# On vacuum energies and renormalizability in integrable quantum field theories

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Abstract: We compute for various perturbed conformal field theories the vacuum energies by means of the thermodynamic Bethe ansatz. Depending on the infrared and ultraviolet divergencies of the models, governed by the scaling dimensions of the underlying perturbed conformal field theory in the ultraviolet, the vacuum energies exhibit different types of characteristics. In particular, for the homogeneous sine-Gordon models we observe that once the conformal dimension of the perturbing scalar field is smaller or greater than 1/2, the vacuum energies are positive or negative, respectively. This behaviour indicates the need for additional ultraviolet counterterms in the latter case. At the transition points we obtain an infinite vacuum energy, which is partly explainable with the presence of several free Fermions in the models studied.

## 1. Introduction

According to the ideas developed first in [1] a large class of massive quantum field theories in 1+1 space-time dimensions can be viewed as perturbed conformal field theories (CFT) with Euclidian action

$$\mathcal{S} = \mathcal{S}_{\mathrm{CFT}} + \lambda \int d^2 x \varphi(x) \ . \tag{1.1}$$

Here $\mathcal{S}_{\mathrm{CFT}}$ denotes a fixed point action, $\varphi(x)$ a scalar field with (left, right) conformal dimension $(\Delta, \Delta)$ and $\lambda$ a coupling constant, scaling with $(1-\Delta, 1-\Delta)$. The great virtue of such theories is that very often they are integrable and can be solved exactly, that is to all orders in perturbation theory. Since the original formulation various non-perturbative techniques have been developed to study such theories with great success. Nonetheless, once the CFT is well investigated one may also employ standard perturbative arguments and unravel the meaning of certain types of behaviour in that more traditional language.

Accordingly, the vacuum expectation value of any local operator $\mathcal{O}$ can then be computed as

$$\langle \mathcal{O}(z,\bar{z})\rangle = Z^{-1} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int d^2 z_1 \ldots d^2 z_n \, \langle \mathcal{O}(z,\bar{z}) \varphi(z_1,\bar{z}_1) \ldots \varphi(z_n,\bar{z}_n) \rangle_{\mathrm{CFT}} \ . \qquad (1.2)$$

Here the normalization factor is in general $Z = \left\langle \exp -\lambda \int d^2 z \varphi(z,\bar{z}) \right\rangle_{\mathrm{CFT}}$, with $\rangle_{\mathrm{CFT}}$ denoting the vacuum state related to $\mathcal{S}_{\mathrm{CFT}}$. In quantum field theories such expressions are plagued by various types of divergencies. First of all one encounters the infinities due to the self-contraction of the fields, which can be regularized fairly easily by a normal ordering prescription. Second, one might have ultraviolet (UV) singularities for $(z - z_i) \to 0$. Here the case $\mathcal{O} = \varphi$ will be important, for which we can approximate with the help of standard CFT operator product expansion the integrals in (1.2) as $\sim \int dz_i \, |z - z_i|^{-2\Delta}$. Thus for $\Delta < 1/2$ the integrals in (1.2) remain finite, whereas for $\Delta > 1/2$ we require in general counterterms to elliminate the divergencies. Third, one might have infrared (IR) singularities for $(z - z_i) \to \infty$. In the infinite plane it is usually an intricate issue to handle them [2, 3]. However, when formulating the theory from the very beginning on a cylinder instead of an infinite plane the integrals in (1.2) will automatically be IR finite for $\Delta > 0$, as the cylinder radius $R$ constitutes a natural cut off. The fourth singularity occurring is related to the fact, that even when the individual integrals in (1.2) are finite the entire series will in general be IR divergent for large $R$.

Supposing now that one is able to compute (1.2) exactly, that is to all orders in perturbation theory, the different types of renormalization quantities should be tractable in that context. In fact, the thermodynamic Bethe ansatz (TBA) [4] is a method which allows such identifications when $\mathcal{O}$ is taken to be the energy operator. The above mentioned arguments hold when recalling [5] that this operator is proportional to the perturbing field $\varphi$. Defining then for the ground state energy $E(R)$ the scaling function $c(R) = -6RE(R)/\pi$ one encounters several types of general behaviours, which can all be brought into the generic form

$$c(r) = c_{\mathrm{eff}} + \mathcal{E}_0 r^2 + \mathcal{E}_0' r^2 \ln r + \sum_{n=1}^{\infty} \mathcal{E}_n \lambda^n f_n(r) \ . \qquad (1.3)$$

Usually one uses the dimensionless parameter $r = R/m$ with $m$ being a mass scale and $\mathcal{E}_0$, $\mathcal{E}_0'$ being finite real numbers. The function $c(r)$ is normalized in such a way that $c(r = 0)$ coincides with the effective central charge $c_{\mathrm{eff}} = c - 24\Delta_{\min}$, with $c$ being the Virasoro central charge of the underlying ultraviolet conformal field theory and $\Delta_{\min}$ the smallest conformal scaling dimension in the model. This constant $c$ has the well known interpretation as the Casimir energy, which is the vacuum energy on the cylinder and becomes zero when mapped onto the plane. Viewing (1.2) as resulting from a partition function, the term $\mathcal{E}_0 r^2$ has to be present in (1.3), since thermodynamics dictates that for large $r$ the energy has to be proportional to the volume. In quantum field theoretic terms both $\mathcal{E}_0 r^2$ and $\mathcal{E}_0' r^2 \ln r$ are related to renormalization issues, characterized by the conformal dimension $\Delta$ as described above. These terms are also needed in order to ensure that $\lim_{r\to\infty} c(r) = 0$, which one expects for a purely massive model. Finally, the $f_n(r)$

result from the integrals in (1.2) and takes on various general forms depending on the regime in which $\Delta$ is valued.

In this paper we will discuss more concretely the precise nature of the expansion (1.3). We will first recall in section 2 how the TBA can be used to compute the vacuum energies and in the following sections we discuss the different regimes for different types of concrete theories, the homogeneous sine-Gordon (HSG) models [6, 7] and affine Toda field theories (ATFT) [8, 9]. These theories probe several regimes for $\Delta$ and exhibit different types of behaviours. In particular for the HSG-models, which are defined in the entire regime $0 < \Delta < 1$, our results will be new. Our conclusions are stated in section 6.

## 2. Vacuum energies from the TBA

Let us briefly recall the main steps of how vacuum energies may be computed [4] (more details on the arguments can also be found in [10]) non-perturbatively with the help of the TBA. One considers a relativistic theory in which the scattering matrices $S_{ij}(\theta)$ for the particles of the type $i$,$j$ with masses $m_i, m_j$ are known as functions of the rapidity $\theta$. Then the entire TBA analysis can be formulated with only two inputs: first the dynamical interaction, which enters via the logarithmic derivative of the S-matrix $\varphi_{ij}(\theta) = -id \ln S_{ij}(\theta)/d\theta$ and an assumption on the statistical interaction, which we choose here to be of fermionic type. The thermodynamic Bethe ansatz equations are then a set of coupled non-linear integral equations

$$r m_i \cosh\theta = \varepsilon_i(\theta, r) + \sum_j [\varphi_{ij} * \ln(1 + e^{-\varepsilon_j})](\theta, r)\,, \tag{TBA}$$

where the pseudo-energies $\varepsilon_i(\theta, r)$ are the unknown quantities. We denote as usual the convolution of two functions by $(f * g)(\theta) := 1/(2\pi) \int d\theta' f(\theta - \theta')g(\theta')$. The scaling parameter is related to the inverse temperature $T$ as $r = m/T$, $m_l \to m_l/m$, with $m$ being an overall mass scale. In [4] it was shown that when taking the sum and difference of the derivatives $d/dr$(TBA) and $d/d\theta$(TBA)$/r$ one may derive a set of coupled linear integral equations for the quantities

$$\psi^i_\pm(\theta, r) = \frac{\partial \varepsilon_i(\theta, r)}{\partial r} \pm \frac{1}{r} \frac{\partial \varepsilon_i(\theta, r)}{\partial \theta}, \tag{2.1}$$

respectively, namely

$$\psi^i_\pm(\theta, r) = m_i\, e^{\pm\theta} + \sum_j [\varphi_{ij} * \frac{1}{e^{\varepsilon_j} \pm 1} \psi^j_\pm](\theta, r)\ . \tag{2.2}$$

