Optimization methods for deadbeat control design: A state space approach

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Abstract

This thesis addresses the synthesis problem of state deadbeat regulator using state space techniques. Deadbeat control is a linear control strategy in discrete time systems and consists of driving the system from any arbitrary initial state to a desired final state in finite number of time steps.

Having described the framework for development of the thesis which is in the form of a lower linear-fractional transformation (LFT), the conditions for internal stability based on the notion of coprime factorization over the set of proper and stable transfer matrices, namely $\mathbb{R}H_\infty$, is discussed. This leads to the derivation of the class of all stabilizing linear controllers, which are parameterized affinely in terms of a stable but otherwise free parameter Q, usually known as the Q-parameterization. In this work, the classical Q-parameterization is generalized to deliver a parameterization for the family of deadbeat regulators.

Time response characteristics of the deadbeat system are investigated. In particular, the deadbeat regulator design problem in which the system must satisfy time domain specifications and minimize a quadratic (LQG-type) performance criterion is examined. It is shown that the attained parameterization for deadbeat controllers leads to the formulation of the synthesis problem in a quadratic programming framework with Q regarded as the design variable. The equivalent formulation of this objective as a quadratic integral in the frequency domain provides the means for shaping the frequency-response characteristics of the system. Using the LMI characterization of the standard H_∞ problem, a new scheme for shaping the system frequency response characteristics by minimizing the infinity norm of an appropriate closed-loop transfer function is introduced. As shown, the derived parameterization of deadbeat compensators simplifies considerably the formulation and solution of this problem.

The last part of the work described in this thesis is devoted to addressing the synthesis problem of deadbeat regulators in a robust way, when the plant is subject to structured norm-bounded parametric uncertainties. A novel approach which is expressed as an LMI feasibility condition has been proposed and analysed.

Notation

$\mathbb{R}H_\infty$ Set of proper and stable rational matrices

X^T Transpose of the matrix X

$\mathcal{F}_l(.,.)$ Lower LFT

$\mathcal{F}_u(.,.)$ Upper LFT

C_l Composition of two LFTs

$R^{n\times m}$ Set of $n \times m$ matrices whose elements all belong to R

K_s Nominal (central) controller in the set of admissible controllers

$\mathbb{R}^{n\times m}$ Set of $n \times m$ real matrices

$\mathbb{C}$ Controllability matrix

S_i i-th controllable subspace

v Controllability (reachability) index

$\mathbb{R}$ Set of real numbers

$\mathbb{C}$ Set of complex numbers

$\mathbb{Z}$ Set of integers

$\mathbb{N}$ Set of natural numbers

$\mathfrak{f}$ Field

$\mathfrak{f}^\mathbb{Z}, \mathfrak{f}\langle x\rangle$ Formal Laurent series in one indeterminate x over $\mathfrak{f}$

$\tau(f)$ Order of the sequence f

$\mathfrak{f}[[x]]$ Formal power series in one indeterminate x over $\mathfrak{f}$

$\mathfrak{f}[x]$ Formal polynomials in one indeterminate x over the field $\mathfrak{f}$

$\partial(f)$ Degree of a polynomial f

$\mathfrak{f}(x)$	Rational fractions or rational sequences in one indeterminate x over $\mathfrak{f}$
	(Formal rational series in one indeterminate x over $\mathfrak{f}$)
$\mathfrak{f}(x)$	Rational functions in $\mathfrak{f}$
d	Indeterminate, delay operator
$\mathbb{R}\{d\}$	Set of recurrent sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}(d)$	Set of rational sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}^o(d)$	Set of causal sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}^+(d)$	Set of stable sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}[d]$	Set of polynomial sequences with one indeterminate d over $\mathbb{R}$
$\left.\begin{array}{l}\mathbb{R}(d)\\\mathbb{R}^o(d)\\\mathbb{R}^+(d)\end{array}\right\}$	Corresponding functions over $\mathbb{R}$
$\left.\begin{array}{l}\mathbb{R}_{lm}(d)\\\mathbb{R}^o_{lm}(d)\\\mathbb{R}^+_{lm}(d)\end{array}\right\}$	Corresponding sequential matrices
$\mathbb{D}$	Closed unit disc in the complex plane
z	The advance shift operator ($z = d^{-1}$)
$\delta^f_m(G)$	Number of finite poles of the rational matrix G
$\delta^\infty_m(G)$	Number of infinite poles of the rational matrix G
$\delta_m(G)$	McMillan degree of the rational matrix G
R_G	Right composite matrix of G
L_G	Left composite matrix of G
$\mathcal{M}(\mathcal{R})$	Set of matrices with elements from $\mathcal{R}$
$U[\mathcal{R}]$	Set of $\mathcal{R}$-unimodular matrices with elements from $\mathcal{R}$
$\rho_H(d)$	Characteristic polynomial of $H(d)$
$\mathfrak{F}(P)$	Family of all causal deadbeat controllers (in the algebraic approach)

R_i	i-th reachable (Krylov) subspace
$s(k)$	Unit step response of a discrete time system
$\mathfrak{D}$	Domain of the optimization problem
X_{opt}	Set of all optimal points
A_f	Shift matrix
I_N	Identity matrix of size N
e_i	i-th column in the identity matrix
w_n	Measurement noise
w_d	Disturbance signal (process noise)
$\mathcal{V}$	Intensity matrix of w_n
$\mathcal{W}$	Intensity matrix of w_d
E	Expectation operator
$\delta(t)$	Delta function
μ	Mean value
σ^2	Variance
$\|.\|_F$	Frobenius norm
$\|.\|_2$	H_2 norm, l_2 norm, Euclidean norm (depending on context)
Δ	Perturbation, Uncertainty
Ω	Compact bounding set of perturbation matrices
$\mathfrak{R}$	Parameter box
$\mathcal{D}$	Generalized stability region
$f_{\mathcal{D}}(z)$	Characteristic function of $\mathcal{D}$
z^*	Conjugate transpose of z
$D(\alpha, r)$	A disk of radius r centred at α

$l_2(-\infty, +\infty)$ Time domain Lebesgue space

$l_2(j\mathbb{R})$ Square integrable functions on the imaginary axis

$\|.\|_\infty$ The H_∞ norm

$\bar{\sigma}$ Largest singular value

$G^\sim$ The parahermitian transpose of the transfer matrix G (shorthand for

$G^T(-s)$ in continuous time and $G^T(z^{-1})$ in discrete time)

$G_\perp$ The orthogonal complement of the transfer matrix G, such that $[G \quad G_\perp]$

or $\begin{bmatrix} G \\ G_\perp \end{bmatrix}$ is all-pass

Γ Hankel operator

$\mathcal{X}_+(H)$ Stable invariant subspace of H

$\mathcal{X}_-(H)$ Antistable invariant subspace of H

$Ric(H)$ The stabilizing solution of an algebraic Riccati equation

ρ Spectral radius

Symbols

$\in$	Belongs to
$f * g$	Convolution of f and g
$:=$	Equal by definition
∞	Infinity
$\cup$	Union
$\cap$	Intersection
$\subset$	Proper subset
$\subseteq$	Subset
$\sim$	Associate
$\sum$	Summation
$\otimes$	Kronecker product (tensor product or direct product)

Abbreviations

MFD   Matrix fractional description

LFT   Linear fractional transformation

BIBO   Bounded Input Bounded Output

SISO   Single Input Single Output

MIMO   Multi Input Multi Output

GCD   Greatest common divisor

det   Determinant

r.c.f   Right coprime factorization

l.c.f   Left coprime factorization

YJBK   Youla-Jabr-Bongiorna-Kucera

PMD   Polynomial matrix descriptions

PMFD   Polynomial matrix fractional description

FST   Finite settling time

TFST   Total finite settling time

CAD   Computer Aided Design

FIR   Finite impulse response

Im   Image or range of a matrix

rank   Rank of a matrix

Ker   Kernel

LTI   Linear time invariant

LP   Linear programming

LQ   Linear quadratic

LQG   Linear quadratic Gaussian

max   Maximum

min   Minimum

lim   Limit

dom   Domain of a function

inf   Infimum

vec   Vectorization operator

LQR   Linear Quadratic Regulator

tr   Trace of a matrix

RMS   Root mean square

LMI   Linear matrix inequality

diag   Diagonal

LPV   Linear parameter varying

Re   Real part of a complex number

LHP   Left half plane

SVD   Singular value decomposition

dist   Distance

Ric   Riccati

Chapter 1

Introduction

From the very early applications of discrete time system theory, a distinctive property of linear discrete time systems, namely their ability to achieve a desired operating regime in finite time in response to an arbitrary set of initial conditions, had received considerable attention. Since 1954, when the problem was first introduced by Bergen and Ragazzini [26], it has intrigued control engineers for many years. The first major contribution to the deadbeat control problem was made by Kalman-[27], who tackled the problem in the state space framework and provided the solution which was in the form of linear state feedback. Since then, his elegant solution has motivated a large body of research in this area.

In this thesis the synthesis problem of deadbeat controller in a state space framework has been investigated. Formally, a state deadbeat controller drives a discrete time system from any arbitrary initial state to a desired final state in finite number of time steps. Without loss of generality, it can be assumed that the final state is the origin of the complex plane. Accordingly, the ability to find a control sequence of finite length for any set of initial conditions, which steers the actual states to the desired state in finite number of control iterations is known as the *deadbeat controllability* property.

It should be noted that the deadbeat nature of the response is an exclusive attribute of discrete time systems and has no correlate in continuous time. This stems from the difference between the form of the solutions to the differential and difference state equations describing continuous and discrete time systems, respectively. For an asymptotically stable system, due to the exponential characteristic of the state equation solution in the continuous time case, the error decays exponentially and finally vanishes only in the limit as time tends to infinity.

Sometimes in the literature, the term "deadbeat" is taken as synonymous to time-optimality, i.e. requiring the states to settle to the final value in the minimum number of

time steps. In this work in order to avoid confusion and emphasize the time optimality characteristic, the controller is referred to as "minimum-time" or "time-optimal deadbeat". So, what is meant by "deadbeat" is the property of achieving the final state in just finite number of time steps.

Probably one of the major drawbacks to the implementation of the deadbeat regulator is its poor robustness and excessive overshoot of control signals. This is natural to expect, since all the states are intended to be driven to the origin in the shortest possible time. However, study of deadbeat compensators offers insight into the properties of linear systems (Glad [143]). Hence, even if we do not aim to implement time optimal control, this may still be used to gain a good understanding of the performance limitations of a given system. On the other hand, Zhao et al. have shown in [30] that a trade-off between the settling time and control signal magnitude can be found.

In the literature, there are many versions of deadbeat control which differ with regards to the type of the problem considered, e.g. tracking, disturbance rejection, etc., and the approach adopted. Generally, approaches that are used to achieve deadbeat regulation fall into two broad areas, namely the state space approach [27, 29, 32-35, 37, 38, 14, 150-152, 155-157, 40-44, 52], and the algebraic or transfer function approach [26, 144-148, 14, 16, 28, 52, 57, 153, 154, 161-164]. It is further possible to classify the state space method into two subdivisions, i.e. the dynamic and spectral approach. The dynamic method first introduced by Kalman, is developed based on the notions of controllability and controllable subspaces. According to the main property of the compensator obtained in this way, i.e. the nilpotency of the closed-loop system matrix, the synthesis problem of deadbeat regulator may be recast as that of assigning a prescribed set of eigenvalues to the origin by means of a linear state feedback. This leads to the spectral treatment of the design problem. The second scheme, the so-called algebraic approach, is of a quite different nature compared to the first one and arises by treating input-output signals as sequences, and accordingly, the system as a mapping between the input and output vector-sequences. The main attribute of this method is the isomorphism between certain classes of formal series in one indeterminate over $\mathbb{R}$, and series expansion of functions over $\mathbb{R}$.

Objectives:

The main objectives of this thesis are:

1. To introduce the theoretical framework on which this thesis has been developed. The considered setting is in the form of a Linear Fractional Transformation (LFT). As it is well-known many synthesis and analysis problems may be recast in this framework. On the other hand, the equivalent reconstruction of the setting as the Q-parameterization delivers considerable simplifications in formulating the constrained deadbeat control design problem.

2. To parameterize the family of all controllers which internally stabilize the closed-loop system and drive the state-vector to zero in a finite number of steps, the so-called deadbeat control scheme, in terms of a free design parameter Q.

3. To formulate and solve deadbeat synthesis problems in order to satisfy pre-specified time domain performance specifications, thus shaping the system's transient response characteristics.

4. To formulate and solve the deadbeat control design problem involving quadratic performance criteria (similar to those arising in LQG control) subject to additional magnitude constraints on selected state and output variables.

5. To minimize the robust worst case performance of deadbeat feedback systems by formulating and solving H_∞ optimal control problems via LMI-based efficient and tractable numerical algorithms.

6. To extend robust stability analysis and synthesis methods to systems described by structured norm-bounded parametric uncertainties within the deadbeat design framework.

In the remaining part of the introduction, an outline of the thesis will be presented.

Chapter 2 introduces the formal framework based on which this thesis is developed. The framework in which synthesis problem of the deadbeat controller is treated is in the form of a lower LFT (Linear Fractional Transformation). It is well-known that many control design problems may be reconfigured in such a setting [1]. Conditions which guarantee the internal stability of system interconnection are investigated; this is perhaps the most fundamental and useful property of control systems. The conditions are first formulated in terms of the state space description of the closed-loop system. However, we also look into the issue of internal stability in a different framework, which is based on the Matrix Fractional Description (MFD) of the constituent systems of the feedback interconnection. The central idea is to consider the set of transfer matrices with a prescribed property as a ring, and then model a given system as the ratio of two transfer matrices in that ring [165, 166, 11, 12, 13, 15, 17, 20]. In this way, the main synthesis problem transforms into designing a feedback system which lies in a desired ring of operators when both the plant and compensator are modelled as a quotient of operators from that ring [11]. What makes this approach appealing is that the design problem results in a complete characterization of all compensators which place the feedback system in the desired ring.

For the purpose of studying internal stability, we will only be concerned with those aspects of the fractional representation theory pertaining to feedback stabilization. To accomplish this, the notions of right and left fractional representation of matrices will be introduced. By imposing the additional requirement of coprimeness, the concepts of the right and left Bezout identity, also known as the Diophantine equation [167, 168, 28, 52], are introduced and connections between the two are established. The relation is referred to as doubly coprime factorization, or generalized Bezout identity.

The ring concerning the internal stability problem is the set of proper and stable rational transfer matrices, namely $\mathbb{R}H_\infty$ [165, 166]. As is shown in [6] and [13], the doubly coprime factorization leads to a parametric characterization of all controllers which internally stabilize a given plant. All admissible compensators can be parameterized as a coprime factorization over $\mathbb{R}H_\infty$, including the elements of the doubly coprime factorization of the system to be stabilized over $\mathbb{R}H_\infty$, and a proper stable but otherwise arbitrary parameter Q. Formulating the plant and controller as an irreducible quotient of two transfer matrices in $\mathbb{R}H_\infty$ does not only catch the usual notion of instability due to

unstable closed-loop poles, but also excludes the possibility of unstable pole-zero cancellations between the plant and controller.

Parameterizing the controller in the above fashion will convert the linear fractional description of the closed-loop map $H_{zw} = \mathcal{F}_l(P, K) = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})$, to an affine parameterization in terms of the design parameter Q, $H_{zw} = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21}$, known as the Q-parameterization or Youla-Jabr-Bongiorno-Kucera (YJBK) parameterization, first developed in [25]. This affine dependence on the parameter Q is exploited to simplify the design procedure by reducing the problem of search or optimization over the set of admissible controllers to a search or unconstrained optimization over Q. It is also shown that all the stabilizing controllers are in the form of a stable observer combined with a stabilizing state feedback.

Chapter 3 considers presenting the state space and algebraic approaches to the design of the deadbeat regulator. The state space method is developed based on the fundamental concept of system controllability. It is shown that the minimum number of time steps needed to transfer any initial state to the origin of the complex plane is equal to the controllability index, defined as the smallest possible integer for which the controllability matrix is full rank. The maximum number of steps though, is equal to the order of the system. The first step in defining a deadbeat controller in this scheme is the selection of n linearly independent columns of the controllability matrix [29, 32, 33, 34, 37]. Various ways to accomplish this have been discussed in [34], [39] and [37]. The non-uniqueness of the selection procedure leads also to the non-uniqueness of the resulting compensator. It is further shown in [27, 34] that the minimum-time deadbeat controller is in the form of a state feedback; this does not follow from an a priori assumption but is forced on us by the requirement that every state is driven to the origin in the minimum time steps. The properties of the resulting closed-loop map are also investigated. As it is noticed the closed-loop system matrix is nilpotent with the index of nilpotency equal to the system controllability index. This, correspondingly, reduces the synthesis problem of deadbeat regulator to that of assigning the eigenvalues to the origin via linear state feedback [45], usually known as the spectral approach. The derived controller through this method is also non-unique [35, 41, 42], owing to the freedom in selecting the Jordan structure of the closed-loop system, and assigned eigenvectors and generalized eigenvectors. As it is

argued in [43] and [44], the freedom may be employed to shape the transient response characteristics of the closed-loop map. Hence, the synthesis problem may be regarded as an eigenstructure assignment, rather than just an eigenvalue assignment, as treated in [155, 156, 157].

O'Reilly in [34] surveys two decades of research in the deadbeat synthesis problem in the state space framework up to 1981. After that date, significant contributions in this area includes the work of Zhao et al. [149, 150] who used the Youla parameterization to design robust one degree of freedom deadbeat controllers. The same authors in [151] and [152] applied the method to the design of two-degrees of freedom compensators. In this chapter we also look into the second major design procedure of the controller, i.e. the algebraic approach. It was first Bergen and Ragazzini [26] who applied the method to attain the solution to the problem of deadbeat tracking. The approach has been promoted ever since mainly by Kucera [144, 145, 14, 16, 146, 147, 148], followed by a number of other researchers like Eichsteadt [153] and Wolovich [154]. Essentially, this approach is based on the fact that in a discrete time framework, the input and output signals may be interpreted as sequences, and accordingly systems are inferred as a uniquely defined mapping between the input and output vector-sequences. To explicate this scheme, a quick review of the basic tools of the algebraic approach within the context of the discrete time systems is given. As it will be apparent, the fundamental attribute of this approach is the isomorphism between certain classes of formal series in one indeterminate over an infinite field, and series expansion of functions over the same field, which in the case of discrete time systems and in general linear dynamical systems is the set of real numbers $\mathbb{R}$.

The notion of sequences when extended to the case of matrices is explained and their categorization while expressed as matrix fractions with elements from corresponding sets of sequences (which are either fields or rings), is also defined. The ability to express sequential matrices as the quotient of two matrices from desired fields or rings, leads to the formulation of the deadbeat controller as the solution to a polynomial Diophantine equation. The solution to the general problem considered in the work, i.e. the deadbeat problem, is first derived. Then the conditions to attain time-optimal deadbeat response are discussed. Milonidis et al. in [52] have extensively elaborated on the approach while additional performance requirements and constraints on the structure of the controller are

imposed. In their work, instead of deadbeat they use the term Finite Settling Time (FST) first coined by Karcanias et al. in [28]. For further references related to the deadbeat regulator design problem in the algebraic approach, the reader is referred to [28, 57, 161, 162, 163, 164].

In the final part of chapter 3 we present a numerical method for constructing a static state feedback which assigns all controllable modes to the origin. This was developed by Van Dooren in [58] and is based on the recursive construction of a unitary transformation, yielding a coordinate system in which the state feedback is computed by merely solving a set of linear equations. The coordinate system is related to the Krylov sequence. An important feature of this numerical method is that the backward stability of the algorithm is guaranteed through application of unitary transformations. Before constructing the state feedback gain, the system is first reduced into block Hessenberg form [61], also known as the staircase model, so that the controllable and uncontrollable subsystems are separated. It is apparent that the problem is feasible if all the uncontrollable modes are already at zero, or equivalently the uncontrollable subsystem is nilpotent. We have programmed the algorithm in MATLAB, and a few examples have been considered. In all of the examples in the following chapters, we will use this algorithm to compute the observer and state feedback gains.

Having discussed the framework to study the deadbeat controller design problem, and the existing approaches to tackle the problem, in chapter 4 we address the synthesis problem of deadbeat regulator subject to time domain constraints. First the input-output mathematical description of a system having deadbeat response is demonstrated. The impulse response of such systems is of finite duration, in other words it is a polynomial in the unit delay operator z^{-1}, hence all the poles are located at the origin of the complex plane. Such systems are frequently known as FIR.

It is well known that the time response of any system includes two parts, the transient and steady state. The transient part generates in response to the poles of the closed-loop system, whereas the steady state response is a result of the poles of the input or forcing function. Clearly, it is desirable that the transient response be sufficiently fast and represent satisfactory damping. Frequently transient response characteristics of a system are analysed in terms of the system response to standard inputs such as unit step, or ramp.

This is in view of the fact that the system response to any arbitrary input may be estimated from its response to such standard inputs. In this chapter, a partial list of typical time domain performance specifications of control systems, including transient response characteristics, is presented. It is shown that when the closed-loop system is described in terms of the parameter Q, these design specifications may be expressed in a multilinear form. This demonstrates the benefits of Q parametrization in the present context; as discussed by Boyd et al. [4] this task is much more complex if the design parameter is chosen as the controller K.

The achieved simplification enables us to recast the synthesis problem of the deadbeat compensator satisfying desired time domain constraints as a Linear Program (LP). However, designing such a controller demands to confine the closed-loop system to be FIR. This can be accomplished by assigning the whole set of the closed-loop poles, which in fact is the union of the poles of the state feedback and the observer as the constituting elements of the controller, to the origin. This in turn leads to all the sub-systems T_{11}, T_{12}, and T_{21} in $H_{zw} = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21}$ be FIR. By restricting T_{11}, T_{12}, and T_{21} to be FIR, having a deadbeat response will now just necessitate to restrict the design parameter Q to be FIR too. In this way, a complete characterization of the family of deadbeat controllers is obtained. It should be noted that restricting Q to be FIR does not involve considerable restriction of the set of stabilising controllers considered, provided the maximum degree of Q is chosen sufficiently large. The problem of parameterizing the family of the state deadbeat regulators was first introduced by Sebakhy et al. [158] for the special case of time-optimal controllers, through minimizing a quadratic performance index. However, their achieved description of the family was overparameterized. It was then Schlegel [159] who gave the description of the family in terms of the minimum number of parameters. Fahmy et al. [35] considered the problem for the more general case of non-time-optimal deadbeat regulators, under the assumption of the invertibility of the transition matrix. In [160], Amin and Elabdalla treated the problem by relaxing the aforementioned assumption.

The second part of this chapter is dedicated to the design problem of deadbeat controller subject to quadratic performance specifications in the form of the standard Linear Quadratic Gaussian (LQG) problem. First, a brief introduction to the LQG problem and

its solution is given. Then, an alternative interpretation is discussed by considering a broader class of problems corresponding to the so-called H_2 optimization framework. This interpretation eliminates the need to incorporate the stochastic ingredient of the LQG, and consequently offers a great deal of flexibility, especially when it is difficult to determine the precise stochastic properties of the signals involved. The relation between LQG and H_2 optimization is observed by recognizing the fact that the LQG performance index is expressible as the system H_2 norm when it is excited by white noise input signals, a notion which is elaborated on. The established relation between the LQG and H_2 optimization problems will be exploited to show that any constraint in the form of LQG imposed on the regulated signals, may be transferred into a quadratic programming with Q as the design parameter. The efficiency of the proposed approaches has been illustrated by means of an example.

Up to this point, the work is based on the assumption that there is no uncertainty present in the plant model. However, real systems always involve some amount of uncertainty. This motivates us to consider the design problem of robust deadbeat controller in chapter 5. As mentioned earlier, achieving deadbeat response necessitates to locate all the closed-loop system poles at the origin of the complex plane. However, due to the existence of uncertainty in the description of the plant it is virtually impossible to assign all poles exactly at the origin for every combination of the uncertain parameters. Consequently, the performance of the system will be adversely affected and hence the deadbeat characteristic of the system response will be lost. As a result, it is desirable to find the smallest possible region around the origin at which all the poles can be placed. The most natural form of such a region, is a circle centred at the origin. So, in this chapter the main purpose is to obtain the circle of minimum radius centred at the origin of the complex plane within which the closed loop poles can be robustly placed. This has been accomplished through a generalization of the Lyapunov stability method, also known as the quadratic stability approach. Before introducing this and its association with the work here, two major classes of perturbations arising in system models are defined, namely the parametric uncertainty and the norm-based uncertainty models. In this chapter, we are mainly interested in the parametric uncertainty model arising when the parameters in the system description are only known approximately or are in error.

Having discussed the inclusion of the uncertainty in the construction of our framework, we will then concisely examine the sensitivity of the eigenvalues to parametric uncertainty. As Wilkinson shows in [93], sensitivity of the poles depends upon the magnitude of their condition number. He also provides an upper bound on the sensitivities of the eigenvalues in terms of the condition number of the eigenvector matrix.

We next turn our attention to the concept of quadratic stability which forms the foundation of deadbeat regulator design problems. It is well-known that quadratic stability analyses the stability of systems in terms of the existence of a positive definite symmetric matrix corresponding to solution of a Lyapunov algebraic equation or inequality [49]. Amato in [100] extends the criterion for stability to the case of linear parameter-varying (LPV) systems when the perturbation, in the form of structured norm-bounded uncertainties, enters just the state matrix, or both the state and input matrices, respectively known as model parameter uncertainty and input connection parameter uncertainty. What is significant about the achieved conditions is that they are formulated as LMI feasibility problems and so they can be investigated via efficient tractable numerical algorithms, e.g. the interior-point method, discussed in [9, 103, 104, 105, 106].

As argued in [95], [116], and [100], the controller which renders the closed-loop system quadratically stable is a linear time-invariant state feedback compensator. Besides stability, requiring desirable system dynamical behaviour compels us to assign the closed-loop poles to specific sub-regions of the complex plane, the so-called generalized stability regions, designated by $\mathcal{D}$. [87] This leads to the notion of the $\mathcal{D}$-stability. Due to the presence of uncertainty, $\mathcal{D}$-stability may be developed in a natural way to that of the quadratic $\mathcal{D}$-stability. Hence, quadratic $\mathcal{D}$-stability extends $\mathcal{D}$-stability to uncertain systems in a similar fashion that quadratic stability extends stability to uncertain systems. The conditions for quadratic $\mathcal{D}$-stability of systems are obtained in [97, 98, 100]. As shown these are based on the concept of the LMI regions, defined in [98]. Again, the conditions are expressed in the form of the LMI feasibility problems.

The above stability conditions have been stated based on continuous time system descriptions. However, they can be readily translated to the discrete time case as well, using the fact that quadratic stability of a discrete time system is equivalent to quadratic $\mathcal{D}$-stability of its continuous time counterpart when $\mathcal{D}$ is the unit disk centred at the origin

of the complex plane. In view of this, the LMI feasibility problem equivalent to the quadratic stability of a discrete time system subject to both model parameter and input connection parameter uncertainties is established. This is then followed by stating the corresponding conditions for quadratic $\mathcal{D}$-stability of a discrete time system when the $\mathcal{D}$ region is considered to be a circle [120, 118]. Kim et al. in [120] use the result to show that the problem of finding the smallest radius is equivalent to an optimization problem subject to the achieved LMI modified accordingly.

As mentioned earlier, in the synthesis problem of a robust deadbeat controller the main aim is quantitatively defining the circular region of minimal radius which contains all eigenvalues of the closed-loop system, and subsequently finding the observer and the state feedback gains which accomplishes this. By invoking Parrott's theorem [137] it is argued that the robust design of a deadbeat controller when the plant is subject to the parametric uncertainties is equivalent to finding the minimum radius for each of the disks encompassing the poles of the state feedback and the observer, and then selecting the maximum as the solution to the problem. This chapter is concluded by an example illustrating the design procedure.

In chapter 6, the final chapter of this work, we will investigate the design problem of the deadbeat controller subject to the H_∞ norm constraints. The H_∞ norm is defined as the l_2 gain of the system. A problem in which the objective is to minimize the H_∞ norm of a system is known as the H_∞ optimization problem. This typically arises from the requirement to reduce the sensitivity of a feedback system against disturbances. It first appeared in the seminal work of Zames [125] and Doyle and Stein in [126]. Other most celebrated examples of control objectives expressible as H_∞ norm constraints are disturbance attenuation, robust control, and the mixed sensitivity problem, as examined in [124, 83, 19, 122]. This chapter first gives an introduction to the H_∞ norm and its interpretation in both the time and frequency domains. The H_∞ norm is an indicator of the worst-case energy of the output for energy bounded inputs and accordingly can be naturally used as a measure of worst case performance.

Based on the general framework described earlier, the H_∞ optimization problem is mathematically formulated and the three main approaches through which the problem can

be tackled are briefly discussed. These can be classified as model-matching, Riccati equation-based, and the LMI approaches. In the first scheme, i.e. model-matching approach [127], using the characterization of the closed-loop system, $H_{zw} = T_{11} + T_{12}QT_{21}$, the H_∞ problem is to match T_{11} to the cascade $T_{12}QT_{21}$, considering Q as the design parameter. It has been shown in [128] that the problem can be formulated in the form of the so-called Nehari extension problem. Treatment of the H_∞ optimization problem in this scheme is both theoretically and computationally very involved. That is why Glover et al. in [136] proposed a new approach which relies on the solution to two algebraic Riccati equations with the same order as that of the system, a method which will also be briefly reviewed.

In our work, deadbeat controllers which satisfy H_∞ norm constraints are synthesized via the LMI approach. This method is chosen mainly due to the existence of efficient and tractable numerical algorithms on which it is based. In this scheme, the H_∞ norm minimization problem is formulated as a standard linear matrix inequality (LMI) feasibility problem [142, 106]. The LMI characterization of the H_∞ problem is the so-called "bounded real lemma". The main aim of this chapter is to show that the Markov parameters of the design parameter Q appear affinely only in the C and D matrices of the state space realization of the closed-loop system H_{zw} which results in an overall linear function of the matrix variables. So, the main problem reduces to finding appropriate C and D matrices such that the LMI condition is satisfied. As an example, the control design procedure is applied to the model of a DC motor.

Achievements:

The main contributions of the thesis are as follows:

- The thesis has provided an affine parameterization of the family of deadbeat regulators in terms of a free parameter Q. According to the mathematical characterization of the closed-loop map as $H_{zw} = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21}$, a deadbeat response necessitates the assignment of all closed-loop poles to the origin of the complex plane (hence, making T_{11}, T_{12}, and T_{21} all FIR) and also confining the free parameter Q to a FIR matrix function. The case of minimum-time deadbeat controller is achieved by setting Q to zero.

- The problem of designing deadbeat regulators subject to the LQG performance specifications has been recast as a quadratic programme. Moreover, it is demonstrated that the problem of shaping the transient response of the closed-loop system and generally satisfying time domain constraints can be reformulated as a linear programming with Q being the design parameter. Both objectives can be addressed in a quadratic programming optimization setting.

- The thesis has proposed a new method for shaping the frequency response of the closed-loop system in terms of its worst case performance, which is quantified by the system H_∞ norm. The problem is stated as an LMI feasibility condition in the form of the bounded real lemma. It is shown that the Markov parameters of the design matrix Q appear affinely in the output part of the closed-loop state-space model (C and D matrices only) without affecting the input part of the model (matrices A and B). This attribute results in simple LMI conditions and an overall efficient algorithm.

