right]^{\gamma} - \varepsilon h, \tag{3}$$

where σ is the constant rate of intertemporal elasticity of substitution, ρ a positive discount rate, n the natural rate of population growth, δ the rate of depreciation of per-capita physical capital k, r its rate of return, ε the rate of depreciation of human capital and w the wage rate.[1] The terms $\bar{u} \in (0,1)$ and $\bar{h}$ are the time t share of time devoted to market work and the time t stock of human capital, aggregated over all the agents in the economy and expressed in per-capita terms—hence the term $\left[(1-\bar{u}) \bar{h}\right]^{\gamma}$ captures the efficiency enhancing external effects of that portion of the human capital stock devoted to its production, and the parameter γ regulates its magnitude. Time not devoted to work for wages is spent accumulating human capital—ν is the maximum rate at which human capital can be accumulated.

The original Lucas formulation does not distinguish between private and social returns in the production of human capital. Empirical evidence from studies on schooling (Harmon and Walker (1999) and Carneiro and Heckman (2003)) suggest that returns to human capital diminish at the individual level. Yet if physical capital plays no role in human capital production, balanced growth is only possible if in aggregate the technology that produces it is linear. Economy-wide activity devoted to human capital production complements private production—we assume spill-overs within the sector account for the difference.

Physical goods are produced by a combination of physical capital and effective labor $\phi = uh$:

$$y = \left(\bar{u}\bar{h}\right)^{\beta} F(k, \phi), \tag{4}$$

where the term $\left(\bar{u}\bar{h}\right)^{\beta}$ captures the efficiency enhancing external effects of that portion of the human capital stock employed in the production sector. We assume that the function $F : R^2 \rightarrow R$ takes the constant returns, Cobb-Douglas form $F(k, \phi) = k^{\alpha}\phi^{1-\alpha}$. Internal factor returns are:

$$r = \left(\bar{u}\bar{h}\right)^{\beta} F_k(k, \phi), \tag{5}$$

$$w = \left(\bar{u}\bar{h}\right)^{\beta} F_{\phi}(k, \phi). \tag{6}$$

[1]Though this formulation of preferences seems restrictive, both the rate of intertemporal elasticity of substitution and the subjective discount rate must be constant to ensure the existence of a balanced growth path. See Palivos *et. al.* (1997).

Unlike Lucas' aggregate external effects, we limit the scope of external effects to be sector-specific. Only the portion of human capital that is employed in the production sector generates spill-over effects for that sector, but this is sufficient to generate both differential rates of steady state growth for the two types capital, and higher rental rates for human capital in rich countries. The most obvious spill-overs are likely to be the result of complementarities between the skills of workers—personnel in a sector interact and learn from each other. Increases in the total stock of knowledge certainly enhance efficiency in the production sector—however, this may be knowledge produced by both domestic and foreign human capital sectors. Restricting spill-overs to be sector-specific obviates the need to distinguish between endogenous domestically produced human capital, and the foreign portion of human capital which is accumulating exogenously.[2]

The present value Hamiltonian that corresponds to the consumer's optimization problem is:

$$\begin{aligned} H\left(c,u,k,h,\lambda,\mu\right) &= e^{(n-\rho)t}\frac{\sigma}{\sigma-1}c^{1-1/\sigma}+\lambda\left[wuh+rk-c-nk\right] \\ &\quad +\mu\left[\left[\left(1-u\right)h\right]^{1-\gamma}\left[\left(1-\bar{u}\right)\bar{h}\right]^{\gamma}-\varepsilon h\right], \end{aligned} \tag{7}$$

where λ and μ are the costate variables for physical and human capital.

Inserting the values from (5) and (6), in place of r and w, the first order necessary conditions for an interior solution to the individual constrained optimization are:

$$\frac{e^{(n-\rho)t}}{c^{1/\sigma}}=\lambda, \tag{8}$$

$$\left(1-\alpha\right)\lambda h\left(\bar{u}\bar{h}\right)^{\beta}k^{\alpha}\left(uh\right)^{-\alpha}=\mu\left(1-\gamma\right)\nu\left(1-u\right)^{-\gamma}h^{1-\gamma}\left[\left(1-\bar{u}\right)\bar{h}\right]^{\gamma}, \tag{9}$$

$$\lambda\left(\alpha\left(\bar{u}\bar{h}\right)^{\beta}k^{\alpha-1}\left(uh\right)^{1-\alpha}-\delta-n\right)=-\dot{\lambda}, \tag{10}$$

$$\mu\left[\left(1-\gamma\right)\nu\left(1-u\right)^{1-\gamma}h^{-\gamma}\left[\left(1-\bar{u}\right)\bar{h}\right]^{\gamma}-\varepsilon\right]+\lambda\left(1-\alpha\right)\left(\bar{u}\bar{h}\right)^{\beta}k^{\alpha}\left(uh\right)^{-\alpha}u=-\dot{\mu}, \tag{11}$$

plus the two transversality conditions,

$$\lim_{t\to\infty}\lambda k=0, \tag{12}$$

$$\lim_{t\to\infty}\mu h=0, \tag{13}$$

and the constraint that u falls within the unit interval. We define the parameter space Θ: $\theta\equiv\left(\alpha,\beta,\gamma,\delta,\varepsilon,\nu,\rho,\sigma,n\right)$, and $\theta\in\Theta$, where $\Theta=R^2_{++}\times R^4_{+}\times[0,1)^3$.

[2]Paul and Siegel (1999) find strong evidence of sizeable increasing returns—two-thirds to almost three-quarters can be ascribed to agglomeration effects—sector specific external effects at the two-digit industry level. Harrison (1998) finds evidence of increasing returns but rejects spillovers between sectors and Benhabib and Jovanovic (1991), demonstrate that the source of any aggregate increasing returns to scale are not associated with the capital input. Finally, Durlauf *et. al.* (2008) finds strong evidence for the existence of production externalites in explaining cross-country differences in per-capita growth.

Setting $\bar{u} = u$ and $\bar{h} = h$, differentiating (8) with respect to time, and substituting into (10), the law of motion for per-capita consumption is:

$$\frac{\dot{c}}{c} = \sigma \left(\alpha k^{\alpha-1} \phi^{1-\alpha+\beta} - \delta - \rho \right). \tag{14}$$

The law of motion for per-capita physical capital is:

$$\frac{\dot{k}}{k} = k^{\alpha-1} \phi^{1-\alpha+\beta} - \frac{c}{k} - \delta - n, \tag{15}$$

Substituting the wage equation into (9) and differentiating with respect to t:

$$\frac{\dot{\mu}}{\lambda} - \frac{\mu}{\lambda^2} \dot{\lambda} = \frac{\alpha (1-\alpha) k^{\alpha-1} \phi^{\beta-\alpha}}{(1-\gamma) \nu} \dot{k} + \frac{(\beta-\alpha)(1-\alpha) k^{\alpha} \phi^{\beta-\alpha-1}}{(1-\gamma)\nu} \dot{\phi}. \tag{16}$$

Substituting (11) and (15) for $\dot{\mu}$ and $\dot{k}$ into (16) yields the law for motion of effective labor:

$$\frac{\dot{\phi}}{\phi} = \frac{\alpha}{\alpha-\beta} \left(\frac{(1-\gamma)\nu - \varepsilon}{\alpha} + \frac{1-\alpha}{\alpha} (n+\delta) - \frac{c}{k} \right). \tag{17}$$

The evolution of the economy is described by the system (14), (15) and (17) in the non-stationary variables c, k and ϕ. To make this system stationary, we define stationary consumption and physical capital: $\widetilde{c} = c\phi^{-\frac{1-\alpha+\beta}{1-\alpha}}, \widetilde{k} = k\phi^{-\frac{1-\alpha+\beta}{1-\alpha}}$. The dynamic system reduces to two stationary laws of motion:

$$\frac{\dot{\widetilde{c}}}{\widetilde{c}} = \sigma \left(\alpha \widetilde{k}^{\alpha-1} - \delta - \rho \right) - \vartheta \left(\frac{(1-\gamma)\nu - \varepsilon}{\alpha} + \frac{1-\alpha}{\alpha} (n+\delta) - \frac{\widetilde{c}}{\widetilde{k}} \right), \tag{18}$$

and

$$\frac{\dot{\widetilde{k}}}{\widetilde{k}} = \widetilde{k}^{\alpha-1} + (\vartheta - 1) \frac{\widetilde{c}}{\widetilde{k}} - \vartheta \frac{(1-\gamma)\nu - \varepsilon}{\alpha} - \frac{n+\delta}{\alpha-\beta}, \tag{19}$$

where $\vartheta = \frac{1-\alpha+\beta}{\alpha-\beta} \frac{\alpha}{1-\alpha}$.

3 Balanced Growth

The balanced growth paths of the economy are the solutions to the equations (18) and (19) when $\dot{\widetilde{c}}$=$\dot{\widetilde{k}}$=0. Differentiating $\phi = uh$ with respect to time: $\dot{\phi} = \dot{u}h + u\dot{h}$, setting $\dot{u} = 0$, and combining the law of motion for human capital in (3) with (17), (18), and (19) yields the steady state fraction of hours devoted to production in the production sector:

$$u^* = \frac{\rho - n - (\eta - \gamma)\nu + \eta\varepsilon}{(1-\eta)\nu}, \tag{20}$$

where $\eta = \frac{(1-\alpha+\beta)(\sigma-1)}{(1-\alpha)\sigma}$ is the product of the curvature of the utility function, and the ratio of the social marginal product of human capital to the private marginal product of human capital. The steady state growth rate of physical output, consumption wages and physical capital is:

$$\kappa = \frac{(1-\alpha+\beta)((1-\gamma)\nu - \varepsilon - \rho + n)}{(1-\alpha)(1-\eta)}, \tag{21}$$

and the steady state growth rate of human capital is $\frac{(1-\gamma)\nu-\varepsilon-\rho+n}{1-\eta}$.

