SOME CANONICAL BASIS VECTORS IN THE BASIC $U_q(\widehat{\mathfrak{sl}}_n)$-MODULE

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1. Introduction

Lascoux, Leclerc, and Thibon [7] presented a fast algorithm for computing the canonical basis of the basic $U_q(\widehat{\mathfrak{sl}}_n)$-module. As far as we know, there is no known nonrecursive method to make this calculation.

We give an explicit formula for a certain class of canonical basis vectors which includes the entire canonical basis in a family of weight spaces of arbitrarily large dimension. We were led to this formula by our interest in Broué's abelian defect group conjecture in the representation theory of finite groups. As predicted by Rouquier in 1992 (see [11]), one can prove Broué's conjecture for certain blocks of the symmetric groups by showing that they are Morita equivalent to blocks of wreath products of symmetric groups [2]. These wreath products are analyzed in [3] and applied to the study of blocks of symmetric groups in [4]. They are the source of the formula we present here.

Putting $q = 1$ when $n = 2$ we recover Corollary 2.6 of [6] using Ariki's Theorem [1].

Let $\mathcal{P}$ be the set of partitions and let

$$\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Q}(q)\lambda$$

be the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$. We identify the basic representation $M(\Lambda_0)$ of $U_q(\widehat{\mathfrak{sl}}_n)$ with the submodule of $\mathcal{F}$ generated by the empty partition $\emptyset$. There is a distinguished basis $\{G(\sigma)\}$ of $M(\Lambda_0)$, called the canonical basis, which is indexed by n-regular partitions σ.

Let κ be an n-core partition with the following property: it may be displayed on a James n-abacus in which the number of beads on each runner is nondecreasing as we go from left to right; suppose that for $1 \le i \le n-1$, there are $d_i \ge 0$ more beads on the i-th runner than there are on the $(i-1)$-th runner. We consider partitions with 'locally small n-quotients': let $\mathcal{P}_\kappa$ be the set of partitions σ with n-core κ and n-quotient $(\sigma^0, \sigma^1, \dots, \sigma^{n-1})$ such that

(1) $|\sigma^{i-1}| + |\sigma^i| + |\sigma^{i+1}| \le d_i + 1$ for all $i = 1, 2, \dots, n-1$ (where $\sigma^n = \emptyset$);
(2) If $|\sigma^i| + |\sigma^{i+1}| = d_i + 1$ (where $\sigma^n = \emptyset$), then $\sigma^j = \emptyset$ for all $0 \le j < i$.

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Note that $\mathcal{P}_\kappa$ contains all the partitions with n-core κ and n-weight less than or equal to $v = 1 + \min_{1\leq i<n} d_i$.

The following is the main theorem of this paper:

Theorem 1.1. *Let σ be an n-regular partition in $\mathcal{P}_\kappa$. Then $G(\sigma) = \sum_\lambda d_{\lambda\sigma}(q)\lambda$, where the sum is over all partitions λ with n-core κ, and*

$$d_{\lambda\sigma}(q) = \sum_{\substack{\mu^1,\dots,\mu^{n-1}\\ \nu^0,\dots,\nu^{n-2}}} \left(\prod_{i=0}^{n-1} c^{(-1)^i\lambda^i}_{\mu^i\nu^i} \prod_{i=1}^{n-1} c^{(-1)^i\sigma^i}_{\nu^{i-1}\mu^i}\, q^{|\nu|} \right),$$

where $(\emptyset, \sigma^1, \dots, \sigma^{n-1})$ and $(\lambda^0, \lambda^1, \dots, \lambda^{n-1})$ are the n-quotients of σ and λ respectively, $\mu^0 = \nu^{n-1} = \emptyset$, $|\nu| = |\nu^0| + \cdots |\nu^{n-1}|$, and conjugation of partitions is indicated by multiplication by -1.

Example 1.2. Let $n = 5$ and $\kappa = (7, 3^2)$, so that $d_1 = d_2 = 0$ and $d_3 = d_4 = 1$. Then $\sigma = (12, 3^2, 2, 1^3) \in \mathcal{P}_\kappa$, with 5-quotient $(\emptyset, (1), \emptyset, \emptyset, (1))$, and

$$G(\sigma) = \sigma + q(12, 3^2, 1^5) + q(7^2, 4, 2, 1^3) + q^2(7^2, 4, 1^5).$$

Remark. By using the fact that $c^\alpha_{\beta\gamma} = 0$ unless $|\alpha| = |\beta| + |\gamma|$ and by replacing some of the dummy summation partitions with their conjugates, it is easy to see that

$$d_{\lambda\sigma}(q) = \sum_{\substack{\mu^1,\dots,\mu^{n-1}\\ \nu^0,\dots,\nu^{n-2}}} \left(\prod_{i=0}^{n-1} c^{\lambda^i}_{\mu^i\nu^i} \prod_{i=1}^{n-1} c^{\sigma^i}_{(\nu^{i-1})'\mu^i} \right) q^{\sum_{i=i}^{n-1} i(|\sigma^i|-|\lambda^i|)},$$

This better formulation is due to Leclerc and Miyachi [8], who obtained independently an analogous closed formula for partitions λ and σ (not necessarily n-regular) having n-core κ and n-weight not more than $v = 1 + \min_{1\leq i<n} d_i$. The main advantage of their expression is that it makes clear that $d_{\lambda\sigma}(q)$ is a monomial. At the same time they have avoided the use of the awkward notations of $(-1)^i\lambda$ and $(-1)^i\sigma$ (which arise naturally when results from the wreath products of symmetric groups are transferred to blocks of symmetric groups [4]).

Because Leclerc and Miyachi's formula is good for n-singular σ, they actually obtain a closed formula for the canonical basis in 'good' weight spaces of the whole of the Fock space, not just of the basic representation. In addition, they transfer their results to other weight spaces via 'Scopes isometries'; in doing so, they show that their formula is valid for many partitions whose n-cores are not of the form κ.