- The thesis has proposed a new method for designing robust deadbeat controllers in the presence of structured norm-bounded parametric uncertainties. The radius of the smallest circular region centered at the origin of the complex plane containing all closed-loop eigenvalues for all possible combinations of the uncertain parameters has been computed. Moreover, the controller which assigns the closed-loop eigenvalues within this region is synthesized via linear state feedback. The design procedure is based on LMI characterization of quadratic stability.

Chapter 2

The general framework and preliminaries

2.1 Introduction:

In this chapter, the general framework based on which this thesis is developed is introduced. The internal and external descriptions of the feedback configuration have been derived. As for any interconnection in the control theory, the most fundamental requirement of internal stability has been discussed. Internal stability is first described based on the state space realization of the system. Through description of the notions of Matrix Fractional Description (MFD) and coprimeness over the set of proper and stable rational matrices, namely the ring $\mathbb{R}H_\infty$, an external characterization of internal stability is established. This in turn, leads to the complete parameterization of the set of all stabilizing compensators. The parameterization is linear fractional in character, and results in an affine characterization of the closed-loop system in a stable but arbitrary design parameter. The state space realizations for the coprime factors of both plant and controller are given. Finally, it is shown that every controller which stabilizes the plant can be realized as an observer-based controller.

2.2 Linear fractional transformation (LFT):

As in any control problem, the first step is to construct a formal framework based on which the problem is treated. It is well-known that many control problems can be formulated in a linear fractional transformation (LFT) framework [1].

Given a complex matrix G block partitioned as:

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)}$$

and two other complex matrices $\Delta_l \in \mathbb{C}^{n_2 \times m_2}$ and $\Delta_u \in \mathbb{C}^{n_1 \times m_1}$, we can formally establish two mappings, namely lower and upper LFT. The lower LFT with respect to Δ_l is defined as the map:

$$\mathcal{F}_l(G, \bullet) : \mathbb{C}^{n_2 \times m_2} \to \mathbb{C}^{m_1 \times n_1}$$

where

$$\mathcal{F}_l(G, \Delta_l) = G_{11} + G_{12}\Delta_l(I - G_{22}\Delta_l)^{-1}G_{21} \qquad (2.2.1)$$

In a similar fashion, an upper LFT with respect to Δ_u is defined as:

$$\mathcal{F}_u(G, \bullet) : \mathbb{C}^{n_1 \times m_1} \to \mathbb{C}^{m_2 \times n_2}$$

where

$$\mathcal{F}_u(G, \Delta_u) = G_{22} + G_{21}\Delta_u(I - G_{11}\Delta_u)^{-1}G_{12} \qquad (2.2.2)$$

Obviously, these two mappings are well-defined provided that the inverses exist. The following representations of $\mathcal{F}_l(G, \Delta_l)$ and $\mathcal{F}_u(G, \Delta_u)$ clearly justifies the terminologies of lower and upper LFTs.

(a)   (b)
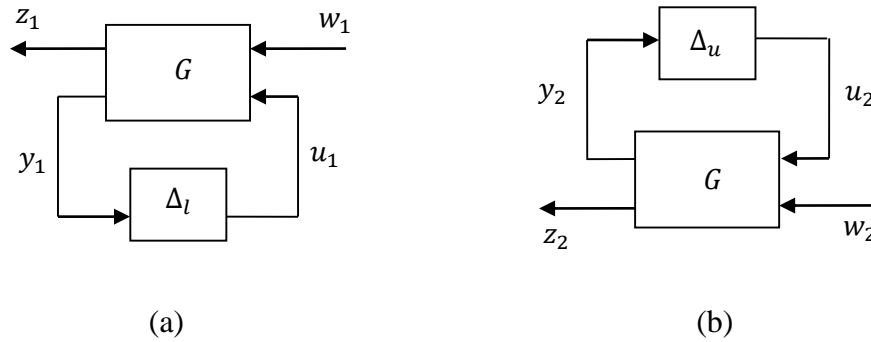
Figure 2.2.1 (a) The lower and (b) the upper LFT

By taking G as a proper transfer matrix, the lower and upper LFTs defined above are simply the closed-loop transfer matrices from w_1 to z_1 and from w_2 to z_2 respectively, i.e. [2]:

$$\mathcal{F}_l(G, \Delta_l) = T_{z_1 w_1} \text{ and } \mathcal{F}_u(G, \Delta_u) = T_{z_2 w_2} \quad\quad (2.2.3)$$

Every synthesis problem can be cast as a lower LFT when G is interpreted as a generalized plant and Δ_l as a controller to be designed. On the other hand, every analysis problem can be formulated as an upper LFT when G is an interconnection matrix with some structured Δ_u representing parametric or unstructured uncertainty [2].

The present work which involves the design problem of deadbeat controller under various constraints is developed based on the lower LFT configuration.

2.3 The general framework:

In view of the arguments presented in the previous section, any control problem can be reconfigured as an LFT model. With this motivation, we consider the configuration in figure 2.3.1 as the fundamental framework in this thesis.
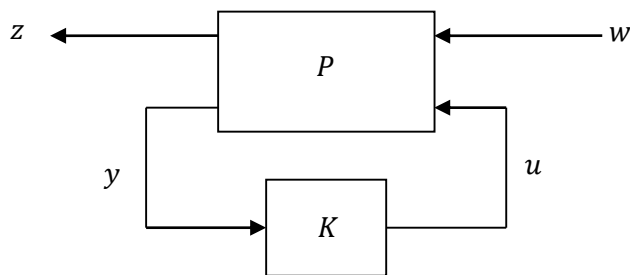
Figure 2.3.1 The general framework

In the illustrated block diagram, P is the generalised plant which admits the following state space description:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{11} w + D_{12} u \qquad (2.3.1)$$
$$y = C_2 x + D_{21} w + D_{22} u$$

Intuitively, the generalised-plant transfer function is partitioned as:

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{21} \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad (2.3.2)$$

The controller K is described by the state space realization:

$$\dot{x}_K = A_K x_K + B_K u_K \qquad (2.3.3)$$
$$y_K = C_K x_K + D_K u_K$$

We make the standard assumption that the realisations of the plant and the controller are both stabilizable and detectable.

With regards to the (vector) signals, the input signal w, referred to as the exogenous input, captures the effect of the environment on the plant. It contains disturbance and actuator's/sensors' noise-signals. w may also contain fictitious inputs injected at any point in the plant. The input signal u denotes the inputs manipulated by the controller. The output vector signal y, known as the measured or sensor outputs, represents the signals accessible to the controller. The regulated variable, denoted by z, as the name suggests, include all the outputs we wish to regulate or control. Basically, it represents every signal about which we express a specification or constraint. As such, it may include internal states or variables, or even components of u and y. [3, 4, 5]

In order to get the closed-loop transfer function, the output feedback control law:

$$u = Ky \qquad (2.3.4)$$

is applied and the equalities $u_K = y$ and $y_K = u$ are imposed. Solving the set of equations in (2.3.1) for z in terms of w, yields the corresponding input-output characterization of the closed-loop interconnection as:

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w := H_{zw}w \qquad (2.3.5)$$

H_{zw}, the closed-loop map from the exogenous inputs w to the regulated variables z, contains every closed-loop transfer function of interest. Having compared the above description with (2.2.1), it can be easily inferred that the closed-loop map is in the form of a lower LFT.

Derivation of H_{zw} necessitates $(I - P_{22}K)$ to be invertible and proper. This, which is known as the "well-posedness" condition ensures that all closed-loop maps are well-defined and proper. In other words, this condition ensures that the feedback system makes sense or is physically realizable. The invertibility of $(I - P_{22}K)$ is a necessary and sufficient condition for well-posedness, and is equivalent to the invertibility of $(I - P_{22}(\infty)K(\infty))$ [1, 5]. In most practical systems the feed-through matrix D_{22} is zero which automatically guarantees the existence of the inverse [6]. Therefore for systems with zero feed-through matrix or simply strictly proper systems, the well-posedness is guaranteed.

2.4 Internal stability of the LFT:

One of the most fundamental issues arising in any control system problem is internal stability of the closed-loop system. The fact that the response of any LTI system is the combination of responses to external inputs and initial conditions, motivates two different but closely related notions of stability, namely BIBO (Bounded Input Bounded Output) stability and internal stability. The concepts of BIBO and internal stability respectively signify the stability of the system in response to external inputs assuming zero initial states, and stability of the system response due to initial conditions while external inputs are considered to be zero.

Internal stability refers to the autonomous system dynamics in the absence of external inputs and so it coincides with the standard notion of asymptotic stability of dynamical systems. Internal stability is a basic requirement for every practical feedback system. This is because all interconnected systems may be unavoidably subject to some nonzero initial conditions and some (possibly small) errors, which in practice cannot be tolerated. Such errors at some points of the closed-loop system may lead to unbounded signals at other points in the interconnection. Through internal stability of the closed-loop system it is ensured that all signals in a system are bounded provided that the injected signals at any locations are bounded.

To analyse the internal stability of the LFT configuration of Figure 2.3.1 in terms of the state space description, consider the following corresponding setup for internal stability:
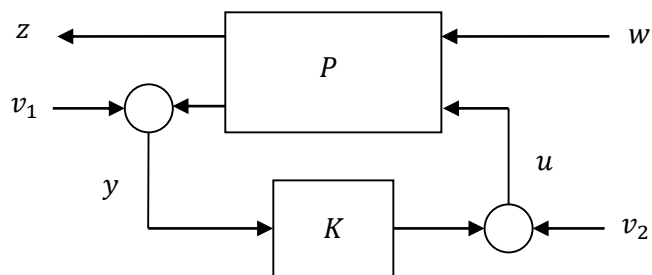
Figure 2.4.1 Setup for internal stability definition

Definition 2.4.1 [8] The LFT interconnection is internally stable if the nine mappings from w, v_1 , and v_2 to z, y, and u are all stable.

In order to limit the number of tedious calculations when deriving the state space realization of the closed-loop transfer matrix, a further assumption is to omit the direct feed-through term D_{22}. As discussed in the preceding section, this will also ensure the well-posedness of the system. We may restore $D_{22} \neq 0$ by a loop shifting argument that absorbs D_{22} into K, in the case that $D_{22} \neq 0$. The procedure to do this is fully discussed in [6] and is known as loop shifting.

Having imposed the assumption, the closed-loop system dynamical equations reduce to:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} B_1 + B_2 D_K D_{21} & B_2 D_K & B_2 \\ B_K D_{21} & B_K & 0 \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \\ C_2 & 0 \\ D_K C_2 & C_K \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} D_{11} + D_{12} D_K D_{21} & D_{12} D_K & D_{12} \\ D_{21} & I & 0 \\ D_K D_{21} & D_K & I \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

$$(2.4.1)$$

Lemma 2.4.2 [6] The LFT $\mathcal{F}_l(P, K)$ is internally stable if and only if the system matrix $\begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}$ is asymptotically stable (Hurwitz).

It should be noted that not every linear fractional transformation is stabilizable. The simplest example which illustrates this is when P_{11} is unstable and $P_{21} = 0$.

Lemma 2.4.3 [5] P is stabilizable if and only if (A, B_2, C_2) is stabilizable and detectable.

Thus, from the assumed stabilizability and detactability of P, the stabilizability and detectability of P_{22} is assured.

The ensuing lemma states that P and P_{22} have identical internal stabilizability properties. This, in turn, leads to the simplification of Figure 2.4.1 to the equivalent configuration of Figure 2.4.2.
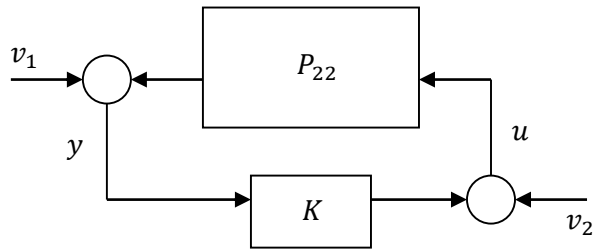
Figure 2.4.2 Equivalent diagram to analyse internal stability

Lemma 2.4.4 [6] K is an internally stabilizing controller for P if and only if it internally stabilizes P_{22}.

A proper controller K which internally stabilizes the plant P is said to be admissible. Moreover, such a plant for which there exists at least one stabilizing controller is called a generalized plant. [9]

2.5 Coprime factorization and internal stability:

In the previous section the internal stability of the LFT configuration was discussed through the system state space description. In this and the successive section however, stability is analysed in a different framework, involving the coprime factorization of the constituent systems of the feedback interconnection over the set of proper and stable transfer matrices. Studying the stability problem in this framework is a special case of a more general approach in which the analysis and synthesis problems are formulated based on the fractional representation of the systems, principally developed in [11, 12, 13, 20]. This approach, which has its roots in abstract algebra, considers in the SISO case the transfer functions with the prescribed properties as a ring H, and then models a given system as the ratio of two transfer functions in H. This casts the synthesis problem as designing a feedback system which lies in a desired ring of operators when both the plant and compensator are modelled as a quotient of operators from that ring [11]. What makes the procedure highly interesting is that the synthesis problem yields a complete characterization of all compensators which place the feedback system in the ring H. This approach could be readily extended to the MIMO systems when the transfer matrix has all its entries in H. The operations of matrix addition and multiplication induced on the set of matrices over H by the associated addition and multiplication operations with H, correspond to parallel and cascade interconnection of such systems [15, 17].

For the purpose of this section, we are only concerned with those aspects of the fractional representation theory pertaining to feedback stabilization. The central idea is that of expressing each constituent elements of the feedback interconnection as the irreducible quotient of two proper and stable elements. [15, 13] This is accomplished by considering

the notion of coprime factorization and its characteristics relevant to internal stability theory, which forms the foundation for developing a parameterization of stabilizing controllers in the next section.

Let R be a ring and $R^{n \times m}$ be the set of $n \times m$ matrices whose elements all belong to R. Every element f of $F^{n \times m}$, the set of $n \times m$ transfer matrices, can be factored as an element of the field of fractions associated with the ring R and expressed as the ratio of two matrices N and D, as $f = ND^{-1}$ where $N, D \in R^{n \times m}$ and $\det D \neq 0$. The pair (N, D) is referred to as a right fractional representation of f. In a similar way, the left fractional representation of every $f \in F^{n \times m}$ is defined as $f = \widetilde{D}^{-1}\widetilde{N}$ where again $\widetilde{N}, \widetilde{D} \in R^{n \times m}$ and $\det \widetilde{D} \neq 0$ [11, 13, 20, 165, 166].

Definition 2.5.1 [13] Two matrices $N, D \in R^{n \times m}$ are called right coprime if there exists matrices $X, Y \in R^{n \times m}$ such that:

$$XD - YN = I_m \qquad (2.5.1)$$

which can be stated equivalently as:

Definition 2.5.2 [19] Two matrices $N, D \in R^{n \times m}$ are right coprime if they have equal number of columns and there exists matrices $X, Y \in R^{n \times m}$ such that:

$$[X \quad Y]\begin{bmatrix} D \\ -N \end{bmatrix} = I_m \qquad (2.5.2)$$

This is equivalent to the matrix $[D^T \quad -N^T]^T$ being left-invertible in $R^{n \times m}$.

The equality (2.5.1) is known as the right Bezout identity or right Diophantine identity. It extends the notion of relatively prime (or coprime) integers, i.e. the Euclid's algorithm, to matrices. If a and b are two integers, i.e. $a, b \in \mathbb{Z}$, there exists $u, v \in \mathbb{Z}$ such that $au + bv = \text{GCD}(a, b)$, with GCD denoting the greatest common divisor of a and b. a and b are called relatively prime (coprime) if their GCD is 1 [6].

Definition 2.5.3 [13] In definition 2.5.2, if D is non-singular, $f = ND^{-1}$ is referred to right coprime factorization (r.c.f) of f.

The notions of left coprime and left coprime factorization can be defined analogously.

Definition 2.5.4 [19] For $f \in F^{n \times m}$, $\widetilde{N}, \widetilde{D} \in R^{n \times m}$ with equal number of rows, and $\widetilde{D}$ non-singular, $f = \widetilde{D}^{-1} \widetilde{N}$ is called the left coprime factorization (l.c.f) of f if there exists matrices $\widetilde{X}, \widetilde{Y} \in R^{n \times m}$ such that:

$$[\widetilde{D} \quad \widetilde{N}] \begin{bmatrix} \widetilde{X} \\ -\widetilde{Y} \end{bmatrix} = I_n \qquad (2.5.3)$$

This is equivalent to the matrix $[\widetilde{D} \quad \widetilde{N}]$ being right-invertible in $R^{n \times m}$.
The Bezout identity corresponding to (2.5.3):

$$\widetilde{D}\widetilde{X} - \widetilde{N}\widetilde{Y} = I_n \qquad (2.5.4)$$

is referred to left Bezout identity or left Diophantine identity.

The ring concerning the internal stability problem being considered here, is the set of proper and stable rational transfer matrices, namely the ring $\mathbb{R}H_\infty$ [19, 165, 166]. The setting in which $R = \mathbb{R}H_\infty$, not only catches the usual notion of instability as the result of existing unstable closed-loop poles, but also excludes the possibility of unstable pole-zero cancellations between the plant and controller. These will become clear as we proceed. From coprime fractional representation over $\mathbb{R}H_\infty$, some significant properties imposed by coprimeness can be inferred, which reveals the benefits of studying stabilization problem in a ring theoretic setting.

In view of Lemma 2.4.4 which establishes the equivalence between the stabilization of the plant P and that of P_{22} (figure 2.3.1), all the subsequent discussion is made about P_{22}. Suppose that P_{22} has a r.c.f over $\mathbb{R}H_\infty$ as $P_{22} = N_r D_r^{-1}$, where $N_r, D_r \in \mathbb{R}H_\infty$. Rewriting the right Bezout identity of (2.5.1) (in accordance to the r.c.f of P_{22}) as $X_r D_r - Y_r N_r = I$, in which $X_r, Y_r \in \mathbb{R}H_\infty$, and using the identity $X_r - Y_r P_{22} = D_r^{-1}$, it can be inferred that D_r^{-1} does not have unstable poles other than the unstable poles of P_{22}. Therefore:

Remark 2.5.5 [15] Instabilities of P_{22} are completely characterized by the denominator of a r.c.f of P_{22}, i.e. the unstable poles of P_{22} are precisely the unstable zeros of D_r.

Theorem 2.5.6 [20] The pairs $N_1, D_1 \in \mathbb{R}H_\infty$ and $N_2, D_2 \in \mathbb{R}H_\infty$ define right coprime factorizations of P_{22} as $P_{22} = N_1 D_1^{-1} = N_2 D_2^{-1}$, if and only if:

$$\begin{bmatrix} D_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} W$$

where $W, W^{-1} \in \mathbb{R}H_\infty$. In other words, W is $\mathbb{R}H_\infty$-unimodular.

Definition 2.5.7 [46] Let $\mathcal{R}$ be a ring and $\mathcal{R}^{l \times m}$ denote the $l \times m$ matrices with elements from $\mathcal{R}$. A matrix $U \in \mathcal{R}^{m \times m}$ is unimodular, if and only if $\det U$ is a unit in $\mathcal{R}$, i.e. it is a matrix in $\mathcal{R}^{m \times m}$ whose inverse belongs to $\mathcal{R}^{m \times m}$ too. Such matrices are termed $\mathcal{R}$-unimodular and designated as $U[\mathcal{R}]$.

Theorem 2.5.6 may be stated in the equivalent form of the following remark.

Remark 2.5.8 [22] A right coprime factorization is unique up to a unimodular common right divisor.

This remark, in turn, leads to the following important observation:

Remark 2.5.9 [17] Cancellation of instabilities, between the numerator and denominator of a r.c.f is not allowed. In this sense, a r.c.f is irreducible.

This makes clear the notion of irreducible quotient of stable elements mentioned earlier. A similar theorem and remarks hold analogously for a l.c.f of $P_{22} = D_l^{-1} N_l$.

It is worth mentioning that for rational transfer matrices existence of right and left coprime factorizations is assured. However, this is not the case in the general ring theoretic formulation [24, 23, 6].

The following theorem establishes the connection between the right and left coprime factorizations.

Theorem 2.5.10 [23] Assume that P_{22} admits both right and left coprime factorization as $P_{22} = N_r D_r^{-1} = D_l^{-1} N_l$, with $N_r, D_r, N_l, D_l \in \mathbb{R}H_\infty$, for which there exists $X_r, Y_r \in \mathbb{R}H_\infty$ such that $X_r D_r - Y_r N_r = I$. Then, there exist $X_l, Y_l \in \mathbb{R}H_\infty$ such that:

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & Y_l \\ N_r & X_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (2.5.5)$$

This is referred to as the doubly coprime factorization or generalized Bezout identity, and is the cornerstone in the parameterization of all stabilizing controllers, as will be shown in the next section.

The notion of coprimeness readily extends to continuous-time as well as discrete-time systems, lumped and distributed systems, and one- and multi-dimensional systems. Therefore, all these situations can be captured within the single framework of stable factorization approach [15, 13].

2.6 Parameterization of all stabilizing controllers:

In the preceding section, the notion of coprime factorization over $\mathbb{R}H_\infty$ was defined. The main intention was to express a system as an irreducible quotient of two rational proper and stable elements in $\mathbb{R}H_\infty$. In the current section, the relation between coprime factorization over $\mathbb{R}H_\infty$ and internal stability of the LFT feedback interconnection of figure 2.3.1 will be discussed. Moreover, parametric characterization of all compensators which stabilize a given plant will be developed. As stated in Lemma 2.4.4, internal stability of P in the basic configuration of figure 2.4.1 is equivalent to that of P_{22}. This accordingly suggested considering P_{22} as the system to be stabilized in the associated configuration of figure 2.4.2. To motivate what follows, take the case where P_{22} is a scalar function with coprime factorization $P_{22} = nd^{-1}$, and let $x, y \in \mathbb{R}H_\infty$ being scalar functions satisfying the Diophantine equation $xd - yn = 1$. We claim that $K = y^{-1}x$ is

a stabilizing controller for P_{22}. To see this, note that in figure 2.4.2 the mapping from $[v_1^T \ v_2^T]^T$ to $[u^T \ y^T]^T$ is:

$$\frac{1}{1 - KP_{22}}\begin{bmatrix} 1 & K \\ P_{22} & KP_{22} \end{bmatrix} = \frac{1}{xd - yn}\begin{bmatrix} xd & yd \\ xn & yn \end{bmatrix} = \begin{bmatrix} xd & yd \\ xn & yn \end{bmatrix}$$

Since all n, d, x, and y elements are in $\mathbb{R}H_\infty$, it is easily inferred that the closed-loop system is internally stable.

Noticing that for any $q \in \mathbb{R}H_\infty$ the following Diophantine equation is satisfied:

$$(x - nq)d - (y - dq)n = 1$$

it follows analogously that:

$$K = \frac{y - dq}{x - nq}$$

is an admissible controller for any $q \in \mathbb{R}H_\infty$. Hence, coprime factorization of P_{22} generates a family of stabilizing controllers over the proper and stable (but otherwise arbitrary) parameter q. The idea can be extended to the general matrix case in the form of following theorem.

Theorem 2.6.1 [6, 13] Suppose P is stabilizable. Let $P_{22} = N_r D_r^{-1} = D_l^{-1} N_l$ be right and left coprime factorizations of P_{22}, and let its corresponding doubly coprime factorization as:

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & D_l \end{bmatrix}\begin{bmatrix} D_r & Y_l \\ N_r & X_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (2.6.1)$$

Then the following statements are equivalent:

1. K internally stabilizes the feedback loop of figure 2.4.1.

2. $K = U_r V_r$, where

$$\begin{bmatrix} U_r \\ V_r \end{bmatrix} = \begin{bmatrix} D_r & Y_l \\ N_r & X_l \end{bmatrix} \begin{bmatrix} -Q \\ I \end{bmatrix}, \quad Q \in \mathbb{R}H_\infty \qquad (2.6.2)$$

3. $K = V_l^{-1} U_l$, where

$$[V_l \quad -U_l] = [I \quad Q] \begin{bmatrix} X_r & -Y_r \\ -N_l & D_l \end{bmatrix}, \quad Q \in \mathbb{R}H_\infty \qquad (2.6.3)$$

4. $K = \mathcal{F}_l(K_S, Q) \qquad (2.6.4)$

in which

$$K_s = \begin{bmatrix} X_r^{-1} Y_r & -X_r^{-1} \\ X_l^{-1} & X_l^{-1} N_r \end{bmatrix} = \begin{bmatrix} U_r V_r^{-1} & -V_l^{-1} \\ V_r^{-1} & V_r^{-1} N_r \end{bmatrix} \qquad (2.6.5)$$

The theorem clearly exhibits the relationship between coprime factorization and stabilizing compensators. In other words, coprime factorization and stabilization are intimately connected. The doubly coprime factorization, defined by generalized Bezout equations, leads to a parametric characterization of all controllers which internally stabilizes a given plant. All stabilizing controllers are expressed as a coprime factorization, including the elements of a doubly coprime factorization of the system to be stabilized and a proper stable but arbitrary parameter. In fact, the doubly coprime factorization is equivalent to the choice of a single stabilizing controller, and the theorem renders the whole set of stabilizing controllers constructed from that single choice, termed K_S. This is sometimes known as the nominal or central controller and the set of stabilizing controllers is obtained through augmenting K_S.

As mentioned in the last part of theorem 2.6.1, stabilizing compensators have the structure of a linear fractional transformation. In regards to the LFT construction of the closed-loop map H_{zw} in (2.3.5) and the composition of two LFTs formulated in [6] and designated by C_l, H_{zw} may be written as:

$$H_{zw} = \mathcal{F}_l(P, K) = \mathcal{F}_l\big(P, \mathcal{F}_l(K_s, Q)\big) = \mathcal{F}_l(C_l(P, K_s), Q) \qquad (2.6.6)$$

Denoting the composition $C_l(P, K_s)$ by T, gives the closed-loop map as:

$$H_{zw} = \mathcal{F}_l(T, Q) \qquad (2.6.7)$$

Accordingly, the closed-loop map description is given by:

$$H_{zw} = \mathcal{F}_l\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, K \right) = \mathcal{F}_l\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix}, Q \right) = T_{11} + T_{12} Q T_{21} \qquad (2.6.8)$$

which is clearly an affine parameterization in the design parameter Q. This is known as the Q-parameterization or Youla-Jabr-Bongiorno-Kucera (YJBK) parameterization, which was first developed in [25]. Thus, the given parameterization of all stabilizing compensators replaces the linear fractional parameterization $\mathcal{F}_l(P, K)$ of the closed-loop maps of interest with the affine parameterization $T_{11} + T_{12} Q T_{21}$. In many synthesis problems, these closed-loop maps form the design objectives of optimization problems with stability as a constraint. The Q-parameterization reduces the problem of search or optimization over the set of stabilizing controllers to a search or unconstrained optimization over the parameter $Q \in \mathbb{R}H_\infty$.

By substituting the parameterized set of stabilizing controllers of (2.6.2) and (2.6.3) in (2.6.1), the equality (2.6.1) can be expressed as:

$$\begin{bmatrix} V_l & -U_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & U_r \\ N_r & V_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (2.6.9)$$

It is evident that, by construction, $U_r, V_r, U_l, V_l \in \mathbb{R}H_\infty$, i.e. all are proper and stable. From the Bezout equations ensuing from (2.6.9) as:

$$V_l D_r - U_l N_r = I \qquad (2.6.10)$$
$$D_l V_r - N_l U_r = I$$

the following lemma can be deduced:

Lemma 2.6.2 [17] Suppose P_{22} admits a r.c.f as $P_{22} = N_r D_r^{-1}$. Then the following statements are equivalent:

1. (P_{22}, K) is internally stable.
2. K admits a l.c.f $K = V_l^{-1} U_l$, with $V_l D_r - U_l N_r = I$.

The analogous lemma when P_{22} admits a l.c.f is inferred by utilizing symmetry and interchanging the role of P_{22} and K. The possibility of interchanging the role of the plant and compensator enables one to also parameterize all plants which are internally stabilizable by a given controller [17].

According to the Diophantine identities in (2.6.10) and definition of coprimeness over $\mathbb{R}H_\infty$, it is easily concluded that there is not any unstable pole-zero cancellations between the plant and controller as required by internal stability.

Based on the parameterization of the stabilizing controllers, the general framework of figure 2.3.1 may be accordingly rearranged as depicted in the following configurations.

(a)   (b)
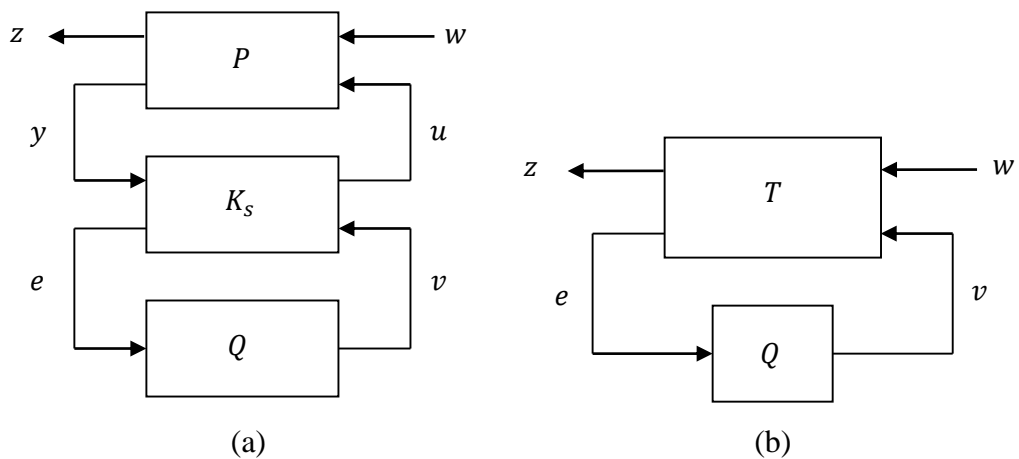
Figure 2.6.1 (a) Q-parameterization as modification to nominal controller (b) Closed-loop configuration for the class of all stabilizing controllers

In the next section, we will derive the state space model of the coprime factors of the plant and the solutions to the associated Bezout equations.

2.7 State space realization of coprime factors and solutions to the Bezout identities:

This section presents a state-space technique for computing the coprime factors over $\mathbb{R}H_\infty$ of P_{22}, as the system to be stabilized. The solutions to the associated Diophantine equations are also derived, leading to a state space realization of the coprime factors of the stabilizing controller. The section is mainly based on the results delivered in [21] and [83].

The approach discussed in [21] and [83], is in fact based on the polynomial matrix descriptions (PMD) of LTI systems, which naturally arises when the dynamical behaviour of a system is described by differential (or difference) equations of order higher than one. The polynomial matrix description of a system is characterized by the matrix quadruple $\{P, Q, R, W\}$ whose entries are real polynomials in the differential (difference) operator. Expressing a transfer matrix as a fraction of two polynomial matrices, called polynomial matrix fractional description (PMFD), can be viewed as representation of internal realizations of the transfer matrix. This is in view of the well-known fact that, right PMFD of a transfer matrix G as $G = RP^{-1}$, corresponds to the controllable PMD of G. Specifically, R and P are polynomial matrices in the differential (or difference) operator, originating from an internal description that is controllable, and signify the transfer matrix numerator and denominator, respectively. Analogously, left PMFD of G as $G = P^{-1}Q$ corresponds to the observable PMD of G. The elements Q and P, respectively, denoting the transfer matrix numerator and denominator, are again polynomial matrices in the differential (difference) operator which originate from an observable internal description. These concepts are extensively discussed in [10], chapter 7. In there, system state space description which involves only first order differential (difference) equations is considered as a special case of PMDs. In other words, PMDs are generalizations of state space descriptions. This has been utilized in [21, 83] to recast the achieved results in terms of state space realizations. First, a relation between the coprime proper and stable factorization of a transfer matrix and coprime polynomial matrix factorization of the same transfer matrix has been developed. This is accomplished via the correspondence of PMFDs to the internal description of the system, although, this advantage appears to be lost in the case of factorizations over $\mathbb{R}H_\infty$. The relation has been expressed in the form

of a theorem and leads to the main results as follows. Note that for the sake of generality, the results are given for the case $D_{22} \neq 0$ in the realization of P_{22}.