Setting the left hand sides of (18) and (19) equal to zero, we solve for balanced growth consumption and capital:

$$\widetilde{c}^* = \left[\frac{(1-\alpha)(n+\delta)}{\alpha} + \frac{(1-\alpha+\beta-\eta)((1-\gamma)\nu-\varepsilon)+(\alpha-\beta)(\rho\ -n)}{\alpha(1-\eta)}\right]\widetilde{k}^*, \tag{22}$$

and

$$\widetilde{k}^* = \left[\frac{\alpha\,(1-\alpha)\,(1-\eta)}{(1-\alpha+\beta-(1-\alpha)\,\eta)\,(n+\delta+(1-\gamma)\,\nu-\varepsilon)-\beta\,(\delta+\rho)}\right]^{\frac{1}{1-\alpha}}. \tag{23}$$

To ensure the existence of interior solutions along the balanced growth path, the representative agent cannot be so impatient that he allocates all available time to immediate production, or so patient that all participation in the labor market is postponed indefinitely as the maximum accumulation of human capital is pursued. Therefore, as in Benhabib and Perli (1994) and Ben-Gad (2003), we use bounds on the discount rate to describe the restrictions on preferences necessary to ensure that the fraction of hours worked is on the unit interval and that the steady state rate of growth is positive.

We define two disjoint subspaces of the parameter space $\Theta_1, \Theta_2 \subset \Theta$:

$$\Theta_1 \equiv \{\theta \in \Theta|\ n+(\eta-\gamma)\,\nu-\eta\varepsilon<\rho<n+(1-\gamma)\,\nu-\varepsilon \text{ and } \eta<1\}, \tag{24}$$

$$\Theta_2 \equiv \{\theta \in \Theta|\ n+(1-\gamma)\,\nu-\varepsilon<\rho<n+(\eta-\gamma)\,\nu-\eta\varepsilon \text{ and } \eta>1\}. \tag{25}$$

Proposition 1 *If $\theta \in \{\Theta_1, \Theta_2\}$, the steady state growth rate in (21) $\kappa > 0$, the steady state fraction of hours worked in (20) $u^* \in (0,1)$, and the steady state stock of physical capital in (23) $\widetilde{k}^* > 0$.*

Proof: See Appendix.

The conditions in Proposition 1 are necessary but not sufficient to ensure the existence of an interior balanced growth path. If for example $\alpha = 0.6$, $\beta = 0.31$, $\gamma = 0.28$, $\delta = 0.03$, $\varepsilon = 0.15$, $\nu = 0.2$, $\rho = 0.03$, $\sigma = 4$, and $n = 0.02$, from (20), (21), and (23) u^*=0.0085, κ=0.0857, $\widetilde{k}^*$=147.355. However, the value generated by (22) for consumption is negative and equal to -0.00185. To ensure positive steady state consumption we define the disjoint subsets $\Theta_1^A, \Theta_1^B \subset \Theta_1$ and $\Theta_2^A, \Theta_2^B \subset \Theta_2$:

$$\Theta_1^A \equiv \{\theta \in \Theta_1 | n+\left(1-\frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\,\nu-\varepsilon)-\zeta<\rho<n+(1-\gamma)\,\nu-\varepsilon \text{ and } \alpha>\beta\}, \tag{26}$$

$$\Theta_1^B \equiv \{\theta \in \Theta_1 | n+(\eta-\gamma)\,\nu-\eta\varepsilon<\rho<n+\left(1-\frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\,\nu-\varepsilon)-\zeta \text{ and } \alpha<\beta\}, \tag{27}$$

$$\Theta_2^A \equiv \{\theta \in \Theta_2 | n+(1-\gamma)\,\nu-\varepsilon<\rho<n+\left(1-\frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\,\nu-\varepsilon)-\zeta \text{ and } \alpha>\beta\}, \tag{28}$$

$$\Theta_2^B \equiv \{\theta \in \Theta_2 | n+\left(1-\frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\,\nu-\varepsilon)-\zeta<\rho<n+(\eta-\gamma)\,\nu-\eta\varepsilon \text{ and } \alpha<\beta\}. \tag{29}$$

where $\zeta = \frac{1-\eta}{\alpha-\beta}\,(1-\alpha)\,(n+\delta)$.

Proposition 2 *The necessary and sufficient condition for the existence of an interior balanced growth path is:* $\theta \in \cup_{i=1,2}^{j=A,B}\Theta_i^j$ *if* $\alpha \neq \beta$, *and* $\theta \in \cup_{i=1,2}\Theta_i$ *if* $\alpha = \beta$.

Proof: See Appendix.

The sets Θ_1 and Θ_2 are separated in Θ by a hyperplane defined by the set Θ_3:

$$\Theta_3 \equiv \{\theta \in \Theta | \rho = n + (1-\gamma)\,\nu - \varepsilon \text{ and } \eta = 1\}\,. \tag{30}$$

If $\theta \in \Theta_3$ the numerators and denominators in both (22) and (23) are both equal to zero, implying the existence of an infinite number of balanced growth paths.

4 Dynamics and Equilibria

The results in the previous section demonstrate the conditions for balanced growth paths to be both interior and unique. If $\alpha = \beta$, the production function is linear in effective labor, and there are no transition paths. Instead, the model's behavior mimics that of the one sector AK models where the growth rate is always constant. In all other cases the equilibrium paths that converge to these growth paths are only unique if the dynamic system has a saddle path structure. To find the local stability properties of the reduced system in the neighborhood of the balanced growth paths, we linearize the system (18) and (19). The Jacobian of the linearized system evaluated along the balanced growth path:

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & -\mathbf{J}_{11} + \frac{\beta(\rho+\delta)\sigma-(1-\alpha+\beta)((1-\gamma)\nu+n+\delta-\varepsilon)}{1-\eta} \\ \frac{\beta}{\alpha(1-\alpha+\beta)}\mathbf{J}_{11} & -\frac{\beta}{1-\alpha+\beta}\mathbf{J}_{11} - \frac{(1-\alpha)(n+\delta)+(1-\alpha+\beta)((1-\gamma)\nu-\varepsilon)}{\alpha-\beta} \end{bmatrix}, \tag{31}$$

where $\mathbf{J}_{11} = \vartheta\frac{\widetilde{c}^*}{\widetilde{k}^*}$, and the value of $\frac{\widetilde{c}^*}{\widetilde{k}^*}$ is defined from (22) (see Appendix). If the eigenvalues of the Jacobian have opposite signs, all competitive equilibria, at least in the neighborhood of the balanced growth path are determinate (locally unique). If both eigenvalues are negative, all paths converge to the balanced growth path and any point in its vicinity qualifies as a competitive equilibrium. Finally, if both eigenvalues are positive, the dynamics are explosive—all paths off the balanced growth path violate conditions (12) and (13).

Proposition 3 *In the neighborhood of all balanced growth paths there exist unique competitive equilibria iff* $\theta \in \Theta_1^A \cup \Theta_2^B$.

Proof: The determinant of $\mathbf{J}$ is:

$$|\mathbf{J}| = \frac{\alpha\,(1-\alpha)\,(1-\eta)\,\sigma}{\alpha-\beta}\widetilde{c}^*\widetilde{k}^{*\alpha-1}. \tag{32}$$

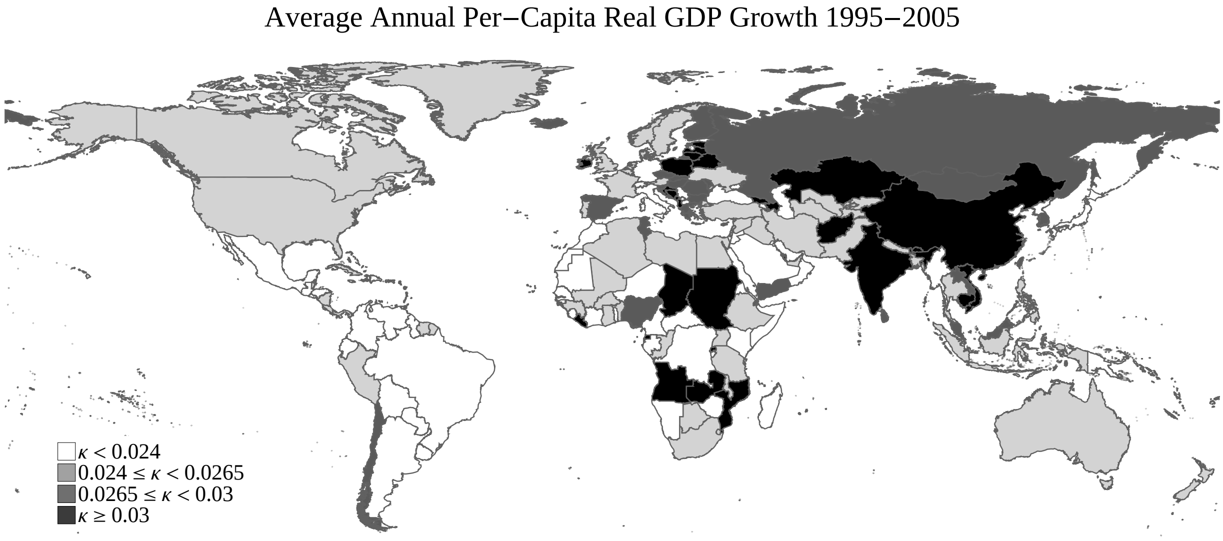

Figure 1: Per-capita real output growth κ, averaged over the decade between 1995-2005. *Source: Penn World Tables,*

which is negative if and only if $\eta < 1$ and $\alpha > \beta$, or $\eta > 1$ and $\alpha < \beta$. If and only if $\theta \in \Theta_1^A \cup \Theta_2^B$ do the eigenvalues of $\mathbf{J}$ have opposite signs. ■

The implication of Proposition 3 is that there are unique balanced growth paths in the portions of the parameter space defined by Θ_1^B and Θ_2^A, but the equilibria in their neighborhoods are either unstable or indeterminate. We rule out the latter.

Proposition 4 *If $\theta \in \Theta_2^A \cup \Theta_1^B$, all balanced growth paths are unstable.*

Proof: The trace of $\mathbf{J}$ is:

$$tr\mathbf{J} = \frac{\rho - n - ((1-\gamma)\nu - \varepsilon)\eta}{1-\eta}. \tag{33}$$

which is positive iff $\theta \in \Theta_1 \cup \Theta_2$ and negative otherwise. From (26)—(29) the determinant (32) is positive iff $\theta \in \Theta_2^A \cup \Theta_1^B$. If the determinant and trace of $\mathbf{J}$ are positive, the eigenvalues of $\mathbf{J}$ are positive as well, and we rule out multiple equilibria (indeterminacy). ■

5 Intertemporal Elasticities of Substitution Greater than One

The vast majority of models in the macroeconomic literature employ preferences characterized by constant intertemporal elasticity of substitution. In the Dynamic Stochastic General Equilibrium literature these elasticities are in turn calibrated with values of σ that typically range between one half and one.[3] By contrast, in much of the endogenous fertility literature (see Barro and Becker

[3]For time additive utility functions, the intertemporal elasticity of substitution is the reciprocal of the Arrow Pratt measure of relative risk aversion, usually assumed in the DSGE literature to fall between one and two.