The strategie is now to solve first the equations (TBA) for $\varepsilon_i(\theta, r)$ and thereafter (2.2) for $\psi^i_\pm(\theta, r)$. Once one has carried out the first step, one can already compute the scaling function

$$c(r) = \frac{3\, r}{\pi^2} \sum_i m_i \int_{-\infty}^{\infty} d\theta\, \cosh\theta\, L_i(\theta, r)\,, \tag{2.3}$$

with $L_i(\theta, r) = \ln(1+e^{-\varepsilon_i(\theta,r)})$. Concerning the status of analytical solutions for (2.2), it is similar as for the TBA itself, that is only for free theories [10] a closed solution was found and for interacting theories (2.2) was only solved in the extreme ultraviolet limit. Numerical solutions exist even less. Once it is solved, one may compute the vacuum expectation value of the trace of the energy momentum tensor, i.e. vacuum energies

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{\pi^2}{3r}\frac{d}{dr}c(r) = \frac{1}{2}\sum_i m_i \left( T^i_+ + T^i_- \right) \tag{2.4}$$

$$= \frac{1}{2}\sum_i m_i \int_{-\infty}^{\infty} d\theta \, \frac{1}{1+e^{\varepsilon_i(\theta,r)}} \left[ \psi^i_+(\theta,r)e^{-\theta} + \psi^i_-(\theta,r)e^{\theta} \right] . \tag{2.5}$$

In a parity invariant theory we have $\varepsilon_i(\theta, r) = \varepsilon_i(-\theta, r)$ and consequently $\psi^i_+(\theta, r) = \psi^i_-(\theta, r)$, $T^i_+ = T^i_- = T^i$ such that matters simplify. We like to keep the treatment here generic for a while as we will also consider below the homogeneous sine-Gordon models, which are not parity invariant.

There exists no systematic way to solve the equations (TBA) and (2.2) analytically, albeit, numerically this is a solvable problem. Nonetheless, it is well known that at the fixed points approximations can be made, such that one can solve (TBA) analytically and hence also obtain analytic expressions for (2.3) at these points ($r = 0$ is one of them). Likewise we expect to be able to solve (2.2) for these values and compute $\langle T^{\mu}{}_{\mu} \rangle$ analytically. Following now essentially the argumentation of [4, 10], we need to make only three assumptions:

**i)** The logarithmic derivative of the scattering matrix in (TBA) admits an expansion of the form

$$\varphi_{ij}(\theta) = -\sum_s \varphi^{(s)}_{ij} e^{-s|\theta|} . \tag{2.6}$$

**ii)** For the first coefficient in (2.6) we presume proportionality to the masses

$$\varphi^{(1)}_{ij} = \rho_{ij} m_i m_j \tag{2.7}$$

for some function $\rho_{ij}$ specific to the particular theory.

**iii)** One assumes that

$$\hat{\varepsilon}_i(\theta) - \varepsilon_i \ll e^{\theta} \qquad \text{for} \quad \theta \ll 0 \tag{2.8}$$

where the $\varepsilon_i$ are the pseudo-energies of the constant TBA equation and the $\hat{\varepsilon}_i(\theta)$ are quantities in the r-independent TBA-equation

$$\varphi_{ij} * \hat{L}_j(\theta) = -\hat{\varepsilon}_i(\theta) + m_i e^{\theta} \tag{2.9}$$

obtained from (TBA) by the shift $\theta \to \theta + \ln(r/2)$, $\varepsilon_i(\theta, r) \to \hat{\varepsilon}_i(\theta)$. This assumption is usually difficult to justify a priori, but is sustained in hindsight by meaningful results or supported by numerical data.

For $\theta \to -\infty$ one can now derive with (2.8) the equation

$$\varphi_{ij} * \hat{L}_j(\theta) = -\varepsilon_i + \frac{1}{2\pi} e^{\theta} \varphi_{ij}^{(1)} T_+^j + \mathcal{O}(e^{\eta\theta}) \tag{2.10}$$

where $\eta \geq 2$. Comparing then (2.9) and (2.10) for the parity invariant case, it follows directly with (2.8) that

$$m_i = \frac{1}{2\pi} \varphi_{ij}^{(1)} T^j \ . \tag{2.11}$$

Finally we deduce the expression for the vacuum expectation value for the energy momentum tensor with (2.8) and (2.4) to

$$\langle T^{\mu}{}_{\mu} \rangle = 2\pi \sum_{i,j} \rho_{ij}^{-1} \ . \tag{2.12}$$

This quantity is of course sensitive to above mentioned renormalization issues and possibly exhibits the distinction between the different regimes quoted. Furthermore, one has the possibility of comparison, as there are various other methods to obtain the vacuum energies, such as the truncated conformal space approach [11] or a matching between the high-energy behaviour of the scattering matrix with a Feynman diagramatic analysis [12].

Let us briefly comment on the different regimes:

$\underline{\mathbf{0 < \Delta < 1/2:}}$ As mentioned in the introduction, in this regime the individual integrals in the expansion (1.2) are UV and IR convergent term by term when formulated on the cylinder. From general arguments one finds for the behaviour in (1.3) that $\mathcal{E}_0' = 0$ and $f_n(r) = r^{2n(1-\Delta)}$ [13]. From (1.3) and (2.4) follows also that we can identify $\langle T^{\mu}{}_{\mu} \rangle |_{r=0} = -\pi^2/3\mathcal{E}_0$. Thermodynamically this term can be seen as the infinite volume energy and field theoretically this corresponds to the sum of all infrared substractions, which achieve the convergence of the sums (1.3) for large $r$.

$\underline{\mathbf{1/2 < \Delta < 1:}}$ Now the individual integrals in the expansion (1.2) are still IR convergent, but cease to be UV convergent. Nonetheless, we may still employ similar arguments as in the previous regime and find again for the behaviour in (1.3) that $\mathcal{E}_0' = 0$ and $f_n(r) = r^{2n(1-\Delta)}$ [13]. From (1.3) and (2.4) follows once more that we can identify $\langle T^{\mu}{}_{\mu} \rangle |_{r=0} = -\pi^2/3\mathcal{E}_0$. However, now the field theoretic interpretation of this term changes. Since we require in this case UV conterterms to make the individual integrals finite, the $\mathcal{E}_0$-term corresponds now to the sum of these UV counterterms and all infrared substractions, which achieve the convergence of the sums (1.3) for large $r$. Indeed, for the concrete models studied below this becomes visible in a change of sign in the transition from the regime $\Delta < 1/2$ to $\Delta > 1/2$.

$\underline{\mathbf{\Delta < 0:}}$ Now the individual integrals in the expansion (1.3) are still UV convergent, but cease to be IR convergent even on a cylinder. General arguments now yield for the behaviour in (1.3) that $\mathcal{E}_0 \neq 0$, $\mathcal{E}_0' \neq 0$ and $f_n(r) = (\alpha + \ln(r))^{-n}$ with $\alpha$ being some constant [14, 15, 16, 17, 18]. One still finds that $\langle T^{\mu}{}_{\mu} \rangle |_{r=0} = -\pi^2/3\mathcal{E}_0$, e.g. [12], but now the interpretation is less obvious as some counterterms also accumulate in the $\mathcal{E}_0'$-term.

$\underline{\mathbf{\Delta = 1/2:}}$ In this case one usually finds free Fermions in the model and $\mathcal{E}_0 \neq 0$, $\mathcal{E}_0' \neq 0$, $f_n(r) = r^n$. Now the vacuum energy is divergent, see e.g. [10] for an analytical expression.

We will investigate some concrete theories.