Theorem 2.7.1 [21, 83] All proper stable right coprime factorizations of $P_{22} = N_r D_r^{-1}$ are achieved by applying stabilizing linear state feedback to a stabilizable and detectable realization of P_{22}.

The linear state feedback considered is $u = Fx + r$ in which F is the state feedback gain matrix such that $(A + B_2 F)$ is Hurwitz, and r is an external input vector known as command or reference input. This renders the state space realizations of N_r and D_r as:

$$N_r = \begin{bmatrix} A + B_2 F & B_2 \\ C_2 + D_{22} F & D_{22} \end{bmatrix} \quad , \quad D_r = \begin{bmatrix} A + B_2 F & B_2 \\ F & I \end{bmatrix} \qquad (2.7.1)$$

Theorem 2.7.2 [21, 83] All proper stable left coprime factorizations of $P_{22} = D_l^{-1} N_l$ are achieved through implementing full-order observers in a stabilizable and detectable realization of P_{22}.

Consequently, the state space model of N_l and D_l are obtained as:

$$N_l = \begin{bmatrix} A + HC_2 & B_2 + HD_{22} \\ C_2 & D_{22} \end{bmatrix} \quad , \quad D_l = \begin{bmatrix} A + HC_2 & H \\ C_2 & I \end{bmatrix} \qquad (2.7.2)$$

in which H is the observer gain, selected so that $A + HC_2$ is Hurwitz. Obviously, the theorem and the results are duals to the theorem 2.7.1 and can be obtained by applying the stabilizing feedback H^T to any controllable and detectable realization of P_{22}^T, although in [21] they have been established through direct constructive proofs.

The following theorem gives the solutions of the Bezout identities corresponding to right and left coprime factorizations of P_{22} as in (2.6.10).

Theorem 2.7.3 [21, 83] All solutions of (2.6.10) can be determined from the coprime factors of a combined state feedback/observer structure.

In view of the doubly coprime factorization of (2.6.9), and lemma 2.6.2 and theorem 2.7.3, it can be readily inferred that the family of all stabilizing compensators is realizable as an observer-based state feedback.

The state space characterization of the solutions to the Diophantine equations, and correspondingly, those of the coprime factors of the stabilizing controllers are obtained as:

$$
V_l = \begin{bmatrix} A + HC_2 & -(B_2 + HD_{22}) \\ F & I \end{bmatrix} \quad , \quad U_l = \begin{bmatrix} A + HC_2 & -H \\ F & 0 \end{bmatrix} \qquad (2.7.3)
$$

$$
V_r = \begin{bmatrix} A + B_2F & -H \\ C_2 + D_{22}F & I \end{bmatrix} \quad , \quad U_r = \begin{bmatrix} A + B_2F & -H \\ F & 0 \end{bmatrix} \qquad (2.7.4)
$$

By substituting (2.7.1), (2.7.3) and (2.7.4) in (2.6.5), the state space description of the "generator" of all stabilizing controllers can be obtained as:

$$
K_S = \left[\begin{array}{c|cc} A + B_2F + HC_2 + HD_{22}F & -H & B_2 + HD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array} \right] \qquad (2.7.5)
$$

An interpretation of the structure of stabilizing controllers (2.7.5) can be obtained via Figure 2.6.1(a): In this diagram e can be interpreted as the output prediction error $\hat{y} - y = (C_2\hat{x} + D_{22}u) - y$ of an observer, and v is just an auxiliary input added before the observer tap to the output of the nominal controller K_s [4]. The controller constructed in this manner is sometimes referred to as observer-based controller. Thus, every controller which stabilizes P can be realized as an observer-based controller.

Referring to the state space realization of the generalized plant in (2.3.2) and relation (2.6.6) which expresses T as the composition of the pair (P, K_s), the state space characterization of T can be obtained as:

$$T = \begin{bmatrix} A + B_2F & -HC_2 & -HD_{21} & B_2 \\ 0 & A + HC_2 & B_1 + HD_{21} & 0 \\ \hline C_1 + D_{12}F & C_1 & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{bmatrix} \qquad (2.7.6)$$

Note that since the state feedback gain matrix F, and the observer gain matrix H are chosen such that $(A + B_2F)$ and $(A + HC_2)$ are Hurwitz, T is stable.
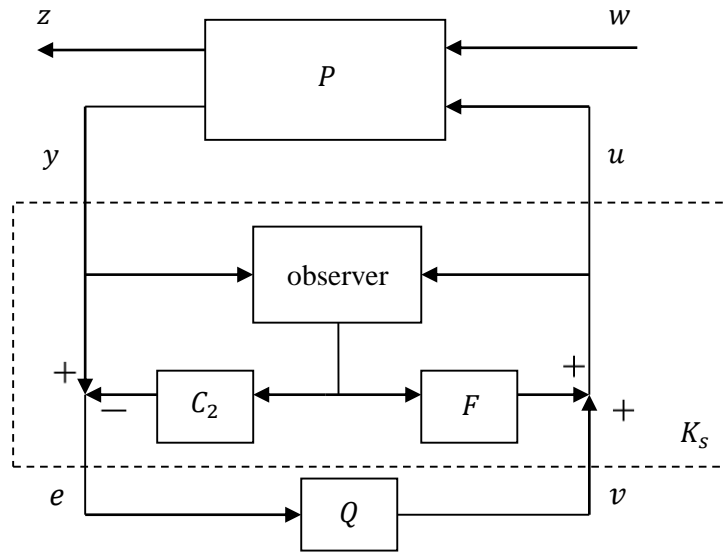
Figure 2.7.1 Interpretation of Q-parameterization for estimated state feedback K_s

It can be shown that in (2.6.8), T_{11} is simply H_{zw} achieved with the central controller K_s, T_{12} is the map from v to z, and T_{21} is the map from w to e. The key to the Q-parameterization is that the closed-loop map from v to e is zero. In other words, Q "sees" no feedback.

2.8 Conclusion:

This chapter presented the general framework on which this thesis is based. The internal and external description of the framework, which is in the form of Linear Fractional Transformation (LFT), were derived. The well-posedness and internal stability conditions

of the interconnection were investigated. It was shown that the controller is stabilizing if and only if it stabilizes the subsystem interconnecting measured outputs and control input. An alternate external characterization of the internal stability was also delivered. This was based on the matrix fractional representation of the plant and controller. It was argued that in the general case, to design a system which lies in a prescribed ring of operators representing desired properties, the plant and controller should be initially modelled as a quotient of operators from that ring. By choosing a model for the plant which is matched to the design criteria, a similar model may be specified for the feedback system constructed from the given plant. This in turn, is used to verify if the feedback system lies in the prescribed ring or equivalently has the desired properties. This approach provides the means for deriving a complete characterization of the family of all compensators which place the feedback system in the prescribed ring. The ring concerning the internal stability property is $\mathbb{R}H_\infty$, i.e. the set of proper and stable rational matrices.

The notions of right and left coprimeness, and the relationship between coprime factorization and stabilizing controllers were reviewed. It was shown that the doubly coprime factorization, defined by generalized Bezout equations, leads to a parametric characterization of all controllers which internally stabilizes a given plant. The parameterization is linear fractional in character. All admissible compensators were formulated as a coprime factorization over $\mathbb{R}H_\infty$, including the elements of the doubly coprime factorization of the system to be stabilized over $\mathbb{R}H_\infty$ and a proper stable but otherwise arbitrary parameter Q. This setting not only captures the usual notion of instability in terms of unstable closed-loop poles, but also excludes the possibility of unstable pole-zero cancellations between the plant and controller.

The state space realizations of the coprime factors of the plant and controller were given. Moreover, it was shown that the admissible controllers have the form of a stable observer combined with a stabilizing state feedback, which in fact can be generated by augmenting a nominal (central) controller.

Recasting the internal stability problem in this framework and the resulting characterization of the stabilizing controllers, reduced the linear fractional description of the closed-loop system to an affine parameterization in terms of the design parameter Q. This in turn, reduces a search or optimization over the set of stabilizing compensators to the search or optimization over the free parameter Q.

In the ensuing chapter, the achieved parameterization for admissible controllers will be naturally extended to parameterize the whole family of state deadbeat regulators which affinely depend on a free FIR parameter Q. In later chapters, the characterization will be exploited to recast the deadbeat controller synthesis problem subject to time and frequency domain constraints as optimization problems.

Chapter 3

Deadbeat controller design; state space and polynomial approaches

3.1 Introduction:

This chapter is dedicated to the revision of the state deadbeat regulator control problem and the two main synthesis approaches, out of the several ones which tackle the problem, namely the state space and the algebraic (transfer function) approach.

Deadbeat control is widely used in the literature. Some authors refer to deadbeat problem as the time-optimal or minimum-time controller, which is steering the states to the origin in minimum number of time steps. So, deadbeat on its own could inherently mean time-optimal. However, deadbeat control may also be inferred as achieving the final state in just finite number of time steps, rather than minimum number of steps. Usually in order to avoid confusion, the regulator is specified as minimum-time or time-optimal deadbeat control where the time-minimality is required. The deadbeat problem without the additional constraint of time optimality is also known as the finite settling time (FST) problem, and was first introduced by Karcanias [28].

In this chapter, both the time-optimal and non-optimal (or FST) deadbeat controllers are considered in parallel. The type of the controller will specifically be determined, unless when it is clear from the context.

It should be noted that deadbeat is an attribute just exclusive to discrete time systems and has no correlate in continuous time systems. This stems from the solution to the differential and difference state equations of the continuous and discrete time systems, respectively. Due to the exponential nature of the state equation solution in the continuous time case, the exponentially decaying error vanishes only in the limit as time goes to infinity.

In this chapter we first introduce the concept of controllability which was the early fundamental contribution to the solution of state deadbeat regulator problem. Within the state space framework, two classes of deadbeat controllers are developed. The first which is based on the concepts of reachability and k-th controllable subspace, as was proposed by Kalman, is known as the dynamic approach. One of the major properties of the controller achieved in this way is the nilpotency of the closed-loop transfer matrix. This feature inspired the second approach, referred to as the spectral approach, in which state deadbeat problem is treated as an eigenvalue assignment problem. The two design classes are discussed in this chapter. The other synthesis procedure based on polynomial algebra and known as the algebraic or transfer function approach, is also presented. This chapter concludes by giving a numerical algorithm for computation of the state deadbeat gain.

3.2 State deadbeat controller- Definition:

In this section, the two problems of the minimum-time and non-minimum-time, equivalently known as the FST or just deadbeat regulation will be formally introduced.

Definition 3.2.1 [29] A linear state feedback controller, generating an input sequence $u(1), u(2), \dots, u(n)$, which forces a linear discrete time system from any arbitrary initial state $x(0)$ to a desired final state $x(d)$ in the minimum number of control iterations n, is called a minimum-time deadbeat regulator compensator. Without loss of generality, it can be assumed that $x(d)$ equals the zero state.

Accordingly, deadbeat controllability is the possibility of finding a control sequence of finite length for any set of initial conditions, which renders the actual state to be equal to the desired state.

Definition 3.2.2 [28] A linear discrete time system exhibits an internal (external) finite settling time response if for a step change in any of its inputs and for any initial condition, all the internal (external) signals settle to a new steady state value in a finite number of time steps. The values of the finite settling time and that of the steady state are left free.

In the case that both the internal and external signals settle to the new steady state value, the problem is referred to as Total Finite Settling Time (TFST).

It is clear from the definitions that FST (or deadbeat) is a generalization of the minimum-time deadbeat problem. The degree of freedom which could be attained through relaxing the time-minimality constraint in deadbeat control, i.e. demanding only that every state be transferred to the origin in at most n steps, may be exploited to apply additional performance criteria, for instance, decreasing control input magnitude.

One of the major hindrances to minimum-time deadbeat controller is that they are usually achieved at the cost of producing large magnitude control signals, which in turn may lead to poor robustness. This is a natural attribute to expect, since all states (if possible) are intended to be driven to the origin in the shortest possible time. As is well known, there is typically a trade-off between excessive control signals and settling time specifications. The magnitude of control signal can be decreased by increasing the settling time. Despite its bad reputation for poor robustness however, investigating deadbeat controllability in a system could reveal intrinsic properties and performance limitations of the system. Therefore, it can be used as a starting point in the synthesis of a better controller [30].

3.3 State deadbeat controller- Dynamic approach:

As stated earlier, there are two major deadbeat controller synthesis schemes in the state space framework; dynamic and spectral approaches. In this section, we first study the dynamic approach which is based on the fundamental system concepts of controllability and reachability. Consider a multi-input LTI discrete time system described by the difference equation of the form:

$$x(k + 1) = Ax(k) + Bu(k), \quad k = 0, 1, 2, \dots \qquad (3.3.1)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. It is assumed that the system matrix A and the input matrix B are of full rank. The assumption that B is full rank corresponds to the fact the

system has no redundant inputs. Any linearly dependent columns corresponding to redundant inputs can always be eliminated.

It is well-known that the system matrix of any discrete time system obtained by sampling a continuous time system is non-singular. However, this is not necessarily true for more general forms of discrete time systems. Here, as we are only interested in studying the basis of the deadbeat synthesis problem, for the sake of simplicity it is assumed that the A matrix is invertible. In the general case, it is important to be able to design a compensator without this restriction. The following definitions and results are standard.

Definition 3.3.1 [31, 33] The system (3.3.1) is completely v-step controllable (controllable to the origin) if there exists a control sequence $u = \{u(k), u(k + 1), \dots, u(k + v - 1)\}$ which steers the states of the system from $x(k)$ to the origin in v time steps.

Definition 3.3.2 [31] The system (3.3.1) is completely v-step reachable (controllable from the origin) if there exists a control sequence $u = \{u(k - v + 1), u(k - v + 2), \dots, u(k)\}$ which steers the states of the system from the origin to $x(k)$ in v time steps.

Lemma 3.3.3 [31, 18] The system (3.3.1) is completely v-step reachable if and only if its controllability matrix is full rank, i.e.:

$$\text{rank } C_v = \text{rank } [B, AB, \dots, A^{v-1}B] = n \text{ for some } v \leq n \qquad (3.3.2)$$

If A is invertible, criterion (3.3.2) will also be a necessary and sufficient condition for v-step controllability. Otherwise, it is just a sufficient (and not necessary) condition for controllability [18, 31].

Lemma 3.3.4 [31] If A is invertible, the system (3.3.1) is v-step controllable if and only if:

$$\text{rank } [A^{-1}B, A^{-2}B, \dots, A^{-v}B] = n \text{ for some } v \leq n \qquad (3.3.3)$$

Depending on the context, v is referred to as the reachability or controllability index (or both) of the system. It is the smallest possible integer for which the controllability matrix is full rank, i.e. $v = \min \{ i \mid \text{rank}(C_i) = n \}$ [32, 33].

Corresponding to the concept of v-step controllability, the i-th controllable subspace S_i which defines the set of i-step controllable states of the system (3.3.1), is defined as: [18, 36]

$$S_i = \left\{ x_0 \mid A^i x_0 \in A^{i-1}B + \cdots + B \right\} = \text{Im} \left[A^{-1}B + A^{-2}B + \cdots + A^{-i}B \right] \tag{3.3.4}$$

Now the deadbeat control problem can be stated as follows:

Definition 3.3.5 [34] Minimum-time deadbeat control problem is that of finding a control sequence $u = \{u(k), u(k+1), \dots, u(k+v-1)\}$ such that the states of the system (3.3.1) at any instant k are driven to the origin in at most $v \leq n$ time steps under the action of u.

The definition is in view of the following lemma:

Lemma 3.3.6 [33] The minimum number of time steps needed to transfer any initial state $x(0)$ to the origin is equal to the controllability index v.

In general, if P is the number of iterations taken to transfer the states to the origin, the maximum achievable value for P is equal to the order of the system n, while the minimum value is the controllability index v [35].

Regarding the definition of the i-th controllable subspace S_i and with respect to lemma 3.3.6, it can be easily seen that S_v is the maximal space of initial states of the system that can be transferred to the origin in minimum number of steps. If $S_v \in \mathbb{R}^n$, then every $x(0) \in \mathbb{R}^n$ can be steered to the origin in at most v time steps.

The following geometric properties of S_i are known [173, 37, 38]:

$${0} = S_0 \subset S_1 \subset \cdots S_v = S_{v+1} = \cdots \qquad (3.3.5)$$

$$S_{v+1} = A^{-1} \left(S_v + \text{Im} \left(B \right) \right) \qquad (3.3.6)$$

From the expression (3.3.5) it is inferred that the transferred initial states to the origin remain in the origin for $i \geq v$.

The first step in development of a state deadbeat controller is the selection of n linearly independent column vectors from the full rank matrix S_v [27, 37, 34, 52]. There are several ways to accomplish this. In [39], Luenberger has established two methods which are based on two different rearrangements for selection of the columns of S_v. In the first method, the selection procedure is contrived such that the n linearly independent vectors are of the form:

$$[A^{-1}b_1, \ldots, A^{P_1-1}b_1, A^{-1}b_2, \ldots, A^{P_2-1}b_2, \ldots, A^{P_r-1}b_m] \qquad (3.3.7)$$

The requisite here is that, no vector of the form $A^k b_i$ is selected unless all lower powers of A times b_i are also included.

In the second method, the vectors are checked over for dependency in the following rearrangement [32, 39]:

$$\left[A^{-1}b_1, A^{-1}b_2, \ldots, A^{-1}b_m, A^{-2}b_1, A^{-2}b_2, \ldots, A^{-i}b_m \right] \qquad (3.3.8)$$

In either of (3.3.7) and (3.3.8), any columns of the input matrix B can be considered as the starting point (Here, without loss of generality, we have assumed that b_1 is selected). This in turn, results in non-uniqueness of the derived set of linearly independent column vectors. In [39], among various ways of selecting n linearly independent columns of S_v, these two specific selection methods are of special interest as they lead to canonical forms composed of fundamental companion matrices located in blocks alongside the diagonal. This feature offers certain design advantages [39].

An alternative selection procedure for choosing n linearly independent columns of S_v is through the ordered selection, expressed in the form of following definition:

Definition 3.3.7 [37] Let $\{Y_k\}$ be a sequence of $n \times m$ matrices and $L_k = \text{rank} [Y_1, \dots, Y_k] - \text{rank} [Y_1, \dots, Y_{k-1}]$. An ordered selection for $\{Y_k\}$ is a sequence of matrices $\{D_k\}$ with D_k of dimension $m \times L_k$ for which:

$$\text{Im} [Y_1 D_1, \dots, Y_k D_k] = \text{Im} [Y_1, \dots, Y_k] \text{ for each } k.$$

The matrix $[Y_1 D_1, \dots, Y_k D_k]$ is full rank. $\{D_k\}$ can be simply chosen by eliminating any column of $[Y_1, \dots, Y_k]$ which is linearly dependent on the set of columns which precedes it.

Owing to the many different ways of choosing the sequence $\{D_k\}$, the ordered selection is nonunique.

Kalman [27] showed that for a single input system, a time-optimal state deadbeat regulator is in the form of a state feedback:

$$u(k) = Fx(k), \quad k = 0, 1, \dots, v - 1 \qquad (3.3.9)$$

The result was then generalized to the multi input systems in [34]. Note that the state feedback nature of the control law is not an a priori assumption but is forced on us by the requirement that every state be driven to the origin in minimum time steps. However, in the general case of the deadbeat regulator, the requirement that the control law is in the form of linear state feedback is imposed as an assumption [27, 34].

As mentioned earlier, the feedback gain matrix is obtained in terms of the n linearly independent columns of S_v [29, 32, 33, 34, 37]. Since the selection procedure can be accomplished in many different ways, this leads to the non-uniqueness of the resulting compensator.

Regarding the state space description of the discrete time system in (3.3.1) and the deadbeat control law in (3.3.9), the controlled closed-loop system is expressible in the following homogenous form:

$$x(k + 1) = (A + BF)x(k) \qquad\qquad (3.3.10)$$

Also, the trajectory from $x(0)$ to the origin is expressed as:

$$x(k) = (A + BF)^k x(0) = A_{cl}^k\, x(0) \qquad\qquad (3.3.11)$$

In the sequel, some of the features of the closed-loop state transition matrix A_{cl}^k are adduced. These properties facilitate the development of the second class of deadbeat regulators based on the eigenvalue assignment.

Property 1: [33, 40] In the optimal-time deadbeat control, the closed-loop system matrix $(A + BF)$ is a nilpotent matrix with index of nilpotency equal to the system controllability (reachability) index v, i.e. $(A + BF)^v = 0$. In the general case of the deadbeat control, the index of nilpotency will be greater than v, but with the maximum value equal to the order of the system n.

Property 2: [33, 34, 40] A_{cl} has all of its eigenvalues at zero, and the eigenvectors span its null space.

Property 3: [35] In the minimum-time deadbeat control, if q and n are respectively the geometric and algebraic multiplicity of the zero eigenvalue, A_{cl} will be similar to a Jordan matrix consisting of $q \leq m$ (the number of inputs) Jordan blocks J_{r_i} of orders r_i, $i = 1, 2, \dots, q$.

$$J = \begin{bmatrix} J_{r_1} & & & & \\ & J_{r_2} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & J_{r_q} \end{bmatrix} \qquad\qquad (3.3.12)$$

where:

$$\begin{cases} q \leq m \\ r_1 + r_2 + \cdots + r_q = n \\ r_1 \geq r_2 \geq \cdots \geq r_q \\ r_1 = v \end{cases} \qquad (3.3.13)$$

Each Jordan block is a nilpotent matrix of order r_i, with ones on the first superdiagonal and zeros elsewhere. r_i's are referred to as the reachability indices of the system (3.3.1). It is evident that in the non-minimum-time deadbeat control, $v < r_1 \leq n$. In this case $A_{cl}^{r_1} = 0$. It is apparent from these properties that the deadbeat synthesis problem corresponds to placement of all the eigenvalues at the origin under the action of a constant state feedback and therefore it is equivalent to a generalized eigenvalue problem. This forms the basis of our discussion in the succeeding section.

3.4 State deadbeat controller- Spectral approach:

As was indicated at the beginning of this chapter, in the state space framework deadbeat synthesis problem can be handled through two different approaches, namely dynamic and spectral approach. As we saw, the design in the dynamic scheme is based on the selection of n linearly independent columns of the controllable subspace. These independent columns provide the means for computation of the control law which is in the form of a static state feedback [34].

The three major attributes of the closed-loop system derived in this way were studied. We observed that the deadbeat system has all its eigenvalues at the origin of the complex plane. In other words, the state transition matrix of the closed-loop system A_{cl} is a nilpotent matrix. This feature of the state matrix enables us to treat the deadbeat synthesis problem as that of assigning a prescribed set of eigenvalues via linear state feedback. Hence, the problem can be restated as follows:

Definition 3.4.1 Assuming a linear state variable feedback control law, determine an $m \times n$ real constant feedback gain matrix F such that the closed-loop system matrix $A_{cl} = (A + BF)$ is a nilpotent matrix.

From reformulation of the problem, it is readily inferred that the design problem is equivalent to a special eigenvalue assignment problem in which to attain the minimality criteria, we aim to assign the closed-loop system matrix a Jordan form of (3.3.12) with given specifications of (3.3.13). This in turn, implies the assignment of any allowable minimum polynomial and any admissible Jordan form of the closed-loop matrix A_{cl}. Specifically, the Jordan form of A_{cl} corresponding to a zero eigenvalue of algebraic multiplicity n and geometric multiplicity q, will have q Jordan blocks J_{r_i} of orders r_i, $i = 1, 2, \dots, q$ where r_1 is equivalent to the controllability index of the system v. Evidently, the number of Jordan blocks can not exceed that of the control inputs m [35]. In this case, A_{cl} is a nilpotent matrix of degree $r_1 = v$, i.e. $A_{cl}^v = 0$, and the closed-loop characteristic and minimum polynomials take the respective forms of z^n and z^v. The general case of $v < r_1 \leq n$, results in a deadbeat or FST controller with which the system settles to the desired value in a finite but not optimum number of time steps. This type of compensator is sometimes referred to as r_1-step deadbeat controller.

As stated by Wonham [45], controllability in the sense of lemma 3.3.4 is equivalent to arbitrary eigenvalue assignment under the action of linear state feedback law (3.3.9). So, the definition 3.4.1 is legitimate under the controllability assumption of the system (3.3.1). It should be clear from the Jordan structure of (3.3.12) that the closed-loop eigenvalues do not uniquely define a closed-loop system. This is owing to the freedom in eigenvalue assignment. For instance, for a fourth-order system with three inputs and the controllability index $v = 2$, the four possible admissible Jordan structures are:

$$
J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
J_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

The forms J_1 and J_2 are achieved through implementation of a minimum-time deadbeat controller, while J_3 and J_4 by 3-step and 4-sted deadbeat controllers, respectively.

Beyond the freedom in selecting the Jordan structure, non-uniqueness of the solution to the deadbeat problem is also attributable to the freedom in selecting the associated set of

assignable eigenvectors and generalized eigenvectors [34, 35, 41, 42]. Freedom in selection of different sets is exploited to shape the transient response characteristics of the system. This is in regard with the fact that, while the overall speed of the closed-loop system response is determined by its eigenvalues, different choices of eigenvector sets corresponding to the specified (zero value) eigenvalues determine different transient responses [43, 44].

Property 3 of the deadbeat compensator which indicates the similarity of the closed-loop state transition matrix A_{cl} to the Jordan structure of (3.3.12), suggests that deadbeat controller may be more readily achieved through transforming the system into Luenberger's canonical form [39]. The solution is developed in [34].

3.5 State deadbeat controller- Algebraic or transfer function approach:

As was mentioned at the beginning of this chapter, a second major procedure to synthesize deadbeat controllers is the algebraic or transfer function approach. This approach rests upon the sequence characterization of discrete time systems, which in turn arises from the fact that in such systems the signals are defined in discrete time instances, i.e. as sequences. In other words, a discrete time system is a system which is stimulated with sequences as its inputs and generates sequences as its outputs. Kalman [47] and Kucera [46] were the pioneers in studying discrete time systems in terms of sequences. This was in view of the dual nature of sequences as formal power series over a field on one hand, and as power series expansions of functions over the same field on the other hand. These two notions coincide, or are isomorphic if and only if the field is an infinite field.

The most general representation of a sequence is as formal Laurent series over a field. Sequences can be classified into sets of rational, recurrent, causal, and stable sequences. Each of these sets forms either a field, or a ring. The notion of sequences can be generalized to the matrices. Matrices whose elements are sequences, are called sequential matrices. These matrices can be expressed in terms of matrix fractions with elements from the aforementioned sets of sequences. The algebraic approach is established based on an isomorphism between series expansions of functions and certain classes of formal series

in one indeterminate, both over an infinite field $\mathfrak{f}$. More specifically, in the case of MIMO lumped LTI discrete time systems the isomorphism is established between rational function matrices over the field of real numbers $\mathbb{R}$, which characterizes the input-output behaviour of such systems, and recurrent or rational sequential matrices in one indeterminate over $\mathbb{R}$.

One of the main advantages of recasting the deadbeat synthesis problem, and even more general performance problems of linear systems in the algebraic framework is that the problem can be reduced to the solution of certain polynomial, or polynomial matrix equations. This in turn, provides the means for the complete parameterization of solutions and allows for the formulation of a CAD-based methodology in design problems. In what follows, we outline some of the basic tools of the algebraic approach within the context of the discrete time systems.

3.5.1 Sequences, polynomials, and classifications:

Given any field $\mathfrak{f}$, the set of integers $\mathbb{Z}$, and the set of natural numbers $\mathbb{N}$, the set of all infinite sequences denoted by $\mathfrak{f}^{\mathbb{Z}}$, can be expressed as [52]:

$$f = \{f_{-n}, f_{-n+1}, \dots, f_{-1}; f_0, \dots, f_k, \dots\}, \quad f_k \in \mathfrak{f} \text{ and } n \in \mathbb{N} \qquad (3.5.1.1)$$

By convention, the elements of negative and non-negative indices are separated by a semicolon.
An $f \in \mathfrak{f}^{\mathbb{Z}}$ can alternatively be represented as [52]:

$$f = \{f_n, f_{n+1}, \dots, f_k, \dots\}, \quad f_k \in \mathfrak{f} \text{ and } n \in \mathbb{Z} \qquad (3.5.1.2)$$

By defining the operations of pointwise addition and convolutory multiplication between the elements of $\mathfrak{f}^{\mathbb{Z}}$, respectively as [46, 52]:

$$h_h = (f + g)_n := f_n + g_n, \quad f, g \in \mathfrak{f}^{\mathbb{Z}}$$

$$h_n = (f * g)_n := \textstyle\sum_{l+m=n} f_l g_m \qquad\qquad (3.5.1.3)$$

$\mathfrak{f}^{\mathbb{Z}}$ forms a field. The zero element of $\mathfrak{f}^{\mathbb{Z}}$ is the sequence $\{0; 0, 0, \dots\}$ and the identity element is the sequence $\{0; 1, 0, 0, \dots\}$ [48].

Definition 3.5.1.1 [46] We call the sequence $\{0; 0, 1, 0, 0, \cdots\}$ an indeterminate and we will denote it by x.

Through the definition of convolutory multiplication in (3.5.1.3) and the definition 3.5.1.1, it can be shown that $x^k := x * x * \dots * x$, $k \in \mathbb{Z}$ is a sequence of zeros except for the element 1 at the k-th position. Therefore, any sequence $f \in \mathfrak{f}^{\mathbb{Z}}$ can be written in the form of a formal Laurent series:

$$f = f_n x^n + f_{n+1} x^{n+1} + f_{n+2} x^{n+2} + \cdots, \quad n \in \mathbb{Z} \text{ fixed} \qquad\qquad (3.5.1.4)$$

It should be emphasized that x is simply an indeterminate over the field f. It represents in no sense an element of f, and in fact it is an element of $\mathfrak{f}^{\mathbb{Z}}$ and serves as the position-maker in the sequence.
As the name implies, the series (3.5.1.4) is formal; it is just an alternative and convenient way of representing the sequence (3.5.1.1). Hence, it should not be interpreted as a function of x and there is no question of convergence whatsoever.

The series expression of (3.5.1.4) will be adopted to represent the elements of $\mathfrak{f}^{\mathbb{Z}}$, denoted by $f(x)$. $\mathfrak{f}^{\mathbb{Z}}$ itself, called the field of formal Laurent series over f, will be designated by $\mathfrak{f}\langle x \rangle$. An important notion in connection with $\mathfrak{f}\langle x \rangle$ is that of order given below.

Definition 3.5.1.2 [46] Given $f = \sum f_k x^k$, a nonzero element of $\mathfrak{f}\langle x \rangle$, the smallest integer n such that x^n appears in the sequence is called the order of f and is denoted by $\tau(f)$.

Using the notion of order, an important subring of $\mathfrak{f}\langle x \rangle$, namely the ring of formal power series can be introduced.

Definition 3.5.1.3 [48] The subset of $f\langle x \rangle$ comprising all sequences in one indeterminate x with nonnegative orders is called the set of formal power series over f and is designated by $f[[x]]$. Thus, a sequence $f \in f[[x]]$ is represented by (3.5.1.4) when $n \in \mathbb{N}$.