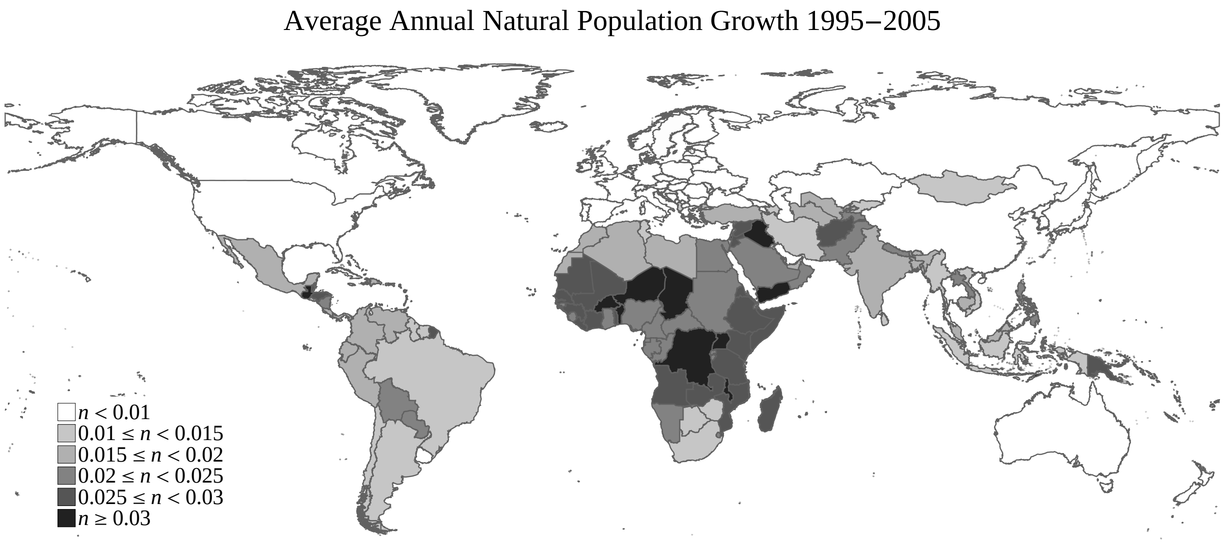

Figure 2: Natural rate of population growth n, averaged over the decade between 1995-2005. *Source: United Nations*

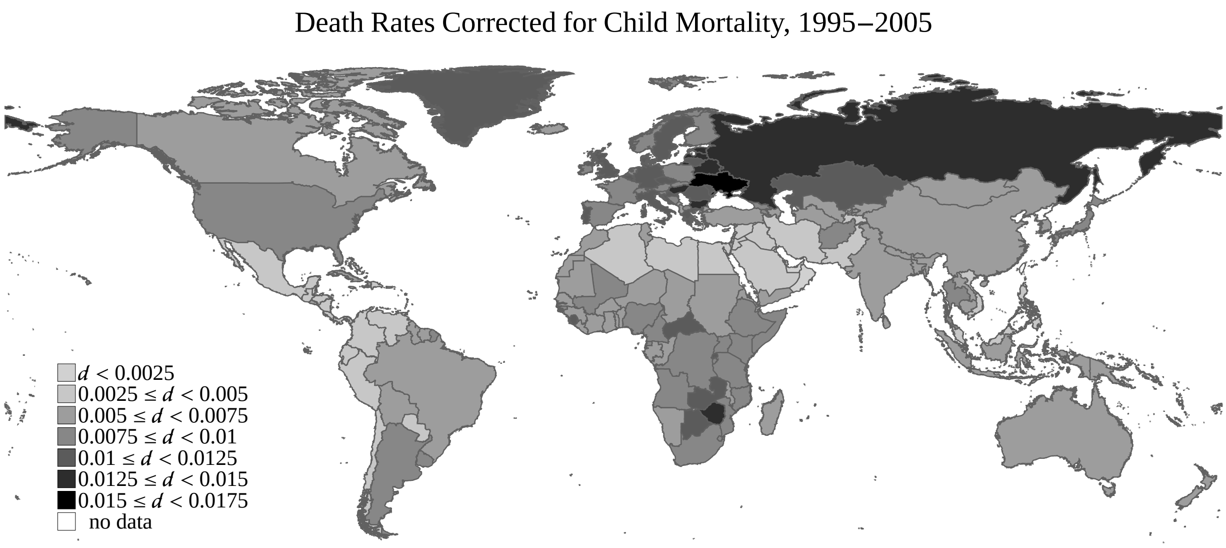

Figure 3: The death rate corrected for the mortality rate of children aged five averaged between 1995-2005. *Source: United Nations Statistical Division and CIA Factbook*

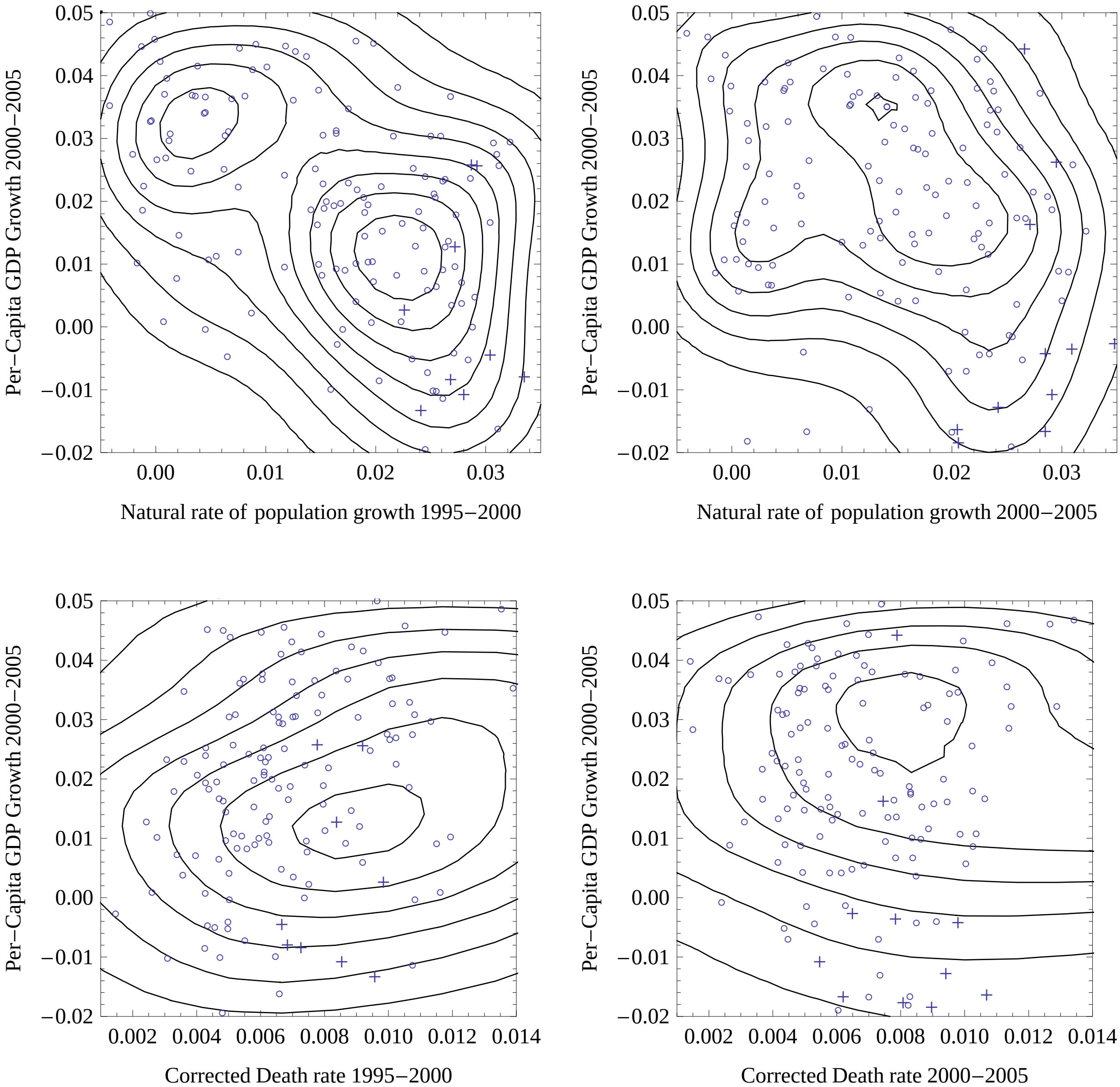

Figure 4: Smoothed kernel density plots for per-capita growth against the natural rate of population growth or the death rate corrected for the mortality rate of children aged five averaged for the years 1995-2000 and 2000-2005. The bandwidth is set using the Silverman criteria. Crosses represent data points for countries with per-capita GDP below \$1,000 and circles data points above \$1,000. *Source: Penn World Tables, United Nations Statistical Division and CIA Factbook*

(1988), (1989)) the intertemporal elasticities of substitution are typically set higher than one. These higher values also appear in some recent empirical studies.

Gruber (2006) estimates the intertemporal elasticity of substitution for individuals in the United States to be two. Hamori (1996) estimates the elasticity for Japanese consumers to be between one and two, and Fuse (2004) estimates the elasticity in Japan to be about four. Attanasio and Weber (1989), Mulligan (2002), Vissing-Jørgensen and Attanasio (2003), Bansal and Yaron (2004), Bansal, Kiku and Yaron (2007), and Hansen, et. al. (2007) all estimate high values for the intertemporal elasticity of substitution for the Epstein-Zin recursive utility function.[4] In terms of the model itself, (21) suggests a connection between the growth rates and two parameters observable in the data. The rate of per-capita economic growth κ, should be negatively [positively] correlated with n the natural rate of population growth, and positively [negatively] correlated with the rate of depreciation for human capital ε if $\eta > 1$ [$\eta < 1$]. Furthermore given the restrictions on the other parameters, $\eta > 1$ [$\eta < 1$] implies that the intertemporal rate of substitution $\sigma > 1$ [$\sigma < 1$]. Indeed, though $\sigma > 1$ implies $\eta > 0$, if $\eta > 1$, the in addition the exponent regulating the human capital external effect β must be positive as well. Though we do not observe human capital depreciation directly, we can use the simple proxy in Figure 3 —the rate of death for people above the age of five.