## 3. $0 < \Delta < 1/2$, minimal affine Toda field theories

These theories have been studied before [10, 19], nonetheless, we recall them here as they are easy examples which illustrate the working of the above formulae and we shall also point out some novel features. We recall first that minimal affine Toda field theories can be realized as perturbations of the coset conformal field theories $\mathbf{g}_1 \otimes \mathbf{g}_1/\mathbf{g}_2$, with $\mathbf{g}_k$ being a simply laced Kac-Moody algebra of rank $\ell$ and level $k$ [20, 21]. The corresponding Virasoro central charges $c$ and conformal dimension of the perturbing operator $\Delta$ are

$$c = \frac{2\ell}{2+h} \qquad \text{and} \qquad \Delta = \frac{2}{2+h} \ , \tag{3.1}$$

respectively. Apart from $h = 2$, i.e. the free Fermion with $\mathbf{g} = A_1$, we always have for the Coxeter number $h > 2$ and are therefore in the stated regime $0 < \Delta < 1/2$. The renormalization issues are handled most easily in this case and the vacuum energies are computable with the above arguments. With regard to assumption i), we recall the expansion of the TBA-kernel for these theories [10, 22, 23]

$$\varphi_{ij}(\theta) = -4 \sum_{s \in \mathcal{E}} \cot \frac{s\pi}{h} x_i(s) x_j(s) e^{-s|\theta|} \ , \tag{3.2}$$

with $\mathcal{E} = \{s + nh\}$, $s$ being an exponent of $\mathbf{g}$, $n \in \mathbb{N}_0$ and $x_i(s)$ are the left eigenvectors of the Cartan matrix. In particular, we have $x_i(1) = m_i/m$, with $m$ being an overall mass scale, which is needed for the assumption ii) to hold. Having therefore the quantity $\varphi_{ij}^{(1)}$ in the form (2.7), we deduce immediately with the help of (2.12)

$$\langle T^{\mu}{}_{\mu} \rangle = m^2 \frac{\pi}{2} \tan \frac{\pi}{h} \ . \tag{3.3}$$

Obviously apart from the free Fermion with $h = 2$, when $\langle T^{\mu}{}_{\mu} \rangle \to \infty$, we have $\langle T^{\mu}{}_{\mu} \rangle > 0$. This result agrees with a similar formula obtained in [10] in terms of the coefficients $\varphi_{11}^{(1)}$ without explicit evaluation and "1" referring to the lightest particle. More concrete case-by-case studies were carried out in [19] for perturbations of $\mathbf{g}_l \otimes \mathbf{g}_k/\mathbf{g}_{k+l}$-coset CFT's (see formulae (3.14) therein). When using the overall mass scale to perform suitable normalizations the formula for $k = l = 1$ in there can be brought into the universal formula (3.3), which is not obvious at first sight. The formulae in [19] are expressed in terms of a mass scale $M$ whose relation with our $m$ varies for every theory as

$$\begin{array}{l}
A_\ell : \ M = m \sin \frac{\pi}{\ell - 1} \\
D_\ell : \ M = m/\sqrt{2} \\
E_6 : \ M = m \sqrt{\sqrt{\frac{3}{2}} \sin \frac{\pi}{12}} \\
E_7 : \ M = m \sqrt{\sin \frac{\pi}{18} / \sin \frac{2\pi}{9}} \\
E_8 : \ M = m \sqrt{2 \sin \frac{\pi}{30} \sin \frac{\pi}{5}} \ .
\end{array} \tag{3.4}$$

The advantage of our formulation relies on the fact that the masses are normalized with respect to the same general mass scale $m$ for all simply laced Lie algebras, which allows for the very compact and generic expression (3.3). Alternatively these results were also confirmed in [24].

## 4. $0 < \Delta < 1$, $\mathbf{g}_k$-homogeneous sine-Gordon models

Let us now consider a theory which is more interesting with regard to the above mentioned problematic, namely the $\mathbf{g}_k$-HSG model [6, 7], with $\mathbf{g}$ being a simple Lie algebra of rank $\ell$ and level $k$. These models can be viewed as perturbed Wess-Zumino-Novikov-Witten (WZNW) [25] coset-models

$$\mathcal{S}_{\text{HSG}} = \mathcal{S}_{\text{WZNW}} + \frac{m^2}{\pi\beta^2} \int d^2x \, \left\langle \Lambda_+, g(\vec{x})^{-1}\Lambda_- g(\vec{x}) \right\rangle \, . \tag{4.1}$$

Here $\langle \, , \, \rangle$ denotes the Killing form of $\mathbf{g}$ and $g(\vec{x})$ a group valued bosonic scalar field. $\Lambda_\pm$ are arbitrary semi-simple elements of the Cartan subalgebra associated with the maximal abelian torus $\mathbf{h} \subset \mathbf{g}$, which have to be chosen not to be orthogonal to any root of $\mathbf{g}$. The parameters $m$ and $\beta$ are the bare mass scale and the coupling constant, respectively. The Virasoro central charge of the coset model and the dimension of the perturbing operator are computed to

$$c = \ell \, \frac{k\, h - h^\vee}{k + h^\vee} \qquad \text{and} \qquad \Delta = \frac{h^\vee}{k + h^\vee} \, , \tag{4.2}$$

with $(h^\vee)\, h$ being the (dual) Coxeter number of $\mathbf{g}$. We note that now the constraint $\Delta < 1/2$ does not automatically hold for each level and the above mentioned complications could arise for some theories in this series when changing from $k > h^\vee$ to $k < h^\vee$. Up to now no indication for a different behaviour of the theories in this two different regimes have been found in the literature. We treat the simply laced and non-simply laced cases separately.

### 4.1 Simply laced HSG-models

As in the original formulation of these models, the algebra $\mathbf{g}$ is assumed to be simply laced. Since for this case the expansion of the kernel $\varphi$ does not appear in the literature, we will start with the scattering matrix, which was found originally in [26] (see [27] for an integral representation). We cast the matrix describing the scattering between the particle of type $A = (a, \tilde{a})$ and $B = (b, \tilde{b})$, with $1 \leq \tilde{a}, \tilde{b} \leq \ell$; $1 \leq a, b < k$ into the form

$$S_{ab}^{\tilde{a}\tilde{b}}(\theta) = \eta_{ab}^{\tilde{a}\tilde{b}} \exp \int \frac{dt}{t} \tilde{K}_{\tilde{a}\tilde{b}}(t) \frac{\sinh(at/k)\sinh[(k-b)t/k]}{\sinh(t/k)\sinh t} e^{-it(\theta + \sigma_{\tilde{a}\tilde{b}})/\pi} \, . \tag{4.3}$$

Here $\eta_{ab}^{\tilde{a}\tilde{b}} = \exp[i\pi\varepsilon_{\tilde{a}\tilde{b}}(2 - I_{A_{k-1}})^{-1}_{\bar{a}\bar{b}}]$ are constant phase factors not relevant for our analysis, $\tilde{K}_{\tilde{a}\tilde{b}}(t) = 2\delta_{\tilde{a}\tilde{b}} \cosh t/k - I_{\tilde{a}\tilde{b}}$ with $I$ being the incidence matrix of $\mathbf{g}$ and the $\sigma$'s are the resonance parameters, which indicate the presence of unstable particles in these models. In order to evaluate the expansion for $\varphi$, we can employ the residue theorem for a contour along the real axis closing up in the positive half of the complex plane encircling all poles on the imaginary axis in the upper half plane. Noting that in (4.3) $t = i\pi n$ are simple poles, except for $t = i\pi nk$ which constitute double poles for $n \in \mathbb{N}$, we deduce for $\sigma_{\tilde{a}\tilde{b}} = 0$

$$\varphi_{ab}^{\tilde{a}\tilde{b}}(\theta) = 2\pi i \sum_{s=1; s \neq nk}^{\infty} \operatorname*{Res}_{t = i\pi s} \left(-\frac{1}{\pi}\right) \tilde{K}_{\tilde{a}\tilde{b}}(t) \frac{\sinh(at/k)\sinh[(k-b)t/k]}{\sinh(t/k)\sinh t} e^{-it\theta/\pi} \tag{4.4}$$

$$= -2 \sum_{s=1; s \neq nk}^{\infty} \tilde{K}_{\tilde{a}\tilde{b}}(i\pi s) \frac{\sin(a\pi s/k)\sin(b\pi s/k)}{\sin(\pi s/k)} e^{-s|\theta|} \, . \tag{4.5}$$