Under the operations of pointwise addition and convolutory multiplication, $f[[x]]$ forms a ring. A simple relation between the ring $f[[x]]$ and the field of formal Laurent series is established and stated in the form of following theorem.

Theorem 3.5.1.4 [48] $f\langle x \rangle$ is the field of quotients, or the field of fractions of the domain $f[[x]]$.

A simplification which follows from formal series representation of the sequences is that the convolutory multiplication turns to the usual multiplication.

By restricting the number of elements in formal power series representation, an important subring of the domain $f[[x]]$ is achieved and defined next.

Definition 3.5.1.5 All formal power series of finite length form the set of formal polynomials in one indeterminate x over the field f and is denoted by $f[x]$. Mathematically, the set is represented as:

$$f[x] = \{ f = f_0 + f_1 x + \cdots + f_n x^n \mid f_k \in f \} \qquad (3.5.1.5)$$

The above definition implies that polynomials are sequences with nonnegative orders with all but a finite number of elements zero. Consequently, polynomials are regarded as algebraic objects with the indeterminate x over the field f, rather than functions of x.

A notion similar to that of the order of a sequence is the degree of a polynomial. If f_n in (3.5.1.5) is nonzero, then n is called the degree of f. It is designated by $\partial(f)$ and is a function from $f[x]$ to $\mathbb{N} \cup \{-\infty\}$. By convention, the degree of the zero polynomial is defined as $-\infty$. The set of polynomials $f[x]$ forms a subdomain of $f[[x]]$ and the units of the set are polynomials of degree zero [46, 50]. The key fact in development of the

algebraic approach in synthesis of discrete time systems, is the isomorphism between the formal polynomials and the polynomial functions.

Taking the field $\mathfrak{f}$, $\mathfrak{f}^{\mathfrak{f}}$ designates the set of all functions from $\mathfrak{f}$ to $\mathfrak{f}$. Let $f = f_0 + f_1 x + \cdots + f_n x^n \in \mathfrak{f}[x]$. We can associate with f a function $\phi(f) : \mathfrak{f} \to \mathfrak{f}$ such that [50]:

$$\phi(f)(u) = f_0 + f_1 u + \cdots + f_n u^n , \quad u \in \mathfrak{f}$$

$f(u)$ is known as the value of f at u. The function ϕ is in fact, a map from $\mathfrak{f}[x]$ to $\mathfrak{f}^{\mathfrak{f}}$ and the Im ϕ is called ring of polynomial functions on $\mathfrak{f}$, whereas Ker ϕ comprises all elements of $\mathfrak{f}[x]$ which vanish identically on $\mathfrak{f}$.

Theorem 3.5.1.6 [50] The map $\phi : \mathfrak{f}[x] \to \mathfrak{f}^{\mathfrak{f}}$ is injective, if and only if $\mathfrak{f}$ is an infinite field.

In view of the preceding theorem, formal polynomials and polynomial functions are isomorphic when they are defined over an infinite field. As a result,

$$f = f_0 + f_1 x + \cdots + f_n x^n, \quad f_n \in \mathfrak{f} \qquad (3.5.1.6)$$

may be regarded either as a finite sequence with x as an indeterminate, or as a polynomial function where x will be a variable in $\mathfrak{f}$. In the latter case, we use x in lieu of the italic x, to emphasize its role as a variable.

An important property of formal polynomials and polynomial functions is that they both are integral domains. Therefore, their fields of fractions can be constructed. The field of fractions of formal polynomials is referred to as rational fractions or rational sequences in one indeterminate x over f, and is denoted by $\mathfrak{f}(x)$. Accordingly, $\mathfrak{f}(x)$ the field of rational functions in $\mathfrak{f}$, is the field of fractions of the polynomial functions. These two fields are isomorphic if and only if f is an infinite field. This is also in view of the following theorem.

Theorem 3.5.1.7 [53] Let $\mathcal{R}$ and $\mathcal{R}'$ be two isomorphic integral domains and let $\hbar$ and $\hbar'$ be their respective fields of fractions. If T_0 is an isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$, then T_0 can be extended in a unique manner to an isomorphism T of $\hbar$ onto $\hbar'$.

With regards to the theorem 3.5.1.4, and the fact that $\mathfrak{f}[x]$ is a subdomain of $\mathfrak{f}[[x]]$, it is readily inferred that rational fractions are isomorphic to a subfield of formal Laurent series $\mathfrak{f}\langle x\rangle$, called field of rational formal Laurent series. Consequently, rational fractions are sequences that can be formulated in the form of formal Laurent series of (3.5.1.4).

To summarize, any rational Laurent series $\mathfrak{f}(x)$ over a field f of the form:

$$f(x) = f_n x^n + f_{n+1} x^{n+1} + f_{n+2} x^{n+2} + \cdots, \quad n \in \mathbb{Z} \text{ fixed} \qquad (3.5.1.4)$$

is treated either as a rational sequence with x as an indeterminate over the field f, or as a rational function in f. Obviously, in the case of f being an infinite field there is no distinction between the two notions.

In the case of discrete time systems and in general linear dynamical systems, the infinite field f is the set of real numbers $\mathbb{R}$. For the study of discrete time systems the indeterminate is designated by d and accordingly, the field of formal Laurent series over $\mathbb{R}$ is termed $\mathbb{R}\langle d\rangle$. An element of $\mathbb{R}\langle d\rangle$ can mathematically be described as:

$$f = f_n d^n + f_{n+1} d^{n+1} + f_{n+2} d^{n+2} + \cdots, \quad n \in \mathbb{Z} \text{ fixed} \qquad (3.5.1.7)$$

According to the infinite nature of $\mathbb{R}$, (3.5.1.7) is inferred either as a sequence over $\mathbb{R}$, or a function in $\mathbb{R}$. In the case of (3.5.1.7) being a rational sequence, it can be considered as the impulse response of a linear, lumped discrete time system. Subsequently, the indeterminate d being a sequence of the form $\{0; 0, 1, 0, \dots\}$ will serve as a position maker the powers of which represent the discrete instances [46]. Alternatively, if the Laurent series (3.5.1.7) is regarded as a rational function in $\mathbb{R}$ with d being a variable, it mathematically describes the transfer function of a linear, lumped discrete time system. Having replaced d with z^{-1}, the series is no more than the z-transform of the impulse response $f = \{f_n, f_{n+1}, \dots, f_k, \dots\}$ [51]. It should be pointed out that in the latter case,

series (3.5.1.7) can be considered as a function with real coefficients from $\mathbb{C}$ to $\mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers and the algebraic closure of $\mathbb{R}$.

In what follows, we concisely describe some of the classifications of the sequences in $\mathbb{R}\langle d \rangle$, which are related to system properties.

Definition 3.5.1.8 [46] A sequence $f \in \mathbb{R}\langle d \rangle$ is called recurrent if there exists nonnegative integers r, s and reals $\lambda_1, \dots, \lambda_r$ such that:

$$f_{j+r} + \lambda_1 f_{j+r-1} + \cdots + \lambda_r f_j = 0, \quad j = n+s, n+s+1, \dots \qquad (3.5.1.8)$$

The set of recurrent sequences forms a field and will be designated by $\mathbb{R}\{d\}$. This field is isomorphic to the field $\mathbb{R}(d)$ of rational sequences. For this reason, we can refer to both fields of recurrent and rational sequences with the same notation $\mathbb{R}(d)$.
A main feature of the recurrent expression (3.5.1.8) is that it has the form of a linear difference equation. The significance of recurrent sequences is attributed to the possibility of expressing them as polynomial fractions.

Definition 3.5.1.9 [46] A recurrent sequence f is said to be causal if it has a nonnegative order. Set of causal sequences is denoted by $\mathbb{R}^o(d)$. A $f \in \mathbb{R}^o(d)$ may be represented as:

$$f = f_0 + f_1 d + f_2 d^2 + \cdots \qquad (3.5.1.9)$$

$\mathbb{R}^o(d)$ is a subdomain of $\mathbb{R}[[d]]$ and its units are sequences of order 0, i.e. of the form:

$$f = f_0 + f_1 d + f_2 d^2 + \cdots, \quad f_0 \neq 0 \qquad (3.5.1.10)$$

Definition 3.5.1.10 [46] Let $f = \{f_0, f_1, \dots, f_k, \dots\} \in \mathbb{R}^o(d)$. f is called stable or Hurwitz if and only if f is absolutely summable, i.e. :

$$\sum_{k=0}^{\infty} |f_k| < \infty$$

The set of stable sequences designated by $\mathbb{R}^+(d)$, is a subdomain of $\mathbb{R}^o(d)$, and is an integral domain. Moreover, it can be easily seen that the set of polynomials $\mathbb{R}[d]$ is a subdomain of $\mathbb{R}^+(d)$. These inclusion properties can be expressed as:

$$\mathbb{R}[d] \subset \mathbb{R}^+(d) \subset \mathbb{R}^o(d) \subset \mathbb{R}(d) \qquad (3.5.1.11)$$

where $\subset$ denotes the subring property.

The notions of causality and stability can also be extended to polynomials. Through defining causal and stable polynomials, we will be able to give a fractional characterization of all recurrent sequences.

Definition 3.5.1.11 [46] Let $f = f_0 + f_1 d + \cdots + f_n d^n \in \mathbb{R}[d]$. Then f is causal if it is a unit of $\mathbb{R}^o(d)$. According to (3.5.1.10), f is causal if and only if $f_0 \neq 0$.

Definition 3.5.1.12 [46, 52] Let $f \in \mathbb{R}[d]$. f is called stable if it is a unit of $\mathbb{R}^+(d)$. This holds true if and only if the roots of f lie outside the closed unit disc $\mathbb{D}$.

The succeeding theorem, states the fractional description of causal and stable sequences in terms of polynomials.

Theorem 3.5.1.13 [53] Let $f = b/a \in \mathbb{R}(d)$ be a coprime polynomial fraction. Then, the ring of stable sequences is quotient ring of $\mathbb{R}[d]$ with a being a stable polynomial. Accordingly, the ring of causal sequences is a quotient ring of $\mathbb{R}[d]$ with a being a causal polynomial.

In the above theorem and the two definitions prior to that, the polynomials are considered as formal polynomials, i.e. algebraic objects with d as an indeterminate over $\mathbb{R}$, rather than functions of d. However, due to the isomorphism between formal polynomials and polynomial functions, and the isomorphism between rational sequences and rational functions, both resulting from the infinite nature of $\mathbb{R}$, the causal and stable rational functions may be defined correspondingly.

Definition 3.5.1.14 [52] Let $f = b/a \in \mathbb{R}(d)$ be the set of rational functions and a, b be polynomial functions over $\mathbb{R}$ with variable $d \in \mathbb{C}$. Then the set of rational functions with denominator a being a polynomial function corresponding to a causal formal polynomial is called the set of causal rational functions and is denoted by $\mathbb{R}^o(d)$. Accordingly, the set of rational functions with denominator a being a polynomial function corresponding to a stable formal polynomial is called the set of stable rational functions and is designated by $\mathbb{R}^+(d)$.

We close this section by giving a frequency domain characterization of the rings of causal and stable rational sequences, which constitutes the basis for the synthesis of the deadbeat compensator.

According to definitions 3.5.1.11 and 3.5.1.12, and theorem 3.5.1.13, a causal rational sequence $f(d)$ is expressible as a coprime polynomial fractions $b(d)/a(d)$ in which $a(d)$ has a nonzero constant term $a_0 \neq 0$, i.e. $a(d)$ has no roots at the origin. If the roots of $a(d)$ or equivalently the poles of $f(d)$ lie outside the closed unit disc $\mathbb{D}$, the coprime polynomial fraction forms a stable rational sequence. In the ensuing section, the notions reviewed in the current section will be extended to the case of matrices.

3.5.2 Sequential matrices and their classification:

This section is mainly an extension of the notion of sequences and their classifications as previously defined, to the case of matrices. The section is in reference to [46]. Generally, matrices with all elements sequences are called sequential matrices. In particular, the set of $l \times m$ matrices whose elements are in $\mathbb{R}(d)$, $\mathbb{R}^o(d)$, and $\mathbb{R}^+(d)$ are respectively classified and designated as rational-sequence $\mathbb{R}_{lm}(d)$, causal-sequence $\mathbb{R}_{lm}^o(d)$, and stable-sequence $\mathbb{R}_{lm}^+(d)$ matrices. d is not a variable, but rather an indeterminate over $\mathbb{R}$ [46]. Any rational sequence matrix $P \in \mathbb{R}_{lm}(d)$, can be recast in the form of a matrix recurrent sequence as:

$$P = P_n d^n + P_{n+1} d^{n+1} + \cdots, \qquad P_k \in \mathbb{R}^{l \times m} \qquad (3.5.2.1)$$

Similar to the order of a sequence, the order of P denoted by $\tau(P)$ is n where $P_n \neq 0$. $\tau(P) \geq 0$ defines a causal sequential matrix, whereas, if $\tau(P) \geq 0$ while $(p_{ij})_k \to 0$ as $k \to \infty$, P will be a stable sequential matrix.

Having defined the set of polynomial-sequence matrices $P[d] \in \mathbb{R}^{l \times m}[d]$ as the set of finite sequence matrices with order greater or equal to zero, the following inclusion property is obvious:

$$\mathbb{R}_{lm}[d] \subset \mathbb{R}_{lm}^+(d) \subset \mathbb{R}_{lm}^o(d) \subset \mathbb{R}_{lm}(d) \qquad (3.5.2.2)$$

A polynomial-sequence matrix is said to be causal if it is $\mathbb{R}^o(d)$-unimodular, i.e. its determinant is expressible as rational fractions with both numerator and denominator causal polynomials.

A stable polynomial-sequence matrix can be defined accordingly as $\mathbb{R}^+(d)$-unimodular, i.e. its determinant is a rational fraction with both numerator and denominator stable polynomials.

Corresponding to the causal and stable polynomials, a polynomial matrix $P(d)$ is causal if and only if $\det P_0 \neq 0$, where P_0 is the constant matrix term of $P(d)$. Also, $P(d)$ will be stable if and only if the roots of $\det P(d)$ lie outside the closed unit disc $\mathbb{D}$. It is well known that any sequential matrix can be expressed as polynomial matrix fractions.

3.5.3 Sequential description of discrete time systems:

In the two preceding sections, a summary of basic concepts of sequences and sequential matrices were given to accommodate the needs of discrete time systems, which in turn enables us to set up a framework in terms of sequences to reformulate the deadbeat synthesis problem. This framework is established based on the isomorphism between recurrent formal Laurent series in one indeterminate d over $\mathbb{R}$, and rational functions over $\mathbb{R}$, which is due to the infinite nature of $\mathbb{R}$. Many concepts and properties stated in this section may be found in a wide range of textbooks, like [22, 46, 51, 138, 20, 139, 140].

One of the main features in discrete time systems is that the processed and generated signals are in the form of sequences. Mathematically speaking, a discrete time system is a transformation $\mathcal{G}$ that uniquely maps a real input vector-sequence u to a real output vector-sequence y, i.e. :

$$y = \mathcal{G}[u], \quad u, y \in \mathbb{R}\langle d \rangle \qquad (3.5.3.1)$$

A key sequence in the study of discrete time systems is the unit sequence $d^0 = \{0; 1, 0, 0, ...\}$, which is mainly known as the impulse sequence or just impulse and is designated by δ. The response of the system $\mathcal{G}$ to the impulse sequence is called the impulse response and termed g, i.e. $g := \mathcal{G}[\delta]$. In control theory, the impulse sequence δ is treated as a unity amplitude signal applied at time zero. In view of our discussion in section 3.4.1, $\delta d^k = d^k$ may be regarded as an impulse applied at time k. The response of the system to this signal is denoted by g^k.

For the general case of an input sequence of the form $u = \{u_p, u_{p+1}, ...\}$, using the formal Laurent series representation of u , the output of a discrete time system is achieved as:

$$y = \mathcal{G}[u] = \mathcal{G}\left[\sum_{k=p}^{\infty} u_k d^k\right] \qquad (3.5.3.2)$$

Since the systems that are dealt with in this work are linear time invariant (LTI) systems for which the principle of superposition applies, the output of a multivariable linear time invariant discrete time system with l outputs and m inputs is computed as:

$$y_i = \sum_{j=1}^{m} g_{ij} * u_j := G * u, \quad i = 1, ..., l \qquad (3.5.3.3)$$

where $*$ and g_{ij} respectively designates the convolutory multiplication and the impulse response of the system at the i-th output due to an impulse at the j-th input. Also, G defines the impulse response matrix of the multivariable system.

Any discrete time system which can be described by the input-output relation of (3.5.3.1) is a system with memory, that is the output at a specific time instance k_0 depends on the input applied before and/or after k_0. A subclass of LTI systems are those in which the output at any time instance k_0 depend on the input only for $k \leq k_0$. Such systems which

arise naturally and are of practical importance are known as the causal systems. The condition for causality of an LTI discrete time system is stated in the form of the following theorem.

Theorem 3.5.3.1 An LTI discrete time system is causal if and only if $\tau(G) \geq 0$.

Another class of discrete time systems of particular importance is that of BIBO stable systems, for which any bounded input signal results in a bounded output signal.

Theorem 3.5.3.2 An LTI discrete time system with input space $\mathcal{U}$ and output space $\mathcal{Y}$ respectively subspaces of infinite sequences $\mathbb{R}^m\langle d \rangle$ and $\mathbb{R}^l\langle d \rangle$, is BIBO stable if and only if:

$$\sum_{K=\tau_{ij}}^{\infty} \left| \left(g_{ij} \right)_k \right| < \infty, \quad i = 1, \dots, l , j = 1, \dots, m$$

where $\tau_{ij} = \tau\left(g_{ij} \right)$.

The aforementioned classes of LTI discrete time systems can be characterised in a sequential framework. Generally, for a multivariable LTI discrete time system with m inputs and l outputs, the impulse response matrix G, is any sequential matrix in one indeterminate d over $\mathbb{R}$, i.e. $G \in \mathbb{R}_{lm}\langle d \rangle$. The systems we consider in our work, are a subclass of LTI systems, namely lumped systems in which the impulse response matrix is a recurrent, or rational sequential matrix. Therefore, $G \in \mathbb{R}_{lm}(d)$. Now, the causal and stable systems can be classified accordingly.

Corollary 3.5.3.3 A lumped LTI discrete time system with $\mathcal{U} \subseteq \mathbb{R}^m\langle d \rangle$ and $\mathcal{Y} \subseteq \mathbb{R}^l\langle d \rangle$ is causal if and only if G is a causal sequential matrix, i.e. $G \in \mathbb{R}_{lm}^o(d)$.

Corollary 3.5.3.4 A causal lumped LTI discrete time system with $\mathcal{U} \subseteq \mathbb{R}^m\langle d \rangle$ and $\mathcal{Y} \subseteq \mathbb{R}^l\langle d \rangle$ is BIBO stable if and only if G is a stable sequential matrix, i.e. $G \in \mathbb{R}_{lm}^+(d)$.

As was mentioned in the previous section, rational sequential matrices can be expressed as polynomial matrix fractions. The conditions for causality and stability can be expressed in terms of MFDs.

Theorem 3.5.3.5 Consider an LTI discrete time system G, and let $(N(d), D(d))$, $\left(\widetilde{N}(d), \widetilde{D}(d)\right)$ be right and left coprime polynomial MFDs of $G(d)$. Then the system is causal if and only if $D(d)$, or $\widetilde{D}(d)$ is causal, i.e. $\det D(0) \neq 0$, or $\det \widetilde{D}(d) \neq 0$.

In addition, the system is stable if and only if $D(d)$, or $\widetilde{D}(d)$ is stable, i.e. the roots of $\det D(d)$, or $\det \widetilde{D}(d)$ lie outside the closed unit disc $\mathbb{D}$.

As it is well-known, the impulse response matrix of a system gives rise to its transfer function matrix. Therefore, another way of characterising the input-output behaviour of a system is through its transfer function.

Definition 3.5.3.6 Consider an LTI discrete time system with impulse response matrix $G = \{G_r, G_{r+1}, \dots, G_k, \dots\}$, $G_k \in \mathbb{R}^{l \times m}$. The z-transform of G is called the transfer function matrix and is denoted by $\tilde{G}(z)$:

$$\tilde{G}(z) = \sum_{k=r}^{\infty} G_k z^{-k}, \quad G_k \in \mathbb{R}^{l \times m}, z \in \mathbb{C} \qquad (3.5.3.4)$$

In order to establish a more precise relationship between $G(d)$ and $\tilde{G}(z)$, assume a system with an impulse response $g = d$. For a general input u of order $\tau(u) = p$, as $u = u_p d^p + u_{p+1} d^{p+1} + u_{p+2} d^{p+2} + \cdots$, the output y is a sequence of the form:

$$y = u_p d^{p+1} + u_p d^{p+2} + u_p d^{p+3} + \cdots$$

As it can be seen, the order of y is $p + 1$, from which it is deduced that what is applied as input at time instance k is shifted forward to the time instance $k + 1$ in the output. This shows that d is a delay operator. By equating $z = d^{-1} = \{1; 0, 0, \dots\}$, z will be an advance shift operator. According to the dual nature of d as an indeterminate or a variable, z can also be interpreted as an indeterminate or a complex variable, while $z = d^{-1}$ represents a bilinear transformation.

Now, from (3.5.3.4) and the definition of impulse response matrix $G(d)$ we have:

$$G(d) = G(z^{-1}) = \tilde{G}(z) \qquad\qquad (3.5.3.5)$$

It should be clear that $G(d)$ and $\tilde{G}(z)$ may be tackled either as rational functions or sequential matrices. To emphasize its sequential nature, $G(d)$ is called the impulse response matrix, whereas the transfer function matrix term is used to distinguish the functional aspect of $\tilde{G}(z)$. Due to isomorphism, $G(d)$ and $\tilde{G}(z)$ are also known as the d-transfer function and z-transfer function, respectively.

As an example, consider the case of a SISO LTI discrete time system with impulse response the rational sequence $g(d)$. In view of the discussion in section 3.5.1, $g(d)$ takes the form of a formal Laurent series over $\mathbb{R}$ and is expressible as a fraction of two coprime polynomials as:

$$g(d) = g_l d^l + g_{l+1} d^{l+1} + \cdots = \frac{b_0 + b_1 d + \cdots + b_m d^m}{a_0 + a_1 d + \cdots + a_n d^n} = \frac{b(d)}{a(d)}$$

$$(3.5.3.6)$$

where $a_n, b_m \neq 0$. If series (3.5.3.6) is regarded as a formal series and considered to be causal, hence describing a causal rational sequence $\{g_l, g_{l+1}, \dots, g_k, \dots\}$, it can be treated as the impulse response of a causal LTI discrete time system. However, if due to isomorphism, series (3.5.3.6) is considered as a causal rational function of d, then it can be treated as the transfer function of a causal LTI discrete time system. This becomes clear by performing the bilinear transformation $z^{-1} = d$ in series (3.5.3.6):

$$\tilde{g}(z) = g_l z^{-l} + g_{l-1} z^{-l-1} + \cdots = z^{n-m} \frac{b_0 z^m + \cdots + b_m}{a_0 z^n + \cdots + a_n} = g(z^{-1})$$

$$(3.5.3.7)$$

$\tilde{g}(z)$ is the z-transform of the causal impulse response $\{g_k\}$ and therefore, transfer function of a causal discrete time system. According to causality condition $a_0 \neq 0$, and in view of the coprimeness of the numerator and denominator, it can be shown that $\tilde{g}(z)$ is a proper rational function in z.

In a similar manner, if (3.5.3.6) is a formal and stable series, thus indicating an absolutely summable sequence $\{g_k\}$, it can be regarded as the impulse response of a BIBO stable discrete time system. Accordingly, $\tilde{g}(z)$ will be the transfer function of such system, and

indeed, is the z-transform of the absolutely summable impulse response $\{g_k\}$. As $d = z^{-1}$, the stable poles of $g(d)$ lying outside the closed unit disc $\mathbb{D}$, corresponds to the stable poles of $\tilde{g}(z)$ which are placed inside the open unit disc $\mathbb{D}$.

In the general case of a MIMO system with m inputs and l outputs, a causal rational matrix $G(d) \in \mathbb{R}^o_{lm}(d)$ representing a system with no poles at zero in the d-plane, corresponds to a proper rational matrix $\tilde{G}(z) \in \mathbb{R}^{l \times m}_{pr}(z)$ which designates a system with no poles at infinity in the z-plane. Similarly, a stable rational matrix $G(d) \in \mathbb{R}^+_{lm}(d)$ whose poles lie outside the closed unit disc, corresponds to a proper and stable rational matrix $\tilde{G}(z) \in \mathbb{R}^{l \times m}_{ps}(z)$ whose poles lie inside the open unit disk $\mathbb{D}$. We have seen that the causality and stability of systems can be verified from the location of the poles of its impulse response matrix $G(d)$ (theorem 3.5.3.5). The peculiarity of the d-plane description is that the forbidden region for stability and causality, which is the closed unit disc in the d-plane, is rather simple.

A measure of the total number of finite as well as infinite poles of $G(d)$, respectively designated by $\delta^f_m(G)$ and $\delta^\infty_m(G)$, is its McMillan degree $\delta_m(G)$, defined as $\delta_m(G) := \delta^f_m(G) + \delta^\infty_m(G)$ [52]. $\delta^f_m(G)$ and $\delta^\infty_m(G)$ can be derived from the Smith-McMillan forms of $G(d)$ over $\mathbb{R}_{pr}(d)$ and $\mathbb{R}[d]$, respectively [10, 20, 22, 54]. Although Smith and Smith-McMillan forms are central in the study of the structure of systems, an alternative way for derivation of $\delta_m(G)$ is through properties of $G(d)$, rather than resorting to the aforementioned decompositions. The approach for computing $\delta_m(G)$ follows from the next theorem.

Theorem 3.5.3.7 [55, 56] Let $G(d) \in \mathbb{R}^{l \times m}(d)$. Consider any right coprime MFD of $G(d)$ over $\mathbb{R}[d]$ as $G(d) = N(d)D^{-1}(d)$, and the right composite matrix $R_G(d)$ defined as $R_G(d) := [N^T(d) \quad D^T(d)]^T$. Then:

$$\delta_m(G) = \partial(R_G(d)) \qquad (3.5.3.8)$$

It should be pointed out that the degree of the composite matrix $R_G(d)$ is invariant of the MFD and is known as the Forney dynamical order of $G(d)$. Moreover, if $(N(d), \ D(d))$ and $(N'(z), \ D'(z))$ are respectively two right coprime MFDs of the d- and z-transfer matrices, with corresponding composite matrices $R_G(d)$ and $R_{\tilde{G}}(z)$, then:

$$\delta_m\big(G(d)\big) = \delta_m\left(\tilde{G}(z)\right) = \partial(R_G(d)) = \partial(R_{\tilde{G}}(z)) \qquad (3.5.3.9)$$

In other words, the McMillan degree is also invariant under the bilinear transformation [55, 56].

The left composite matrix of $G(d)$ corresponding to a left MFD over $\mathbb{R}[d]$ as $G(d) = \tilde{D}^{-1}(d)\tilde{N}(d)$, can be determined in a similar manner as $L_G(d) := \begin{bmatrix} \tilde{N}(d) & \tilde{D}(d) \end{bmatrix}$. Note that in the general definition of the right and left composite matrices, the MFDs are not necessarily coprime.

3.5.4 Deadbeat controller synthesis in an algebraic framework:

During three former sections, the aim was to construct the foundations necessary for recasting the synthesis of the state deadbeat compensator in the algebraic framework. In this section, the design procedure and its main results will be presented. The results will be developed first for the general case of the deadbeat or finite settling time problem, from which the results regarding the specific case of the minimum-time deadbeat regulation will be extracted.

In the algebraic or transfer function approach the main control configuration which is used is in the form of a unity feedback also known as the one-parameter feedback configuration, as depicted in figure 3.5.4.1. Although such a configuration may not be ideal one for complicated design problems, it can still accommodate several control problems, e.g. tracking and disturbance rejection.
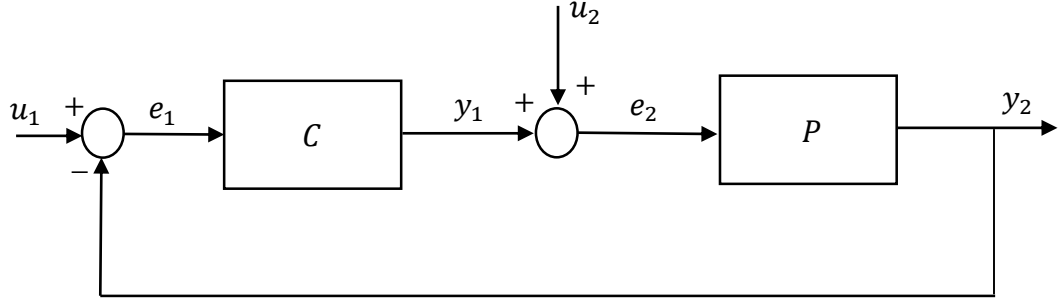
Figure 3.5.4.1 The unity feedback configuration

In the figure, consider the plant P and the controller C with respective d-transfer functions as $P \in \mathbb{R}^{l \times m}(d)$, and $C \in \mathbb{R}^{m \times l}(d)$. The externally applied vector inputs are denoted by u_1 and u_2, while y_1 and y_2 show the output signals. All signals are vector sequences in d. By $\mathcal{R}$ we will denote $\mathbb{R}[d]$ or any quotient ring of $\mathbb{R}[d]$ whose field of fractions is $\mathbb{R}(d)$, and $\mathcal{M}(\mathcal{R})$ is the set of matrices with elements from $\mathcal{R}$. Moreover, $U[\mathcal{R}]$ designates the set of $\mathcal{R}$-unimodular matrices with elements from $\mathcal{R}$. Depending on whether u_1, or u_2 is the external control signal, the construction correspondingly represents either a feedback or cascade compensation.

The unity feedback system referred to as the pair (P, C), can be described by either of the two following transfer function matrices:

$$H(P, C) \quad : \quad e := H(P, C)u$$
$$W(P, C) \quad : \quad y := W(P, C)u \qquad (3.5.4.1)$$

$H(P, C)$ and $W(P, C)$ are related, and their relationship is expressed in the form of the following lemma.

Lemma 3.5.4.1 [20] Suppose that the pair (P, C) is well-formed. Then:

$$W(P, C) = F(H(P, C) - I) \quad , \quad F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \qquad (3.5.4.2)$$

and so, $W(P, C) \in \mathcal{M}(\mathcal{R})$ if and only if $H(P, C) \in \mathcal{M}(\mathcal{R})$.

Although from the above lemma it is inferred that either transfer functions can be used, it is usually common to use the error transfer function matrix $H(P, C)$ [52].