On the other hand, Theorem 1.1 covers the canonical basis vectors corresponding to n-regular partitions in Leclerc and Miyachi's 'good' weight spaces, as well as many others (like Example 1.2) lying in weight spaces which are not addressed in their presentation.

By Ariki's Theorem [1] the polynomials $d_{\lambda\sigma}(q)$ evaluated at $q = 1$ give decomposition numbers of Hecke algebras at complex n-th roots of unity. In [4] it is further shown that if p is prime and $n = p > v$ then these numbers also give p-decomposition numbers of symmetric groups, in accordance with a conjecture of James. Moreover the polynomials $d_{\lambda\sigma}(q)$ describe the Loewy series and Jantzen filtrations of Specht modules. In describing these connections we are reversing the direction of discovery; we just want to emphasize that the formula we present here arose from consideration of Broué's conjecture in the case of the symmetric groups.

This paper is organised as follows: in section 2, we introduce the background theory needed in this paper; then in section 3, we prove some lemmas and set up the machineries required to prove the main theorem in section 4.

2. Preliminaries

In this section, we give a brief account of the background theory which we require, and introduce some notations and conventions which will be used throughout this paper.

Let $\mathfrak{h}$ be an $(n+1)$-th dimensional $\mathbb{Q}$-vector space. The algebra $U_q(\widehat{\mathfrak{sl}}_n)$ may be defined as the associative algebra with 1 that is generated by $\{e_i, f_i \mid i = 0, 1, \ldots, n-1\} \cup \mathfrak{h}$ subject to certain relations, for which we refer the interested reader to [7, Section 4.1]. If k is a positive integer, write $f_i^{(k)}$ for $f_i^k/[k]!$, where

$$[k]! = [k][k-1]\cdots[1], \qquad [k] = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

Write $U_{\mathbb{Q}}^-$ for the sub-$\mathbb{Q}[q, q^{-1}]$-algebra generated by $f_i^{(k)}$ for all positive integers k and for all $i = 0, 1, \ldots, n-1$.

We identify the basic representation $M(\Lambda_0)$ of $U_q(\widehat{\mathfrak{sl}}_n)$ with the submodule of the Fock space $\mathcal{F}$ generated by the empty partition. The ring automorphism $P \mapsto \overline{P}$ of $U_q(\widehat{\mathfrak{sl}}_n)$, defined by

$$\overline{q} = q^{-1}, \quad \overline{q^h} = q^{-h}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i$$

for all $h \in \mathfrak{h}$ and $i = 0, 1, \ldots, n-1$, induces an involution $v \mapsto \overline{v}$ on $M(\Lambda_0)$ defined by

$$\overline{P\emptyset} = \overline{P}\emptyset$$

for all $P \in U_q(\widehat{\mathfrak{sl}}_n)$.

Let $A \subseteq \mathbb{Q}(q)$ be the subring of rational functions without pole at $q = 0$. The lower crystal lattice L of the Fock Space $\mathcal{F}$ is

$$L = \bigoplus_{\lambda \in \mathcal{P}} A\lambda.$$

There is a distinguished basis $\{G(\sigma)\}$ of $M(\Lambda_0)$, called the canonical basis or the lower global crystal basis, which is indexed by the set of all n-regular partitions σ and has the following characterisation (see, for example, [7, Theorem 6.1]):

(1) $G(\sigma) \in U_{\mathbb{Q}}^- \emptyset$.
(2) $G(\sigma) \equiv \sigma \pmod{qL}$.
(3) $\overline{G(\sigma)} = G(\sigma)$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ be a partition, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$. We write $|\lambda| = \sum_{i=1}^{s} \lambda_i$ and $l(\lambda) = s$. We denote its conjugate partition by $-\lambda$. This allows us to write $(-1)^i\lambda$ to mean λ if i is even and the conjugate of λ if i is odd. As the Fock space $\mathcal{F}$ has the set of partitions as a basis, there is some danger of confusion, but we will avoid this by using this notation only for partitions indexing Littlewood-Richardson coefficients.

Any partition may be displayed on a James n-abacus (see, for example, [5, Section 2.7]). This display is only unique when the number of beads used is fixed. In this paper, we fix an n-core κ and consider partitions whose n-core is κ. In the abacus display of any such partition we will always use $l(\kappa) + nW$ beads for a sufficiently large W. Note that a bead representing a nonzero part of the partition will always lie in the same runner regardless of W, and the end node corresponding to this bead has residue i if and only if it lies on the j-th runner, where $j \equiv l(\kappa) + i \pmod{n}$. In view of this, for $j = 0, 1, \ldots, n-1$, we define an element $\mathfrak{f}_j$ of $U_q(\widehat{\mathfrak{sl}}_n)$ by $\mathfrak{f}_j = f_i$ where $j \equiv l(\kappa) + i \pmod{n}$. If λ is a partition with n-core κ, we shall also denote its n-quotient by $(\lambda^0, \lambda^1, \ldots, \lambda^{n-1})$, where λ^i is the partition read off from the i-th runner of the abacus display of λ.

We call a bead in an abacus display addable if the position to its immediate right is unoccupied (if the bead is in the rightmost runner we mean the position a row lower in the leftmost runner). We write $\lambda \xrightarrow{i:k} \mu$ if the abacus display of μ is obtained from that of λ by sliding k addable beads on the $(i-1)$-th runner of the display of λ into the i-th runner. Let B_k denote the set of k beads which are moved. We define $N(\lambda,\mu)^+$ and $N(\lambda,\mu)^-$ as follows:

$$N(\lambda,\mu)^+ = \sum_c \#\{b \in B_k \mid b \text{ is above } c\},$$
$$N(\lambda,\mu)^- = \sum_d \#\{b \in B_k \mid b \text{ is above } d\},$$

where the sums are over all beads $c \notin B_k$ on the $(i-1)$-th runner of λ, and over all beads d on the i-th runner of λ respectively.

We then define

$$N(\lambda,\mu) = N(\lambda,\mu)^+ - N(\lambda,\mu)^-.$$

We write $N(\xrightarrow{i:k})$, $N(\xrightarrow{i:k})^+$, and $N(\xrightarrow{i:k})^-$ for $N(\lambda,\mu)$, $N(\lambda,\mu)^+$, and $N(\lambda,\mu)^-$, respectively, when it is clear from the context what λ and μ are.