The desired coefficient (2.7) follows from this directly to

$$\varphi^{(1)}_{\tilde{a}\tilde{b}} = 2\tilde{K}_{\tilde{a}\tilde{b}}(i\pi) m_a m_b / m^2 \sin(\pi/k) \tag{4.6}$$

where $m_a^{\tilde{a}} = m_a\, m^{\tilde{a}}$ with $m_a = \sin a\pi/k$ being the masses of $A_{k-1}$-affine Toda field theory and $m^{\tilde{a}}$ are $\ell$ free mass scales. We choose them here to be all equal $m^{\tilde{a}} = m \;\forall\; \tilde{a}$. Finally we derive from this a closed expression for the vacuum expectation value for the trace of energy-momentum tensor

$$\left\langle T^{\mu}{}_{\mu} \right\rangle = \pi m^2 \sin(\pi/k) \sum_{\tilde{a},\tilde{b}=1}^{\ell} \left[ \tilde{K}^{-1}(i\pi) \right]_{\tilde{a}\tilde{b}} \; . \tag{4.7}$$

We are not aware of a generic formulation for $\tilde{K}^{-1}(i\pi)$ and analyse therefore the expression (4.7) below in more detail case-by-case. We can summarize our findings as

$$\left\langle T^{\mu}{}_{\mu} \right\rangle \begin{cases} > 0 & \text{for } k > h \;\; \equiv \;\; \Delta < 1/2 \\ \to \infty & \text{for } k = h \;\; \equiv \;\; \Delta = 1/2 \\ < 0 & \text{for } k < h \;\; \equiv \;\; \Delta > 1/2 \end{cases} . \tag{4.8}$$

In many cases we can attribute the divergence for $\Delta = 1/2$ to the presence of free Ferminons. The change of sign when going from $\Delta < 1/2$ to $\Delta > 1/2$ reflects the fact that besides the IR counterterms, which achieve the convergence of the sums (1.2) for large $r$, needed in both cases in the latter we also require UV conterterms to make the individual integrals in (1.2) finite.

#### 4.1.1 $(\mathrm{A}_\ell)_k$-HSG model

For $\mathrm{A}_\ell$ the Coxeter number is $h = \ell + 1$. The inverse of the matrix relevant in (4.7) can be cast in this case into a simple formula

$$\left[ \tilde{K}^{-1}_{A_\ell}(i\pi) \right]_{\tilde{a}\tilde{b}} = \frac{\sin(\tilde{a}\pi/k)\sin[(h-\tilde{b})\pi/k]}{\sin(\pi/k)\sin(h\pi/k)} \qquad \text{for } \tilde{a} \le \tilde{b}. \tag{4.9}$$

Computing the sums over both entries then yields after some algebra

$$\left\langle T^{\mu}{}_{\mu} \right\rangle = \frac{\pi m^2}{2\tan^2 \pi/2k} \left[ \tan\frac{h\pi}{2k} - h \tan\frac{\pi}{2k} \right] \; . \tag{4.10}$$

Hence, the condition $\langle T^{\mu}{}_{\mu} \rangle > 0$ becomes

$$\tan\frac{h\pi}{2k} > h\tan\frac{\pi}{2k} \tag{4.11}$$

or equivalently, when expanding the tan,

$$\frac{4}{\pi}\frac{h}{k}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - (h/k)^2} > h\frac{4}{\pi}\frac{1}{k}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - (1/k)^2} \; . \tag{4.12}$$

It is easily seen that (4.12) holds term by term once $h/k < 1$, hence establishing the first inequality in (4.8). Similar arguments show that the opposite inequality holds in the regime $h/k > 1$. We comment more on the case $k = h$ below.

### 4.1.2 $(\mathrm{D}_\ell)_k$-HSG model

For $\mathrm{D}_\ell$ the Coxeter number is $h = 2\ell - 2$ and by evaluating (4.7) similarly as in the previous subsection, we find

$$\langle T^\mu_{\ \mu} \rangle = \frac{\pi m^2 \sin\frac{\pi}{k}\left[2 - (2+h)\cos\frac{h\pi}{2k}\right] + 2\pi m^2 \sin\frac{h\pi}{2k}}{\sin^2\frac{\pi}{2k}\cos\frac{h\pi}{2k}}. \tag{4.13}$$

The condition $\langle T^\mu_{\ \mu}\rangle > 0$ is now equivalent to

$$\sin\frac{\pi}{k}\left[(2+h) - \frac{2}{\cos\frac{h\pi}{2k}}\right] < 2\tan\frac{h\pi}{2k}. \tag{4.14}$$

Expanding the left and right hand side of this inequality yields by similar arguments as in the previous subsection once more the relation (4.8).

### 4.1.3 $(\mathrm{E}_6)_k$-HSG model

For $\mathrm{E}_6$ the Coxeter number is $h = 12$ and we find

$$\langle T^\mu_{\ \mu} \rangle = 2\pi m^2 \frac{\sum_{p=1}^{4} \tau_p \sin p\pi/k}{2\cos 4\pi/k - 1} \qquad \vec{\tau} = (4,4,5,3). \tag{4.15}$$

We see that the numerator is $< 0$ for $k = 2$ and $> 0$ for $k > 2$. The denominator is $> 0$ for $k = 2$, $k > 12$ and $< 0$ for $2 < k < 12$. The denominator vanishes for $k = 12$. Hence the relation (4.8) holds.

### 4.1.4 $(\mathrm{E}_7)_k$-HSG model

For $\mathrm{E}_7$ the Coxeter number is $h = 18$ and we find

$$\langle T^\mu_{\ \mu} \rangle = \frac{\pi m^2 \sum_{p=1}^{7} \tau_p \sin p\pi/k}{\cos\pi/k(4\cos 6\pi/k - 2)} \qquad \vec{\tau} = (9,18,20,22,17,12,7)\ . \tag{4.16}$$

We observe now that the numerator is $< 0$ for $k < 4$ and $> 0$ otherwise. The denominator on the other hand is $> 0$ for $k = 3, k > 18$ and $< 0$ otherwise except for $k = 2, 18$ in which case it is zero. Hence (4.8) holds also in this case.

### 4.1.5 $(\mathrm{E}_8)_k$-HSG model

For $\mathrm{E}_8$ the Coxeter number is $h = 30$ and we find

$$\langle T^\mu_{\ \mu} \rangle = \frac{\pi m^2 \sum_{p=1}^{7} \tau_p \sin p\pi/k}{\cos 8\pi/k + \cos 6\pi/k - \cos 2\pi/k - 1/2} \qquad \vec{\tau} = (4,8,12,12,13,10,7,4)\ . \tag{4.17}$$

We see that the numerator is $< 0$ for $k < 5$ and $> 0$ otherwise. The denominator is $> 0$ for $k = 2, 3, 4; k > 30$ and $< 0$ otherwise except for $k = 30$ in which case it is zero. Hence (4.8) holds also in this case.

### 4.2 Non-simply laced HSG-models

Now we allow the algebra $\mathbf{g}$ to be also non-simply laced. In this case the scattering matrix is slightly more complicated as the symmetry between the exchange of long and short roots is lost. It can be restored by the use of the symmetrizers $t_{\tilde{a}}$ of the incidence matrix of $\mathbf{g}$, i.e. $t_{\tilde{a}} I_{\tilde{a}\tilde{b}} = t_{\tilde{b}} I_{\tilde{b}\tilde{a}}$, with $t_{\tilde{a}} = 2/\alpha_{\tilde{a}}^2$ and the length of long roots normalized to $\alpha_l^2 = 2$. In [28] the scattering between the particle of type $A = (a, \tilde{a})$ and $B = (b, \tilde{b})$, with $1 \leq \tilde{a}, \tilde{b} \leq \ell$; $1 \leq a, b < k_{\tilde{a}} = t_{\tilde{a}} k$ was proposed to be described by

$$S_{ab}^{\tilde{a}\tilde{b}}(\theta) = \eta_{ab}^{\tilde{a}\tilde{b}} \exp \int \frac{dt}{t} \tilde{K}_{\tilde{a}\tilde{b}}(t) \frac{\sinh(at/k_{\tilde{a}}) \sinh[(1 - b/k_{\tilde{b}})t]}{\sinh(t/k_{\tilde{a}\tilde{b}}) \sinh t} e^{-it(\theta + \sigma_{\tilde{a}\tilde{b}})/\pi} \ . \tag{4.18}$$