The mathematical description of the error transfer matrix in view of the Schur formula [141] can be obtained as:

$$H(P,C) = \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} = \begin{bmatrix} I - P(I + CP)^{-1}C & -P(I + CP)^{-1} \\ (I + CP)^{-1}C & (I + CP)^{-1} \end{bmatrix}$$

$$(3.5.4.3)$$

$$= \begin{bmatrix} (I + PC)^{-1} & -(I + PC)^{-1}P \\ C(I + PC)^{-1} & I - C(I + PC)^{-1}P \end{bmatrix}$$

Clearly, the closed-loop system is well-formed if $t(d) := \det(I + PC) = \det(I + CP)$ is a nonzero element of $\mathbb{R}(d)$. An alternative characterization of the mathematical description of the closed-loop system may be obtained by substituting the plant and controller by their coprime MFDs over $\mathbb{R}[d]$, i.e.

$$P = N_P D_P^{-1} = \widetilde{D}_P^{-1} \widetilde{N}_P \qquad \text{and} \qquad C = N_C D_C^{-1} = \widetilde{D}_C^{-1} \widetilde{N}_C \qquad (3.5.4.4)$$

which gives [46]:

$$H(P,C) = \begin{bmatrix} D_C \\ N_C \end{bmatrix} \widetilde{\Delta}_{cl}^{-1} [\widetilde{D}_P \quad -\widetilde{N}_P] + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

$$(3.5.4.5)$$

$$= \begin{bmatrix} -N_P \\ D_P \end{bmatrix} \Delta_{cl}^{-1} [\widetilde{N}_C \quad -\widetilde{D}_C] + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\Delta_{cl} := \widetilde{N}_C N_P + \widetilde{D}_C D_P \qquad \text{and} \qquad \widetilde{\Delta}_{cl} := \widetilde{N}_P N_C + \widetilde{D}_P D_C \qquad (3.5.4.6)$$

Theorem 3.5.4.2 [46] Δ_{cl} and $\widetilde{\Delta}_{cl}$ are associates, i.e.:

$$\Delta_{cl} \sim \widetilde{\Delta}_{cl}$$

This implies that Δ_{cl} and $\widetilde{\Delta}_{cl}$ share the same nonunit invariant polynomials.

In a commutative ring R with two elements $a, c \in R$, a and c are called associates if $a = cu$ and u is a unit of R [46].

The relation between the characteristic polynomial of the closed-loop system with minimally realized plant and controller designated by $\rho_H(d)$, and Δ_{cl} and $\widetilde{\Delta}_{cl}$ is established in the form of the following corollary.

Corollary 3.5.4.3 [46] The characteristic polynomial $\rho_H(d)$ when the plant and controller are minimally realized, is given by:

$$\rho_H(d) \sim \det\Delta_{cl} \sim \det\widetilde{\Delta}_{cl}$$

As it is shown in [46], the corollary 3.5.4.3 implies that the elementary pole-polynomials of $H(P, C)$ are associate to the nonunit invariant polynomials of Δ_{cl} and $\widetilde{\Delta}_{cl}$. Therefore, the pole structure of the feedback system can be described by either of the $\det\Delta_{cl}$ or $\det\widetilde{\Delta}_{cl}$.

As discussed earlier, two important features of the feedback configuration which need to be considered are those of the stability and well-posedness. These can be discussed in view of the above theorem and corollary. The closed-loop system of figure 3.5.4.1 is externally stable if $\Delta_{cl} \sim \widetilde{\Delta}_{cl}$ are stable polynomial matrices. Moreover, under the stabilizability and detectability assumptions for both the plant and controller, the closed-loop system is internally stable if and only if it is BIBO stable [20]. As we will see, in this framework the set of admissible controllers is also expressible in the form of a YJBK parameterization.

The physical realizability of the closed-loop system can be examined through verification of its well-posedness. Well-posedness condition implies that every subsystem of the interconnection is well-posed, and all the transfer functions from any input to any output are well-defined and causal. With $P, C \in \mathcal{M}(\mathbb{R}^0(d))$, and their corresponding coprime

MFDs over $\mathbb{R}[d]$ as given in (3.5.4.4), the feedback interconnection is well-posed if and only if $\det \Delta_{cl}(0) \sim \det \tilde{\Delta}_{cl}(0) \neq 0$ [20, 46].

For a strictly causal plant i.e. $P(0) = 0$, which is a discrete analog of a strictly proper system, the unity feedback will be well-posed for any causal controller C. For such a plant, any stabilizing controller C is causal and the closed-loop system is well-posed for any C. This is in view of the argument made in the former section that, in the d-plane causality is a special case of stability.

Having briefly discussed the framework and also the notions of stability and well-posedness which are two requisites of the closed-loop system, we now give the solution to the deadbeat (FST) problem and the parameterization of deadbeat controllers.

Lemma 3.5.4.4 [52] A causal discrete time system with impulse response $G(d)$, is deadbeat if and only if $G(d)$ is a polynomial matrix in d, or equivalently $G(d)$ is of finite duration. Such systems are known as Finite Impulse Response (FIR) systems and exhibit a finite settling time response to almost any recurrent input and not just to step inputs.

Therefore, the unity feedback system of figure 3.5.4.1 exhibits an FST response if and only if $H(P,C) \in \mathcal{M}(\mathbb{R}[d])$, or more precisely $H(P,C) \in \mathbb{R}^{(l+m) \times (l+m)}[d]$.

The succeeding theorem provides the solution to the MIMO finite settling time problem.

Theorem 3.5.4.5 [52, 57] Consider the unity feedback configuration of figure 3.5.4.1, with the plant P and the compensator C having coprime MFDs over $\mathbb{R}[d]$ as given in (3.5.4.4). The solution to the deadbeat problem exists if and only if:

$$\Delta_{cl} := \tilde{N}_c N_P + \tilde{D}_c D_P \in U(\mathbb{R}[d]) \qquad (3.5.4.7)$$

or equivalently:

$$\tilde{\Delta}_{cl} := \tilde{N}_P N_C + \tilde{D}_P D_C \in U(\mathbb{R}[d]) \qquad (3.5.4.8)$$

Moreover, the family of all causal deadbeat controllers is parameterized as:

$$\mathfrak{F}(P) = \{\, (N_C, D_C) : \ N_C = X + D_P R \,, \quad D_C = Y - N_P R \,,$$
$$R \in \mathcal{M}(\mathbb{R}[d]) \ \text{ and } \ |Y(0) - N_P(0)R(0)| \neq 0 \quad \text{if} \quad N_P(0) \neq 0\} \qquad (3.5.4.9)$$

$$\mathfrak{F}(P) = \{\, (\tilde{N}_c, \tilde{D}_c) : \ \tilde{N}_c = \tilde{X} + S\tilde{D}_P, \quad \tilde{D}_c = \tilde{Y} - S\tilde{N}_P \,,$$
$$S \in \mathcal{M}(\mathbb{R}[d]) \quad \text{and} \quad \left|\tilde{Y}(0) - S(0)\tilde{N}_P(0)\right| \neq 0 \quad \text{if} \quad \tilde{N}_P(0) \neq 0\} \qquad (3.5.4.10)$$

where S and R are arbitrary, and $X, Y, \tilde{X}, \tilde{Y}$ are appropriate polynomial matrices satisfying the following generalized Bezout identity:

$$\begin{bmatrix} -\tilde{X} & \tilde{Y} \\ \tilde{D}_P & \tilde{N}_P \end{bmatrix} \begin{bmatrix} -N_P & Y \\ D_P & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (3.5.4.11)$$

Two stated conditions in (3.5.4.9) and (3.5.4.10) are imposed to ensure causality of the controllers, as not all controllers $C \in \mathfrak{F}(P)$ are physically realizable. In the special case where the plant possesses a delay, i.e. $N_P(0) = 0$, the whole $\mathfrak{F}(P)$ family is causal [52]. The computation of the family $\mathfrak{F}(P)$ is accomplished through first computing a particular solution of the Diophantine equation:

$$\tilde{N}_c N_P + \tilde{D}_c D_P = I \qquad (3.5.4.12)$$

or:

$$\tilde{N}_P N_C + \tilde{D}_P D_C = I \qquad (3.5.4.13)$$

The set of deadbeat (FST) controllers can then be parameterized in a YJBK format. One way of achieving a solution to the above Diophantine equation is by reducing the problem to a standard linear algebra problem over $\mathbb{R}$ using Sylvester matrices. The procedure has been elaborated in [52], [57] and [163]. It is developed in such a way that provides the means to study the McMillan degree properties of the family of causal deadbeat controllers and consequently, characterizing the family based on the McMillan degree of the compensators.

Particular solutions of the expressions (3.5.4.12) and (3.5.4.13) are those of fixed, and minimum column and minimum row complexity solutions, which have been extensively discussed in [52] and [57]. According to theorem 3.5.4.5, the deadbeat controller shifts the poles of $H(P,C)$ to the infinity in the d-plane, and therefore externally stabilizes the closed-loop system. Under the stabilizability and detectability assumptions for both the plant and compensator, the feedback system is internally stable and all the controllable and observable eigenvalues are shifted to the origin of the z-plane. Moreover, the system exhibits an external FST response (definition 3.2.2). In the case that both subsystems are controllable and observable, the closed-loop configuration represents a total (internal as well as external) FST response [57].

For the case of SISO discrete time systems, it can be readily shown that the theorem 3.5.4.5 reduces to the following theorem.

Theorem 3.5.4.6 [164] In unity feedback configuration of figure 3.5.4.1, let $P(d) = \frac{n_P}{d_P}$ and $C(d) = \frac{n_C}{d_C}$. Then the solution of the deadbeat problem exists if and only if:

$$\delta(P,C) = n_P n_C + d_P d_C \in \mathbb{R} - \{0\} \qquad (3.5.4.14)$$

Moreover, the family of all causal deadbeat controllers is given by:

$$\mathfrak{F}(P) = \{(n_C, d_C) : \ n_C = x + td_P \ , \ d_C = y - tn_P \ , t \in \mathbb{R}[d] \text{ and}$$
$$y(0) - t(0)n_P(0) \neq 0 \text{ if } n_P(0) \neq 0\} \qquad (3.5.4.15)$$

where x and y is a particular solution pair of the Diophantine equation:

$$n_P x + d_P y = 1 \qquad (3.5.4.16)$$

The algebraic approach is explicated by means of the following example for the case of a SISO system.

Example 3.5.4.1 In figure 3.5.4.1, consider that the plant is described by its transfer function as [164, 52]:

$$P(d) = \frac{n_P(d)}{d_P(d)} = \frac{-0.0132d - 0.0139d^2}{1 - 2.1889d + 1.1618d^2}$$

and the controller is in the following form:

$$C(d) = \frac{n_C(d)}{d_C(d)}$$

Regarding theorem 3.5.4.6, the family of all deadbeat compensators satisfy the Diophantine equation:

$$(-0.0132d - 0.0139d^2)n_C + (1 - 2.1889d + 1.1618d^2)d_C = 1 \qquad (3.5.4.17)$$

A particular solution pair, designated by (x, y), to the above Diophantine equation may be achieved through reducing $[n_P \quad d_P]$ to its Smith form. Since n_P and d_P are coprime, there exists a 2×2 $\mathbb{R}[d]$-unimodular matrix U such that [46, 22]:

$$[n_P \quad d_P]U = [1 \quad 0] \qquad (3.5.4.18)$$

Such a U is:

$$U = \begin{bmatrix} -105.38 + 66.685d & 105.38 - 230.67d + 122.43d^2 \\ 1 + 0.7978d & 1.3911d + 1.4648d^2 \end{bmatrix}$$

Hence, according to (3.5.4.18), one particular solution of the equation (3.5.4.17) is:

$$x = -105.38 + 66.685d$$
$$y = 1 + 0.7978d$$

Correspondingly, since the plant has a delay, i.e. $n_P(0) = 0$, the family of all causal deadbeat compensators is parameterized as:

$$\mathfrak{F}(P) = \{(n_C, d_C) : n_C = x + td_P , d_C = y - tn_P , t \in \mathbb{R}[d]$$

An alternative approach for computing the solution to the Diophantine equation may be found in [52, 57, 164].

Based on the feedback construction of figure 3.5.4.1 and its mathematical description, for a step change at u_1 when $n_c = x$, the output y_2 is obtained as:

$$y_2 = 1.3911d + 1.9756d^2 + 1.0487(d^3 + d^4 + \cdots)$$

which equivalently may be described by the following sequence:

$$y_2 = \{0;\ 1.3911, 1.9756, 1.0487, 1.0487, \dots\}$$

It is easy to see that the output signal y_2 settles to the final value of 1.0487 from zero after two time steps. ∎

Having discussed the solution to the general problem of deadbeat control, the solution to the special case of the minimum-time deadbeat control synthesis problem, in which the states are transferred to the origin in not just finite but minimum number of time steps, may be stated in the form of the following theorem.

Theorem 3.5.4.7 [57] In the feedback configuration of figure 3.5.4.1 where the plant and compensator are both assumed to be controllable and constructible, the family of all causal controllers $C = N_C D_C^{-1}$ whose right composite matrix $R_C = [N_C^T \quad D_C^T]^T$ has the least possible column degrees is exactly the family of the minimum-time state deadbeat regulators.

Column degrees of the right composite matrix $R_C = [N_C^T \quad D_C^T]^T$ are less than or equal to $v - 1$, where v is the controllability index of the plant [52].

It should be noted that constructibility refers to the ability of determining the states from knowledge of the current and past outputs and inputs. While observability always implies constructibility, the reverse relation holds only when the state transition matrix of the system is nonsingular. Conditions for the constructibility may be found in [18].

In the next section, we present a numerical algorithm for synthesizing the deadbeat regulator via the spectral approach. In later chapters, this will be used to compute the state and observer gains in the observer-based deadbeat controllers.

3.6 A numerical algorithm for eigenvalue assignment:

According to the discussion in sections 3.3 and 3.4, the synthesis problem of state deadbeat regulator is expressible as a special eigenvalue assignment problem in which all the eigenvalues are clustered at the origin. In order to attain the time minimality criteria, the closed-loop system matrix A_{cl} is assigned a Jordan form of (3.3.12) with given specifications of (3.3.13). It was observed that the state transition matrix of the closed-loop system A_{cl} is a nilpotent matrix for which the index of nilpotency is the controllability index of the system v, i.e. $A_{cl}^v = 0$.

In what follows, we present a numerical algorithm developed in [58] which constructs a static state feedback of the form:

$$u(k) = Fx(k) \qquad (3.6.1)$$

for a multi-input LTI discrete time system, described by the difference equation:

$$x(k+1) = Ax(k) + Bu(k), \quad k = 0, 1, 2, \dots \qquad (3.6.2)$$

The dimension of the state and input vectors are assumed to be n and m, respectively. The numerical algorithm renders a state feedback of the form (3.6.1), such that the resulting closed-loop system has a nilpotent state transition matrix A_{cl} as $A_{cl}^i = (A + BF)^i$ for some minimal power i. Therefore, all the eigenvalues of A_{cl} will be placed at the origin of the complex plane and the solution to the homogenous part of the closed-loop system $x(k+1) = (A + BF)x(k)$ dies out after i steps.

The algorithm is based on the recursive construction of a unitary transformation, yielding a coordinate system in which the static state feedback F is computed by merely solving a

set of linear equations. The coordinate system is related to the Krylov sequence $A^{-1}B, A^{-2}B, A^{-3}B, \ldots$.

Evidently, one of the most-noticed aspects of any numerical algorithm is that of the stability of the algorithm, called the backward stability. The name follows from the fact that application of the algorithm induces an error known as the backward error, which can be interpreted as an error on the data. Therefore, backward stability is a property of an algorithm, which is in contrast to the other source of the error i.e. conditioning of the problem. Conditioning is exclusively associated with a problem and the specific data for that problem. A good algorithm therefore, is the one that is backward stable as the size of the generated errors is not affected by the algorithm, but mainly by the condition of the problem. An unstable algorithm may results in generation of large errors even when the problem is well conditioned [59].

In the algorithm, what makes the application of unitary transformations peculiar is that by exclusively implementing unitary transformations the backward stability of the algorithm is guaranteed. A sequence of such transformations will also be backward stable since the norm of each transformation is unit. Exceptions, however, have to be made in exploiting unitary transformations, e.g. in cases where a special structure is apparent in the data like sparse matrices [59, 60].

It is well known that arbitrary eigenvalue assignment under the action of control law given in (3.6.1) is equivalent to the controllability of the system [45]. As a result, the solution to the problem of regulator synthesis may be more readily achieved by transforming the state space model of the system into a canonical form which separates the assignable and unassignable parts of the system, or more precisely the controllable and uncontrollable subsystems. A numerically effective way is reducing the pair (A, B) to a block Hessenberg form through orthogonal similarity transformations. The process constructs an orthogonal matrix V such that:

$$V^T A V = V^{-1} A V = H \text{ and } V^T B = \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad (3.6.3)$$

where H is an upper Hessenberg matrix, i.e. $h_{ij} = 0$ for $i > j + 1$ (In a lower Hessenberg matrix $h_{ij} = 0$ for $j > i + 1$). (3.6.3) is called the controller-Hessenberg form of (A, B),

or the staircase form, and the pair $(H, \tilde{B})$ is known as the controller-Hessenberg pair of (A, B) [61]. This form can be achieved using Householder's or Given's method. In [61] and [62] a procedure for reduction to the controller-Hessenberg form using Householder's transformation and QR factorization with column pivoting is given. The algorithm is usually known as staircase algorithm and transforms the pair (A, B) to:

$$[V^T B|\ V^T A V] = [\tilde{B}|\ H] = \begin{bmatrix} B_1 & H_{1,1} & H_{1,2} & H_{1,3} & \cdots & H_{1,i} \\ & H_{2,1} & H_{2,2} & H_{2,3} & \cdots & H_{2,i} \\ & 0 & & & & \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ & 0 & & 0 & H_{i,i-1} & H_{i,i} \end{bmatrix} \qquad (3.6.4)$$

By construction, B_1 and all the sub-diagonal blocks are of full rank. The controllability, and controllable and uncontrollable subsystems are determined from the rank property of $H_{i,i-1}$. The block $H_{i,i-1}$ has either full rank, signifying that the pair (A, B) is controllable, or $H_{i,i-1}$ is a zero matrix signifying that the system is uncontrollable [61, 63]. In the case of uncontrollability, the following sub-matrix represent the controllable sub-system whose spectrum can be assigned arbitrarily:

$$\begin{bmatrix} H_{1,1} & \cdots & & H_{1,i-1} \\ H_{2,1} & \cdots & & H_{2,i-1} \\ 0 & & & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & H_{i-1,i-2} & H_{i-1,i-1} \end{bmatrix} \begin{matrix} \}r_1 \\ \}r_2 \\ \\ \vdots \\ \}r_{i-1} \end{matrix} \qquad (3.6.5)$$

In (3.6.3) the transformation V reduces the j-th controllable subspace $S_j(A, B)$ to [59]:

$$S_j(V^T A V, V^T B) = \text{Im} \begin{bmatrix} I_{d_j} \\ 0 \end{bmatrix} \qquad (3.6.6)$$

where $d_j = r_1 + r_2 + \cdots + r_j$, for $j = 1, \dots, i$. Therefore, the columns of V form orthogonal bases for the growing subspace $S_j(A, B)$. Orthogonality of the transformation V ensures stable numerical behaviour of the algorithm [63].

The first step in the numerical algorithm for the synthesis of the deadbeat regulator is reduction of the system to its staircase model as in (3.6.7) and (3.6.8), hence the solvability of the problem can be verified. The problem is solvable if all the uncontrollable modes are already at zero, in other words the uncontrollable subsystem is nilpotent.

It should be pointed out that due to the similarity of closed-loop matrices $(A + BF)$ and $(V^T A V + V^T B F V)$, transforming the system to its staircase model does not affect the design problem formulation.

$$[V^T B| \; V^T A V] = \begin{bmatrix} B_c & A_c & * \\ 0 & 0 & A_{\bar{c}} \end{bmatrix} = \qquad (3.6.7)$$

$$\begin{bmatrix} B_1 & A_{1,1} & A_{1,2} & & \cdots & A_{1,i} & A_{1,i+1} \\ & A_{2,1} & A_{2,2} & & \cdots & A_{2,i} & A_{2,i+1} \\ 0 & 0 & & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & A_{i,i-1} & A_{i,i} & A_{i,i+1} \\ 0 & 0 & & 0 & 0 & 0 & A_{i+1,i+1} \end{bmatrix} \begin{matrix} \}r_1 \\ \}r_2 \\ \vdots \\ \\ \}r_i \\ \}n_{i+1} \end{matrix} \qquad (3.6.8)$$
$$\quad m \quad r_1 \quad r_2 \qquad r_{i-1} \quad r_i \quad n_{i+1}$$

In (3.6.7), A_c and $A_{\bar{c}}$ respectively designates the controllable and uncontrollable subsystems, and their dimensions are equal to that of the associated controllable and uncontrollable subspaces. Also, i denotes the controllability index of the pair. In the sequel, the problem is assumed to be solvable. So, $A_{\bar{c}}$ is irrelevant and can be omitted. As stated before, by construction B_1 and the off-diagonal blocks $A_{j,j-1}$ in (3.6.8) have full row rank r_j, $j = 1, \dots, i$. The last n_{i+1} rows correspond to the uncontrollable subsystem. According to the rank properties of B_1 and the off-diagonal blocks, one can check that the dimension of the reachable subspace, also known as the Krylov subspace, of the whole system is equivalent to that of the controllable subsystem [59, 60]:

$$\dim R_j(A, B) = \dim R_j(A_c, B_c) = d_j \qquad (3.6.9)$$

where:

$$d_j = r_1 + r_2 + \cdots + r_j, \quad r_j = 0 \text{ for } j > i \qquad (3.6.10)$$

From the definitions of controllability in lemma 3.3.4, the j-th controllable subspace S_j in (3.3.4), and the fact that the closed-loop matrix $A_{cl} = (A + BF)$ can be chosen to be invertible, it follows that:

$$R_j = A_{cl}^j S_j \qquad (3.6.11)$$

which in view of the expression (3.3.6) can equivalently be indicated as:

$$S_j(A_{cl}, B) = \text{Im}\left[A_{cl}^{-1} B, \ A_{cl}^{-2} B, \ \ldots, \ A_{cl}^{-j} B \right] \qquad (3.6.12)$$

A feedback matrix F is a solution to deadbeat control problem if [37, 58]:

$$(A + BF)S_j \subset S_{j-1} \quad j = 1, \ldots, i \qquad (3.6.13)$$

The condition in (3.6.13) is just sufficient. However, to achieve the additional property of time minimality one seeks a feedback gain which satisfies that condition.

Now, if F is a solution of (3.6.13), and U is a unitary transformation partitioned in i blocks of r_i columns:

$$U = [U_1| \ldots |U_i]$$
$$\quad r_1 \qquad r_k$$

such that:

$$S_j = \text{Im}\left[U_1 \big| \ldots \big| U_j \right]$$

then [58, 38]:

90

$$U'(A + BF)U = \begin{bmatrix} 0 & A_{1,2} & \cdots & & A_{1,i} \\ & 0 & & & \vdots \\ & & \ddots & & \\ & & \ddots & & \\ & 0 & & & A_{i-1,i} \\ & & & & 0 \end{bmatrix} \begin{matrix} \}r_1 \\ \}r_2 \\ \\ \\ \}r_{i-1} \\ \}r_i \end{matrix} \qquad (3.6.14)$$

$$\begin{matrix} r_1 & r_2 & & & r_i \end{matrix}$$

The unitary transformation matrix U is the product of i block diagonal matrices $\widehat{U}_j$ acquired in a recursion of i steps, each constructed in such a way that zeroes out the diagonal and the sub-diagonal blocks in the staircase model of (3.6.8). The state feedback gain F can also be computed simultaneously.

The specific construction of (3.6.14) follows from the fact that in the new coordinate system the controllable subspaces are spanned by:

$$S_j = \mathrm{Im} \begin{bmatrix} I_{d_i} \\ 0 \end{bmatrix}, \quad j = 1, \dots, i \qquad (3.6.15)$$

The algorithm has been coded in MATLAB. The first part of the code computes the staircase model through which the controllability of the system is verified. If the system is uncontrollable, it gives the number of uncontrollable modes and defines whether the system is "beatable to death". This is accomplished through comparison of the uncontrollable modes with some tolerance. If they could be regarded as zero, the outputs from the first part of the code are delivered to the second part, which computes the state feedback gain matrix of (3.6.1) such that the closed-loop state transition matrix A_{cl} is nilpotent.

The code has been applied on two examples and the results are included in the following paragraphs.

Example 3.6.1 Take the case of the pair (A, B) as:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The staircase model of the pair is:

$$H = \begin{bmatrix} 2.3333 & -0.4714 & 0 \\ 0.9428 & 0.6667 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} -1.7321 & -1.7321 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It is clear that the system is uncontrollable, and the dimension of the uncontrollable subsystem is one. However, as this mode is almost zero, the system can be regarded as beatable to death in two time steps. The deadbeat gain F which transfers all the eigenvalues of the closed-loop system matrix $A_{cl} = A + BF$ to zero is computed as:

$$F = \begin{bmatrix} -0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 \end{bmatrix} \quad \blacksquare$$

Example 3.6.2 Consider the pair:

$$A = \begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & -2 & 1 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$$

This system is uncontrollable with one uncontrollable mode which is close enough to zero to be regarded as zero. The system is beatable to death in two steps, and its stair case form is:

$$H = \begin{bmatrix} -0.4000 & 2.6533 & -1.0488 & 3.4785 \\ -0.6633 & 1.6727 & -3.2485 & 2.1930 \\ 0 & -2.2998 & -1.2727 & -3.0151 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} -1.3416 & -2.2361 \\ 1.4832 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The state feedback gain F, which locates the closed-loop system eigenvalues at the origin is achieved as:

$$F = \begin{bmatrix} -0.7500 & 2.0000 & -1.5000 & -0.7500 \\ 1.2632 & -1.3158 & 1.1579 & 1.2632 \end{bmatrix} \quad \blacksquare$$

3.7 conclusion:

In this chapter the problem of state deadbeat control problem, which is that of finding an input sequence steering the states of a linear discrete time system from any arbitrary initial condition to the origin of the complex plane in finite number of time steps, was addressed. The two main synthesis approaches to tackle the problem, namely the state space and the algebraic (or transfer function) approaches, were examined. Within the state space framework, two classes of deadbeat regulator synthesis procedures, i.e. dynamic and spectral approaches, were reviewed. Investigating the main features of the closed-loop state transition matrix attained in the dynamic approach, specifically its nilpotency, revealed that the problem is equivalent to the generalized eigenvalue problem in which all the eigenvalues are assigned to the origin. This in turn, inspired the second approach, referred to as the spectral approach, in which state deadbeat problem is treated as an eigenvalue assignment problem. It was observed that the deadbeat regulator was in the form of a static state feedback. In the minimum-time deadbeat control, the nature of the control law is forced on us by the requirement that every state be driven to the origin in minimum time steps, whereas in the general case of the deadbeat regulator it was a priori assumption.

Development of the dynamic approach rests upon choosing n (system dimension) linearly independent columns of controllability matrix, for which three major procedures were given. From the non-uniqueness of the selection procedure, the non-uniqueness of the deadbeat regulator gain was inferred. It was argued that the minimum number of time steps to transfer any initial state to the origin is equal to the controllability index of the system. By relaxing the time-optimality constraint, the number of time steps increases up to the maximum value equal to the order of the system.

In the second part of this chapter, state deadbeat regulator problem was reformulated in an algebraic set up. This was in regards to one of the main features in discrete time systems in which the processed and generated signals are discrete (i.e. time sequences). So first, a brief introduction to the concepts of sequences and sequential matrices were given to accommodate the needs of sequential description of discrete time systems. The key fact in development of the algebraic approach is the isomorphism between rational sequences in an indeterminate d and rational functions with d being a variable, over an

infinite filed. In the case of linear dynamical system the infinite field is the set of real numbers $\mathbb{R}$. The rational sequences were identified as the impulse response of a lumped linear discrete time system, whereas rational functions mathematically described the transfer function of such systems. According to the fact that rational sequential matrices are expressible as polynomial matrix fractions, causality and stability conditions were discussed in terms of MFDs.

The results concerning the design of deadbeat regulator in algebraic framework were first given for the general case of Finite Settling Time (FST) problem, from which those regarding the specific case of time-optimal deadbeat regulation were extracted. It was observed that in a discrete time system, necessary and sufficient conditions for all the internal signals to settle to a new steady state value in a finite and not just minimum number of time steps is that the system is FIR. The family of all deadbeat controllers were derived as the solution to a polynomial Diophantine equation. This is accomplished through first computation of a particular solution of Diophantine equation, and then the whole set of FST controllers was parameterized in a YJBK format.

It is clear that the state space and algebraic approaches are connected. Nett in [15] shows for the general case of a synthesis problem how the two approaches can be reconciled by establishing the equivalence of left and right coprime factorizations with the solution of a state feedback and output injection stabilization problem. For the special case of deadbeat regulator design problem Kucera in [171] and [172] examines the close parallels between the two techniques. In both schemes it was observed that the solution to the design problem of state deadbeat controller is achievable as the solution to Diophantine equations which accordingly led to the parameterization of all deadbeat regulators in a YJBK fashion. Although the state space approach is more convenient and easier to understand, the major advantage of the algebraic approach over the state space technique may be attributed to the possibility of parameterizing the family of deadbeat controllers based on their McMillan degree, which is regarded as a measure for the complexity of the controller. This is especially true in the case of deadbeat synthesis problem which usually results in controllers with high McMillan degrees which in the state space approach can be tackled through rather complicated balanced-truncation or Hankel-norm based techniques. The other two problems which are more easily treated via the transfer function

approach are those of the strong and simultaneous stabilization as addressed in [163, 161, 57, 52].

This chapter was concluded by giving a numerical algorithm for assigning all the eigenvalues to the origin of the complex plane. The algorithm which first reduces the system to its staircase model, is based on the recursive construction of a unitary transformation, yielding a coordinate system in which the state feedback is computed by merely solving a set of linear equations. The coordinate system was related to the Krylov sequence.

Chapter 4

Deadbeat controller design with time domain constraints

4.1 Introduction:

This chapter studies the design of deadbeat controllers with time domain constraints, through application of Linear and Quadratic Programming. In section 3.6, a numerical algorithm to design a static state feedback which assigns all the eigenvalues to the origin in the minimum number of time steps was presented. According to the Q-parameterization of the stabilized closed-loop systems introduced in chapter two, and the fact that a deadbeat system has all its modes at the origin, the algorithm may be applied to design deadbeat systems.

As we saw earlier, in Q-parameterization, the set of closed-loop eigenvalues is the union of those of the state feedback and the observer as the constituent elements of the observer-based admissible controllers. With regard to this, requiring the system to be minimum-time deadbeat compels to compute F and H, i.e. the gains of the state feedback and observer respectively, such that the poles of $A + B_2 F$ and $A + H C_2$ are assigned to the origin in the optimal time. This in turn, is equivalent to locating all the poles of T (expression (2.7.6)) in the closed-loop system map $H_{zw} = \mathcal{F}_l(T, Q) = T_{11} + T_{12} Q T_{21}$, and hence those of T_{11}, T_{12}, and T_{21} at the origin. Accordingly, T_{11}, T_{12}, and T_{21} subsystems will all be minimum-time deadbeat systems.

Recasting the deadbeat system characterization as the LFT interconnection of T and the parameter Q, transforms the synthesis problem of deadbeat regulator to the design of Q such that the performance specifications are satisfied. However, including the design parameter Q into the closed-loop system description, will result in the system to be non-minimum-time deadbeat.

With respect to section 3.2, relaxing the time-minimality constraint on steering the states to the origin in the optimal-time deadbeat controllers, yields the more general class of deadbeat or FST compensators. The achieved degree of freedom, which is the result of relaxing the time-minimality constraint, can be exploited to exert time or frequency domain performance specifications.

In section 3.5.4, it was shown that a causal discrete time system represents deadbeat response if and only if its impulse response is of finite duration, i.e. it is a polynomial. As mentioned before, this type of compensators are known as Finite Impulse Response (FIR).