We do not need the explicit action of each element of $U_q(\widehat{\mathfrak{sl}}_n)$ on the Fock space $\mathcal{F}$ in this paper, but only the following result:

Lemma 2.1. Let λ be a partition with n-core κ and let k be a positive integer. Viewing λ as an element in the Fock space $\mathcal{F}$, we have

$$\mathfrak{f}_i^{(k)}(\lambda) = \sum_{\lambda \xrightarrow{i:k} \mu} q^{N(\lambda,\mu)} \mu.$$

Proof. This is just a recasting of [10, Lemma 6.16]. □

Let $\mathfrak{S}_r$ denote the symmetric group of degree r and let χ^λ be the irreducible character of $\mathfrak{S}_r$ corresponding to a partition λ of r. If $s + t = r$ and μ and ν are partitions of s and t, let $c^\lambda_{\mu\nu}$ denote the multiplicity of χ^λ in $\mathrm{Ind}_{\mathfrak{S}_s \times \mathfrak{S}_t}^{\mathfrak{S}_r}(\chi^\mu \otimes \chi^\nu)$. We refer the reader to [9, I.9] for a combinatorial description of $c^\lambda_{\mu\nu}$, which is known as a Littlewood-Richardson coefficient. By convention, we define $c^\lambda_{\mu\nu}$ to be 0 when $|\lambda| \neq |\mu| + |\nu|$.

We gather some useful results involving Littlewood-Richardson coefficients:

Lemma 2.2. Let λ, γ, δ and τ be partitions.

(1) $c^{\lambda}_{\gamma\delta} = c^{\lambda}_{\delta\gamma} = c^{-\lambda}_{(-\gamma)(-\delta)}$.
(2) $\sum_{\alpha\in\mathcal{P}} c^{\lambda}_{\alpha\gamma} c^{\alpha}_{\tau\delta} = \sum_{\alpha\in\mathcal{P}} c^{\lambda}_{\alpha\delta} c^{\alpha}_{\tau\gamma} = \sum_{\alpha\in\mathcal{P}} c^{\lambda}_{\alpha\tau} c^{\alpha}_{\gamma\delta}$.
(3) Let k be a positive integer. Then

$$\sum_{j=0}^{k} \sum_{\mu,\nu} c^{\tau}_{\mu\nu} c^{\gamma}_{\nu(j)} c^{\delta}_{\mu(k-j)} = \sum_{\eta} c^{\eta}_{\tau(k)} c^{\eta}_{\gamma\delta},$$

where μ, ν and η runs over all partitions in their respective sums.

Proof. (1) is straightforward.

For (2), observe that each sum in the equation gives the multiplicity of χ^{λ} in $\mathrm{Ind}^{\mathfrak{S}_{|\lambda|}}_{\mathfrak{S}_{|\gamma|}\times\mathfrak{S}_{|\delta|}\times\mathfrak{S}_{|\tau|}}(\chi^{\gamma}\otimes\chi^{\delta}\otimes\chi^{\tau})$.

For (3), set $s = |\gamma|$ and $t = |\delta|$. It's easy to check that both sides are zero unless $|\tau| = s + t - k$. Note also that if $c^{\gamma}_{\nu(j)} > 0$, then $j \le s$, and if $c^{\delta}_{\mu(k-j)} > 0$, then $j \ge k - t$.

We consider the symmetric group $\mathfrak{S}_{s+t}$ along with Young subgroups $\mathfrak{S}_s \times \mathfrak{S}_t$ and $\mathfrak{S}_k \times \mathfrak{S}_{s+t-k}$. We have, by Mackey's formula,

$$\begin{aligned}&\mathrm{Res}^{\mathfrak{S}_{s+t}}_{\mathfrak{S}_s\times\mathfrak{S}_t}\mathrm{Ind}^{\mathfrak{S}_{s+t}}_{\mathfrak{S}_k\times\mathfrak{S}_{s+t-k}}(\chi^{(k)}\otimes\chi^{\tau}) = \\ &\quad\sum_{g}\mathrm{Ind}^{\mathfrak{S}_s\times\mathfrak{S}_t}_{g(\mathfrak{S}_k\times\mathfrak{S}_{s+t-r})g^{-1}\cap\mathfrak{S}_s\times\mathfrak{S}_t}\mathrm{Res}^{g(\mathfrak{S}_k\times\mathfrak{S}_{s+t-k})g^{-1}}_{g(\mathfrak{S}_k\times\mathfrak{S}_{s+t-k})g^{-1}\cap\mathfrak{S}_s\times\mathfrak{S}_t} g(\chi^{(k)}\otimes\chi^{\tau}),\end{aligned}$$

where g runs over an appropriate set of double coset representatives. We can take this set to be $\{g_j\}$, where for $\max(0, k-t) \le j \le \min(k, s)$, we choose g_j to be an element of $\mathfrak{S}_{s+t}$ conjugating $\mathfrak{S}_j \times \mathfrak{S}_{k-j} \times \mathfrak{S}_{s-j} \times \mathfrak{S}_{t-k-j}$ to $\mathfrak{S}_j \times \mathfrak{S}_{s-j} \times \mathfrak{S}_{k-j} \times \mathfrak{S}_{t-k-j}$. Thus

$$\begin{aligned}&\mathrm{Res}^{\mathfrak{S}_{s+t}}_{\mathfrak{S}_s\times\mathfrak{S}_t}\mathrm{Ind}^{\mathfrak{S}_{s+t}}_{\mathfrak{S}_k\times\mathfrak{S}_{s+t-k}}(\chi^{(k)}\otimes\chi^{\tau}) = \\ &\sum_{j=\max(0,k-t)}^{\min(k,s)}\mathrm{Ind}^{\mathfrak{S}_s\times\mathfrak{S}_t}_{\mathfrak{S}_j\times\mathfrak{S}_{s-j}\times\mathfrak{S}_{t-j}\times\mathfrak{S}_{t-k-j}}\mathrm{Res}^{\mathfrak{S}_k\times\mathfrak{S}_{s+t-k}}_{\mathfrak{S}_j\times\mathfrak{S}_{k-j}\times\mathfrak{S}_{s-j}\times\mathfrak{S}_{t-k-j}}(\chi^{(k)}\otimes\chi^{\tau}).\end{aligned}$$

By computing, in terms of Littlewood-Richardson coefficients, the multiplicity of $\chi^{\gamma}\otimes\chi^{\delta}$ on both sides of this equation, we arrive at the desired equality. □

3. Setup

We set up the machineries required to prove the main theorem in this section.