Here $\eta_{ab}^{\tilde{a}\tilde{b}}$ are once more constant phase factors not relevant for our analysis. Furthermore, one needs the quantity $k_{\tilde{a}\tilde{b}} = k \max(t_{\tilde{a}}, t_{\tilde{b}})$ and the matrix $\tilde{K}$ with entries $\tilde{K}_{\tilde{a}\tilde{b}}(t) = 2\delta_{\tilde{a}\tilde{b}} \cosh t/k_{\tilde{a}} - I_{\tilde{a}\tilde{b}} t_{\tilde{b}} / \max(t_{\tilde{a}}, t_{\tilde{b}})$. Note that in comparison with [28] we interchanged the long and short roots, that is we have taken the $t$'$s$ to be left and not the right symmetrizers of the incidence matrix. A similar analysis as in the previous subsection leads now to the following expansion of the TBA-kernel

$$\varphi_{ab}^{\tilde{a}\tilde{b}}(\theta) = -2 \sum_{s=1; s \neq n k_{\tilde{a}\tilde{b}}}^{\infty} \tilde{K}_{\tilde{a}\tilde{b}}(i\pi s) \frac{\sin(a\pi s/k_{\tilde{a}}) \sin(b\pi s/k_{\tilde{b}})}{\sin(\pi s/k_{\tilde{a}\tilde{b}})} e^{-s|\theta|} \ , \tag{4.19}$$

such that

$$\varphi_{\tilde{a}\tilde{b}}^{(1)} = 2\tilde{K}_{\tilde{a}\tilde{b}}(i\pi) m_a^{\tilde{a}} m_b^{\tilde{b}} / m^2 \sin(\pi/k_{\tilde{a}\tilde{b}}) \ . \tag{4.20}$$

Here the masses are assumed to renormalize with an overall factor and are therefore expected to be the same as in the semi-classical analysis [29], that is $m_a^{\tilde{a}} = m \sin a\pi/k_{\tilde{a}}$. The overall mass scales associated with each colour are once more choosen to be the same. Thus we finally deduce

$$\left\langle T^{\mu}{}_{\mu} \right\rangle = \pi m^2 \sum_{\tilde{a}, \tilde{b}=1}^{\ell} \left[ \hat{K}^{-1} \right]_{\tilde{a}\tilde{b}} \ . \tag{4.21}$$

where $\hat{K}_{\tilde{a}\tilde{b}} = \tilde{K}_{\tilde{a}\tilde{b}}(i\pi) \sin(\pi/k_{\tilde{a}\tilde{b}})$. As in the previous case, we are not aware of a generic formulation for $\hat{K}^{-1}$ and analyse therefore (4.21) in more detail case-by-case. Our findings are summarized as

$$\left\langle T^{\mu}{}_{\mu} \right\rangle \begin{cases} > 0 & \text{for } k > h^{\vee} \ \equiv \ \Delta < 1/2 \\ \to \infty & \text{for } k = h^{\vee} \ \equiv \ \Delta = 1/2 \\ < 0 & \text{for } k < h^{\vee} \ \equiv \ \Delta < 1/2 \end{cases} , \tag{4.22}$$

with similar interpretations as in (4.8). We establish (4.22) in more detail case-by-case.

#### 4.2.1 $(\mathrm{G}_2)_k$-HSG model

The dual Coxeter number for $\mathrm{G}_2$ is $h^{\vee} = 4$ and the symmetrizers are taken to be $t_1 = 3, t_2 = 1$. With these data we compute from (4.21)

$$\left\langle T^{\mu}{}_{\mu} \right\rangle = 2\pi m^2 \frac{\sin \pi/k + \sin 4\pi/3k}{2\cos 4\pi/3k - 1} \ . \tag{4.23}$$

Obviously, the numerator is $> 0$ for $k \geq 2$, whereas the denominator is $< 0$ for $k = 2, 3$ and otherwise $> 0$ except for $k = 4$ when it is zero. Evidently this agrees with (4.22).

#### 4.2.2 $(\mathrm{F}_4)_k$-HSG model

The dual Coxeter number for $\mathrm{F}_4$ is $h^\vee = 9$ and the symmetrizers are taken to be $t_1 = t_2 = 1$ and $t_3 = t_4 = 2$. From (4.21) we compute

$$\langle T^\mu_{\ \mu} \rangle = 2\pi m^2 \frac{2\sum_{p=1}^{6} \sin p\pi/2k - \sin 3\pi/2k}{2\cos 3\pi/k - 1} \ . \tag{4.24}$$

The numerator is $> 0$ for $k \geq 2$, whereas the denominator is $< 0$ for $2 \leq k < 9$ and otherwise $> 0$ except for $k = 9$ when it is zero. Evidently this agrees with (4.22).

#### 4.2.3 $(\mathrm{B}_\ell)_k$-HSG model

The dual Coxeter number for $\mathrm{B}_\ell$ is $h^\vee = 2\ell - 1$ and the symmetrizers are taken to be $t_1 = t_2 = \ldots = t_{\ell-1} = 2$ and $t_\ell = 1$. We find now for even rank $\ell$

$$\langle T^\mu_{\ \mu} \rangle = -\frac{\pi m^2}{\cos\frac{\pi}{2k}} \frac{\sum\limits_{p=2}^{h^\vee-1} \sin\frac{\pi p}{2k} + \frac{\ell}{2}\sin\frac{\pi h^\vee}{2k} + 2\cos\frac{\pi}{2k}\sum\limits_{p=1}^{(\ell-2)/2}(\ell-2p-1)\sin\frac{\pi(1+h^\vee-4p)}{2k}}{1 + 2\sum_{p=1}^{\ell/2}(-1)^p\cos\frac{\pi p}{k}}, \tag{4.25}$$

whereas for odd $\ell$ we obtain

$$\langle T^\mu_{\ \mu} \rangle = \frac{\pi m^2}{\cos\frac{\pi}{2k}} \frac{\sum\limits_{p=1}^{h^\vee-1} \sin\frac{\pi p}{2k} + \frac{\ell}{2}\sin\frac{\pi h^\vee}{2k} + 2\cos\frac{\pi}{2k}\sum\limits_{p=1}^{(\ell-3)/2}(\ell-2p-1)\sin\frac{\pi(1+h^\vee-4p)}{2k}}{1 + 2\sum_{p=1}^{\ell/2}(-1)^p\cos\frac{\pi p}{k}}. \tag{4.26}$$

Once more we confirm (4.22). As the details are rather involvolved we drop them here.

#### 4.2.4 $(\mathrm{C}_\ell)_k$-HSG model

The dual Coxeter number for $\mathrm{C}_\ell$ is $h^\vee = \ell + 1$ and the symmetrizers are taken to be $t_1 = t_2 = \ldots = t_{\ell-1} = 1$ and $t_\ell = 2$.

$$\langle T^\mu_{\ \mu} \rangle = \frac{\pi m^2 (i)^\ell}{\cos\frac{\pi}{2k}} \left[\frac{\sum_{p=1}^{h^\vee-1}(p-1)\sin\frac{\pi p}{2k} + \frac{\ell}{2}\sin\frac{\pi h^\vee}{2k}}{1 + 2\sum_{p=1}^{\ell/2}(-1)^p\cos\frac{\pi p}{k}}\right], \quad \text{for} \ \ \ell \ \ \text{even} \tag{4.27}$$

$$\langle T^\mu_{\ \mu} \rangle = \frac{\pi m^2}{\cos\frac{\pi h^\vee}{2k}} \left[\sum_{p=2}^{h^\vee-1}(p-1)\sin\frac{\pi p}{2k} + \frac{\ell}{2}\sin\frac{\pi h^\vee}{2k}\right], \quad \text{for} \ \ \ell \ \ \text{odd.} \tag{4.28}$$

Once more we confirm (4.22) and drop the details for the same reasons as in the previous subsection.

## 5. $\Delta = 1/2$, $g_{h^\vee}$-homogeneous sine-Gordon model

The case $\Delta = 1/2$ is very special as then the vacuum energy diverges in the extreme UV limit. Such type of behviour is well known from free Fermions in form of logarithmic ultraviolet singularities, meaning that (2.4) yields $\langle T^{\mu}{}_{\mu}\rangle \to \infty$ for $r \to 0$. Explicit analytic formulae for the free Fermion $c(r)$-function can be found in [10]. Indeed in many cases we can make this connection quite explicit. It suffices to present an examples to illustrate this point.