In this chapter (and occasionally in later chapters), in order to emphasize the finite nature of deadbeat systems' impulse response, we frequently use the term FIR rather than deadbeat. Based on the mathematical description of deadbeat systems, it is shown that deadbeat controller synthesis problem with typical time domain performance specifications in the LFT framework of figure 2.3.1 or the equivalent construction of figure 2.6.1, is expressible as a Linear Programming (LP) problem. In addition, it is shown that more stringent LQG-type constraints may be recast as quadratic programming with the Q parameter as the design variable. This is accomplished through the established relation between the LQG and H_2 problem.

4.2 Input-output description of deadbeat systems:

As it is known, the pulse transfer function of a system may be defined by the rational function [64]:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad , \quad N \geq M$$

(4.2.1)

in which z^{-1} designates the unit delay operator. (4.2.1) is expressible in terms of the following difference equation:

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \cdots - a_n y(k-N)$$
$$+ b_0 x(k) + b_1 x(k-1) + \cdots + b_M x(k-M) \qquad (4.2.2)$$

By assuming that the coefficients a_i are all zero, (4.2.1) reduces to:

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \qquad (4.2.3)$$

which corresponds to the difference equation:

$$y(k) = b_0 x(k) + b_1 x(k-1) + \cdots + b_M x(k-M) \qquad (4.2.4)$$

It is clear that the assumption $a_i = 0$, $i = 1, \dots, N$ is equivalent to regarding all the poles to be zero. In view of the nilpotency property in deadbeat systems, it can be readily inferred that (4.2.3) describes the pulse transfer function of a deadbeat system. Accordingly, the expression (4.2.4) formulates the impulse response of such systems, which is of finite length. This mathematically justifies our earlier declaration that deadbeat systems are FIR.

Based on the conventional notation, the transfer function of an M-tap FIR system (i.e. a system with an impulse response of length M, and hence the order $M - 1$) with Markov parameters $h(k)$ can be represented as:

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k} \qquad (4.2.5)$$

4.3 Transient response and time domain constraints:

It is well known that the response of any system includes two parts, i.e. transient and steady state responses. The transient response can be attributed to the poles of the closed-loop system, whereas the steady state response arises from the poles of the input or forcing function. When a system is subject to inputs or disturbances, because of its energy storage feature, it cannot respond instantaneously which in turn gives rise to the transient response. Clearly, it is desirable that the transient response is sufficiently fast and exhibits

satisfactory damping. Frequently, transient response characteristics of a system are analysed in terms of the system response to standard inputs such as unit step, or ramp. This is in view to the fact that the system response to arbitrary inputs may be estimated from its response to such standard inputs.

In this section, we introduce a partial list of typical performance specifications of control systems in terms of their transient response to a unit step. As it is known, step inputs are easy to generate and sufficiently drastic to provide useful information on both the transient and steady state response traits. Some of the common time domain quantities which describe transient response characteristics of a discrete time system with unit step response denoted by $s(k)$, are as follows [49, 65, 66, 3]:

- Percentage overshoot is defined as:

$$\max \frac{s(k) - s_\infty}{s_\infty} \times 100 \qquad (4.3.1)$$

This is an indication of the extent to which the output exceeds its steady state value before settling down.

- Rise time is defined as:

$$\min\{k \geq 0 \mid s(k) = 0.5 s_\infty\} \qquad (4.3.2)$$

This is a measure of speed of the system, or equivalently present time delay in response of the system.

- 5% settling time is defined as:

$$\min\{k \geq 0 \mid |s(k') - s_\infty| \leq 0.05 s_\infty \; for \; k' \geq k\} \qquad (4.3.3)$$

Settling time is an indication of the required time for the system response to settle to its steady state value.

- Percentage (adverse) undershoot is defined as:

$$\max\left\{-\frac{s(k)}{s_\infty} \times 100\right\} \qquad (4.3.4)$$

- 5% dead time is defined as:

$$\max\{k \geq 0 | \ |s(k')| \leq 0.05s_\infty \ \ for \ \ 0 \leq k' < k\} \qquad (4.3.5)$$

This is a measure of the time needed for the system to react.

- Decoupling:

To clarify this performance characteristic, consider the case of a MIMO system with n_c command inputs and n_c regulated variables. The diagonal entries of the $n_c \times n_c$ system transfer function are the transfer functions from the inputs to their associated regulated variable. Each of these transfer functions may be required to meet different specifications, like the ones we have discussed so far. The off-diagonal entries of the system transfer function are the transfer functions from the command signals to other regulated variables, which are known as the command interaction transfer functions. In order to minimize the disruption of regulated variable by command inputs other than their associated ones, it is generally desirable that the command interaction transfer functions be "small". A mild constraint on command interaction is asymptotic decoupling expressed as:

$$\lim_{k \to \infty} s(k) \text{ is diagonal} \qquad (4.3.6)$$

The above constraint implies that for constant command inputs, the effect on each regulated variable due to the other commands converges to zero, in other words, for constant command inputs there is no steady state interaction.

A stronger condition to restrict the command interaction is an envelope constraint on each entry of $s(k)$:

$$s_{min}(k) \leq s(k) \leq s_{max}(k), \ \ \forall k \geq 0 \qquad (4.3.7)$$

in which $s_{min}(k)$ and $s_{max}(k)$ are matrices specified by the designer.

An extreme form of the envelope constraint is to require that the step response transfer matrix be diagonal, or equivalently all the off-diagonal step responses are zero. This is known as exact or complete decoupling, which prohibits any command interaction at all.

- Slew rate [69,70]:

Another typical specification in terms of the step response of the system is slew rate limits, which is expressible as:

$$\left|\frac{\Delta S(K)}{\Delta k}\right| = |h(k)| \leq M_{slew}, \quad \forall k \geq 0 \qquad (4.3.8)$$

(4.3.8) implies that the slew rate constraint limits the maximum rate of change of a signal per unit of time. It is one of the most common constraints enforced on the control signals in systems.

In practical applications, because of the physical constraints imposed by actuators, control signals should always be limited. The limitations may be expressed as magnitude constraint on the control action as:

$$u_{min} \leq u(k) \leq u_{max}$$

However, as in most cases it is the rapid fluctuations in the control signal that may cause most damage to the actuator not the size of the control signal amplitude, it is more appropriate to limit the control signal slew rate rather than its magnitude. Large control signal derivatives or inter-sample variations may harm actuators, or at the very least causes expensive wearing of actuators.

The slew rate constraint on control action may be described by the following inequalities:

$$\Delta u_{min} \leq \frac{\Delta u}{\Delta k} \leq \Delta u_{max} \qquad (4.3.9)$$

4.4 Linear Programming (LP):

This section introduces the Linear Programming problem [109, 67, 68]. In section 4.5 this optimization procedure will be applied to design deadbeat controller subject to time domain constraints.

Generally, the standard form of an optimization problem may be formulated as:

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0 , \quad i = 1, \dots, m \\
& h_i(x) = 0 , \quad i = 1, \dots, p
\end{aligned}
\qquad (4.4.1)
$$

in which $f_0(x)$, $f_i(x)$, and $h_i(x)$ designates objective function, inequality constraints, and equality constraints, respectively. The domain $\mathfrak{D}$ of the optimization problem (4.4.1) is the intersection of objective and all equality and inequality functions and is defined as:

$$
\mathfrak{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i
\qquad (4.4.2)
$$

This is the set of points for which the objective and all constraint functions are defined. Note that dom denotes the domain of a function; for a function $f: \mathbb{R}^p \to \mathbb{R}^q$, the domain of $f(x)$ is the subset of $\mathbb{R}^p$ of points x for which $f(x)$ is defined.
A point $x \in \mathfrak{D}$ is said to be feasible and is the one that satisfies the whole set of inequality and equality constraints. The set of all feasible points is also known as the feasible set or the constraint set. The optimization problem is feasible if the feasible set is nonempty, and infeasible otherwise.

Clearly, in an optimization problem the aim is to find the optimal value p^* where:

$$
p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, i = 1, \dots, m, ; h_i(x) = 0, i = 1, \dots, p \}
\qquad (4.4.3)
$$

The point x^* is an optimal point if it is feasible and $f_0(x^*) = p^*$. The set of all optimal points X_{opt}, is called the optimal set and defined as:

$$X_{opt} = \{\, x \mid f_i(x) \le 0, i = 1, \ldots, m, ; h_i(x) = 0, i = 1, \ldots, p\, ,; f_0(x) = p^* \}$$

In the case that the objective function is identically zero, if the feasible set is nonempty the optimal value is zero. But, if the feasible set is empty the optimal value is ∞, which agrees with the standard convention that the infimum of an empty set is ∞. A prerequisite in any optimization problem is to solve the feasibility problem, which is that of determining whether or not the constraints are consistent, and if so, find a point that satisfies them. The feasibility problem can be expressed as:

find $\quad\quad x$

subject to $\quad f_i(x) \le 0, \quad i = 1, \ldots, m \quad\quad\quad$ (4.4.4)

$\quad\quad\quad\quad h_i(x) = 0, \quad i = 1, \ldots, p$

In the optimization problem of (4.4.1), when the objective and constraint functions are restricted to be affine, the problem is referred to as Linear Program (LP) and can be formulated in the form:

minimize $\quad c^T x + d$

subject to $\quad Gx \le h \quad\quad\quad$ (4.4.5)

$\quad\quad\quad\quad Ax = b$

Since the constant term d does not affect the optimal or feasible set, it is common to omit it [67].

Figure 4.4.1 illustrates a geometric interpretation of an LP [67]. The shaded polyhedron represents the feasible set of the LP. The dashed lines are level curves of the linear objective function $c^T x$, which are orthogonal to c. The optimal value x^*, is the farthest point in the feasible set and in the direction of $-c$.
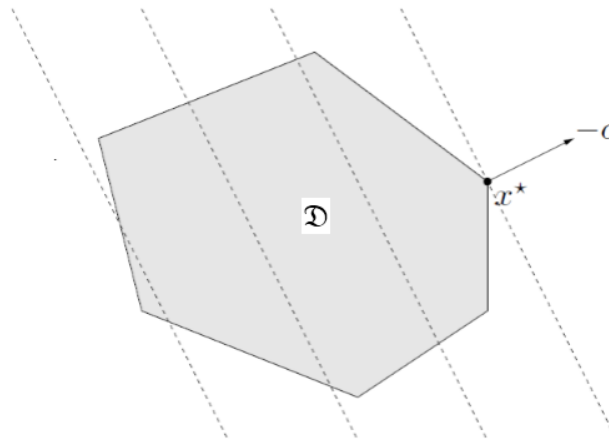
Figure 4.4.1 The geometric interpretation of an LP

One of the significant properties of a Linear Programming problem is that the constraint functions form a convex polyhedron (obtained as the intersection of a finite number of half-spaces and hyperplanes), and the optimum value occurs at an extreme point of the polyhedron. Given that a solution exists, it is guaranteed to be a unique solution value. However, the problem may have multiple optimal solutions, when the level curves are parallel to the edges of the polyhedron [68, 109].

4.5 Deadbeat controller design with time domain constraints using linear programming:

In chapter 2, it was mentioned that most control synthesis problems can be reconstructed as a lower LFT illustrated in figure 4.5.1.
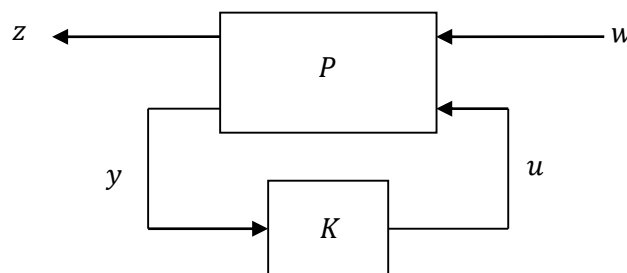

Figure 4.5.1 LFT framework

Having partitioned the generalized plant P as in (2.3.2), the closed-loop map H_{zw} from the exogenous inputs w to the regulated variables z, was described as:

$$H_{zw} = \mathcal{F}_l(P, K) = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}) \qquad (4.5.1)$$

The parameterization of all stabilizing controllers was also derived. It was observed that the set of stabilizing controllers is expressible as a coprime factorization, involving elements of the doubly coprime factorization of the system to be stabilized and a stable but arbitrary parameter Q. In fact, the doubly coprime factorization was equivalent to the choice of a single stabilizing controller K_s, and the whole set of stabilizing controllers being parameterized as $K = \mathcal{F}_l(K_s, Q)$ was generated through augmenting that central controller K_s. It was shown that every controller which stabilizes P, can be realized as an observer-based controller.

The linear fractional nature of the stabilizing controllers enabled us to replace the linear fractional parameterization of the closed-loop map $\mathcal{F}_l(P, K)$, with a parameterization which is affine in the parameter Q:

$$H_{zw} = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21} \qquad (4.5.2)$$

This transformation simplifies the problem by reducing the search over a free parameter $Q \in \mathbb{R}H_\infty$, instead of an optimisation over an implicitly defined set.

The expression (4.5.1), although simply described analytically, translates simple design specifications on the closed-loop map to complicated constraints on the controller K in a linear fractional way. This is in contrast to the equivalent expression of (4.5.2) in which design specifications can be expressed affinely in term of the free parameter Q. This has been extensively elaborated in [4].

The above attribute of the closed-loop map of (4.5.2) may be exploited to simplify the design problem of deadbeat controllers subject to time domain constraints. To proceed, consider the general feedback configuration of figure 4.5.2, in which P is the plant and K is the controller. p and q are the dimensions of the control input u and the measured outputs y, respectively. Let m_j and n_i be the dimensions of the z_j and w_i respectively. The dimension of the closed-loop map H_{zw}, which by definition contains every closed-

loop map of interest, is $m \times n$. In this construction, Q will be a stable $p \times q$ transfer matrix.

In order to impose a constraint on any transfer function describing a specific input-output channel, first we should be able to extract description of the mapping of interest.
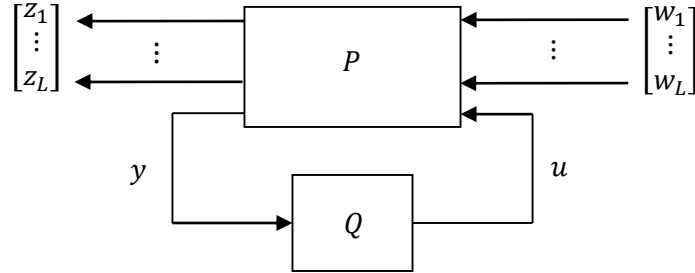

Figure 4.5.2 LFT framework with L channels of exogenous inputs and regulated outputs

This can be achieved through pre- and post-multiplying H_{zw} with selection matrices $L_j \in \mathbb{R}^{m_j \times m}$ and $R_i \in \mathbb{R}^{n \times n_i}$ (shaped with zeros and ones) as [71]:

$$H_{z_j w_i} = L_j H_{zw} R_i \qquad (4.5.3)$$

Having a deadbeat closed-loop system, as was discussed in the second chapter, demands that all the eigenvalues be at the origin. According to the state space characterization of $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix}$ in (2.7.6), the eigenvalues of T are the union of those of $A + B_2 F$ and $A + H C_2$. By employing the presented algorithm in section 3.6, all eigenvalues of T, and accordingly those of T_{11}, T_{12}, and T_{21}, can be readily assigned to the origin. In view of the discussion in section 4.2, this is equivalent to making T_{11}, T_{12}, and T_{21} all FIR. However, making the entire closed-loop system deadbeat compels to restrict the free parameter Q to be FIR, too. In other words, Q is described by a finite-length impulse response:

$$Q(z) = \sum_{k=0}^{N-1} Q_k z^{-k} \qquad (4.5.4)$$

Confining Q to be an N-tap FIR system corresponds to choosing the basis $\{z^0, z^{-1}, \dots, z^{N-1}\}$.

The state space characterization of Q can be easily extracted. To do so, let Q_{rs} designates the individual SISO FIR component systems in Q, with Markov parameters $\{q_{rs}(0), q_{rs}(1), \dots, q_{rs}(N-1), 0, 0, \dots\}$. Regarding the state space realization of the function z^{-n} as:

$$z^{-n} = \begin{bmatrix} A_f & e_1 \\ e_n^T & 0 \end{bmatrix}$$

in which $A_f \in \mathbb{R}^{(N-1)\times(N-1)}$ is a shift matrix made up of all zeros except for ones on the first subdiagonal, and e_1 and e_n are respectively the first and last columns of $I_{(N-1)}$, Q_{rs} may be realized as:

$$\begin{bmatrix} A_{Q_{rs}} & B_{Q_{rs}} \\ C_{Q_{rs}} & D_{Q_{rs}} \end{bmatrix} = \begin{bmatrix} A_f & e_1 \\ q_{rs} & q_{0,rs} \end{bmatrix} \qquad (4.5.5)$$

where q_{rs} is a row matrix as $[q_{rs}(1), \dots, q_{rs}(N-1)]$ and $q_{0,rs} = q_{rs}(0)$ [71, 73, 74, 72].

Restricting the design parameter Q and the transfer matrices T_{11}, T_{12}, and T_{21} to be FIR results in turning imposed time-domain constraints on any regulated output into linear constraints on the Markov parameters of Q [73, 74, 75, 4]. This in turn, can be exploited to recast the deadbeat controller design with time domain constraints as an LP. To see this, take the case that for some fixed input vector of the signal w, the first M samples of the corresponding j-th output z_j are required to remain between some given bounds. w can be, for instance, a vector $w = [0 \ 0 \cdots w_i \ 0 \ \cdots 0]$ in which w_i is a step. Clearly, by linearity the general case can be recovered using superposition. The output z_j is expressible as $z_j = T_{zw}^{ji} w = T_{11}^{ji} w + T_{12}^{j\cdot} Q T_{21}^{\cdot i} w$, in which scalar transfer functions $T_{12}^{j\cdot}$ and $T_{21}^{\cdot i}$ denote row and column vectors with entries T_{12}^{ji} and T_{21}^{ji}, respectively. To simplify the notation, let $a = T_{11}^{ji} w$, $b = T_{12}^{j\cdot}$, and $c = T_{21}^{\cdot i} w$ with their associated power series expansion as $a = \sum_k a_k z^{-k}$, $b = \sum_k b_k z^{-k}$, and $c = \sum_k c_k z^{-k}$. Correspondingly, z_j can be written as:

$$z_j = a + bQc = \sum_{k=0}^{M-1}\left(a_k + \sum_{r=0}^{k}\sum_{s=0}^{r} b_{r-s} Q_s c_{k-r}\right) z^{-k} \qquad (4.5.6)$$

The expression $(a + bQc) = a_k + \sum_{r=0}^{k} \sum_{s=0}^{r} b_{r-s} Q_s c_{k-r}$ clearly shows that the constraints on z_j are translated into constraints on Q_s's. To make this lucid, suppose that z_j is required to be bounded as $\alpha \leq z_j \leq \beta$. In view of (4.5.6), this is equivalent to:

$$\alpha - \sum_{k=0}^{M-1} a_k \leq \sum_{s=0}^{r} Q_s \left(\sum_{k=0}^{M-1} \sum_{r=0}^{k} b_{r-s} c_{k-r} \right) \leq \beta - \sum_{k=0}^{M-1} a_k \qquad (4.5.7)$$

which in turn equates to:

$$\gamma \leq d^T Q \leq \delta \qquad (4.5.8)$$

where

$$d_s = \sum_{k=0}^{M-1} \sum_{r=0}^{k} b_{r-s} c_{k-r} , \quad s = 0, \ldots, r \qquad (4.5.9)$$

In the next section, the deadbeat control design problem subject to more stringent time domain specifications in the form of a quadratic (LQG-type) performance criterion will be addressed.

4.6 Time domain constraints in LQG framework:

In the preceding section, the deadbeat controller design problem with time domain constraints was investigated. It was observed that linear constraints exerted on the regulated outputs are translated into linear constraints on the Q parameter. This attribute could in turn be exploited to recast the problem of deadbeat controller design with temporal constraints as a linear program with Q as the design parameter.

An alternate form of time domain constraints, which are more stringent, may be expressed in the form of LQ (linear quadratic) paradigm with quadratic time domain performance criteria. Including the measurement noise w_n and disturbance signals (process noise) w_d, and modelling them as stochastic processes with known statistical properties in the LQ paradigm, introduces the LQG (Linear Quadratic Gaussian) design method. As the name suggests, in LQG the signals w_d and w_n are assumed to be uncorrelated zero-mean white

Gaussian stochastic processes with constant intensity matrices $\mathcal{W}$ and $\mathcal{V}$, respectively. So, w_d and w_n are white noise processes with covariances [80]:

$$E\{w_d(t)w_d(\tau)^T\} = \mathcal{W}\delta(t - \tau)$$
$$E\{w_n(t)w_n(\tau)^T\} = \mathcal{V}\delta(t - \tau) \qquad (4.6.1)$$
$$E\{w_d(t)w_n(\tau)^T\} = 0$$

where E is the expectation operator and $\delta(t - \tau)$ is a delta function.

In a continuous-time system with state space description:

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw_d(t)$$
$$y(t) = Cx(t) + w_n(t) \qquad (4.6.2)$$
$$z(t) = Dx(t)$$

the objective in LQG control problem is to find an optimal control $u(t)$ which minimizes the performance index [79]:

$$J = E\left\{lim_{T\to\infty} \frac{1}{T}\int_0^T [x^T Sx + u^T Ru]dt\right\} \qquad (4.6.3)$$

The common forms for the weighting matrices S ad R are respectively $S = D^T D$ and $R = \rho I$, where the parameter ρ is real and non-negative. As a result, (4.6.3) will take the form:

$$J = E\left\{lim_{T\to\infty} \frac{1}{T}\int_0^T [z^T z + \rho u^T u]dt\right\} \qquad (4.6.4)$$

in which the first term in the integral measures the accumulated deviation of the output from zero, while the second term measures the input size or actuator authority. Therefore, the whole term in the integral designates the quadratic error expression.

The choice of the constant weighting matrices S and R, which are respectively positive semidefinite and positive definite matrices, reflects the trade-off between the requirement of regulating the outputs and the expenditure of the control energy.

It is well established that the solution to the LQG problem, known as the separation theorem or certainty equivalence principle, consists of first determining the optimal control to a deterministic LQR (Linear Quadratic Regulator) problem, followed by the second step of finding an optimal estimate $\hat{x}$ of the state x, such that:

$$E\big[[x - \hat{x}]^T[x - \hat{x}]\big]$$

is minimized. This term serves as a measure for how well $\hat{x}$ approximates x. The optimal state estimate may be achieved from a Kalman filter [65, 79, 80].

The solution to the LQG problem, is in the form of state feedback law $u(t) = Kx(t)$. It minimizes the rate at which the integrated generalized square error:

$$\int_0^T E[x^T S x + u^T R u]dt$$

approaches ∞. Clearly, due to the presence of white noise disturbance, the state and the input cannot be driven to 0. Subsequently, the integrated generalized quadratic error does not converge to a finite number as $T \to \infty$. The rate at which the error approaches ∞ may be given as $\lim_{T \to \infty} \frac{1}{T}\int_0^T E[x^T S x + u^T R u]dt$. This limit equals the steady-state mean square error:

$$\lim_{t \to \infty} E[x^T S x + u^T R u] \qquad (4.6.5)$$

Thus, the state feedback minimizes the steady-state mean square error [65].

The LQG problem may be tackled through a different interpretation. It has been shown to be a special case of a broader class of problems, which has become known as H_2 optimization. This interpretation eliminates the need to incorporate the stochastic ingredients of the LQG and reduces role of the intensity matrices $\mathcal{V}$ and $\mathcal{W}$ from describing the white noise processes to that of design parameters. This interpretation offers a great deal of flexibility, especially that in many applications it is difficult to establish the precise stochastic properties of disturbance and noise signals [65].

The relation between the LQG and H_2 optimization is observed by recognizing that the LQG performance index may be represented as a system H_2 norm. The relation has been extensively elaborated in the literature, e.g. [9, 65, 79, 80, 81], and will be briefly discussed in what follows.

Consider the general framework of figure 4.5.1 with the associated closed-loop map description H_{zw}, as in (4.5.1). The H_2 norm of H_{zw}, designated by $\|H_{zw}\|_2$, is defined as:

$$\|H_{zw}\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{tr}[H_{zw}(j\omega)^* H_{zw}(j\omega)]d\omega \qquad (4.6.6)$$

in which tr specifies the trace of a matrix, the asterisk denotes conjugate transpose. In regard to the Parseval's theorem, (4.6.6) is equal to the H_2 norm of the impulse response of H_{zw}, termed h_{zw}:

$$\|h_{zw}\|_2^2 = \|H_{zw}\|_2^2 = \int_0^{+\infty} \mathrm{tr}[h_{zw}(\tau)^T h_{zw}(\tau)]d\tau = \int_0^{+\infty} \sum_{ij} \left|h_{zw}^{ij}(\tau)\right|^2 d\tau \qquad (4.6.7)$$

By changing the order of integration and summation in (4.6.7), it may be expressed in the following equivalent form:

$$\|h_{zw}\|_2^2 = \sum_{ij} \int_0^{+\infty} \left|h_{zw}^{ij}(\tau)\right|^2 d\tau \qquad (4.6.8)$$

(4.6.8) implies that the H_2 norm can be interpreted as the total 2-norm of the outputs resulting from applying unit impulses to each input channel one after another. Regarding to the fact that the 2-norm of a signal is the square root of its energy, it may also be inferred that the squared H_2 norm coincides with the total output energy in the impulse response of the system. This elucidates the deterministic interpretation of the H_2 norm.

H_2 norm, however, may also be given a stochastic interpretation. To see this, suppose that in the feedback configuration of figure 4.5.1, the exogenous input w is a white noise of unit intensity i.e. $E[w(t)w(\tau)^T] = I\delta(t - \tau)$. The expected power in the error signal z then will be achieved as:

$$E\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T z(t)^T z(t)dt\right] = \text{tr}E[z(t)z(t)^*] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\text{tr}[H_{zw}(j\omega)H(j\omega)^*]d\omega$$

$$= \|H_{zw}\|_2^2 \qquad (4.6.9)$$

which with respect to the Parseval's theorem is equal to the H_2 norm of the closed-loop map impulse response, i.e. $\|h_{zw}\|_2^2$. Now it is easy to see that the H_2 norm minimization (known as H_2 problem) amounts to the minimization of the output (error) RMS value (Root Mean Square value) of z, when the system is driven by a unit intensity white noise input. This (in view of the expression (4.6.5)) establishes the equivalence between the H_2 and the LQG problems.

To illustrate the reverse relation, which is the capability of casting the LQG problem into an equivalent H_2 optimization problem, take the following stochastic system:

$$\dot{x}(t) = Ax(t) + Bu(t) + w_d$$
$$y(t) = Cx(t) + w_n(t)$$

where w_d and w_n are uncorrelated white noise signals as described in (4.6.1). In the general framework of figure 4.5.1, define an error signal z as:

$$z = \begin{bmatrix} S^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}$$

and represent the stochastic inputs as a function of the exogenous input w, being a white noise process of unit intensity, as:

$$\begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} & 0 \\ 0 & V^{\frac{1}{2}} \end{bmatrix} w$$

The LQG cost function of (4.6.3) will accordingly turn to:

$$J = E\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T z(t)^T z(t)dt\right]$$

which in view of the equalities in (4.6.9) is the H_2 norm of the closed-loop map, $\|h_{zw}\|_2^2$.

So, in this section the relation between the LQG and H_2 optimization problems was established. It was observed that the LQG performance index which aims to minimize the RMS value of the error signal (the squared-power of which represents the average power of the signal), is expressible as the system H_2 norm when it is excited by white noise input signals. On the other hand, the H_2 norm was given two deterministic and stochastic interpretations. The stochastic interpretation enabled us to equate the H_2 problem with that of minimizing the RMS value (or equivalently the power) of the system response to white noise inputs.

In the succeeding section, the equivalence between the problems of LQG and H_2 optimizations will be employed to design a deadbeat controller subject to the quadratic time domain constraints.

4.7 Deadbeat controller design with LQG performance criteria:

In the previous section, it was observed that solving the LQG problem amounts to minimizing the H_2 norm of the closed-loop system. In this section, the equivalence will be exploited to design the deadbeat controller with LQG constraints. As we will see, the problem reduces to a quadratic programming, with quadratic constraints imposed on the Q parameter.

In the general framework of figure 4.5.1, suppose that the closed-loop transfer function H_{zw} is an $(N + 1)$-tap FIR system with Markov parameters H_k:

$$H_{zw}(z) = \sum_{k=0}^{N} H_k z^{-k} \qquad (4.7.1)$$

and the exogenous input w is an uncorrelated zero-mean white noise signal with variance σ^2, denoted by $e(n)$. The output $z(k)$ will correspondingly be:

$$z(k) = H_0 e(k) + H_1 e(k-1) + \cdots + H_N e(k-N) \qquad (4.7.2)$$

According to the discrete-version of (4.6.9):

$$\|z(k)\|_2^2 = \text{tr} E[z(k)z(k)^T] \qquad (4.7.3)$$

in which $E[z(k)z(k)^T]$ is expressible as:

$$E[z(k)z(k)^T] = H_0 E[e(k)e(k)^T]H_0^T + \cdots + H_N E[e(k-N)e(k-N)^T]H_N^T$$

$$= \sigma^2(H_0 H_0^T + \cdots + H_N H_N^T)$$

Therefore,

$$\|z(k)\|_2^2 = \sigma^2 \sum_{i=0}^{N} \text{tr}[H_i H_i^T] \qquad (4.7.4)$$

Square root of the term $\text{tr}[H_i H_i^T]$ is known as the Frobenius norm and is designated by $\|H_i\|_F$. It can be readily shown that $\|H_i\|_F = \|\text{vec}(H_i)\|_2$.

For the sake of simplicity and without loss of generality, we can assume that $\sigma^2 = 1$ (i.e. white noise has unit intensity). As a result, the H_2 norm of the regulated output may be formulated as:

$$\|z(k)\|_2^2 = \sum_{i=0}^{N} \|H_i\|_F^2 = \sum_{i=0}^{N} \|\text{vec}(H_i)\|_2^2 \qquad (4.7.5)$$

It is known from chapter 2 that the closed-loop map of the general configuration of figure 4.5.1 is expressible in terms of the stable, but otherwise arbitrary, parameter Q:

$$H_{zw} = T_{11} + T_{12} Q T_{21}$$

Moreover, it was discussed earlier that making the entire closed-loop system deadbeat, compels us to confine all the terms in the closed-loop map to be FIR. Having described T_{11}, T_{12}, T_{21}, and the parameter Q respectively as $(m+1)$-, $(p+1)$-, $(q+1)$-, and $(n+1)$-tap FIR systems:

$$T_{11} = T_{11_0} + T_{11_1}z^{-1} + \cdots + T_{11_m}z^{-m}$$

$$T_{12} = T_{12_0} + T_{12_1}z^{-1} + \cdots + T_{12_p}z^{-p} \qquad (4.7.6)$$

$$T_{21} = T_{21_0} + T_{21_1}z^{-1} + \cdots + T_{21_q}z^{-q}$$

$$Q = Q_0 + Q_1z^{-1} + \cdots + Q_nz^{-n}$$

the Markov parameters H_i's may be computed as:

$$H_i = \hat{T}_{11_i} + \sum_{k=0}^{i}\sum_{j=0}^{i-k} T_{12_{i-j-k}}Q_kT_{21_j}, \quad i = 0, 1, \ldots, max\{m, p+q+l\}$$

$$(4.7.7)$$

in which:

$$\hat{T}_{11_i} = \begin{cases} T_{11_i} & i = 1, 2, \ldots, m \\ 0 & i \geq m \end{cases} \qquad (4.7.8)$$

Consequently, the H_2 norm of the regulated output can be achieved as the following:

$$\|z(k)\|_2^2 = \sum_{i=0}^{N}\|vec(H_i)\|_2^2 =$$

$$\sum_{i=0}^{N}\left\|vec(\hat{T}_{11_i}) + \sum_{k=0}^{i}\left[\sum_{j=0}^{i-k}\left(T_{21_j}\right)^T \otimes T_{12_{i-j-k}}\right]vec(Q_k)\right\|_2^2 \qquad (4.7.9)$$

which in turn, results in the following compact description:

$$\|z(k)\|_2^2 = \sum_{i=0}^{n}\left\|\tilde{T}_{11_i} + \Pi_i\tilde{Q}\right\|_2^2 \qquad (4.7.10)$$

The equality (4.7.10) is concluded via the following substitutions:

$$\tilde{T}_{11_i} = vec(\hat{T}_{11_i}) \quad \text{and} \quad \Pi_i = \sum_{k=0}^{i}\left[\sum_{j=0}^{i-k}\left(T_{21_j}\right)^T \otimes T_{12_{i-j-k}}\right] \qquad (4.7.11)$$

From the expression (4.7.10), it can be readily inferred that any constraint in the form of LQG imposed on the regulated signals, is equivalent to a quadratic programming with Q as the design parameter.