Figure 4 presents the relationship between per-capita growth κ and both the natural rate of population growth and the rate of death for people above the age of five, together with their corresponding smoothed kernel density plots. The relationship in (21) refers only to balanced growth, not growth associated with convergence. To account for this without removing all but the richest countries from the sample we restrict ourselves to the 123 countries that experienced growth rates that averaged between -2% and +5% during the periods 1995-2000 and 2000-2005. There is a discernible though weak pattern in three of the four panels, negative relationships in both 1995-2000 and 2000-2005 for the relationship between per-capita growth and the natural rate of population growth, and a positive relationship between per-capita growth and the corrected death rate for 1995-2000, though this relationship breaks down for the data between 2000-2005.

To further investigate the relationship in (21) we use the same data and regress κ on both n and the corrected death rate and also on the interaction of each with an indicator variable $\mathrm{I}_{y>1000}$ that takes the value of one for per-capita income above \$1000 (represented by crosses rather than circles in Figure 4) and zero otherwise—the results are presented in Table 1. Although the regressions

[4] In Benhabib *et. al.* (2006) the two sector real business cycle model generates impulse responses that match US data if the intertemporal elasticity of substitution is 'too high', equal to 14. Similarly, Tversky and Kahneman (1992) estimate the coefficient of relative risk aversion to be 0.22, but in their non-expected utility framework, its reciprocal cannot be automatically treated as the intertemporal elasticity of substitution.

	(1)		(2)		(3)		(4)		(5)	
	1995 -2000	2000 -2005	1995 -2000	2000 -2005	1995 -2000	2000 -2005	1995 -2000	2000 -2005	1995 -2000	2000 -2005
constant	0.0318***	0.0274***	0.0312***	0.0263***	0.0033	0.0187***	0.0015	0.01806***	0.0262***	0.0381***
	(13.6856)	(0.0025)	(13.4764)	(10.6006)	(0.8222)	(4.3102)	(0.4028)	(4.3814)	(4.1744)	(6.2205)
pop. growth	-0.7710***	-0.5357***	-1.0760***	-0.9493***					-0.6871***	-0.6883***
	(-6.3068)	(-3.7467)	(-5.4528)	(-4.5173)					(-4.5541)	(-4.231)
pop. growth $\times \mathrm{I}_{y>1000}$			0.3778*	0.5559***						
			(1.9550)	(2.6332)						
death rate					2.4064***	0.1241	-0.0918	-2.3442***	0.6322	-1.2412*
					(4.1370)	(0.2045)	(-0.1211)	(-2.7257)	(0.9503)	(-1.900)
death rate $\times \mathrm{I}_{y>1000}$							2.9843***	2.7738***		
							(4.6695)	(3.8565)		
R^2	0.2474	0.1040	0.2706	0.1529	0.1240	0.0003	0.2586	0.1106	0.2530	0.1301

Table 1: Per-Capita growth rates regressed on natural rate of population growth and the death rate corrected for the mortality rate of children aged five averaged for the years 1995-2000 and 2000-2005. Ordinary Least Squares, T-statistics in parentheses, * indicates statistical significance at the 10% level, ** at the 5% level, and *** at the 1% level. *Source: Penn World Tables, United Nations Statistical Division and CIA Factbook*

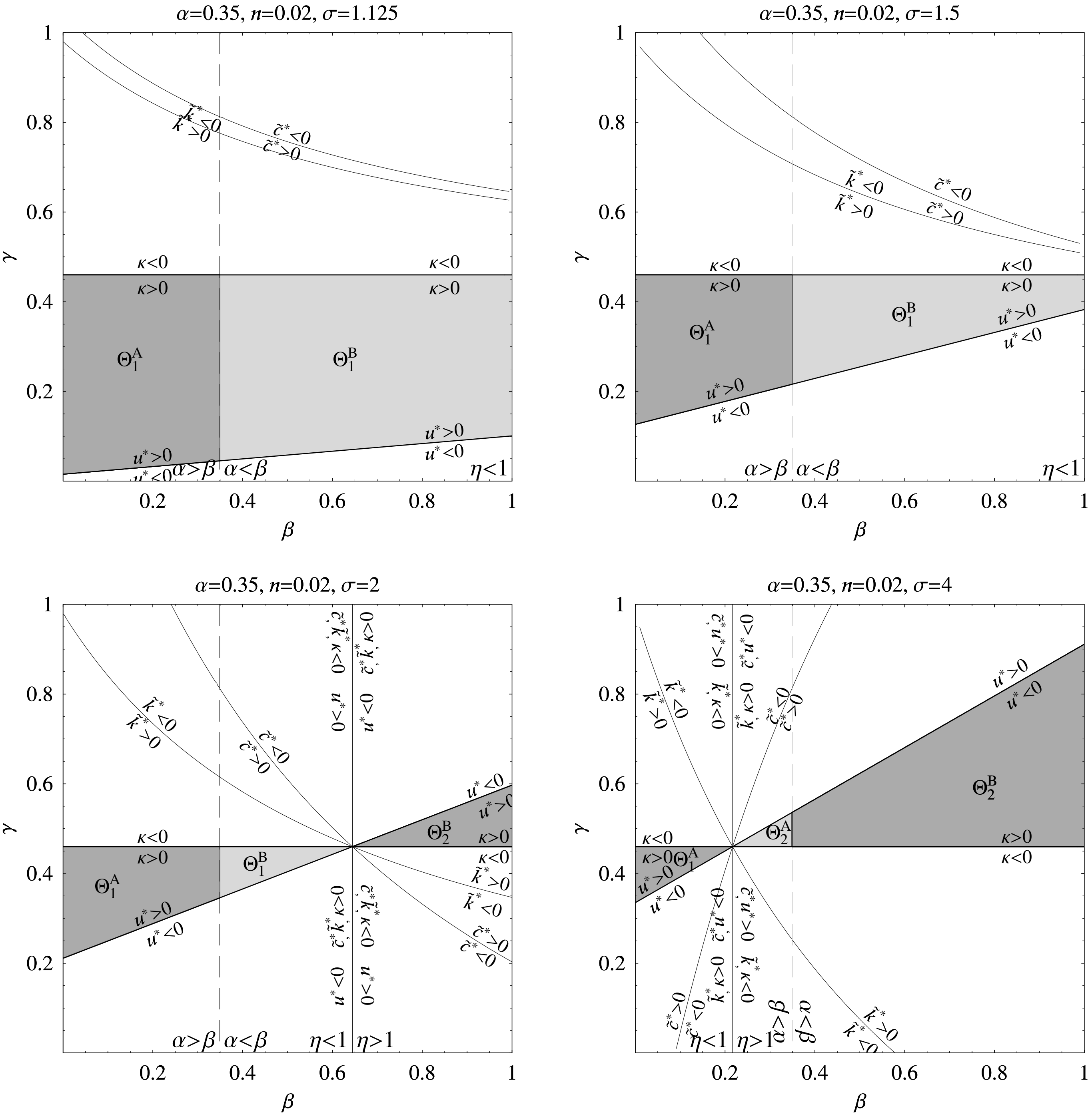

Figure 5: The parameter space for $\alpha = 0.35$ and $n = 0.02$.

suffer from high degrees of collinearity—n and the corrected death rate are not only correlated with the interaction terms but with each other—a clear pattern emerges. In every case the coefficient on n is significantly below zero, ranging between about -.5 and -.75 for all the countries in the sample. With the indicator variable included as an interaction term the result is similar for most of the countries in the sample but approximately -1 for the extremely poor countries with per-capita incomes below $1,000. In keeping with the pattern that emerges in the two upper panels of Figure 4, these results suggests the value of η to be greater than one. The coefficients on the corrected death rate are not always significant but for the period 1995-2000 the coefficient is 2.4064 and significant in regression (2) and the interaction term in (5) is 2.9843 and significant. For the period 2000-2005 the coefficient in (2) is not statistically significant but in (5) the coefficient on the corrected death rate is statistically significant and negative, -2.3442 for the group of countries with per-capita incomes below $1,000 but adding together the two coefficients, positive and equal to 0.4296 for countries with per-capita incomes above $1,000.

As one final test we consider how the changes in the growth rates themselves relate to the changes in both the natural population growth rates and the corrected death rates which allows us to isolate the relationship in (21) as it applies to the countries in the sample themselves rather than as comparisons across the different countries. There are no statistically significant results in Table 2 when $\triangle\kappa$ is regressed on either $\triangle n$ or the change in the corrected death rate alone. Nonetheless, interacting $\triangle n$ with the indicator variable generates a similar pattern to that in Table 1 for countries with per-capita incomes greater than $1,000. However for the poorest countries in regression (2) the relationship is reversed. Interacting the indicator function $\mathrm{I}_{y>1000}$ on the corrected death rate does not yield statistically significant results, but interacting the variable with a new indicator function $\mathrm{I}_{y>4000}$ that takes the value of one if per-capita output is greater than $4,000. Again we see the same pattern emerges—regression (5) implies that the value of η is likely to be greater than one, but only for countries above a certain income threshold.