### 5.1 $(A_\ell)_{\ell+1}$-HSG theories

Let us have a closer look at the $(A_\ell)_{\ell+1}$-HSG theories in order to see how the Fermions arise in there. Obviously for $h = k$ the expression (4.10) yields $\langle T^{\mu}{}_{\mu}\rangle \to \infty$. Already in [27] it was noticed that the $(A_2)_3$-HSG model decomposes into four free Fermions when the resonance parameter vanishes. From the fact that the central charge (4.2) becomes in general $\ell^2/2$ for $(A_\ell)_{\ell+1}$-HSG models, one might suspect that they always decompose completely into $\ell^2$ free Fermions for vanishing resonance parameters, such that each Fermion contributes 1/2 to the central charge. However, this is not quite the case as the following argument shows.

In order to count the Fermions, identified here simply with the amount of particles which contribute 1/2 to the central charge, we recall the constant TBA equations, which arise from (TBA) after some standard manipulations. For the $(A_\ell)_{\ell+1}$-HSG models they take on the form

$$x_a^{\tilde{a}} = \prod_{b,\tilde{b}=1}^{\ell} (1 + x_b^{\tilde{b}})^{N_{ab}^{\tilde{a}\tilde{b}}} \qquad \text{with} \qquad N_{ab}^{\tilde{a}\tilde{b}} = \delta_{ab}\delta_{\tilde{a}\tilde{b}} - \left(K_{A_\ell}^{-1}\right)_{\tilde{a}\tilde{b}} (K_{A_\ell})_{ab} \ . \tag{5.1}$$

Solving these equations for the $x_a^{\tilde{a}}$ =exp$(-\varepsilon_a^{\tilde{a}})$ yields the effective central charge as

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{a,\tilde{a}=1}^{\ell} \mathcal{L}\left(\frac{x_a^{\tilde{a}}}{1 + x_a^{\tilde{a}}}\right) = \frac{\ell^2}{2} \tag{5.2}$$

with $\mathcal{L}(x) = \sum_{n=1}^{\infty} x^n/n^2 + \ln x \ln(1-x)/2$ denoting Rogers dilogarithm. The solutions of (5.1) are very simple in this case

$$x_a^{\tilde{a}} = \frac{\sin[\pi\tilde{a}/(1+\ell)]}{\sin[\pi a/(1+\ell)]} \ . \tag{5.3}$$

Therefore we have $x_a^a = x_a^{\ell+1-a} = 1$ and since $\mathcal{L}(1/2) = \pi^2/12$ it follows from this that each of the particles $(a,a), (a, \ell+1-a)$ for $1 \leq a \leq \ell$ contributes 1/2 to the effective central charge in (5.2). Hence in the $(A_\ell)_{\ell+1}$-HSG models we have always $2\ell$ or $2\ell - 1$ free Fermions when $\ell$ is odd or even, respectively. The remaining particles can be organized without exceptions in pairs $(a,\tilde{a}), (\tilde{a}, a)$. Noting with (5.3) that obviously $x_a^{\tilde{a}} = (x_{\tilde{a}}^a)^{-1}$ and recalling the fact that $\mathcal{L}(x) + \mathcal{L}(1-x) = \pi^2/6$ explains then that the central charge has to be an integer or a semi-integer for these models.

In general, it is less straightforward for the other algebras to identify particles which directly contribute 1/2 to the central charge. In fact, mostly the particles occur in pairs, triplets or higher multiplets contributing integers or semi-integer values to $c$.

## 6. $\Delta < 0$, affine Toda field theories

Affine Toda field theories related to simply laced and non-simply laced Lie algebras have a quite different behaviour due to the fact that in the first case all masses renormalize with an overall factor, which is not the case in the latter (see e.g. [30]). As a result of this, the strong-weak duality observed for ATFT related to simply laced algebras is broken for those associated with non-simply laced Lie algebras. Despite the fact that there exists a uniform formulation, we will treat them here separately as this will be more transparent.

### 6.1 Simply laced

ATFT are quite well studied examples of integrable models, which can be viewed in the spirit of (1.1) which was noted first in [21, 20]

$$\mathcal{S}_{ATFT}=\int d^2x \frac{1}{2}(\partial_\mu \vec{\varphi})^2 + \mu \sum_{i=0}^{\ell} n_i e^{\beta\ \vec{\alpha}_i \cdot \vec{\varphi}} \ . \tag{6.1}$$

The fixed point part of the action $\mathcal{S}_{CFT}$ corresponds to the conformal Toda field theories when the sum over the simple roots $\vec{\alpha}_i$ starts at $i=1$. The $\mu, \beta$ are real parameters and the $n_i$ are the Kac labels related to the negative of the highest root $\vec{\alpha}_0 = -\sum_{i=1}^{\ell} n_i \vec{\alpha}_i$. The Virasoro central charge of the conformal Toda field theories and the conformal dimension of the perturbing operator $V = \mu n_0 e^{\beta\ \vec{\alpha}_0 \cdot \vec{\varphi}}$ have been computed in [21]

$$c = \ell + \frac{4\ell h(h+1)}{B(2-B)} \qquad \text{and} \qquad \Delta = 1 - \frac{2h}{2-B} \ , \tag{6.2}$$

where we use the effective coupling[1] $0 \leq B = 2\beta^2/(\beta^2 + 4\pi) \leq 2$. Since $2h > 1 - B/2$ is always true we are in the regime $\Delta < 0$ and expect the above mentioned complications with regard to renormalization to arise. To establish that, we recall first [10, 22, 23]

$$\varphi_{ij}(\theta) = -2 \sum_{s \in \mathcal{E}} \sin \frac{s\pi B}{2h} \sin \frac{s\pi(2-B)}{2h} / \sin \frac{s\pi}{h} x_i(s) x_j(s) e^{-s|\theta|} \ , \tag{6.3}$$

and deduce thereafter from (2.7) and (2.12)

$$\left\langle T^{\mu}{}_{\mu} \right\rangle = \frac{\pi m^2 \sin(\pi/h)}{\sin(\pi B/2h) \sin[\pi(2-B)/2h]} \ . \tag{6.4}$$

Clearly, as $0 \leq B \leq 2$ we have $\langle T^{\mu}{}_{\mu} \rangle > 0$. Up to an overall mass re-scaling of $m \to 2m$, this agrees precisely with the results in [12], which were obtained by matching the high-energy behaviour of the scattering matrix with a Feynman diagramatic analysis.

It is very interesting to note that by an analytic continuation from real to purely complex coupling we can also reach the regime for $\Delta > 0$ for (6.4) and observe similar phenomena as for the HSG-models[2]. For $h = 2$ this means we continue from sinh-Gordon

[1] Confusion arises sometimes due to different conventions. For instance we can relate our notations to the ones used in [31] by a simple rescaling of the fields $\varphi = \varphi_F/\sqrt{4\pi}$ compensated by a scaling of the coupling constant $\beta = b_F/\sqrt{4\pi}$. In addition, one takes the effective coupling constant to be $B = 2B_F$.

[2] We are grateful to Al. B. Zamolodchikov for pointing this out to us.

to sine-Gordon. Following for this case the argumentation of Destri and De Vega [12], we relate the sinh-Gordon coupling $\beta$ to the sine-Gordon coupling $\tilde{\beta}$ via $\beta \to i\tilde{\beta}/\sqrt{2}$ according to the standard conventions. Also we replace the breather mass scale $m$ with the soliton mass scale $\tilde{m}$ via $m^2 \to 4\tilde{m}^2 \sin^2 \pi B/2$ such that we end up with the simple formula

$$\langle T^{\mu}{}_{\mu} \rangle = \pi \tilde{m}^2 \tan \frac{\pi}{2} \left( \frac{\Delta}{\Delta - 1} \right) \;, \tag{6.5}$$

where $\Delta = \tilde{\beta}^2/8\pi$ is the conformal dimension of the perturbing cos-term in the model. This agrees also with [32]. Note that in the previous argument we considered sinh-Gordon as a perturbed Liouville theory, whereas now we perturb the free theory rather than complex Liouville. Analysing (6.5) in more detail one observes

$$\langle T^{\mu}{}_{\mu} \rangle \begin{cases} < 0 & \text{for} \quad \frac{2n-2}{2n-1} < \Delta < \frac{2n-1}{2n} \\ \to \infty & \text{for} \quad \Delta = \frac{2n-1}{2n} \\ > 0 & \text{for} \quad \frac{2n-1}{2n} < \Delta < \frac{2n}{2n+1} \\ = 0 & \text{for} \quad \Delta = \frac{2n}{2n+1} \end{cases} \tag{6.6}$$

with $n \in \mathbb{N}$. Note that in particular for $n = 1$ we have as for the homogeneous sine-Gordon model at $\Delta = 1/2$ a transition point at which the sign changes by passing through a singularity. Moreover, precisely this value corresponds to the free Fermion point, which in this case is a very explicit example for the free Fermion picture advocated above. However, in this case the structure is more complicated as first of all we have an infinite number of such points rather than just one as in the HSG-models. In addition $\langle T^{\mu}{}_{\mu} \rangle$ is not always divergent at these points, but can also vanish.