It is worth mentioning that $\|z(k)\|_2^2$, when the system is driven by Gaussian white noise signal, also represents the variance of the regulated signal $z(k)$:

$$\sigma^2 = E(z(k)^2) - (E(z(k)))^2 = E(z(k)^2) - \mu^2 \qquad (4.7.12)$$

where σ and μ respectively denote the variance and the mean, or in the case of a stochastic signal the expected value. This is in view of the fact that the output resulting from a system with impulse response $g(n)$ which is driven by a white noise with mean μ_x, has mean μ_y with μ_x as its factor [82]:

$$\mu_y = \mu_x \sum_{n=-\infty}^{+\infty} g(n) \qquad (4.7.13)$$

Since in LQG, the noise is in the form of an uncorrelated zero-mean white noise signal, the output correspondingly will be a stochastic signal of zero mean. According to (4.7.12), this implies that:

$$\sigma^2 = E(z(k)^2)$$

The term on the right hand side of the above equality, when the number of outputs is l, can be given as:

$$E(z(k)^2) = \sum_{j=1}^{l} E[z_i(k)^2] = \mathrm{tr}E[z(k)z(k)^T] \qquad (4.7.14)$$

which in turn is the H_2 norm of $z(k)$ [83].

The design procedures in sections 4.6 and 4.7 are represented by means of the following example.

Example 4.7.1 This example considers the design problem of a deadbeat controller to reduce the vibrations in a one-storey scaled-down building model caused by earthquake while additional constraints on the magnitude and slew rate of the control input are satisfied. The design is first accomplished for the case where the earthquake signal is modelled as an impulse. However, in the second part, the controller is designed under a

real earthquake signal. Model structure of the building is based on the example given in [77] and is depicted in figure 4.7.1.
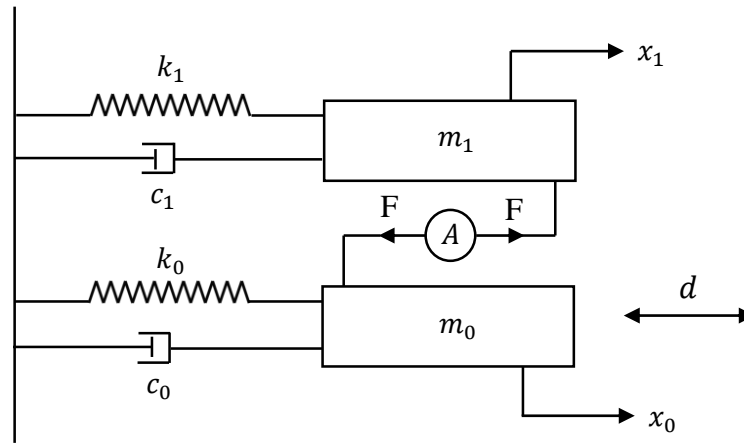

Figure 4.7.1 Model structure of an one-storey building

Parameters describing the construction are as follows:

m_0 = base mass = $5\ kg$

c_0 = base damping coefficient = $100\ N/m/s$

k_0 = base stiffness = $16000\ N/m$

m_1 = first floor mass = $1.72\ kg$

c_1 = first floor damping coefficient = $0.078\ N/m/s$

k_1 = first floor stiffness = $2600\ N/m$

In the figure, displacement of the base and first floor from the equilibrium point is designated by x_0 and x_1, respectively. The external acceleration exerted by the earthquake is also denoted by d. The actuator, specified by A, applies an equal and opposite force F on the two masses along the illustrated directions. Electrical model of the actuator is represented in figure 4.7.2.
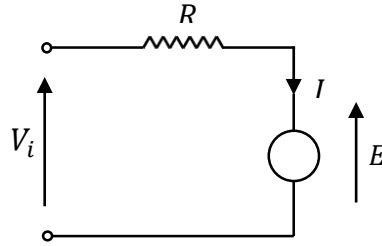
Figure 4.7.2 The electrical model of the actuator

The voltage V_i represents the input to the linear actuator as a result of which the actuator's output force F is generated. E designates the back-emf while I denotes the current. The following actuator parameters have been assumed:

k_f = force constant = $2\ N/A$

k_e = electric constant = $2\ v/m/s$

R = electric resistance = $1.5\ \Omega$

The actuator force F and back-emf E are computed as:

$\text{F} = k_f I$

$E = k_e v = V_i - IR$

where v is the relative linear velocity at the two ends of the actuator.

Based on the above description for the model building and the actuator, system model equations when the state vector is defined as $x = \begin{bmatrix} x_0 & x_1 & \dot{x}_0 & \dot{x}_1 \end{bmatrix}$ and the measured variable is the acceleration of first floor, i.e. $y = \ddot{x}_1$, are given as:

$\dot{x} = Ax + Bu + Gd$

$y = Cx + Du$

in which the input signal u is the actuator voltage V_i. Matrices describing the system state space model are parameterized as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{k_0+k_1}{m_0} & \dfrac{k_1}{m_0} & -\dfrac{c_0+c_1}{m_0}+\dfrac{k_f k_e}{m_0 R} & \dfrac{c_1}{m_0}+\dfrac{k_f k_e}{m_0 R} \\ \dfrac{k_1}{m_1} & -\dfrac{k_1}{m_1} & \dfrac{c_1}{m_1}+\dfrac{k_f k_e}{m_1 R} & -\dfrac{c_1}{m_1}+\dfrac{k_f k_e}{m_1 R} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -\dfrac{k_f}{m_0 R} & \dfrac{k_f}{m_1 R} \end{bmatrix}^T$$

$$C = \begin{bmatrix} \dfrac{k_1}{m_1} & -\dfrac{k_1}{m_1} & \dfrac{c_1}{m_1}+\dfrac{k_f k_e}{m_1 R} & -\dfrac{c_1}{m_1}+\dfrac{k_f k_e}{m_1 R} \end{bmatrix}$$

$$D = \begin{bmatrix} \dfrac{k_f}{m_1 R} \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$$

In order to synthesize the deadbeat controller, the continuous time model, illustrated in figure 4.7.3, was discretized with sampling interval of $T_s = 0.01\ s$.
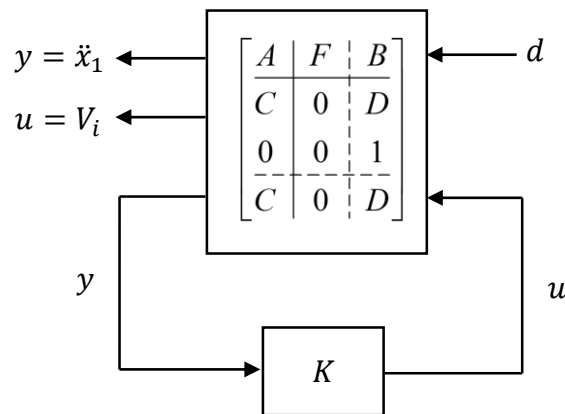

Figure 4.7.3 The generalized regulator

The equivalent discrete system (zero-order-hold equivalent) is depicted in figure 4.7.4.
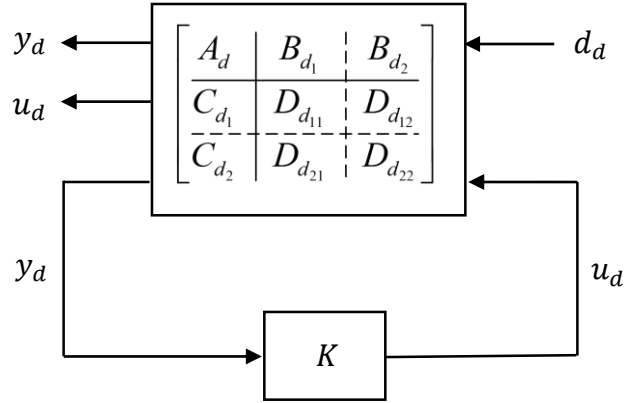
Figure 4.7.4 Discrete model

As mentioned earlier, this example aims to compute a deadbeat control law such that the rms acceleration $y = \ddot{x}_1$ of the first storey is minimized, while the control input u and its slew rate are constrained within realistic limits. The magnitude of the control input and its slew rate are restricted between $+20$ and -20 volts and between $+20$ and -20 volts/s, respectively According to the discussion in section 4.5, the synthesis problem can be expressed as a Linear Program which is described below:

$$\min \max |y(k)|, \quad k = 0, \dots, N-1$$

such that:

$$|u(k)| \le \varepsilon = 20 \text{ volts}, \quad k = 0, \dots, N-1$$
$$|u(k+1) - u(k)| \le \varepsilon_1 = 20 \text{ volts}, \quad k = 0, \dots, N-2$$

where N represents the number of samples.

The example was simulated in MATLAB. For d chosen to be an impulse signal, the regulated variables of the closed-loop system which are the first floor acceleration and the control signal u, were plotted. The results are shown in figures 4.7.5 and 4.7.6. The achieved optimum value was $2.047 \ m/s^2$.
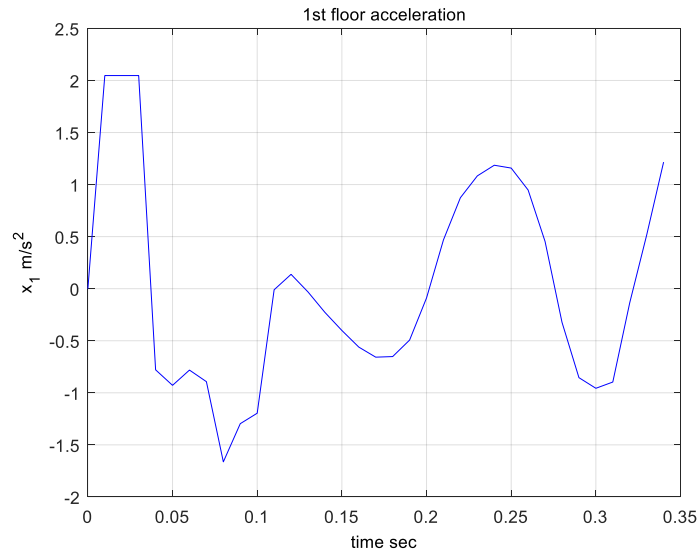
Figure 4.7.5 The first floor acceleration
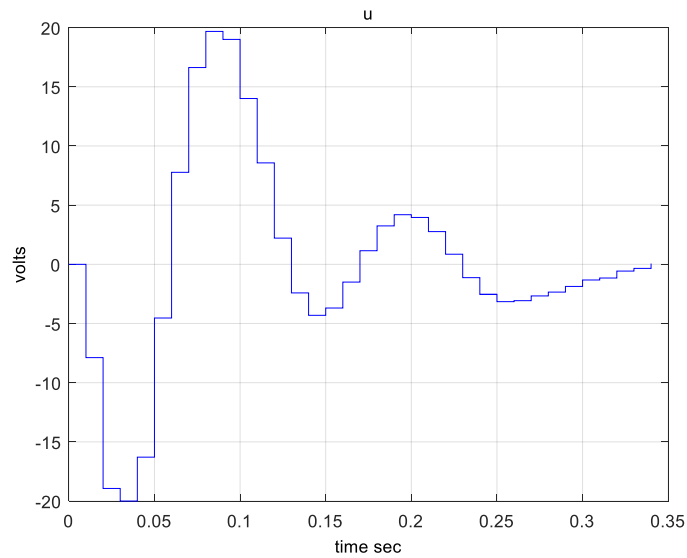
Figure 4.7.6 The control input u

As it is clear from figure 4.7.6, the magnitude of control input u is bounded within the desired limits.

In the next step of the design, the system was simulated with data for the earthquake signal. In MATLAB, the file quake.mat contains 200Hz data from the October 17, 1989 Loma Prieta earthquake in the Santa Cruz mountains. This file was used to synthesize the

controller under realistic conditions. In order to see the advantages achieved through application of the deadbeat controller in the system under consideration, the results were compared with those from the open-loop system and a controller designed via LQR. The LQR problem formulated and solved was:

$$J[u] = \sum_{k=0}^{\infty} [x(k)^T \quad u(k)^T] \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

in which Q, R and N matrices were suitably selected. Simulation results are given in figures 4.7.7-4.7.10.
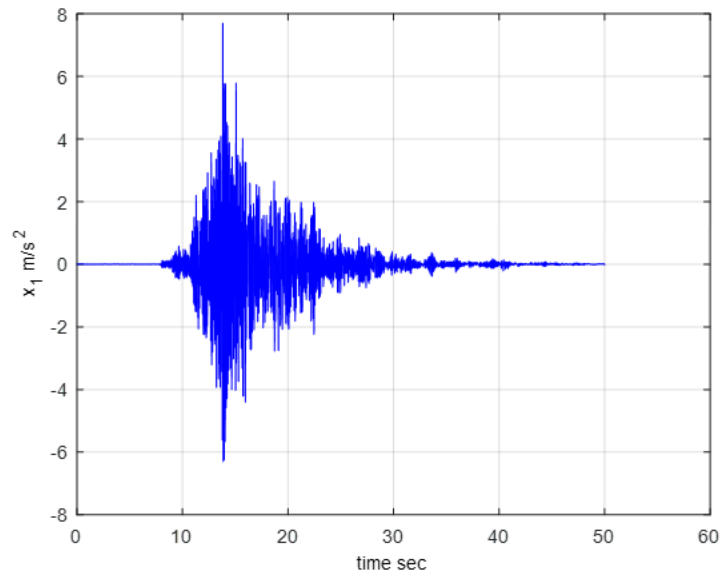
Figure 4.7.7 First floor acceleration- open-loop system
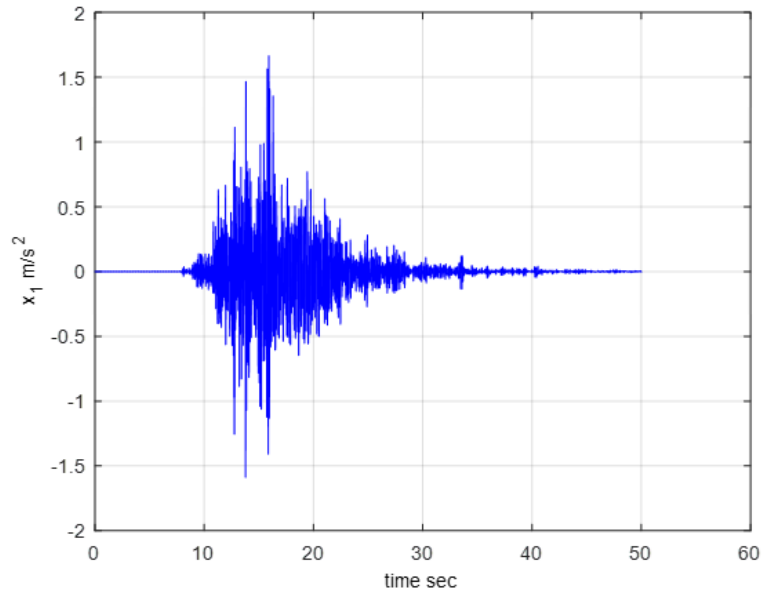
Figure 4.7.8 First floor acceleration- LQR design
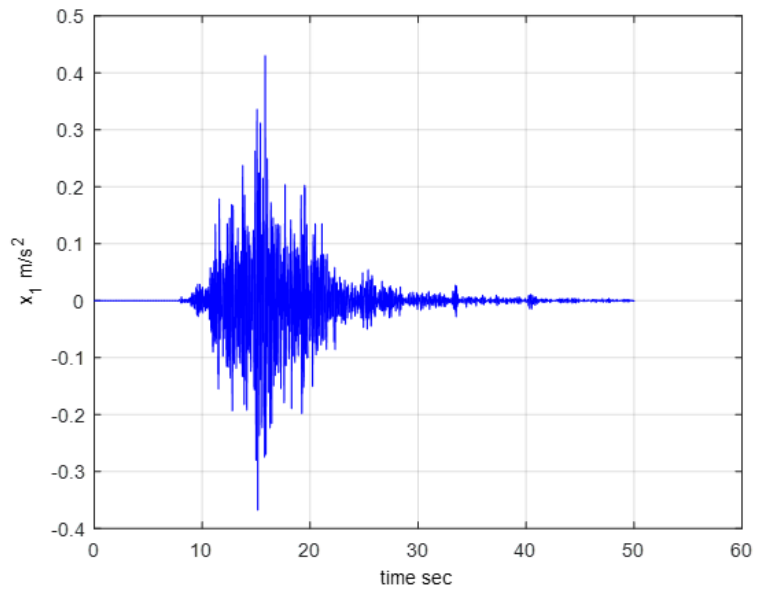
Figure 4.7.9 First floor acceleration- deadbeat control
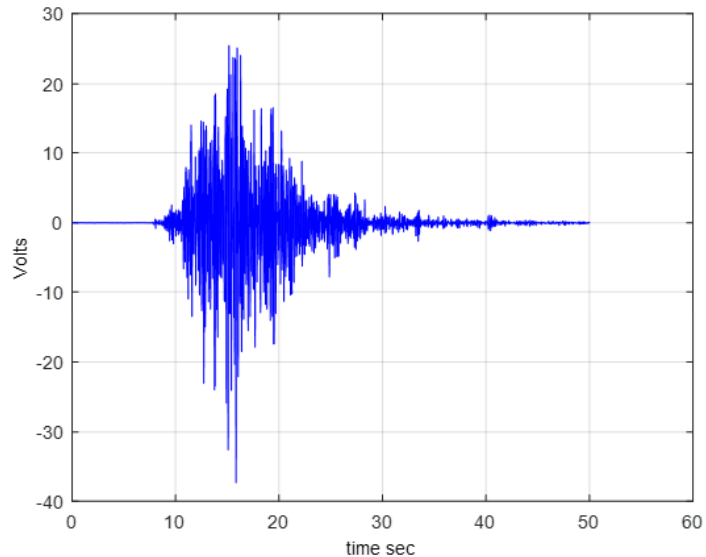
Figure 4.7.10 Deadbeat control input

As figures 4.7.7-4.7.9 show, the peak acceleration with deadbeat response is almost 17 times lower than peak response of open-loop system and almost 4 times lower than LQR design for comparable peak values of control signal shown in figure 4.7.10.

4.8 Conclusion:

This chapter addressed the problem of deadbeat controller design with time domain constraints. First, the mathematical description of deadbeat systems was briefly reviewed. It was observed that having a control system with finite settling time responses compels to restrict not only T_{11}, T_{12}, and T_{21} but also the free parameter Q to be deadbeat. In terms of their impulse responses, this was equivalent to making them all FIR.

A partial list of typical time domain performance specifications in terms of the system response to unit step was derived. Based on a concise description of linear programming, it was shown that time domain performance specifications translate into constraints on the parameter Q. This attribute was then exploited to recast the deadbeat controller design with transient and steady state time domain constraints as an LP with Q being the design parameter.

More stringent time domain performance criteria in the form of LQG were also considered. The problem was tackled through a different interpretation of LQG in which

it is regarded to be a special case of a broader class of problems known as H_2 optimization. This interpretation eliminates the need to incorporate the stochastic ingredient of the LQG. The relation between the LQG and H_2 optimization is observed by recognizing that the LQG performance index which aims to minimize the RMS value of the error signal (the squared-power of which represents the average power of the signal), is expressible as the system H_2 norm when it is excited by white noise input signal. On the other hand, H_2 norm was given two deterministic and stochastic interpretations. The stochastic interpretation enabled us to equate the H_2 optimization problem with that of minimizing the RMS value (or equivalently the power) of the system response to white noise inputs. In the case of deterministic inputs, squared H_2 norm coincides with the total output energy in the impulse response of the system. The relation between the LQG and H_2 problems was then employed to translate the problem of deadbeat controller design with LQG performance criteria into a quadratic programming with the Q parameter as the design variable. This chapter concluded by an example to represent the design procedures. So, basically in this chapter the main achieved result was showing that the synthesis problem of deadbeat regulator subject to the transient and steady state performance specifications is expressible as a linear program. Moreover, the synthesis problem when the controller is to satisfy quadratic performance requirements can be recast as a quadratic program.

Chapter 5

Robust pole placement, LMI approach

5.1 Introduction:

In designing any control system, stability is regarded as the minimum requirement. In most practical situations though, a good controller should also represents acceptable dynamical behaviour, e.g. sufficiently fast and well-damped time responses. It is well known that a customary way of defining many properties of system dynamics is through root clustering. However, for LTI systems achieving the exact location of the poles might be difficult, due to the fact that no mathematical model can exactly describe a physical system. Further to the uncertainty arising from lack of understanding of the physical process which amounts to non-exact characterization of the system, deliberate negligence of the system dynamics in the modelling process, in order to make a number of simplifications, may also introduce uncertainty into the system description.

As was argued in earlier chapters, attaining deadbeat response, which is the main aim of this thesis, requires locating all the eigenvalues at the origin of the complex plane. However, due to the existence of uncertainty, performance of the system might be adversely affected and hence, the deadbeat characteristic of the system response will be lost. In this chapter, we are concerned with the synthesis problem of robust deadbeat controller. First, different sources of uncertainty, their classifications and ways of representing them will be reviewed. The sensitivity of eigenvalues to parametric uncertainty and conditions for minimising it, will then be briefly examined. This is followed by revising the general concept of quadratic stability in the form of the Lyapunov theorem, based on which the conditions for quadratic stability of systems subject to structured norm-bounded parametric uncertainties entering both the state and input matrices, are derived. Notion of the quadratic stability will then be extended to obtain the criteria for quadratic stability of the systems in the generalized stability regions, the so-called quadratic $\mathcal{D}$-stability problem. This is accomplished through introduction of

the LMI (Linear Matrix Inequality) regions. All the conditions are derived first based on the continuous time system descriptions. However, in later sections they will be accommodated to the case of discrete time systems, which are the main focus of this work. In the light of achieved results, the robust deadbeat compensator for systems perturbed by structured parametric uncertainty will be synthesised.

5.2 System uncertainty and its classifications:

Real systems always involve some amount of uncertainty. So, it is natural that the robustness of system properties be one of the main concerns in any analysis and synthesis problem. System uncertainty and the issue of robustness have been greatly discussed in many references like [1, 2, 5, 65, 79, 83-85, 99] from which the following revision is taken.

Basically, uncertainty in the plant model have several origins based on which sources of model uncertainty may be classified into two main classes, namely parametric uncertainty and unstructured uncertainty corresponding to neglected or unmodelled dynamics.

Parametric uncertainty arises from the parameters in system description which are only known approximately or are in error. Because of existing nonlinearities or changes in the operating conditions, there are also varying parameters in the linear model of the system. Imperfections in measurement devices may also introduce uncertainty in the model through giving rise to uncertainty on the manipulated inputs.

There are situations in which despite the availability of a detailed model, it is simpler to consider a model of lower-order and instead, regard the neglected dynamics as uncertainty. Even without deliberate negligence of system dynamics, the structure and the model order may be unknown, especially at high frequencies. Obviously these two cases amount to the neglected and unmodelled dynamics, respectively. Finally, sometimes the controller computed from the design problem may differ from the one implemented in practice. In this case, uncertainty might be included in the form of unmodelled dynamics, to account for inaccuracies resulting from controller implementation or its order reduction.

One or several sources of the two aforementioned perturbation classes may be combined together to form a single lumped perturbation of a chosen structure, which can be considered as the third class of uncertainty, the so-called lumped uncertainty.

Dealing with perturbations in any analysis and synthesis problem requires the uncertainty to be quantified. In order to quantify parametric uncertainty, it is assumed that the equations characterising the dynamical behaviour of a system are known but there is uncertainty about the precise values of parameters in these equations, typically described by an interval of possible values. In the case of a discrete time system, difference equations adapted for representing the uncertainty in the parameters of the system may be characterized as:

$$x(k + 1) = A(p)x(k) + B(p)u(k), \quad p \in \Re$$

in which the uncertain parameter vector p is not known a priori. It is only known that it belongs to the bounding set $\Re$, which is the set of bounding parameters and generally defined as a hyper-rectangle in the parameter space, the so-called parameter box [81].

In the above description, each uncertain parameter p_i is assumed to belong to a known interval. It is not difficult to see that the parametric uncertainty is real in nature and models the perturbation in a structured manner. That is why this type of perturbation is also known as structured uncertainty. In comparison, since the other two classes of uncertainty are somewhat less precise and subsequently more difficult to quantify, they are referred to as unstructured uncertainty. The frequency domain is particularly well suited for these two types to be dealt with in, and are usually quantified by norm bounds.

Although at first sight it seems that the parametric uncertainty is easier to handle, but it is often avoided. In modelling this type of perturbation, a large effort is required to describe the exact model structure which in turn restrains the unmodelled dynamics to be dealt with. On the other hand, as structured uncertainty is real in nature, mathematically and numerically it is more difficult to be dealt with, especially when it comes to controller synthesis. These motives justify translating the parametric uncertainty into complex perturbations. Clearly, this introduces conservatism into the model, but as it is shown in

[79], the conservatism can be reduced by lumping these perturbations into a single complex one.

In robustness analysis and synthesis problems as in most other problems, usually the starting point is an illustration of the system in hand. In robustness problems, the customary way is pulling out the uncertain perturbations into a block diagonal matrix as:

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix} \qquad (5.2.1)$$

in which each Δ_i designates a specific source of uncertainty. Then, the uncertainty could be combined with the general framework of figure 2.3.1, to form the general configuration for controller synthesis as in figure 5.2.1 [79, 8, 1].
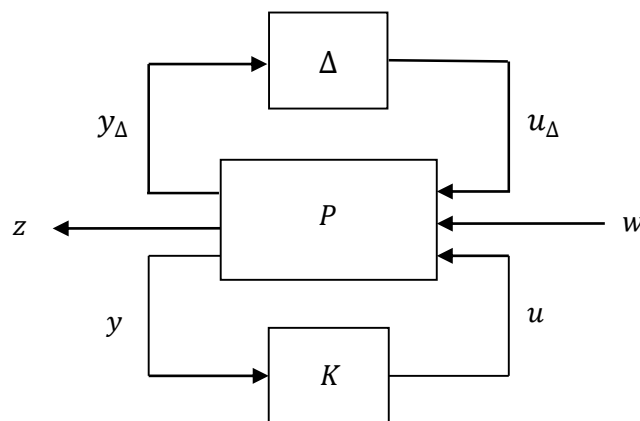
Figure 5.2.1 The general configuration for robust controller synthesis

Alternatively, combining the generalized plant and compensator as a lower LFT to get an equivalent block N, gets the so-called $N\Delta$-structure which may be utilized for robust analysis. The construction is illustrated in figure 5.2.2 [79, 1].
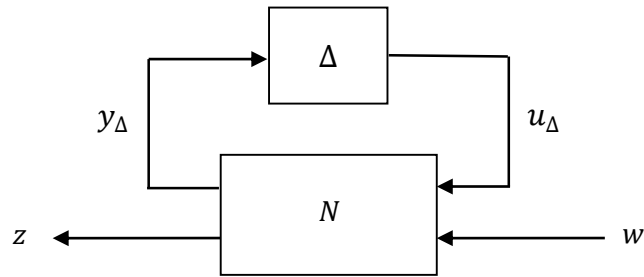
Figure 5.2.2 The $N\Delta$-structure for robust analysis

5.3 Investigating the sensitivity of eigenvalues to perturbations:

Among various approaches to achieve a better system response, a major one is considered to be closed-loop pole assignment, and it is well known that many characteristics of system dynamics may be defined in this way. Since physical systems always involve some amount of uncertainty, the exact locations of the poles might be difficult to attain. Hence, different techniques in robust control theory which deal with various control problems in which plant modelling uncertainty or exogenous signal uncertainty is a dominant issue, were employed to locate the poles within a prescribed region. Besides existence of uncertainty, other motivations for seeking root clustering in specific regions are that of the asymptotic and relative stability and achieving desirable transient response, e.g. by assigning the eigenvalues in a shifted left half plane to adjust the minimal decay rate of the system [86, 87]. In this chapter though, the incentive is robustly assigning the eigenvalues to the origin of the complex plane, as the deadbeat response requirement implies.

Historically, two main approaches to robust control problem have emerged; frequency domain technique, and time domain technique. Here, we tackle the problem of discrete time robust eigenvalue assignment in the time-domain framework. However, first it is worth examining the sensitivity of eigenvalues to parametric uncertainty and conditions for minimizing it.

A nominal LTI discrete time system with dynamic state equation:

$$x(k + 1) = Ax(k) + Bu(k) \qquad (5.3.1)$$

perturbed by linear time varying perturbations entering both the state and input matrices, may be described by difference equation of the form:

$$x(k + 1) = A(\xi(k))x(k) + B(\psi(k))u(k) \qquad (5.3.2)$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and A and B are matrices of appropriate dimensions. The vector-valued functions $\xi(k) \in \mathcal{T}$ and $\psi(k) \in \mathcal{S}$ represent the time varying parametric perturbations, referred to as model parameter uncertainty and input connection parameter uncertainty, respectively. In the general description of (5.3.2), it is assumed that the pre-specified uncertainty restrain sets $\mathcal{T}$ and $\mathcal{S}$ are compact. [89, 90].

The general characterization (5.3.2) of a perturbed system may be simplified as:

$$x(k + 1) = (A + \Delta A)x(k) + (B + \Delta B)u(k) \qquad (5.3.3)$$

in which A and B are the nominal values, while ΔA and ΔB designate uncertainty. The bounding set of perturbation matrices is defined to be Ω such that $(\Delta A, \Delta B) \in \Omega$, where Ω is a compact set in $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$.

It is well known that behaviour of the system (5.3.1), governed by the eigenvalues of the system matrix, may be modified through application of a state feedback control $u(k) = Fx(k) + v(k)$. This, in turn, yields the modified dynamic system $x(k + 1) = (A + BF)x(k) + Bv(k)$ with a desired set of poles $\ell = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, closed under complex conjugation. Such state feedback gain matrix F exists if and only if the pair (A, B) is controllable [45]. In the case of single input systems $(m = 1)$, as is declared in [92], the gain matrix F can be shown to be unique. When the number of independent inputs is equal to that of the states, the system is always completely controllable and any given set of closed-loop eigenvalues is achievable by feedback. In the case that there are fewer inputs than states, the solution to F is in general underdetermined with many degrees of freedom. These extra degrees of freedom may be exploited to exert additional requirements, like insensitivity of eigenvalues to perturbations in the coefficient matrices of the closed-loop system equation.