What can we conclude? Again (21) refers only to balanced growth, whereas the data includes countries at various stages of development. Nonetheless the above results surely indicate that it would be unwise to rule out parameterizations consistent with a value of η greater than one—at least possible for all but the poorest countries. And that implies intertemporal elasticities of substitution greater than one (and also the exisitence of at least some human capital spillovers in the physical production process). The question remains under what circumstances the two-sector endogenous growth model can cope with these higher elasticities?[5]

[5] Jones *et. al.* (2005) simulate the behavior of the endogenous growth model with elastic labor supply and fluctuations. Though they include depreciation in both sectors of the economy, the production function for human capital is linear at both the private and social levels—in their model the intertemporal elasticity of substitution

	(1)	(2)	(3)	(4)	(5)
constant	-0.0018	-0.0021	0.0002	4x10^{-5}	0.0002
	(-0.8922)	(-1.0950)	(0.1455)	(0.0247)	(0.1533)
$\triangle$pop growth	-1.3216	8.1965*			
	(-1.6445)	(1.9792)			
$\triangle$pop growth $\times \mathrm{I}_{y>1000}$		-9.8278**			
		(-2.3413)			
$\triangle$death rate			0.4804	-13.5313	-4.3679*
			(0.3149)	(-1.0236)	(-1.9734)
$\triangle$death rate $\times \mathrm{I}_{y>1000}$				14.2227	
				(1.0671)	
$\triangle$death rate $\times \mathrm{I}_{y>4000}$					8.6984***
					(2.9444)
R^2	0.0219	0.0646	0.0008	0.0102	0.0681

Table 2: Changes in per-capita growth rates regressed on changes in the natural rate of population growth and changes in the death rate corrected for the mortality rate of children aged five. Differences between averages for the years 1995-2000 and 2000-2005. Ordinary Least Squares, T-statistics in parentheses, * indicates statistical significance at the 10% level, ** at the 5% level, and *** at the 1% level. *Source: Penn World Tables, United Nations Statistical Division and CIA Factbook*

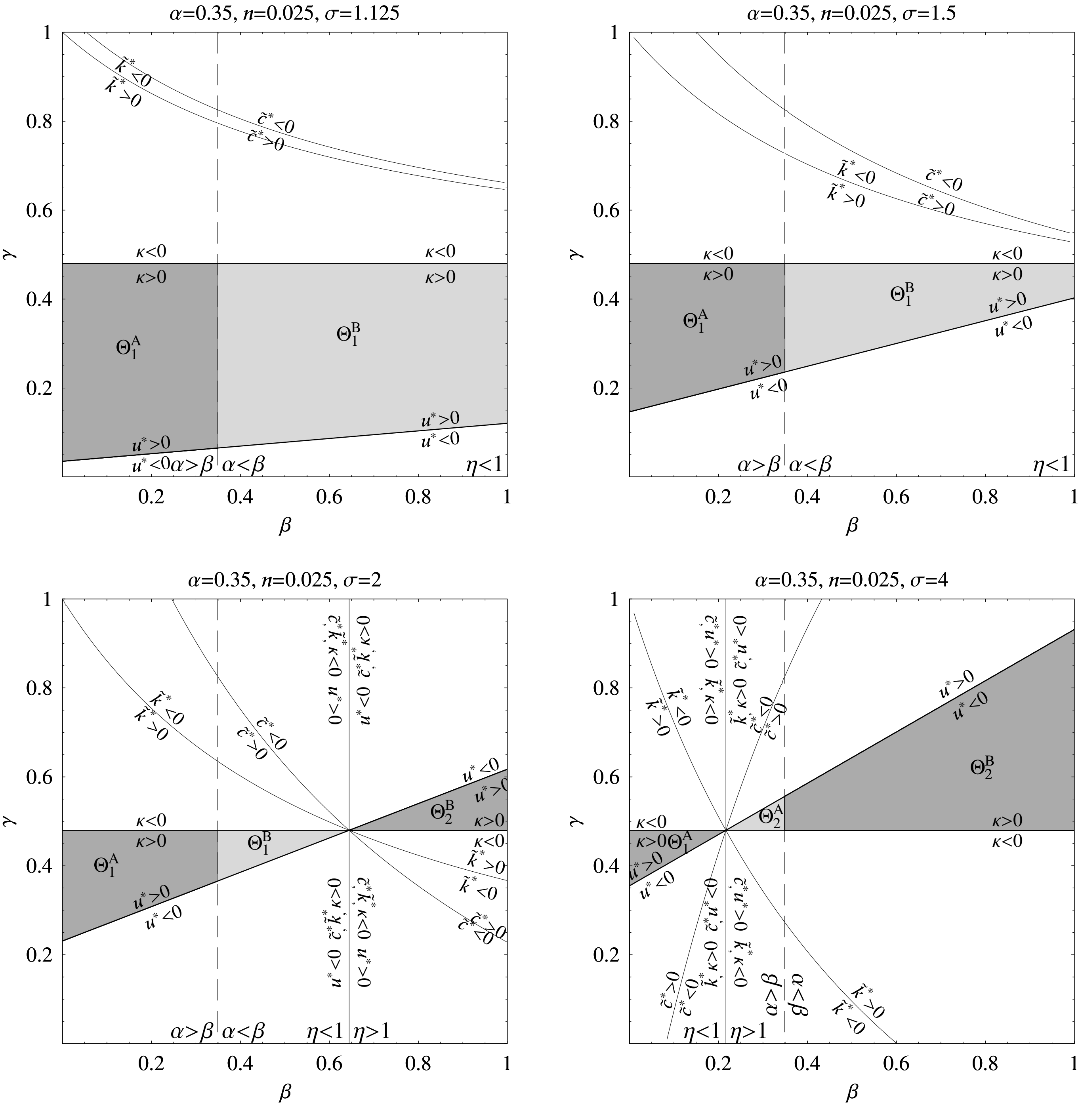

Figure 6: The parameter space for $\alpha = 0.35$ and $n = 0.025$.

In Figure 5, we vary the magnitude of both external effects β and γ along the unit interval, for values of the intertemporal elasticity of substitution σ equal to 1.125, 1.5, 2, and 4, while holding the other parameters of the model fixed. We set the share of capital in output α equal to 0.35, and the subjective discount rate ρ equal to 0.03. The rates of depreciation for physical capital δ is set to equal 0.1, reflecting the low end of Bu's (2006) estimates for a selection of less developed countries. Similarly, we set the rate of population growth to n=0.02, to approximate the recent experience of large parts of the developing world as illustrated in Figure 2.

We know little about the rate of individual human capital depreciation outside the United States and Western Europe. Keane and Wolpin (1997) estimate the rate of individual human capital depreciation for white U.S. males to be 0.096 for blue collar workers and 0.365 for white collar workers. Wu (2007) estimates the rate of human capital depreciation for U.S. workers to be between 0.116 and 0.181, depending on gender and race. In addition, at the aggregate level, we must also consider the rate at which cohorts leave the workforce. As a proxy we use the death rate but correct this for the mortality rate of children aged five. Combining the lower of the values for individual workers in the United States—about 0.11, together with the corrected death rates in Figure 3 most common in the developing world—about 0.015, yields a value for human capital depreciation of ε equal to 0.125. Finally, we set the value of ν to 0.25, double the value of human capital depreciation. The human capital of an individual who spends no time working in the production sector, but devotes all of his or her time to human capital accumulation, will see it increase by 15% over the course of a year—roughly consistent with various estimates for the returns to schooling (see Harmon and Walker (1999) and Card (2001)).

In the upper left-hand panel of Figure 5 we consider the conditions for the existence of an interior balanced growth path, fixing the intertemporal elasticity of substitution to σ=1.125. The value of η is less than one, as long as $\beta < -\frac{1-\alpha}{1-\sigma}$, so throughout the portion of the parameter space under consideration, $\eta < 1$. Both the areas shaded in dark and light gray, denoted Θ_1^A and Θ_1^B, respectively, represent the combinations of β and γ along the unit interval that satisfy all the necessary and sufficient conditions for interior balanced growth paths. However, from Proposition 3 only the former, shaded in dark gray, Θ_1^A, represents economies characterized by unique saddle-path stable equilibria. This area is bounded from below by the constraint $u^* > 0$, which begins where the value of γ equals 0.0156 and rises linearly as the value of β increases. To ensure the existence of an interior balanced growth path, the value of the external effect in the human capital sector, γ, must be positive, but no greater than $1 - \frac{\rho-n+\varepsilon}{\nu}$—here equal to 0.46—otherwise the constraint that the growth rate is positive, $\kappa > 0$ is violated. However since for $\sigma = 1.125$ and all values of $\beta < 1$,

cannot be much higher than 1.11. Nonetheless it only at this upper bound that the simulated standard deviation of the consumption-output ratio approaches that observed in US data.

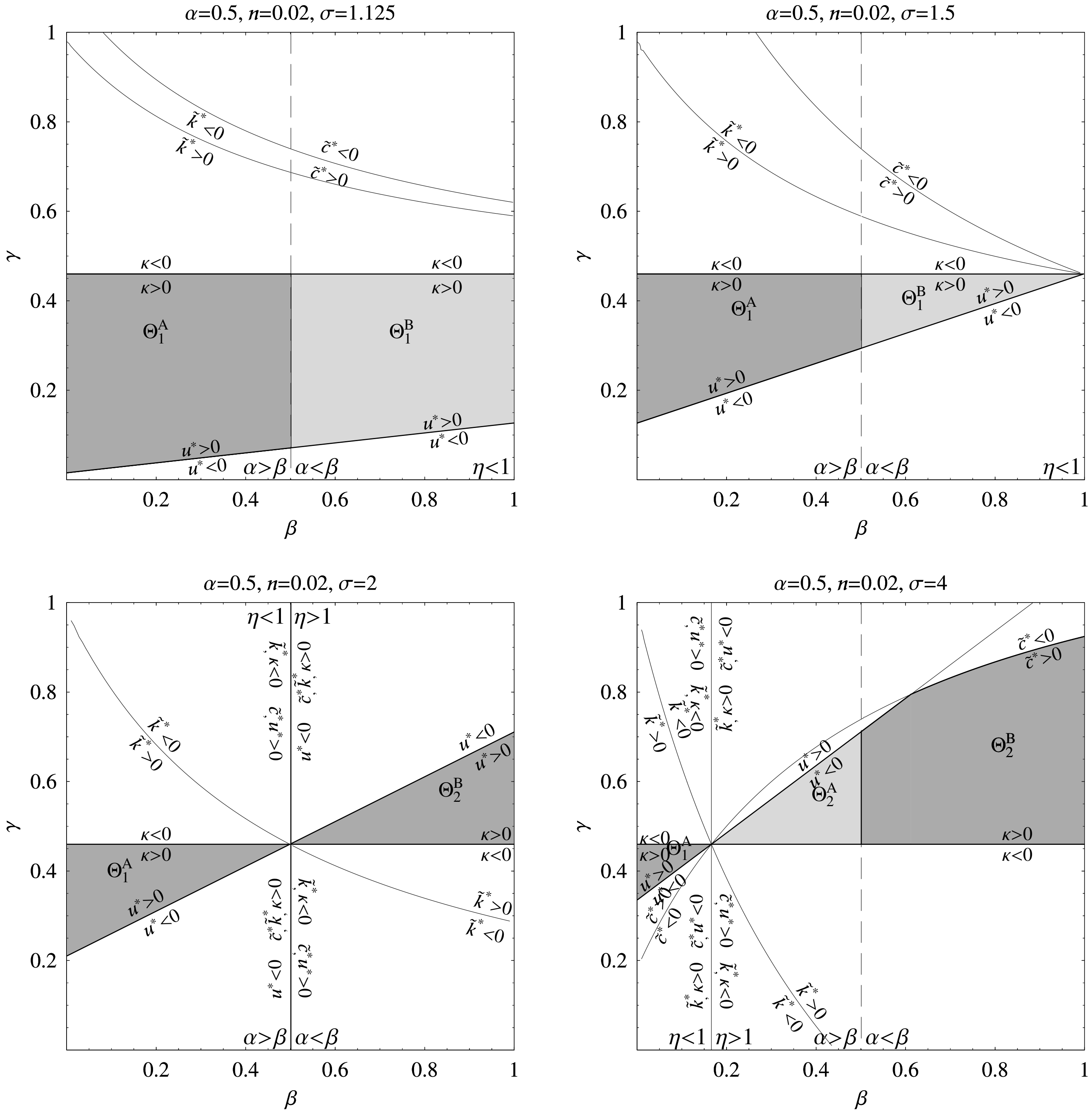

Figure 7: The parameter space for $\alpha = 0.5$ and $n = 0.02$.