### 6.2 Non-simply laced

It is known, that the above mentioned complication of mass renormalization is reconciled if one views ATFT's in terms of dual pairs of Lie algebras. Since simply laced Lie algebras are self-dual, this picture does not yield anything new for that case. The dual pairs of non-simply laced Lie algebras are $(G_2^{(1)}, D_4^{(3)})$, $(F_4^{(1)}, E_6^{(2)})$, $(B_\ell^{(1)}, A_{2\ell-1}^{(2)})$ and $(C_\ell^{(1)}, D_{\ell+1}^{(2)})$. Each algebra of these pairs allows for a description of the form (6.1) related to each other by the strong-weak duality transformation $\beta \to 4\pi/\beta$, where the untwisted algebras relate to the weak coupling limit. The vacuum energies associated to all non-simply laced affine Toda theories were stated in [18]. As in there no details were presented on how they were obtained, it will be instructive to show that the procedure outlined in section 2, also holds in this case.

Let us first of all see what we have to expect with regard to the arguments outlined above and compute the Virasoro central charge and the dimension of the perturbing operator. According to [18] we have

$$c = \ell + 12\vec{Q}^2 \qquad \text{with} \qquad \vec{Q} = \beta \vec{\rho} + \frac{1}{\beta} \vec{\rho}^{\vee}, \tag{6.7}$$

with $(\vec{\rho}^{\vee})$ $\vec{\rho}$ being the (dual) Weyl vector of the untwisted Lie algebra given by half the sum of the positive (co)roots. Note that when evaluating (6.7) for the simply laced case yields precisely (6.2), but for the non-simply laced case it differs from the expressions in [21] by the use of $\vec{\rho}^\vee$ rather than always $\vec{\rho}$. The confomal dimension of a spinless primary field $V_{\vec{a}}(x) = e^{(\vec{Q}+\vec{a})\cdot\vec{\varphi}(\vec{x})}$ in the underlying CFT is $\Delta(\vec{a}) = (\vec{Q}^2 - \vec{a}^2)/2$, such that the perturbing field $\mu n_0 V_{(\beta\vec{\alpha}_0 - \vec{Q})}(x)$ has conformal dimension

$$\Delta(\beta\vec{\alpha}_0 - \vec{Q}) = \beta\vec{\alpha}_0.\vec{Q} - \frac{\beta^2\vec{\alpha}_0^2}{2}, \tag{6.8}$$

$\vec{\alpha}_0$ defined as in the previous section, that is being the negative of the highest root. It will turn out that these dimension will always be smaller zero. We will compute the precise values for some concrete examples below.

Unlike to the previous cases the expansion for the kernel $\varphi$ does not appear in the literature, we therefore start here with the scattering matrix, which can be cast into the universal form [33, 34]

$$S_{ij}(\pm\theta > 0) = \exp\left[\mp 8 \int \frac{dt}{t} \sinh(\vartheta_h t)\sinh(t_j\vartheta_H t)\left[K^{-1}(t)\right]_{ij} e^{\pm it\theta/\pi}\right], \tag{6.9}$$

where $\vartheta_h = (2-B)/2h$, $\vartheta_H = B/2H$ with $h$ being the Coxeter number of the untwisted algebra and $H$ its dual Coxeter number $h^\vee$ multiplied by the twist of the second algebra. The effective coupling is now generalized to $B = 2H\beta^2/(H\beta^2 + 4\pi h)$. The $t_i$ are the symmetrizers of the incidence matrix of the untwisted algebra $t_i I_{ij} = t_j I_{ji}$, with $t_i = 2/\vec{\alpha}_i^2$ and the length of long roots normalized to $\vec{\alpha}_l^2 = 2$. Also needed in (6.9) is the inverse of the $q$-deformed Cartan matrix $K_{ij}(t) = \left(q\bar{q}^{t_j} + q^{-1}\bar{q}^{-t_j}\right)\delta_{ij} - [I_{ij}]_{\bar{q}}$ with deformation parameters $q = \exp(t\vartheta_h)$, $\bar{q} = \exp(t\vartheta_H)$ and $[I_{ij}]_{\bar{q}} = (\bar{q}^{I_{ij}} - \bar{q}^{-I_{ij}})/(\bar{q} - \bar{q}^{-1})$.

In order to evaluate the expansion for $\varphi$, we can employ once again the residue theorem for a contour along the real axis closing up in the positive half of the complex plane encircling all poles on the imaginary axis in the upper half plane. Recalling that $\det K(t) = \prod_{s\in\mathcal{E}} 4\cosh\left[(t+i\pi s)/2h\right]\cosh\left[(t-i\pi s)/2h\right]$, we know the positions of all poles and it follows from the integral representation (6.9) that the TBA kernels admit a series expansion of the form

$$\varphi_{ij}(\theta) = 16\, i \sum_{s\in\mathcal{E}} \operatorname*{Res}_{t\to i\pi s}\left[\sinh(\vartheta_h t)\sinh(t_j\vartheta_H t\,)\check{K}(t)_{ij}\,/\,\det K(t) e^{it\theta/\pi}\right] \ . \tag{6.10}$$

We do not have a closed formula for the cofactors $\check{K}$, but for the sake of our argument it will be sufficient here to present some examples.

### 6.2.1 $(G_2^{(1)}, D_4^{(3)})$-ATFT

Let us first compute (6.7) and (6.8). We carry out the computations in terms of the quantities of the untwisted algebra $G_2^{(1)}$ for which we have two simple roots $\vec{\alpha}_1$ and $\vec{\alpha}_2$ normalised as $\vec{\alpha}_2^2 = 2 = 3\vec{\alpha}_1^2$. Furthermore, the Weyl vector, its dual and the negative of the highest root are given by

$$\vec{\rho} = 5\vec{\alpha}_1 + 3\vec{\alpha}_2 \ , \qquad \vec{\rho}^\vee = 5\vec{\alpha}_1 + \vec{\alpha}_2 \quad \text{and} \quad \alpha_0 = -3\vec{\alpha}_1 - 2\vec{\alpha}_2 \ . \tag{6.11}$$

These realizations allow to compute the quantities needed in (6.7) and (6.8), that is $3\vec{\rho}^2 = 14$, $3\vec{\rho}^\vee.\vec{\rho}^\vee = 26$ and $3\vec{\rho}.\vec{\rho}^\vee = 3\vec{\rho}^\vee.\vec{\rho} = 8$. It follows therefore

$$c = 2 + 32\left[\frac{13 + 3B(B-3)}{B(2-B)}\right] \quad \text{and} \quad \Delta = \frac{3B+2}{B-2}. \tag{6.12}$$

Clearly for $0 < B < 2$ we have $-\infty \leq \Delta \leq -1$.

To proceed further we need the (generalized) Coxeter number for this theory, which are $h = 6$ and $H = 12$. The symmetrizers are $t_1 = 3$ and $t_2 = 1$. Evaluating (6.10) and reading off the first order coefficient we obtain

$$\varphi_{ab}^{(1)} = -8\sqrt{3}\frac{\sin\frac{(2-B)\pi}{12}\sin\frac{B\pi}{8}}{\cos\frac{\pi}{6}(1-\frac{B}{4})}\frac{m_a m_b}{m^2} \qquad a,b = 1,2, \tag{6.13}$$

where we normalized the masses to

$$m_1 = m\cos\frac{\pi}{6}(1+\frac{B}{4}) \qquad \text{and} \qquad m_2 = m\ . \tag{6.14}$$

We deduce then with (2.12)

$$\langle T_\mu^\mu \rangle = \frac{\pi m^2 \cos\frac{\pi}{6}(1-\frac{B}{4})}{4\sqrt{3}\sin\frac{(2-B)\pi}{12}\sin\frac{B\pi}{8}}. \tag{6.15}$$

Agreement with the results in [18] is achieved by changing to the conventions used in there. For this one needs to re-define the effective coupling to $B \to B' = 3B/(4+B)$ and introduce a "floating" Coxeter number $H' = (1-B')h + B'h^\vee$.