We first define some useful elements of $U^{-}_{\mathbb{Q}}$. For $1 \le a \le n-1$ and $k \ge 1$, let

$$F_{a,k} = \mathfrak{f}^{(k)}_a \mathfrak{f}^{(k)}_{a+1} \cdots \mathfrak{f}^{(k)}_{n-2} \mathfrak{f}^{(k)}_{n-1} \mathfrak{f}^{(k)}_{a-1} \mathfrak{f}^{(k)}_{a-2} \cdots \mathfrak{f}^{(k)}_{1} \mathfrak{f}^{(k)}_{0} \in U^{-}_{\mathbb{Q}}.$$

Lemma 3.1. Let λ be a partition with n-core κ and n-quotient $(\lambda^0, \lambda^1, \ldots, \lambda^{n-1})$. In the abacus display of λ, for each $i = 0, 1, \ldots, n-1$, let the bottom bead and the top empty position on the i-th runner lie on rows b_i and g_i respectively. Suppose that, for some integer a with $1 \leq a \leq n-1$, we have

$$b_i < g_{a-1} \quad \text{for all } 0 \leq i \leq a-2; \tag{1}$$

$$b_a < g_j \quad \text{for all } a+1 \leq j \leq n-1; \tag{2}$$

$$b_{a-1} < g_a \tag{3}$$

Let $1 \leq k \leq g_a - b_{a-1}$. Then

$$F_{a,k}(\lambda) = \sum_{j=0}^{k} q^j \left(\sum_{\alpha,\beta} \lambda(\alpha,\beta) \right),$$

where α runs over all partitions obtained by adding j nodes in distinct columns of λ^{a-1} and β runs over all partitions obtained by adding $k-j$ nodes in distinct rows of λ^a; and $\lambda(\alpha,\beta)$ denotes the partition with n-core κ and n-quotient $(\lambda^0,\dots,\lambda^{a-2},\alpha,\beta,\lambda^{a+1},\dots,\lambda^{n-1})$.

Remark. We may rewrite this formula using Littlewood-Richardson coefficients:

$$F_{a,k}(\lambda) = \sum_{j=0}^{k} q^j \left(\sum_{\alpha,\beta\in\mathcal{P}} c^{\alpha}_{\lambda^{a-1}(j)} c^{\beta}_{\lambda^a(1^{k-j})} \lambda(\alpha,\beta) \right).$$

Proof. By using Lemma 2.1 repeatedly, we have

$$F_{a,k}(\lambda) = \sum q^{?} \lambda(a),$$

where the sum is over all sequences $(\lambda(0),\lambda(1),\dots,\lambda(a-1),\lambda(n-1),\lambda(n-2),\dots,\lambda(a+1),\lambda(a))$ such that

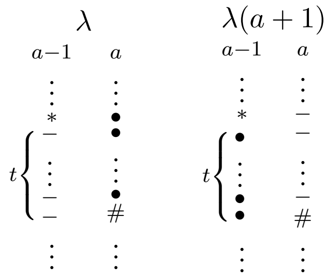
$$\lambda \xrightarrow{0:k} \lambda(0) \xrightarrow{1:k} \lambda(1) \xrightarrow{2:k} \cdots \xrightarrow{a-1:k} \lambda(a-1)$$
$$\xrightarrow{n-1:k} \lambda(n-1) \xrightarrow{n-2:k} \lambda(n-2) \xrightarrow{n-3:k} \cdots \xrightarrow{a+1:k} \lambda(a+1) \xrightarrow{a:k} \lambda(a),$$

and the exponent of q is $\sum_{i=0}^{n-1} N(\xrightarrow{i:k})$.

Note that, to obtain $\lambda(a-1)$ from λ, we first move k addable beads from the $(n-1)$-th to the 0-th runner, and then we move k addable beads from the 0-th runner to the 1-st runner, and so on, continuing this way until we move k addable beads from the $(a-2)$-th runner to the $(a-1)$-th runner. A little thought shows that, by condition (1), the k beads moved from the $(n-1)$-th runner of λ must lie on or below row $g_{a-1}-1$, and that they are moved to the 0-th runner (each moving down a row when doing so), then to the 1-st runner, and so on, until they reach the $(a-1)$-th runner.

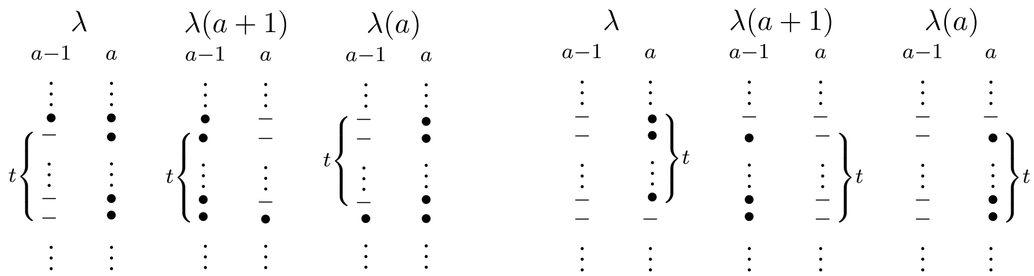
Now, to obtain $\lambda(a+1)$ from $\lambda(a-1)$, we first move k addable beads from the $(n-2)$-th runner to the $(n-1)$-th runner, and then k addable beads from the $(n-3)$-th runner to the $(n-2)$-th runner, and so on, continuing this way until we move k addable beads from the a-th runner to the $(a+1)$-th runner. By condition (2), the k addable beads from the $(n-1)$-th runner of λ that were moved must in fact be lying on or above row b_a; moving these beads frees up k positions on that runner and this get filled up by beads on their left, and this in turn frees up k positions which get filled up by beads on their left, and so on, until k beads from the a-th runner are moved to their right.