$\theta \in \Theta_1$, unique saddle path stable equilibria that converge to an interior balanced growth path are not possible, unless the external effects β and γ, are no larger than 0.35 and 0.46, respectively.

Raising the value of σ to 1.5, narrows the range of values for β and γ that support the existence of interior balanced growth paths. The value of γ, in the upper right-hand panel of Figure 5, must be at minimum 0.127. Raise the value of σ above 1.65 and components of Θ_2 appear for values of β <1. For example, in the lower left-hand panel of Figure 5, $\sigma = 2$—if $\beta = 0.65$, then η=1, and the set that corresponds to interior balanced growth paths reduces to a single point within the separating hyperplane Θ_3. Beyond this point, as the value of β grows beyond 0.65, the range of values of γ consistent with interior balanced growth paths expands within the region defined by Θ_2^B. The regions that support saddle-path equilibria, Θ_1^A and Θ_2^B, are separated in Θ by Θ_2^B, the region associated with unstable dynamics.

Finally, raise the value of σ to 4, the highest of recent estimates of the intertemporal elasticity of substitution cited above, and the area of Θ_1^A shrinks. Furthermore, for all values of $\beta < \alpha$, η is less than one. Hence the region Θ_1^B disappears, and the region Θ_2^A, where the balanced growth path is also unstable, emerges instead. The point on the separating hyperplane Θ_3, where η =1, is $\{\beta, \gamma\} = \{0.217, 0.46\}$.

A necessary and sufficient condition that ensures $u^* > 0$, is that $\sigma < \frac{(1-\alpha+\beta)(\nu-\varepsilon)}{(1-\alpha+\beta)(\nu-\varepsilon)-(1-\alpha)(\rho-n+\gamma\nu)}$. Given $\nu > \varepsilon$ and $\rho > n$, this condition is satisfied for all $\sigma < 1$, even if external effects are absent from both sectors. Raising the value of σ above one and beyond, the curvature of the human capital production function, regulated by the value of the parameter γ, becomes critical. Furthermore, the higher the rate of population growth, the higher the degree of curvature required as well. In the absence of external effects in either the human capital or the production sector, the aforementioned upper bound on σ reduces to $\frac{\nu-\varepsilon}{\nu-\varepsilon-\rho+n}$. Therefore in our example if n=0.02, the upper bound for σ is 1.187, and if n=0.025 the upper bound drops to 1.042.

During the two decades between 1985 and 2005 the annual rate of population growth for each of the countries in South America averaged 0.0183 per year, implying an upper bound of 1.103. The rate of population growth in South Asia averaged 0.0222, corresponding to an upper bound of 1.067; in the Middle East it averaged 0.0239, corresponding to an upper bound of 1.051; in Central America it averaged 0.0248, corresponding to an upper bound of 1.043; and in Sub-Saharan Africa 0.0262, corresponding to an upper bound of 1.031.[6] Furthermore, once we introduce increasing

[6] *South Asia:* Afghanistan, Bangladesh, Bhutan, India, Maldives, Nepal, Pakistan, Sri Lanka; *Middle East*: Algeria, Bahrain, Egypt, Gaza, Iran, Iraq, Israel, Jordan, Kuwait, Lebanon, Libya, Mauritania, Morocco, Oman, Palestinian Territories, Qatar, Saudi Arabia, Sudan, Syria, Tunisia, Turkey, United Arab Emirates, Yemen, West Bank, Western Sahara; *Sub-Saharan Africa*: Angola, Benin, Botswana, Burkina Faso, Burundi, Cameroon, Cape Verde, Central African Republic, Chad, Comoros, Côte d'Ivoire, Democratic Republic of Congo, Djibouti, Equatorial

returns at the social level generated by human capital external effects, these upper bounds drop lower still.

Writing our necessary and sufficient conditions as bounds on the curvature of the human capital production function, an interior balanced growth path only exists for $\eta < 1$, if $\frac{(\nu-\varepsilon)\eta+n-\rho}{\nu} < \gamma < \frac{\nu-\varepsilon+n-\rho}{\nu}$; or for $\eta > 1$, when $\frac{\nu-\varepsilon+n-\rho}{\nu} < \gamma < \frac{(\nu-\varepsilon)\eta+n-\rho}{\nu}$. If $\beta = 0$ and $\sigma = 1.125$, the former bound that corresponds to the rate of population growth in South America is $0.009 < \gamma < 0.453$; for South Asia, $0.024 < \gamma < 0.469$; for the Middle East, $0.031 < \gamma < 0.476$; for Central America, $0.035 < \gamma < 0.479$; and for Sub-Saharan Africa, $0.040 < \gamma < 0.485$. More generally, comparing the panels in Figure 5 where n=0.02, with the panels in Figure 6 where n=0.025, the only difference is that all the admissible areas that correspond to interior balanced growth paths are shifted vertically by 0.02. The higher the rate of population growth, the greater the degree of curvature in the human capital production required if the intertemporal elasticity of substitution is greater than one.

In Figure 7 we restore the rate of population growth to n=0.02, but raise the share of capital in the production of physical output α, to 0.5. In the upper left-hand panel, σ=1.125 and if $\beta = 0$, the value of γ must once again fall between 0.016 and 0.46. Here the constraint that ensures $u^* > 0$ possesses a larger slope in the size of the external effect β, so the range of the parameter set consistent with the existence of interior balanced growth paths, is narrower than in Figure 5. If $\alpha = 0.5$, $\beta = 1$, and σ=1.5, then η=1, so Θ_3 is the very edge of the upper right-hand panel of Figure 7.

In the lower left-hand panel of Figure 7, we set σ=2 so that $\alpha = 1/\sigma$, as it is in the version of the model investigated in Xie (1994). Like Xie (1994) here too we do not encounter any unstable balanced growth paths, both Θ_1^B and Θ_2^A disappear. However, because we rule out intersector spill-overs, so that only the human capital employed in the production sector generates positive external effects there, there is no region characterized by indeterminacy either, and all interior balanced growth paths are saddle path stable.

Finally, in the lower right-hand panel of Figure 7, for σ=4, the size of Θ_1^A reduces to a relatively small region, while the size of Θ_2^A, the range of parameter values that correspond to unstable dynamics expands when compared to its counterparts in Figure 5 and 6. Also here, the binding constraint is no longer either just $\kappa > 0$ or $u^* > 0$, but rather for high values of β and γ, $c^* > 0$ as well. This again demonstrates why Proposition 1, or merely restricting the parameter space to $\theta \in \{\Theta_1, \Theta_2\}$ is a necessary, but not at all a sufficient condition for the existence of an interior

Guinea, Eritrea, Ethiopia, Gabon, Gambia, Ghana, Guinea, Guinea Bissau, Kenya, Lesotho, Liberia, Madagascar, Malawi, Mali, Mauritania, Mauritius, Mayotte, Mozambique, Namibia, Niger, Nigeria, Republic of Congo, Rwanda, São Tomé Principe, Senegal, Seychelles, Sierra Leone, Somalia, South Africa, Swaziland, Tanzania, Togo, Uganda, Zambia, Zimbabwe.

balanced growth path, and the further restriction imposed in Proposition 2 is necessary.

6 Calibrating the Model for a Given Growth Rate

To better understand the nature of the parameter space and how it relates to empirically relevant rates of growth, we can solve (21) for one of the deep parameters of the model, then redefine the balanced growth path in terms of the steady state per-capita rate of growth κ. But which parameter should we replace? We are interested in analyzing the behavior of the model for different values of β, γ, ε, and σ, and the values of α, δ, ρ, and n are all parameters that can be easily calibrated using widely available data, as indeed can the growth rate κ. By contrast, since there is very little direct evidence available that can be used to set the value of ν, the maximum possible growth rate for human capital at the social level, if every moment is devoted to its production (abstracting from its rate of depreciation), we chose its value in Section 5 by inferring its value indirectly. Alternatively we can solve (21) for ν:

$$\nu = \frac{(1-\alpha)(1-\eta)\kappa + (1-\alpha+\beta)(\rho+\varepsilon-n)}{(1-\alpha+\beta)(1-\gamma)}, \tag{34}$$

and then substituting (34) in (20) yields steady state hours worked:

$$u^* = \frac{(1-\alpha+\beta)(\kappa+(\rho-n+\gamma\varepsilon-\kappa)\sigma)+(1-\alpha)\gamma\kappa\sigma}{(1-\alpha+\beta)(1+(\rho-n+\varepsilon)\sigma)-\beta\kappa\sigma}, \tag{35}$$

in (22) yields steady state consumption:

$$\widetilde{c}^* = \left[\frac{1}{\alpha}(\delta+\rho+\frac{\kappa}{\sigma}) - n - \delta - \kappa\right]\widetilde{k}^*, \tag{36}$$

and (23) yields steady state physical capital:

$$k^* = \left(\frac{\alpha\sigma}{\kappa+(\rho+\delta)\sigma}\right)^{\frac{1}{1-\alpha}}, \tag{37}$$

all in terms of the steady state growth rate κ.