### 6.2.2 $(F_4^{(1)}, E_6^{(2)})$-ATFT

For $F_4^{(1)}$ we normalize the four simple roots to $\vec{\alpha}_1^2 = \vec{\alpha}_2^2 = 2\vec{\alpha}_3^2 = 2\vec{\alpha}_4^2 = 2$. The Weyl vector, its dual and the negative of the highest root are in this case given by

$$\begin{aligned} \vec{\rho} &= 16\vec{\alpha}_1 + 30\vec{\alpha}_2 + 42\vec{\alpha}_3 + 22\vec{\alpha}_4\ , \qquad \vec{\rho}^\vee = \vec{\rho} + 22\vec{\alpha}_4\ , \\ \text{and} \qquad \vec{\alpha}_0 &= -2\vec{\alpha}_1 - 3\vec{\alpha}_2 - 4\vec{\alpha}_3 - 2\vec{\alpha}_4 \end{aligned} \tag{6.16}$$

such that $\vec{\rho}^2 = 39$, $\vec{\rho}^\vee.\vec{\rho}^\vee = 402$ and $\vec{\rho}.\vec{\rho}^\vee = \vec{\rho}^\vee.\vec{\rho} = 55$. With this we find

$$c = 4 + 12\left[\frac{1608 + B(331B - 1388)}{(2-B)B}\right] \quad \text{and} \quad \Delta = \frac{16+B}{B-2}. \tag{6.17}$$

Therefore $-\infty \leq \Delta \leq -8$.

For this theory we have $h = 12$, $H = 18$, $t_1 = t_2 = 1$ and $t_3 = t_4 = 2$. The ratios between the masses of the four particles in the theory are

$$\frac{m_4}{m_1} = 2\sin\frac{\pi}{4}(1+\frac{B}{18}), \quad \frac{m_3}{m_1} = 1 + 2\cos\frac{\pi}{6}(1-\frac{B}{6}), \quad \frac{m_2}{m_1} = 2\cos\frac{\pi}{12}(1-\frac{B}{6}). \tag{6.18}$$

We choose the normalization such that $m_1 = m$ and obtain from (6.10)

$$\varphi_{ab}^{(1)} = -8\sqrt{3}\frac{\sin\frac{(2-B)\pi}{24}\sin\frac{B\pi}{18}}{\cos\frac{\pi}{4}(1-\frac{B}{18})}\frac{m_a m_b}{m^2} \qquad a,b = 1,2,3,4. \tag{6.19}$$

Therefore with (2.12) we get$894/4 = \frac{447}{2}$

$$\langle T^{\mu}_{\mu} \rangle = \frac{\pi m^2 \cos\frac{\pi}{4}(1-\frac{B}{18})}{4\sqrt{3}\sin\frac{(2-B)\pi}{24}\sin\frac{B\pi}{18}}. \tag{6.20}$$

We can match with the formulae in [18] by $B \to B' = 4B/(6+B)$, $\tilde{H} = 3(4-B')$, $m_1 \to m_1'$, $m_2 \to m_3'$, $m_3 \to m_4'$ and $m_4 \to m_2'$.

### 6.2.3 $(B_2^{(1)}, A_3^{(2)})$-ATFT

Let us now present the simplest example of the family $(B_\ell^{(1)}, A_{2\ell-1}^{(2)})$. In general, we choose for the algebra $B_\ell^{(1)}$ the normalizations $\vec{\alpha}_i^2 = 2$ for $i = 1, \ldots, \ell-1$ and $\vec{\alpha}_\ell^2 = 1$. Then we have

$$2\vec{\rho} = 3\vec{\alpha}_1 + 4\vec{\alpha}_2, \qquad 2\vec{\rho}^\vee = 3\vec{\alpha}_1 + 8\vec{\alpha}_2 \qquad \text{and} \qquad \vec{\alpha}_0 = -\vec{\alpha}_1 - 2\vec{\alpha}_2, \tag{6.21}$$

from which we compute $12\vec{\rho}^2 = 30$, $\vec{\rho}^\vee.\vec{\rho}^\vee = \vec{\rho}^2 + 72$ and $\vec{\rho}.\vec{\rho}^\vee = \vec{\rho}^2 + 4$. Therefore

$$c = 2 + 8\left[\frac{447 + 24B(4B-17)}{B(2-B)}\right] \quad \text{and} \quad \Delta = \frac{B+4}{B-2}\ .$$

Hence $-\infty \leq \Delta \leq -2$

For this theory we have $h = 4$, $H = 6$, $t_1 = 1$ and $t_2 = 2$. The masses satisfy

$$\frac{m_1}{m_2} = 2\sin\frac{\pi}{4}(1+\frac{B}{6}), \tag{6.22}$$

and we choose $m_1 = m$. Evaluating (6.10) we obtain now

$$\varphi_{ab}^{(1)} = -8\frac{\sin\frac{(2-B)\pi}{8}\sin\frac{B\pi}{6}}{\sin\frac{\pi}{4}(1+\frac{B}{4})}\frac{m_a m_b}{m^2} \qquad a,b = 1,2, \tag{6.23}$$

and therefore with (2.12)

$$\langle T^{\mu}_{\mu} \rangle = \frac{\pi m^2 \sin\frac{\pi}{4}(1+\frac{B}{4})}{4\sin\frac{(2-B)\pi}{8}\sin\frac{B\pi}{6}}. \tag{6.24}$$

Defining once more $B \to B' = 4B/(6+B)$ and $H = 4-B'$ we find agreement with [18]. The previous results also hold for the $(C_2^{(1)}, D_3^{(2)})$-theory by exchanging the roles of particles 1 and 2, since the Dynkin diagrams of $B_2^{(1)}$ and $C_2^{(1)}$ are identical up to the exchange of the short and the long root.

These examples are sufficient to support the validity of the approach outlined in section 2.

## 7. Conclusions

We used the thermodynamic Bethe ansatz to compute vacuum energies $\langle T^{\mu}_{\mu} \rangle$ for various types of perturbed conformal field theories. Despite the fact, that the models considered exhibit different general behaviours, the assumption i)-iii), needed for the validity of the approximations in the TBA, hold in all cases.

The general behaviour of $\langle T^\mu_\mu \rangle$ is shown to be sensitive to IR and UV-conterterms, whose presence can be characterized by the conformal scaling dimension $\Delta$ of the perturbing operator. In the regime $0 < \Delta < 1/2$, realized by minimal ATFT and $\mathbf{g}_k$-HSG models for $k > h^\vee$, the quantity $\langle T^\mu_\mu \rangle$ can be identified with the IR-counterterms needed to compensate the divergencies in the perturbative series expansion (1.2), when viewed on a cylinder. In contrast, in the regime $1/2 < \Delta < 1$, realized by $\mathbf{g}_k$-HSG models for $k < h^\vee$, the quantity $\langle T^\mu_\mu \rangle$ can be associated to the sum of the aforementioned IR counterterms and UV counterterms needed to guarantee the finiteness of the individual integrals in the expansion. In the models studied here these additional counterterms, when passing from $\Delta < 1/2$ to $\Delta > 1/2$ show up in a change of sign in $\langle T^\mu_\mu \rangle$. It would be extremely interesting to verify this assertion by some explicit perturbative computations for the HSG-models. For the regime $\Delta < 0$, realized here by the ATFT (simply laced as well as non-simply laced) $\langle T^\mu_\mu \rangle$ constitutes a mixture of several types of counterterms, less obvious to disentangle. The divergence of $\langle T^\mu_\mu \rangle$ at $\Delta = 1/2$ can be attributed to the occurrence of free Fermions, for which such type of behaviour is well known from explicit analytical expressions. However, we were not able to identify the free Fermions in all $\mathbf{g}_{h^\vee}$-HSG models, which can be viewed as perturbed CFT's with $\Delta = 1/2$. This needs further investigations.

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