Thus the net result of getting from λ to $\lambda(a+1)$ is to move k beads from the a-th runner between row $g_{a-1}-1$ and b_a (both inclusive) of λ to the $(a-1)$-th runner, with each bead moving down a row when doing so. The following diagram illustrates a string of t $(\leq k)$ consecutive beads on the a-th runner of λ that is moved to the $(a-1)$-th runner of $\lambda(a+1)$ (we assume that the bead, if any, that is immediately above or below the string is not moved).



Note that each bead on the $(a-1)$-th runner of λ is not addable by condition (3) and remains not addable in $\lambda(a+1)$, unless it is a bead at position $*$ as shown in the diagram. Moreover, since $t \leq k \leq b_a - g_{a-1}$, if there is a bead at position $*$, then there is also be a bead at position $\#$ in the diagram, unless $t = k = g_a - b_{a-1}$.

We consider first the case where there is a bead at position $\#$ whenever there is a bead at position $*$. Then each string of t consecutive beads moved makes at most t beads on the $(a-1)$-th runner of $\lambda(a+1)$ addable. Thus to obtain $\lambda(a)$, each string of t beads moved must make exactly t beads on the $(a-1)$-th runner of $\lambda(a+1)$ addable. This means that the positions marked by $*$ and $\#$ are either both occupied or both empty. These two possible scenarios are illustrated in the diagrams below:



Thus, to get from λ to $\lambda(a)$, we make a combination of moves of the following types:

I. on the $(a-1)$-th runner, move a bead of λ, that is immediately above a string of t empty positions of λ, t positions down the runner;
II. on the a-th runner, move a string of t beads of λ, that is immediately above an empty position of λ, each a position down the runner.

Taken together the Type I moves correspond to adding nodes to distinct columns of λ^{a-1}, while the type II moves correspond to adding nodes in distinct rows to λ^a.

Next we consider the exceptional case, in which there is a bead at position $*$ and the position $\#$ is empty. In this case, $t = k = g_a - b_{a-1}$ and we have a string of $k+1$ addable beads on the $(a-1)$-th runner of $\lambda(a+1)$. To get from $\lambda(a+1)$ to $\lambda(a)$ we move k of these beads into the a-th runner. If the first of these $k+1$ beads is not moved, then $\lambda(a)$ is obtained from λ by a type II move. If the last of these $k+1$ beads is not moved, then $\lambda(a)$ is obtained from λ by a type I move. If any of the middle beads is not moved, then $\lambda(a)$ is obtained from λ by a type I move together with a type II move.

Thus we have seen that in general $\lambda(a)$ is obtained by a combination of type I and type II moves.

We now show that any $\lambda(\alpha,\beta)$ (which can be obtained from λ by a series of type I and type II moves) occurs as a $\lambda(a)$. For each type I move of a bead from λ to obtain $\lambda(\alpha,\beta)$, from row r to s say, note that, for λ, the positions between row r and row $s-1$ (both inclusive) on runners l for $l \geq a$ are all occupied by condition (2), while the positions between row $r+1$ to s (both inclusive) on runners m for $m < a$ are all empty by conditions (1) and (3). Similarly, for each type II move of a string of t beads of λ, starting from row r say, each down a position to obtain $\lambda(\alpha,\beta)$, note that, for λ, the positions between row r and row $r+t-1$ (both inclusive) on runners l for $l \geq a$ are all occupied by condition (2), while the positions between row $r+1$ to $r+t$ (both inclusive) on runners m for $m < a$ are all empty by conditions (1) and (3). Thus $\lambda(\alpha,\beta)$ will be expressible as $\lambda(a)$ as long as the ending position of each type I move lies above the ending position of the top bead of each type II move, but this is true by our choice of k.

We now compute the exponent of q.

For $i \neq a-1$, the contributions of each position on the i-th runner in the sums defining $N(\xrightarrow{i+1:k})^+$ and $N(\xrightarrow{i:k})^-$ are equal. This is also true for $i = a-1$, except for the positions occupied by the top and bottom left beads of $\lambda(a+1)$ in the diagram of the first scenario; the bottom position is not occupied in $\lambda(a-2)$ so does not contribute to $N(\xrightarrow{a-1:k})^-$, while the top position does not contribute to $N(\xrightarrow{a:k})^+$ as the bead of $\lambda(a+1)$ at this position is being moved, and the contribution by the bottom position to $N(\xrightarrow{a:k})^+$ is t more than that by the top position to $N(\xrightarrow{a-1:k})^-$ since the bottom position is below t more beads which are moved than the top position is. We thus conclude that the coefficient of $\lambda(\alpha,\beta)$ in the sum is $q^{|\alpha|-|\lambda^{a-1}|}$. □

Let the n-core partition κ be displayed on an abacus with $l(\kappa)+nW$ beads, where W is chosen to be 'sufficiently large'. Suppose that on this abacus display, there are r_i beads on its i-th runner $(0 \leq i < n)$. For each $i = 1,\ldots,n-1$, assume that $d_i = r_i - r_{i-1} \geq 0$.

Lemma 3.2. Let σ and λ be partitions with n-core κ, and n-quotients $(\sigma^0,\sigma^1,\ldots,\sigma^{n-1})$ and $(\lambda^0,\lambda^1,\ldots,\lambda^{n-1})$ respectively, such that for some partitions $\mu_1,\ldots,\mu_{n-1}$ and $\nu_0,\ldots,\nu_{n-2}$,

$$\prod_{i=0}^{n-2} c^{(-1)^i\lambda^i}_{\mu^i\nu^i} \prod_{i=1}^{n-1} c^{(-1)^i\sigma^i}_{\nu^{i-1}\mu^i} \neq 0,$$

where $\mu^0 = \nu^{n-1} = \emptyset$. Then for any integer a such that $1 \leq a \leq n-1$, the top empty position on the a-th runner of λ lies at least $d_a + 1 - |\sigma^{a-1}| - |\sigma^a| - |\sigma^{a+1}|$ rows below the bottom bead of the $(a-1)$-th runner, and at least $d_a - |\sigma^a| - |\sigma^{a+1}|$ rows below the top empty position on the $(a-1)$-th runner.