Clearly from (37), if α,δ,κ,ρ, and σ are all positive, $k^* > 0$. We redefine the parameter space $\bar{\Theta}$: $\bar{\theta} \equiv (\alpha,\beta,\gamma,\delta,\varepsilon,\kappa,\rho,\sigma,n)$, and $\bar{\theta} \in \bar{\Theta}$, where $\bar{\Theta} = R^2_{++} \times R^4_{+} \times [0,1)^3$ and define the subsets $\bar{\Theta}_1, \bar{\Theta}_2, \bar{\Theta}_3 \subset \bar{\Theta}$:

$$\bar{\Theta}_1 \equiv \left\{\bar{\theta} \in \bar{\Theta} |\ \rho > n - \gamma\varepsilon - \frac{(1-\alpha)\gamma\kappa}{1-\alpha+\beta} - \frac{1-\sigma}{\sigma}\kappa\right\}, \tag{38}$$

$$\bar{\Theta}_2 \equiv \left\{\bar{\theta} \in \bar{\Theta} |\ \rho > \alpha(n+\delta+\kappa) - \delta - \frac{\kappa}{\sigma}\right\}, \tag{39}$$

$$\bar{\Theta}_3 \equiv \left\{\bar{\theta} \in \bar{\Theta} |\ \rho > n - \varepsilon + \frac{\beta\kappa}{1-\alpha+\beta} - \frac{\kappa}{\sigma}\right\}. \tag{40}$$

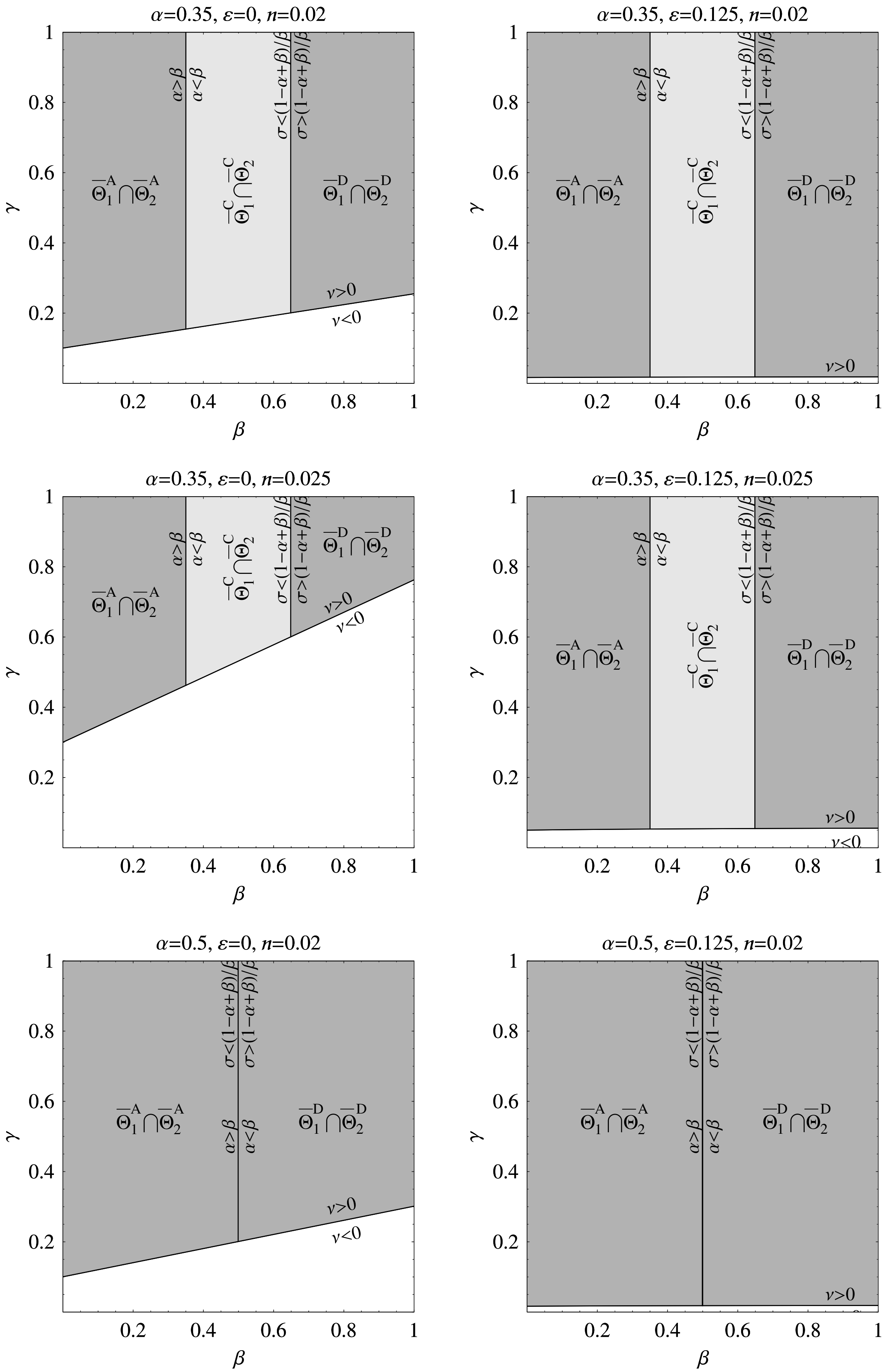

Figure 8: The parameter space for δ=0.1, κ=0.025, ρ=0.03 and σ=2.

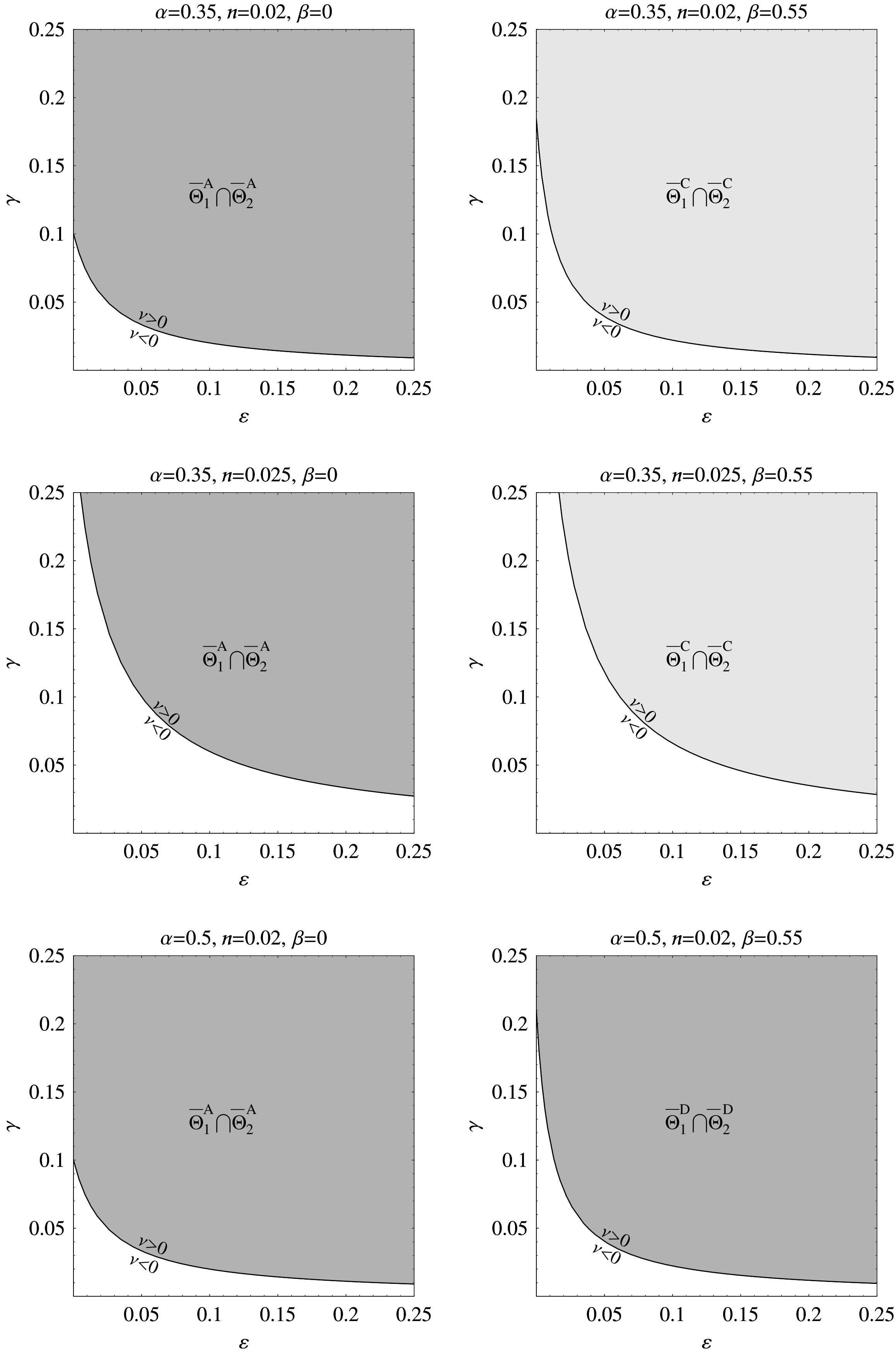

Figure 9: The parameter space for δ=0.1, κ=0.025, ρ=0.03 and σ=2.

Proposition 5 *The necessary and sufficient condition for the existence of an interior balanced growth path is:* $\bar{\theta} \in \bar{\Theta}_1 \cap \bar{\Theta}_2$.

Proof: From (34), $\nu > 0$ iff $\bar{\theta} \in \bar{\Theta}_1$. From (35), $u^* > 0$ iff $\bar{\theta} \in \left(\bar{\Theta} \backslash \bar{\Theta}_1\right) \cup \bar{\Theta}_3$. However $\bar{\Theta}_3 \subset \bar{\Theta}_1$. Finally, from (36), $\widetilde{c}^* > 0$ iff $\bar{\theta} \in \bar{\Theta}_2$. ■

We further subdivide $\bar{\Theta}_1$ and $\bar{\Theta}_2$:

$$\bar{\Theta}_1^A \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 | \ \alpha \geq \beta, \ \eta < 1\right\}, \tag{41}$$

$$\bar{\Theta}_1^B \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 | \ \alpha > \beta, \ \eta > 1\right\}, \tag{42}$$

$$\bar{\Theta}_1^C \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 | \ \alpha < \beta, \ \eta < 1\right\}, \tag{43}$$

$$\bar{\Theta}_1^D \equiv \left\{\bar{\theta} \in \bar{\Theta}_1 | \ \alpha \leq \beta, \ \eta > 1\right\}, \tag{44}$$

$$\bar{\Theta}_2^A \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 | \ \alpha \geq \beta, \ \eta < 1\right\}, \tag{45}$$

$$\bar{\Theta}_2^B \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 | \ \alpha > \beta, \ \eta > 1\right\}, \tag{46}$$

$$\bar{\Theta}_2^C \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 | \ \alpha < \beta, \ \eta < 1\right\}, \tag{47}$$

$$\bar{\Theta}_2^D \equiv \left\{\bar{\theta} \in \bar{\Theta}_2 | \ \alpha \leq \beta, \ \eta > 1\right\}. \tag{48}$$

Proposition 6 *If the parameter values* $\bar{\theta} \in \left(\bar{\Theta}_1^A \cap \bar{\Theta}_2^A\right) \cup \left(\bar{\Theta}_1^D \cap \bar{\Theta}_2^D\right)$, *there is a neighborhood of the balanced growth path in which there exists a unique competitive equilibrium.*

Proof: Follows directly from Propositions 3 and 5. ■

In Figure 8, we set the values of $\delta = 0.1$, and $\rho = 0.03$, fix the intertemporal elasticity of substitution to σ=2, and vary the magnitudes of both external effects β and γ. The per-capita steady state rate of output growth is set to $\kappa = 0.025$, which approximates the average per-capita growth rates between 1995 and 2005 in developed countries such as United Kingdom, at 0.0269; Australia, at 0.0282; Canada, at 0.0271; Spain, at 0.374; or Sweden, at 0.0288 (see Figure 1). We choose these values because the far higher growth rates experienced in many developing countries are not likely to represent steady state growth. In the upper two and lower two panels the population growth rate is $n = 0.02$, the share of physical capital to $\alpha = 0.35$, in the upper and lower panels, and to $\alpha = 0.5$ in the lower two panels. In the middle two panels we set the rate of population growth to $n = 0.025$. In the panels on the left-hand side, the rate of depreciation in the human capital sector is $\varepsilon = 0$, and $\varepsilon = 0.125$ in the panels on the right-hand side.