Proof. The bottom bead on the $(a-1)$-th runner of λ lies on row $r_{a-1}+\lambda_1^{a-1}$, while for $i = a, a-1$, the top empty position on the i-th runner is on row $r_i + 1 - l(\lambda^i)$. Thus, the top empty position on the a-th runner is $d_a + 1 - l(\lambda^a) - \lambda_1^{a-1}$ and $d_a - l(\lambda^a) + l(\lambda^{a-1})$ rows below the bottom bead and the top empty position of the $(a-1)$-th runner respectively. Now,

$$\begin{aligned}\lambda_1^{a-1} + l(\lambda^a) &\leq |\lambda^{a-1}| + |\lambda^a| \\ &= |\mu^{a-1}| + |\nu^{a-1}| + |\mu^a| + |\nu^a| \\ &\leq |\nu^{a-2}| + |\mu^{a-1}| + |\nu^{a-1}| + |\mu^a| + |\nu^a| + |\mu^{a+1}| \\ &= |\sigma^{a-1}| + |\sigma^a| + |\sigma^{a+1}|;\end{aligned}$$

$$\begin{aligned}l(\lambda^a) - l(\lambda^{a-1}) &\leq |\lambda^a| \\ &= |\mu^a| + |\nu^a| \\ &\leq |\nu^{a-1}| + |\mu^a| + |\nu^a| + |\mu^{a+1}| \\ &= |\sigma^a| + |\sigma^{a+1}|.\end{aligned}$$

The lemma thus follows. □

Corollary 3.3. *Let σ and let λ be partitions satisfying the hypothesis in Lemma 3.2. Let i be an integer such that $\sigma^l = \emptyset$ for all $0 \leq l < i$.*

(1) *If $|\sigma^i| + |\sigma^{i+1}| + |\sigma^{i+2}| \leq d_{i+1} + 1$ and $|\sigma^j| + |\sigma^{j+1}| \leq d_j$ for all $i+2 \leq j < n$, then the top $r_i + \lambda_1^i - 1$ positions on each runner of λ to the right of the i-th runner are all occupied.*
(2) *If $|\sigma^i| + |\sigma^{i+1}| \leq d_i$, then the top $r_i - l(\lambda^i)$ positions on each runner of λ to the left of the i-th runner contains all the beads in that runner.*

Proof.

(1) Note that the bottom bead on the i-th runner of λ lies on row $r_i + \lambda_1^i$. Since $|\sigma^i| + |\sigma^{i+1}| + |\sigma^{i+2}| \leq d_{i+1} + 1$, we see that the top $r_i + \lambda_1^i - 1$ positions on the $(i+1)$-th runner are all occupied by Lemma 3.2. Since $|\sigma^j| + |\sigma^{j+1}| \leq d_j$ for all $j \geq i+2$, Lemma 3.2 shows that the top $r_i + \lambda_1^i - 1$ positions all occupied on each j-th runner for $j \geq i+2$.
(2) Note that the top empty position on the i-th runner of λ lies on row $r_i - l(\lambda_i) + 1$. Since $\sigma^{i-1} = \emptyset$, and $|\sigma^i| + |\sigma^{i+1}| \leq d_i$, Lemma 3.2 shows that the bottom bead on the $(i-1)$-th runner lies above row $r_i - l(\lambda_i) + 1$. Since $\sigma^l = \emptyset$ and $d_l \geq 0$ for all $l < i$, we draw the same conclusion for the other runners on the left of the i-th runner.

□

In view of the above lemma, we consider the set of partitions σ with n-core κ and n-quotient $(\sigma^0, \sigma^1, \ldots, \sigma^{n-1})$ such that

(1) $|\sigma^{i-1}| + |\sigma^i| + |\sigma^{i+1}| \leq d_i + 1$ for all $i = 1, 2, \ldots, n-1$ (where $\sigma^n = \emptyset$);
(2) if $|\sigma^i| + |\sigma^{i+1}| = d_i + 1$ (where $\sigma^n = \emptyset$), then $\sigma^j = \emptyset$ for all $0 \leq j < i$.

We denote this set by $\mathcal{P}_\kappa$.

Remark. All partitions with n-core κ and n-weight less than or equal to $v = 1 + \min_{1\le i<n} d_i$ are elements of $\mathcal{P}_\kappa$.

Lemma 3.4. Let $\sigma \in \mathcal{P}_\kappa$. Then σ is n-regular if, and only if, $\sigma^0 = \emptyset$.

Proof. Let i be the least index such that $\sigma^i \neq \emptyset$.

If $\sigma^0 = \emptyset$, then $i > 0$, and it's clear that the top r_0 positions on each runner to the left of the i-th runner are all occupied. By Lemma 3.2 and condition (1) of $\mathcal{P}_\kappa$, the top $r_0 - 1$ positions on the i-th runner are all occupied. By condition (2) of $\mathcal{P}_\kappa$, we see that $|\sigma^l| + |\sigma^{l+1}| \le d_l$ for all $i < l \le n-1$. Hence Lemma 3.2 shows that the top $r_0 - 1$ positions on each runner to the right of the i-th runner are all occupied. This shows that σ is n-regular.

Conversely, if $\sigma^0 \neq \emptyset$, then Corollary 3.3 applied with $\sigma = \lambda$ shows that the top $r_0 + \sigma^0_1 - 1$ positions on each runner j for $j > 0$ are all occupied. Thus σ is n-singular. □

4. Proof of main theorem

We are now able to prove the main theorem. We first state a proposition and describe how to deduce the theorem from it.