What emerges in each of the six panels is that given this high rate of intertemporal substitution, interior balanced growth paths only emerge if there is at least some curvature in human capital production at the private level. How much curvature is required, depends directly on both the rate of depreciation in that sector and the population's growth rate, and inversely on the magnitude of returns to scale at the social level in the production sector. By contrast, the relative share of physical capital in the production process has only a small impact on the admissible range of parameters that support balanced growth, but once again substantially affects the model's dynamic behavior.

Consider the left-hand panels of Figure 8, where $\varepsilon = 0$. For both instances where $n = 0.02$, balanced growth only emerges if the value of γ surpasses 0.1, and then only if there are no external effects in the production sector. Raise the population growth rate to $n = 0.025$, and this threshold rises to 0.3. Furthermore, in the absence of any depreciation in the human capital sector, the degree of concavity necessary to ensure the existence of balanced growth rises steeply, as we increase the size of β. Contrast this with the behavior of the model if we introduce a degree of depreciation in the human capital sector. First, the threshold value of γ drops precipitously, to only 0.0167 if $n = 0.02$, and to 0.05 if $n = 0.025$. Second, these thresholds no longer rise quite so dramatically as the values of β increase.

We can further see the trade-offs between concavity at the private level in the production of human capital, and the rate of depreciation in that sector, in the quasi-concave relationship between ε and γ in the panels of Figure 9 that correspond to the necessary condition for $\nu > 0$ in (34). Again, there is a striking contrast between the required degree of concavity or depreciation, or combination of both, that support interior balanced growth paths for $n = 0.02$ and $n = 0.025$.

Returning to Figure 8, the set of parameters $\bar{\theta} \in \bar{\Theta}$ that support unique saddle path equilibria are confined to the subsets $\bar{\Theta}_1^A \cap \bar{\Theta}_2^A$ and $\bar{\Theta}_1^D \cap \bar{\Theta}_2^D$, and these areas are separated by the set $\bar{\Theta}_1^C \cap \bar{\Theta}_2^C$, that contracts to cover a narrower range of values for β, as α increases. In the lower panels of Figure 8 where $\alpha = 0.5$ and hence $\alpha = 1/\sigma$ this region vanishes. In Figure 9, the regions of the parameter space that correspond to balanced growth paths as both γ and ε vary along the unit interval all correspond to $\bar{\Theta}_1^A \cap \bar{\Theta}_2^A$, regions that are also saddle path stable if $\beta = 0$. By contrast if $\beta = 0.55$, and $\alpha = 0.35$, the relevant region corresponds to $\bar{\Theta}_1^C \cap \bar{\Theta}_2^C$ which is not saddle path stable. However if $\alpha = 0.5$ then the relevant region is $\bar{\Theta}_1^D \cap \bar{\Theta}_2^D$, which once again is saddle-path stable. The two-sector endogenous growth model can accommodate intertemporal elasticities of substitution at the upper bound of estimates we find in the empirical literature along with relatively high rates of population growth, and still generate valid balanced growth paths characterized by saddle path stable local dynamics, provided the human capital accumulation process is augmented by small degrees of external effects and depreciation.

7 Conclusion

The Uzawa-Lucas two sector endogenous growth model accommodates two important observations: there are large differences in the rental rates for human capital (wage for a given skill level) across countries, and also differences between the growth rates of physical and human capital within each country. Hence the need to understand precisely what combinations of parameter values, and steady state growth rates, the model can and cannot accommodate.

Unfortunately, the usefulness of the model in its original form, is somewhat hampered by its inability to accommodate preferences characterized by high intertemporal elasticities of substitution, particularly if the rate of population growth is high as well. If once this limitation was considered relatively benign, newer research indicates that high values of these elasticities can not be ruled out. Our empirical results also suggest that an endogenous growth model that is consistent with at least some of the most widely available cross-country data for most countries is likely to be of only limited use, unless it can cope with high values of the intertemporal elasticity of substitution. Once we include external effects and depreciation, a remedy for this problem emerges—the Uzawa-Lucas two sector endogenous growth model can now accommodate a far wider range of parameterizations than previously thought.

8 Appendix

8.1 Proof of Proposition 1

If $\eta < 1$ then from (21) $\kappa > 0$ iff $\rho < n + (1-\gamma)\,\nu - \varepsilon$ which implies $\rho < n + (1-\gamma)\,\nu - \eta\varepsilon$ and $u^* < 1$ from (20). Furthermore if $\eta < 1$ and $\rho < n + (1-\gamma)\,\nu - \varepsilon$ then $\rho < (n + (1-\gamma)\,\nu - \varepsilon)\left(1 + \frac{(1-\alpha)(1-\eta)}{\beta}\right) + \frac{(1-\alpha)(1-\eta)\delta}{\beta}$ which from (23) implies $\widetilde{k}^* > 0$. If $\eta > 1$ then from (21) $\kappa > 0$ iff $\rho > n + (1-\gamma)\,\nu - \varepsilon$ which implies $\rho > n + (1-\gamma)\,\nu - \eta\varepsilon$ and $u^* < 1$ from (20). Furthermore if $\eta > 1$ and $\rho > n + (1-\gamma)\,\nu - \varepsilon$ then $\rho > (n + (1-\gamma)\,\nu - \varepsilon)\left(1 + \frac{(1-\alpha)(1-\eta)}{\beta}\right) + \frac{(1-\alpha)(1-\eta)\delta}{\beta}$ which from (23) implies $\widetilde{k}^* > 0$. Finally, from (20), $u^* > 0$ iff $\eta < 1$ and $\rho < n + (\eta-\gamma)\,\nu - \eta\varepsilon$, or $\eta > 1$ and $\rho > n + (\eta-\gamma)\,\nu - \eta\varepsilon$. ■

8.2 Proof of Proposition 2

From (22), for a positive valued $\widetilde{k}^*$, then $\widetilde{c}^* > 0$ iff $\rho\frac{\alpha-\beta}{1-\eta} > \frac{\alpha-\beta}{1-\eta}n - \left(1 - \frac{\alpha-\beta}{1-\eta}\right)((1-\gamma)\nu - \varepsilon) - (1-\alpha)\,(n+\delta)$. Assume $\alpha \neq \beta$. If $\frac{\alpha-\beta}{1-\eta} > 0$ then $\widetilde{c}^* > 0$ iff $\theta \in \Theta_1 \cup \Theta_2$ and $\rho > n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-$

$\gamma)\nu - \varepsilon) - \frac{1-\eta}{\alpha-\beta}(1-\alpha)(n+\delta)$, hence $\theta \in \Theta_1^A \cup \Theta_2^B$. If $\frac{\alpha-\beta}{1-\eta} < 0$, then $\widetilde{c}^* > 0$ iff $\theta \in \Theta_1 \cup \Theta_2$ and $\rho < n + \left(1 - \frac{1-\eta}{\alpha-\beta}\right)((1-\gamma)\nu - \varepsilon) - \frac{1-\eta}{\alpha-\beta}(1-\alpha)(n+\delta)$, hence $\theta \in \Theta_1^B \cup \Theta_2^A$. Finally, if $\alpha = \beta$ then $\widetilde{c}^* = \left[\frac{(1-\alpha)(n+\delta)+((1-\gamma)\nu-\varepsilon)}{\alpha}\right]\widetilde{k}^*$ and if $\theta \in \Theta_1 \cup \Theta_2$ then $\widetilde{c}^* > 0$. ■

8.3 The linearized dynamic system

The non-linear dynamic system is linearized:

$J_{11} = \sigma\left(\alpha \widetilde{k}^{*\alpha-1} - \rho\right) - \vartheta\left(\frac{(1-\gamma)\nu}{\alpha} - 2\frac{\widetilde{c}^*}{\widetilde{k}^*}\right)$

$J_{12} = \alpha(\alpha-1)\sigma \widetilde{k}^{*\alpha-1} - \vartheta \frac{\widetilde{c}^*}{\widetilde{k}^*}$

$J_{21} = (\vartheta - 1)\frac{\widetilde{c}^*}{\widetilde{k}^*}$

$J_{22} = \alpha \widetilde{k}^{*\alpha-1} - \vartheta \frac{(1-\gamma)\nu}{\alpha}$

Along the Balanced Growth Path: $\sigma\left(\alpha \widetilde{k}^{*\alpha-1} - \rho\right) = \vartheta\left(\frac{(1-\gamma)\nu}{\alpha} - \frac{\widetilde{c}^*}{\widetilde{k}^*}\right)$, substituting $\vartheta = \frac{1-\alpha+\beta}{\alpha-\beta}\frac{\alpha}{1-\alpha}$ and rewriting in terms of parameters using

$\frac{\widetilde{c}^*}{\widetilde{k}^*} = \left[\rho - \frac{(1-\alpha+\beta)(\alpha-1/\sigma)}{(1-\alpha)(\alpha-\beta)}(1-\gamma)\nu\right]\frac{(\alpha-\beta)}{\alpha(1-\eta)}$ we get J in (31) in Section 4.

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