For $\sigma \in \mathcal{P}_\kappa$, let

$$H(\sigma) = \sum_{\lambda} \sum_{\substack{\mu^1,\dots,\mu^{n-1}\\ \nu^0,\dots,\nu^{n-2}}} \left(\prod_{i=0}^{n-1} c^{(-1)^i\lambda^i}_{\mu^i\nu^i} \prod_{i=1}^{n-1} c^{(-1)^i\sigma^i}_{\nu^{i-1}\mu^i}\, q^{|\nu|} \right) \lambda \in \mathcal{F}.$$

Proposition 4.1. Let $\tau \in \mathcal{P}_\kappa$ be an n-regular partition with n-quotient $(\emptyset, \tau^1, \dots, \tau^{n-1})$ such that $|\tau^i| + |\tau^{i+1}| \le d_i$ for all $i = 1, 2, \dots, n-1$ (where $\tau^n = \emptyset$). Let a and k be positive integers such that $\tau^i = \emptyset$ for all $i < a$, and $k \le d_a + 1 - |\tau^{a+1}| - |\tau^a|$. Then

$$F_{a,k}(H(\tau)) = \sum_{\eta} H(\tau_\eta)$$

where the sum is over all partitions η which can be obtained by adding k nodes in distinct columns of τ^a, and τ_η denotes the partition with n-core κ and n-quotient $(\emptyset, \tau^1, \dots, \tau^{a-1}, \eta, \tau^{a+1}, \dots, \tau^{n-1})$.

Remark.

(1) Note that if $H(\tau_\eta)$ is a summand in the above sum, then τ_η is an n-regular partition in $\mathcal{P}_\kappa$.

(2) We can rewrite the sum as

$$F_{a,k}(H(\tau)) = \sum_{\eta \in \mathcal{P}} c^{\eta}_{\tau^a(1^k)} H(\tau_\eta).$$

Proof of Theorem 1.1 using Proposition 4.1. We need to show that $H(\sigma) \in U^-_{\mathbb{Q}}\emptyset$, $H(\sigma) \equiv \sigma \pmod{qL}$, and $\overline{H(\sigma)} = H(\sigma)$.

In the definition of $H(\sigma)$, suppose we have a term in the sum which is nonzero modulo qL. Then the exponent of q must be 0, so we need $\nu^i = \emptyset$ for $i = 0, 1, \dots, n-2$. Then for each of the Littlewood-Richardson coefficients in the product to be nonzero we must have $\lambda^i = (-1)^i\mu^i = \sigma^i$ for $i = 0, 1, \dots, n-1$. So there is only one term which is nonzero modulo qL, and it is equal to σ.

We prove that $H(\sigma) \in U_{\mathbb{Q}}^{-}\emptyset$ and $\overline{H(\sigma)} = H(\sigma)$ by induction on σ according to a total order $\prec$ on the set of n-regular partitions of $\mathcal{P}_\kappa$ which we define as follows: $\sigma \prec \tau$ if either of the following holds:

- σ has n-weight strictly less that the n-weight of τ
- σ and τ have the same n-weight and there exists some a such that $\sigma^a < \tau^a$ in the lexicographic order and such that $\sigma^i = \tau^i$ for $i = 1, \ldots, a-1$.

The smallest element in this order is the n-core κ itself, and it's clear that $H(\kappa) = \kappa$ is an element of the canonical basis (see, for example, Theorem 6.8 of [7]).

So assume σ has positive n-weight. Let a be the least integer such that $\sigma^a \neq \emptyset$. Let τ^a be the partition obtained by removing the last column of σ^a, and let τ be the partition with n-core κ and n-quotient $(\emptyset, \sigma^1, \ldots, \sigma^{a-1}, \tau^a, \sigma^{a+1}, \ldots, \sigma^{n-1})$. Because τ has n-weight strictly less than that of σ, we have $\tau \prec \sigma$. Hence by induction hypothesis, $H(\tau) \in U_{\mathbb{Q}}^{-}\emptyset$ and $\overline{H(\tau)} = H(\tau)$.

Let k be the number of nodes in the last column of σ^a. As $\overline{F_{a,k}} = F_{a,k}$, we have $\overline{F_{a,k}(H(\tau))} = F_{a,k}(H(\tau))$. Now since $\sigma \in \mathcal{P}_\kappa$, we see that τ, a and k satisfy the hypotheses of Proposition 4.1. So applying Proposition 4.1 to τ, we see that the partitions τ_η appearing in the sum all have n-weight equal to the n-weight of σ. Moreover the partitions η which occur satisfy $\eta \leq \sigma^a$, and σ^a occurs exactly once. Therefore

$$F_{a,k}(H(\tau)) = H(\sigma) + \sum_{\tilde{\sigma}} H(\tilde{\sigma})$$

where the sum is over some partitions $\tilde{\sigma}$ satisfying $\tilde{\sigma} \prec \sigma$. By induction hypothesis, we have $H(\tilde{\sigma}) \in U_{\mathbb{Q}}^{-}\emptyset$ and $\overline{H(\tilde{\sigma})} = H(\tilde{\sigma})$ for each of these $\tilde{\sigma}$. We conclude that $H(\sigma) \in U_{\mathbb{Q}}^{-}\emptyset$ and $\overline{H(\sigma)} = H(\sigma)$ as desired. $\square$

It remains, then, to prove Proposition 4.1. Using Lemma 3.2 and corollary 3.3 , we see each λ with nonzero coefficient in $H(\tau)$ satisfies the hypothesis of Lemma 3.1, and after applying this lemma to each such λ, the lefthand side of the equation in the Proposition becomes

$$F_{a,k}(H(\tau)) = \sum_{\lambda^0,\ldots,\lambda^{n-1}} \sum_{\substack{\mu^1,\ldots,\mu^{n-1} \\ \nu^0,\ldots,\nu^{n-2}}} \sum_{j=0}^{k} \sum_{\alpha,\beta} \left(\prod_{i=0}^{n-1} c_{\mu^i\nu^i}^{(-1)^i\lambda^i} \prod_{i=1}^{n-1} c_{\nu^{i-1}\mu^i}^{(-1)^i\tau^i} \right)$$
$$c_{\lambda^{a-1}(j)}^{\alpha} c_{\lambda^a(1^{k-j})}^{\beta} q^{|\nu|+j} \lambda(\alpha,\beta)$$

Let ρ be a partition with n-core κ and n-quotient $(\rho^0, \rho^1, \ldots, \rho^{n-1})$. Then the coefficient of ρ in the above sum is

$$\sum_{\substack{\mu^1,\ldots,\mu^{n-1} \\ \nu^0,\ldots,\nu^{n-2}}} \sum_{j=0}^{k} \sum_{\gamma,\delta} \left(\prod_{\substack{0\leq i\leq n-1 \\ i\neq a,a-1}} c_{\mu^i\nu^i}^{(-1)^i\rho^i} \prod_{i=1}^{n-1} c_{\nu^{i-1}\mu^i}^{(-1)^i\tau^i} \right) c_{\gamma(j)}^{\rho^{a-1}} c_{\delta(1^{k-j})}^{\rho^a} c_{\mu_{a-1}\nu_{a-1}}^{(-1)^{a-1}\gamma} c_{\mu_a\nu_a}^{(-1)^a\delta} q^{|\nu|+j}$$

On the other hand, the coefficient of ρ in the righthand side of the equation in the Proposition is

$$\sum_{\eta} c^{\eta}_{\tau^a(1^k)} \sum_{\substack{\mu^1,\dots,\mu^{n-1}\\ \nu^0,\dots,\nu^{n-2}}} \left(\prod_{i=0}^{n-1} c^{(-1)^i\rho^i}_{\mu^i\nu^i} \prod_{\substack{1\le i\le n-1\\ i\ne a}} c^{(-1)^i\tau^i}_{\nu^{i-1}\mu^i} \right) c^{\eta}_{\nu^{a-1}\mu^a} q^{|\nu|}.$$

Thus it suffices to show that

$$(*) \quad \sum_{j=0}^{k}\sum_{\gamma,\delta}\sum_{\mu^a,\nu^{a-1}} c^{(-1)^a\tau^a}_{\nu^{a-1}\mu^a} c^{\rho^{a-1}}_{\gamma(j)} c^{\rho^a}_{\delta(1^{k-j})} c^{(-1)^{a-1}\gamma}_{\mu^{a-1}\nu^{a-1}} c^{(-1)^a\delta}_{\mu^a\nu^a} q^{|\nu^{a-1}|+j} = \sum_{\eta}\sum_{\mu^a,\nu^{a-1}} c^{(-1)^{a-1}\rho^{a-1}}_{\mu^{a-1}\nu^{a-1}} c^{(-1)^a\rho^a}_{\mu^a\nu^a} c^{\eta}_{\tau^a(1^k)} c^{\eta}_{\nu^{a-1}\mu^a} q^{|\nu^{a-1}|}.$$

Note that

$$\begin{aligned}\sum_{\gamma} c^{\rho^{a-1}}_{\gamma(j)} c^{(-1)^{a-1}\gamma}_{\mu^{a-1}\nu^{a-1}} &= \sum_{\gamma} c^{(-1)^{a-1}\rho^{a-1}}_{\gamma,(-1)^{a-1}(j)} c^{\gamma}_{\mu^{a-1}\nu^{a-1}}\\ &= \sum_{\gamma} c^{(-1)^{a-1}\rho^{a-1}}_{\mu^{a-1}\gamma} c^{\gamma}_{(-1)^{a-1}(j),\nu^{a-1}},\end{aligned}$$

using Lemma 2.2(1,2). Similarly,

$$\sum_{\delta} c^{\rho^a}_{\delta(1^{k-j})} c^{(-1)^a\delta}_{\mu^a\nu^a} = \sum_{\delta} c^{(-1)^a\rho^a}_{\nu^a\delta} c^{\delta}_{(-1)^a(1^{k-j}),\mu^a}.$$

Thus $(*)$ is true if and only if the following equation holds:

$$\begin{aligned}\sum_{j=0}^{k}\sum_{\gamma,\delta}\sum_{\mu^a,\nu^{a-1}} c^{(-1)^a\tau^a}_{\nu^{a-1}\mu^a} c^{(-1)^{a-1}\rho^{a-1}}_{\mu^{a-1}\gamma} c^{(-1)^a\rho^a}_{\nu^a\delta} c^{\gamma}_{(-1)^{a-1}(j),\nu^{a-1}} c^{\delta}_{(-1)^a(1^{k-j}),\mu^a} q^{|\nu^{a-1}|+j}\\ = \sum_{\eta}\sum_{\mu^a,\nu^{a-1}} c^{(-1)^{a-1}\rho^{a-1}}_{\mu^{a-1}\nu^{a-1}} c^{(-1)^a\rho^a}_{\mu^a\nu^a} c^{\eta}_{\tau^a(1^k)} c^{\eta}_{\nu^{a-1}\mu^a} q^{|\nu^{a-1}|}.\end{aligned}$$

By calling the dummy summation variables ν^{a-1} and μ^a on the righthand side of this last equation γ and δ respectively, and comparing the two sides of the equation so obtained, we see that it remains to show that

$$(\dagger) \quad \sum_{j=0}^{k}\sum_{\mu^a,\nu^{a-1}} c^{(-1)^a\tau^a}_{\nu^{a-1}\mu^a} c^{\gamma}_{(-1)^{a-1}(j),\,\nu^{a-1}} c^{\delta}_{(-1)^a(1^{k-j}),\,\mu^a} q^{|\nu^{a-1}|+j} = \sum_{\eta} c^{\eta}_{\tau^a(1^k)} c^{\eta}_{\gamma\delta} q^{|\gamma|}.$$

Note that, on the lefthand side, we only need to let ν^{a-1} run over partitions such that $|\nu^{a-1}|+j=|\gamma|$. We then deduce $(\dagger)$ using Lemma 2.2(1,3). This completes the proof of Proposition 4.1 and hence Theorem 1.1.

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