In order to examine the sensitivity of eigenvalues to uncertainties, let X_j and Y_j, $j = 1, 2, \dots, n$, be respectively the associated right and left eigenvectors of $\lambda_j \in \ell$, which is an eigenvalue of the closed-loop system matrix $A_{cl} = (A + BF)$. If A_{cl} has n linearly

independent eigenvectors, i.e. it is non-defective, the sensitivity of eigenvalue λ_j, designated by s_j, to the existing perturbations in the components A, B, and F hinge upon the magnitude of its condition number c_j, where [93]:

$$c_j = \frac{1}{s_j} = \frac{\|Y_j\|_2 \|X_j\|_2}{\left|Y_j^T X_j\right|} \geq 1 \qquad (5.3.4)$$

By stacking the eigenvectors to form the eigenvector matrix $\boldsymbol{X} = [X_1, X_2, \ldots, X_n]$, an upper bound on the sensitivities of the eigenvalues may be defined as [93]:

$$\max_j c_j \leq K_2(\boldsymbol{X}) \equiv \|\boldsymbol{X}\|_2 \|\boldsymbol{X}^{-1}\|_2 \qquad (5.3.5)$$

in which $K_2(\boldsymbol{X})$ is the condition number of the matrix $\boldsymbol{X}$. Evidently, $K_2(\boldsymbol{X})$ establishes a measure for the robustness of eigenvalues to the existing perturbations. Based on this, the robust eigenvalue assignment problem may be recast as the problem of selecting independent vectors X_j, $j = 1, 2, \ldots, n$, such that the closed-loop system has the desired set of ℓ eigenvalues while the conditioning of the eigenproblem is minimized. It is not difficult to see that the degrees of freedom available in the selection of the matrix $\boldsymbol{X}$ are reflected precisely by those available in the choice of the feedback gain matrix F. The only restriction in assigning the eigenvectors arises from the fact that the closed-loop system matrix A_{cl} has to be non-defective. This restriction just imposes simple conditions on the multiplicity of the poles which may be assigned [88].

To conclude the section, we remark that $K_2(\boldsymbol{X})$ is not the only possible measure of the sensitivity of the eigenvalues which may be defined [88].

5.4 Quadratic stability:

As was argued in earlier chapters, attaining a deadbeat response demands the placement of all the eigenvalues at the origin of the complex plane. However, due to the existence of uncertainty in the mathematical description of the system, it is almost impossible to assign all the closed-loop poles exactly to the origin for every model in the uncertainty

set. As a result, robust performance is adversely affected and hence the deadbeat characteristic of the system response will be lost for the closed-loop systems corresponding to (almost all) uncertain parameters. This reveals the significance of the robust pole placement. In this section, we consider a quick revision of the notion of quadratic stability. This forms the foundation for the discussion in the subsequent section which addresses a specific notion of stability, known as quadratic $\mathcal{D}$-stability and which in fact is the counterpart of quadratic stability in the context of pole clustering.

In the field of uncertain systems, as is well-known, one of the major tools in stability analysis is Lyapunov theory [108, 110, 111]. The Lyapunov approach which deals with both linear and nonlinear systems described by differential or difference equations, consists of two methods of stability analysis, called the first method and the second or direct method. The distinction between the two methods arises from the fact that the first method depends on finding approximate solutions to the differential or difference equations, while in the direct method such knowledge is not necessary. That is the reason the second method is of great importance in practice.

The second method of Lyapunov is based on a generalization of the fact that "*if the system has an asymptotically stable equilibrium state, then the stored energy of the system displaced within a domain of attraction (i.e. the largest region of asymptotic stability) decays with increasing time until it finally assumes its minimum value at the equilibrium state*" [49, p. 322].

In order to capture the notion of energy function in systems with descriptions which are purely mathematical, a fictitious energy function known as the Lyapunov function is introduced. Generically speaking, a Lyapunov function $V(x, t)$ is a positive definite scalar function which is continuous together with its first partial derivative in a region around the origin. It is a monotonically decreasing function along the system trajectories. For a system described by $\dot{x}(t) = f(x, t)$, this property can be verified just through examining the negativity of the Lyapunov function derivative $\dot{V}(x, t) = \partial_x V(x, t) f(x, t) + \partial_t V(x, t)$ and without knowing the system trajectories. Evidently, when $V(x, t)$ is regarded as the generalized energy, $\dot{V}(x, t)$ could be interpreted as the associated generalized dissipation. The relation between energy function and Lyapunov function has been explored in [9].

As is extensively discussed in the literature, the simplest positive definite function for a Lyapunov function is of the quadratic form $V(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$. However, in general, Lyapunov functions are not necessarily of a simple quadratic form.

The Lyapunov stability theory for a discrete time system may be summarized as follows:

Theorem 5.4.1 [49] For the discrete time system $x((k + 1)T) = f(x(kT))$ with the state vector $x \in \mathbb{R}^n$, and $f(x)$ an n-vector with the property $f(0) = 0$, suppose that there exists a scalar function $V(x)$ continuous in x such that:

1. $V(x) > 0$ for $x \neq 0$
2. $\Delta V(x) < 0$ for $x \neq 0$, in which
   $\Delta V(x(kT)) = V(x(k + 1)T) - V(x(kT)) = V(f(x(kT))) - V(x(kT))$
   is the first difference of $V(x)$ along the system trajectories.
3. $V(0) = 0$
4. $V(x) \to \infty$ as $\|x\|_2 \to \infty$

Then the equilibrium state $x = 0$ is asymptotically stable in the large and $V(x)$ is a Lyapunov function.

The above Lyapunov theorem for LTI discrete time systems, which are the main focus of this work, with quadratic Lyapunov functions can be stated in the form of succeeding theorem:

Theorem 5.4.2 [49] For the LTI discrete time system $x(k + 1) = Ax(k)$ where A is a constant nonsingular matrix, let us select a quadratic Lyapunov function $V(x(k)) = x^T(k)\mathrm{P}x(k)$. According to the Lyapunov difference:

$$\Delta V(x(k)) = V(x(k + 1)) - V(x(k)) = x^T(k)(A^T\mathrm{P}A - \mathrm{P})x(k)$$
$$= -x^T(k)Qx(k) \tag{5.4.1}$$

the system is asymptotically stable if and only if for any given positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P such that the matrix equality

$A^T \mathrm{P} A - \mathrm{P} = -Q$, known as the Lyapunov algebraic equation, is satisfied. This notion of stability is also referred to as the quadratic stability.

As can be seen, in the case of linear systems the stability conditions obtained from a particular Lyapunov function are both sufficient and necessary. Bearing in mind that systems are always subject to some type of uncertainty, it will be natural to look into the generalization of the Lyapunov theorem for the case of perturbed systems. This issue will be addressed in the next section.

5.5 Quadratic stability of continuous time systems with structured norm-bounded parametric uncertainty entering the state matrix:

The preceding section reviewed the general concept of quadratic stability. In the forthcoming sections, we will study the generalization of quadratic stability for the case of systems subject to parametric uncertainty. Although we are mainly interested in the results for the perturbed discrete time systems, we first consider continuous time systems and then the results will be extended to the case of discrete time systems.

An LTI continuous system subject to parametric uncertainty may be characterized by the most basic form, i.e. when perturbation enters only the state matrix, the so-called model parameter uncertainty, as:

$$\dot{x}(t) = A(p)x(t) \qquad (5.5.1)$$

with $p \in \mathfrak{R} \subset \mathbb{R}^q$ being the vector of uncertain parameters, and $\mathfrak{R}$ denoting the parameter box. Such systems are referred to as linear parameter varying (LPV) systems, whose state space matrices are fixed functions of some vector of varying parameters p [112, 113]. Evidently, the above description encompasses a collection of an infinite number of systems. For any given $p \in \mathfrak{R}$, (5.5.1) yields a system of differential equations with constant coefficients. On the other hand, if p is a vector-valued function of time belonging to a certain functional space $\mathcal{S}$, for any $p(.) \in \mathcal{S}$ the description (5.5.1) defines a system of differential equations with time-varying coefficients [100].

Definition 5.5.1 [90, 95, 101, 102] The system (5.5.1) is quadratically stable in $\Re$ if and only if there exists a positive definite matrix P, such that for all $p \in \Re$:

$$A^T(p)\mathrm{P} + \mathrm{P}A(p) < 0 \qquad (5.5.2)$$

In view of the Lyapunov algebraic equation for continuous time systems [49], it is readily seen that the quadratic stability of the system (5.5.1) is reduced to the question of whether there is a quadratic Lyapunov function in the form $V(x) = x^T\mathrm{P}x$, which establishes quadratic stability.

It should be noted that the concept of quadratic stability requires existence of a single quadratic Lyapunov function P for all the systems in the uncertainty domain. The existence of a fixed Lyapunov function for all possible choices of the uncertain parameters ensures the closed-loop system stability for time-varying perturbations [94, 102].

The elementary parametric uncertainty description of (5.5.1) may be recast in a more customary fashion as what follows [95, 96]:

$$\dot{x}(t) = (A + D\Delta E)x(t) \qquad (5.5.3)$$

In the above characterization of uncertainty, known as structured norm-bounded uncertainty, the matrices $D \in \mathbb{R}^{n \times p}$ and $E \in \mathbb{R}^{q \times n}$ are known real matrices that characterize the structure of the perturbations and take into account scaling factors, while the unknown matrix Δ designates the modelling uncertainty. The uncertainty is assumed to be bounded and normalized as follows:

$$\Delta \in \boldsymbol{\Delta} := \{\Delta: \|\Delta\|_2 \leq I\} \qquad (5.5.4)$$

The idea behind the presumed structure of (5.5.3) for uncertainty, which in fact is a generalization of the more intuitive uncertainty representation $\dot{x}(t) = (A + \Delta A)x(k)$, is twofold. First, there are many physical systems in which the uncertainty may be modelled in this manner. Moreover, a linear interconnection of a nominal plant with the uncertainty

Δ results in a description of the form (5.5.3) [94]. As a matter of fact, system (5.4.3) is equivalent to the classical feedback interconnection of the linear system:

$$\dot{x}(t) = Ax(t) + Dw_\Delta(t) \qquad (5.5.5)$$
$$z_\Delta(t) = Ex(t)$$

and the perturbation:

$$w_\Delta(t) = \Delta z_\Delta(t) \qquad (5.5.6)$$

A larger class of uncertainties may be captured by assuming a nonzero direct feed-through matrix $L \in \mathbb{R}^{q \times p}$, as:

$$\dot{x}(t) = Ax(t) + Dw_\Delta(t) \qquad (5.5.7)$$
$$z_\Delta(t) = Ex(t) + Lw_\Delta(t)$$

and the perturbation Δ specified in (5.5.6). The resultant feedback system:

$$\dot{x}(t) = (A + D\Delta(I - L\Delta)^{-1}E)x(t) = (A + D(I - LH)^{-1}\Delta E)x(t) \qquad (5.5.8)$$
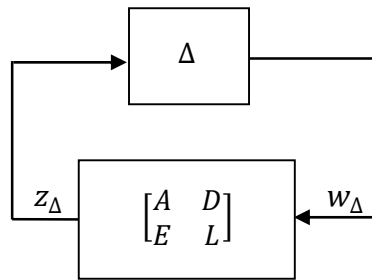
is depicted in figure 5.5.1.

Figure 5.5.1 Closed-loop feedback interconnection of the system (5.4.3) and the uncertainty characterized in (5.5.4) with nonzero feed-through matrix L (as in (5.4.7))

In view of definition 5.4.1, system (5.5.8) is quadratically stable if and only if there exist a positive definite matrix P such that for all $\|\Delta\|_2 \leq I$:

$$(A + D\Delta(I - L\Delta)^{-1}E)^T \mathrm{P} + \mathrm{P}(A + D\Delta(I - L\Delta)^{-1}E) < 0 \qquad (5.5.9)$$

The condition (5.5.9) may be expressed equivalently in the form of following theorem.

Theorem 5.5.2 [100] Necessary and sufficient condition for quadratic stability of system (5.5.8) is the existence of a positive definite matrix P such that:

$$\begin{bmatrix} A^T\mathrm{P} + \mathrm{P}A + E^TE & \mathrm{P}D + E^TL \\ D^T\mathrm{P} + L^TE & -(I - L^TL) \end{bmatrix} < 0 \qquad (5.5.10)$$

What is interesting about the foregoing theorem is that the condition for quadratic stability is indicated in the form of an LMI feasibility problem. As is well known, solving LMIs is a convex optimization problem [141]. Moreover, LMI problems can be solved via efficient tractable numerical algorithms, e.g. interior-point methods [9, 103, 104, 105, 106]. This is especially beneficial for solving problems lacking analytical solution.

One of the major implications of the inequality condition (5.5.10) is that $I - L^TL > 0$, from which it can correspondingly be inferred that:

$$\|L\Delta\|_2 \leq \|L\|_2\|\Delta\|_2 \leq \|L\|_2 < 1 \qquad (5.5.11)$$

This ensures that for all $\|\Delta\|_2 \leq 1$, $I - L\Delta$ is nonsingular, which is exactly the requirement for well-posedness of the system (5.5.8). Therefore, the condition for quadratic stability automatically guarantees the closed-loop system well-posedness.

In view of the Schur complement [141], the criterion in theorem 5.5.2 is also expressible as:

$$\begin{bmatrix} A^T\mathrm{P} + \mathrm{P}A & \mathrm{P}D & E^T \\ D^T\mathrm{P} & -I & L^T \\ E & L & -I \end{bmatrix} < 0 \qquad (5.5.12)$$

which again is in the form of an LMI feasibility condition. The discussion in this section centred around stability of continuous time systems with perturbed state matrix. The problem may be expanded to the case of uncertain systems in which the perturbation influences both the state and input matrices. Stability of such systems is the topic of the following section.

5.6 Quadratic stability of continuous time systems with structured norm-bounded parametric uncertainty entering both the state and input matrices:

Generically, a continuous time system in which both state and input matrices are subject to parametric uncertainty, respectively known as model parameter uncertainty and input connection parameter uncertainty, is described in the form of a linear parameter varying (LPV) system whose state space matrix is fixed function of some vector of varying parameters p, as [112, 113]:

$$\dot{x}(t) = A(p)x(t) + B(p)u(t) \qquad (5.6.1)$$

where $x(t) \in \mathbb{R}^n$, $\mathrm{u}(t) \in \mathbb{R}^m$, and $p \in \mathfrak{R} \subset \mathbb{R}^q$, with $\mathfrak{R}$ being the parameter box. Due to the same reasons argued in section 5.5, the perturbations can be expressed as structured norm-bounded uncertainties [95, 96, 107]:

$$[\Delta A \quad \Delta B] = D\Delta(I - L\Delta)^{-1}[E_1 \quad E_2] \qquad (5.6.2)$$

with $\|L\|_2 < 1$, and $\Delta \in \mathbb{R}^{p \times q}$ any uncertain matrix satisfying:

$$\Delta \in \boldsymbol{\Delta} := \{\Delta \colon \|\Delta\|_2 \leq I\} \qquad (5.6.3)$$

Quadratic stability of perturbed system (5.6.1) may be achieved via application of a memoryless linear time-invariant state-feedback compensator.

Definition 5.6.1 [100] System (5.6.1) is said to be quadratically stabilizable via linear state feedback control $u = Kx$, $K \in \mathbb{R}^{m \times n}$ if and only if the closed-loop system:

$$\dot{x}(t) = (A(p) + B(p)K)x(t) \qquad (5.6.4)$$

is quadratically stable.

According to the debate in [114], quadratic stability through dynamic time-varying linear state feedback compensator infers quadratic stability through static time-invariant linear state feedback compensator. However, Petersen in [115] by making use of a contradictory example has argued that quadratic stabilizability of a linear system subject to parametric uncertainties described in (5.6.1) via a nonlinear controller does not necessarily imply quadratic stabilizability by a linear controller.

By imposing additional assumptions of (5.6.2) and (5.6.3) on the structure of uncertainty, it is shown in [95] and [116] that quadratic stability via nonlinear control implies quadratic stability via linear control. This declaration is indicated in the form of the following theorem:

Theorem 5.6.2 [96] For the system (5.6.1) subject to norm-bounded uncertainties with the structure described in (5.6.2) and (5.6.3), the following statements are equivalent:

(i)   The system admits a control Lyapunov function.
(ii)   The system is quadratically stabilizable.
(iii)   The system is quadratically stabilizable via linear time-invariant (memoryless) control.

It should be pointed out that existence of a control Lyapunov function is just a necessary condition for quadratic stability of the system with the general characterization of (5.6.1).

Having discussed the nature of controller which renders the system with assumed norm-bounded uncertainties quadratically stable, the closed-loop interconnection illustrated in figure 5.6.1, can be achieved as (see (5.5.5)-(5.5.7)):

$$\dot{x}(t) = (A + BK)x(t) + Dw_\Delta(t)$$
$$z_\Delta(t) = (E_1 + E_2 K)x(t) + Lw_\Delta(t) \qquad\qquad (5.6.5)$$
$$w_\Delta = \Delta z_\Delta$$

With regards to the inequality condition (5.5.12), the necessary and sufficient condition for quadratic stability of the system (5.6.1) subject to norm-bounded uncertainties described in (5.6.2) and (5.6.3), can be expressed as the existence of a positive definite matrix P such that the following LMI feasibility condition is satisfied [100]:

$$\begin{bmatrix} (A + BK)^T P + P(A + BK) & PD & (E_1 + E_2 K)^T \\ D^T P & -I & L^T \\ (E_1 + E_2 K) & L & -I \end{bmatrix} < 0 \qquad\qquad (5.6.6)$$

The notion of quadratic stability may be extended, in a natural way, to the case when the eigenvalues should be assigned to the sub-regions of the complex plane, the so-called generalized stability regions. The problem, known as the quadratic $\mathcal{D}$-stability, is dealt with in the two ensuing sections.
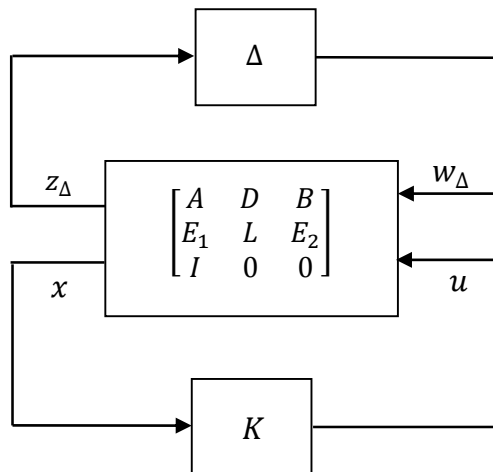

Figure 5.6.1 Closed-loop feedback configuration of perturbed system (5.6.5) with state feedback

5.7 Quadratic $\mathcal{D}$-stability of continuous time systems with structured norm-bounded parametric uncertainty entering the state matrix:

Basically, the standard Lyapunov theorem concerns the asymptotic behaviour of the states in both discrete and continuous time systems. However, as was mentioned earlier, besides stability, requiring desirable system dynamical behaviour compels us to assign the closed-loop poles to specific sub-regions of the complex plane, designated by a set $\mathcal{D}$ [87]. Over and above, in practice it is appealing to have robust performance which in turn necessitates robust assignment of the modes to the generalized stability region $\mathcal{D}$. Such problems are known as robust $\mathcal{D}$-stability problems [97]. In this section, robust $\mathcal{D}$-stability will be addressed based on the generalization of the Lyapunov theorem. This leads to a necessary and sufficient condition for $\mathcal{D}$-stability in the generalized stability regions. When a system is subject to uncertainty, the notion of $\mathcal{D}$-stability may be developed in a natural way to that of the quadratic $\mathcal{D}$-stability. Hence, quadratic $\mathcal{D}$-stability extends $\mathcal{D}$-stability to uncertain systems in a similar fashion as quadratic stability extends stability to uncertain systems.

The generalization of the Lyapunov theory was accomplished through introduction of Linear Matrix Inequality (LMI) regions.

Definition 5.7.1 [98] A subset $\mathcal{D}$ of the complex plane is called an LMI region if there exists a symmetric matrix Λ, and a matrix Θ such that:

$$\mathcal{D} = \{z \in \mathbb{C}: \ f_{\mathcal{D}}(z) < 0\} \qquad (5.7.1)$$

with the matrix-valued function $f_{\mathcal{D}}(z)$:

$$f_{\mathcal{D}}(z) = \Lambda + z\Theta + z^*\Theta^T \qquad (5.7.2)$$

that is referred to as the characteristic function of $\mathcal{D}$.

A wide variety of typical clustering regions may be characterised in the form of LMI regions. As an example, take the case of left half plane $\mathrm{Re}(z) < -a$, which is cast as an LMI region by $f_{\mathcal{D}}(z) = z + z^* + 2a < 0$.

Another favourable example is the disk centred at $(-q, 0)$ with radius r that corresponds to the LMI region described by:

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -r & q+z \\ q+z^* & -r \end{bmatrix} < 0 \qquad (5.7.3)$$

with:

$$\Lambda = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (5.7.4)$$

Based on the definition of LMI regions, the condition for $\mathcal{D}$-stability of a definite system may be expressed as follows:

Theorem 5.7.2 [97] A real matrix A is said to be $\mathcal{D}$-stable if and only if there exists a positive definite matrix P such that:

$$M_{\mathcal{D}}(A, \mathrm{P}) = \Lambda \otimes \mathrm{P} + \Theta \otimes (\mathrm{P}A) + \Theta^T \otimes (A^T \mathrm{P}) < 0 \qquad (5.7.5)$$

In view of the LMI characterization of usual LHP (left half plane), it is readily seen that the inequality (5.7.5) is in fact the generalization of the Lyapunov theorem, in that by substituting $\Lambda = \Theta = 1$, it reduces to $A^T \mathrm{P} + \mathrm{P}A < 0$.

When the state matrix A is subject to model parameter uncertainty, the above theorem accordingly converts to the next theorem.

Theorem 5.7.3 [98, 97] The uncertain system $\dot{x}(t) = A(p)x(t)$ is quadratically $\mathcal{D}$-stable if and only if there exists a positive definite matrix P such that for all $p \in \mathfrak{R}$:

$$M_{\mathcal{D}}(A(p), \mathrm{P}) = \Lambda \otimes \mathrm{P} + \Theta \otimes (\mathrm{P}A(p)) + \Theta^T \otimes (A^T(p)\mathrm{P}) < 0 \qquad (5.7.6)$$

For the case of an autonomous continuous time system perturbed by structured norm-bounded uncertainty, which were discussed in section 5.5 and described as:

$$\dot{x}(t) = (A + D\Delta(I - L\Delta)^{-1}E)x(t), \quad \Delta \in \mathbb{R}^{p \times q} \text{ and } \|\Delta\|_2 \leq I \qquad (5.7.7)$$

results similar to that of the theorem 5.7.3 are achievable in form of the succeeding theorem.

Theorem 5.7.4 [97, 100] Necessary and sufficient condition for quadratic $\mathcal{D}$-stability of the uncertain system (5.7.7), is the existence of a positive definite matrix P such that:

$$M_{\mathcal{D}}(A(p), \text{P}) = \Lambda \otimes \text{P} + \Theta \otimes \big(\text{P}(A + F\Delta(I - L\Delta)^{-1}E)\big)$$
$$+ \Theta^T \otimes ((A + F\Delta(I - L\Delta)^{-1}E)^T \text{P}) < 0 \qquad (5.7.8)$$

Investigating quadratic $\mathcal{D}$-stability of the system (5.7.7) could also be accomplished through examining criteria expressed in terms of LMIs, derived directly from (5.7.8). In this way, testing stability conditions can be efficiently tackled by exploiting LMI problem peculiarities and their solvers [9, 103, 104, 105, 106]. However, it should be noted that the LMI-based condition will only be a sufficient condition for quadratic $\mathcal{D}$-stability, as stated in the next theorem.

Theorem 5.7.5 [97] Suppose that the characteristic function of the generalized stability region $\mathcal{D}$ (see (5.7.2)), has a matrix Θ of rank κ and factorized as $\Theta = \Theta_1^T\Theta_2$, in which $\Theta_1, \Theta_2 \in \mathbb{R}^{\kappa \times h}$ are full row rank matrices (such a factorization is readily obtained from the SVD of Θ). Then, the system (5.7.7) is said to be quadratically $\mathcal{D}$-stable if there exist positive definite matrices $\text{P} \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{\kappa \times \kappa}$, such that:

$$\begin{bmatrix} M_{\mathcal{D}}(A, \text{P}) & \Theta_1^T \otimes (\text{P}D) & (\Theta_2^T V) \otimes E^T \\ \Theta_1 \otimes (D^T\text{P}) & -V \otimes I_p & V \otimes L^T \\ (V\Theta_2) \otimes E & V \otimes L & -V \otimes I_q \end{bmatrix} < 0 \qquad (5.7.9)$$

Although the above linear matrix inequality is just sufficient condition, in the case that Θ is of rank one the matrix V reduces to a scalar, which without loss of generality could be set to one. This in turn, leads the inequality (5.7.9) to be not just sufficient but also necessary condition [100]. This fact will be deployed to extract the condition for

quadratically assigning the modes of discrete time systems when $\mathcal{D}$ is a circular region. According to (5.7.4), for a circular region the matrix Θ is of rank one.

It should be pointed out that, (5.7.9) indicates:

$$\begin{bmatrix} V \otimes I_p & -V \otimes L^T \\ -V \otimes L & V \otimes I_q \end{bmatrix} > 0$$

which in turn guarantees the positive definiteness of $I - L^T L$. This, subsequently, ensures that $I - L\Delta$ in (5.7.7) is invertible for all $\|\Delta\|_2 \leq I$.

In the next section, the quadratic $\mathcal{D}$-stability problem will be extended to the case of continuous time systems which are subject to both structured norm-bounded model and input connection parameter uncertainties.

5.8 Quadratic $\mathcal{D}$-stability of continuous time systems with structured norm-bounded parametric uncertainty entering both the state and input matrix:

In section 5.6, quadratic stability of continuous time systems subject to structured norm-bounded model and input connection parameter uncertainties was studied. In the current section, this notion of stability will be extended to the generalized stability regions $\mathcal{D}$. Consider the perturbed system:

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \qquad (5.8.1)$$

with uncertainties characterized as:

$$[\Delta A \quad \Delta B] = D\Delta(I - L\Delta)^{-1}[E_1 \quad E_2], \quad \Delta \in \mathbb{R}^{p \times q}, \|\Delta\|_2 \leq I, \text{ and } \|L\|_2 < 1 \qquad (5.8.2)$$

In view of the definition 5.6.1, the system (5.8.1), (5.8.2) is quadratically $\mathcal{D}$-stabilizable via linear state feedback control $u = Kx$, if and only if there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that the closed-loop system (5.6.5) is quadratically $\mathcal{D}$-stable.

According to theorem 5.7.5, the condition for quadratic $\mathcal{D}$-stabilizability can be stated in the form of the ensuing theorem.

Theorem 5.8.1 [100] The system (5.8.1), (5.8.2) is quadratically $\mathcal{D}$-stable via state feedback paradigm $u = Kx$, if there exist positive definite matrices $P \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{\kappa \times \kappa}$, and a matrix $K \in \mathbb{R}^{m \times n}$ such that:

$$\begin{bmatrix} M_{\mathcal{D}}(A + BK, P) & \Theta_1^T \otimes (PD) & (\Theta_2^T V) \otimes (E_1 + E_2 K)^T \\ \Theta_1 \otimes (D^T P) & -V \otimes I_p & V \otimes L^T \\ (V\Theta_2) \otimes (E_1 + E_2 K) & V \otimes L & -V \otimes I_q \end{bmatrix} < 0 \qquad (5.8.3)$$

where κ is the rank of matrix Θ, factorized as $\Theta = \Theta_1^T \Theta_2$. As mentioned before, such a factorization is readily attainable from the SVD (singular value decomposition) of Θ. [97]

It is explicit that the above condition is not in the form of an LMI. However, it can be transformed into a linear matrix inequality by letting $V = I_\kappa$ and $K = YS^{-1}$ [9, 106], in which $S = P^{-1}$ is positive definite. That way, (5.8.3) will accordingly convert to the subsequent LMI based feasibility problem [100]:

$$\begin{bmatrix} M_{\mathcal{D},Lin}(A, S, Y) & \Theta_1^T \otimes D & \Theta_2^T \otimes (SE_1^T + Y^T E_2^T) \\ \Theta_1 \otimes D^T & -I_{\kappa p} & I_\kappa \otimes L^T \\ \Theta_2 \otimes (E_1 S + E_2 Y) & I_\kappa \otimes L & -I_{\kappa q} \end{bmatrix} < 0 \qquad (5.8.4)$$

where:

$$M_{\mathcal{D},Lin}(A, S, Y) = \Lambda \otimes S + \Theta \otimes (AS + BY) + \Theta^T \otimes (SA^T + Y^T B^T) \qquad (5.8.5)$$

For the case that Θ is a matrix of rank one, e.g. when $\mathcal{D}$ is a circular region or simply the LHP, the linear inequality of (5.8.4) turns to both sufficient and necessary condition for quadratic $\mathcal{D}$-stability of the system (5.8.1)-(5.8.2).

Having investigated the conditions for quadratic stability of continuous time systems subject to structured norm-bounded parametric uncertainties when the stability region is the LHP or any generalized stability region in the complex plane formulated as an LMI

region, we are now in a position to examine the conditions for the case of discrete time systems, which are the main focus of this work. As we will see, quadratic stability of discrete time systems is equivalent to the quadratic $\mathcal{D}$-stability of continuous time systems when the $\mathcal{D}$ region is the unit circle centred at the origin of the complex plane. The results will then be utilized to design a robust deadbeat controller, i.e. a controller which robustly assigns the eigenvalues to the origin of the complex plane.

5.9 Quadratic stability of discrete time systems with structured norm-bounded parametric uncertainty:

So far, the notion of quadratic stability and the conditions to achieve it has been investigated for the case of continuous time systems subject to structured norm-bounded uncertainties, characterized in (5.6.2). The concept, which on its own is developed as the generalization of the Lyapunov stability theorem for uncertain systems, was then extended to the generalized stability regions $\mathcal{D}$ in the complex plane, the so-called LMI regions.

In this section the criteria to attain quadratic stability in discrete time systems perturbed by norm-bounded model parameter and input connection parameter uncertainties will be examined. The conditions will then be applied to synthesize a robust deadbeat compensator which robustly assigns the poles near the origin of the complex plane.

The most basic form of a perturbed discrete time system is when it is subject to parametric uncertainty entering only the state matrix as:

$$x(k + 1) = A(p)x(k) \qquad (5.9.1)$$

Here, $p \in \mathfrak{R} \subset \mathbb{R}^q$ designates the vector of uncertain parameters.
The system (5.9.1) is said to be quadratically stable in $\mathfrak{R}$, if and only if there exists a matrix $P > 0$, such that for all $p \in \mathfrak{R}$: [117]

$$A^T(p)PA(p) - P < 0 \qquad (5.9.2)$$

In view of the Schur complement [141], the above expression can be equivalently restated as the solvability of a feasibility problem with LMI constraint as:

$$\begin{bmatrix} -P & PA(p) \\ A^T(p)P & -P \end{bmatrix} < 0 \qquad (5.9.3)$$

Referring to the theorem 5.7.3, and also the described LMI characterization of a circle in (5.7.3)-(5.7.4), it can be easily seen that (5.9.3) is in fact the condition for quadratic $\mathcal{D}$-stability of the continuous time counterpart of the system (5.9.1), i.e. $\dot{x}(t) = A(p)x(t)$, when the stability region is considered to be the unity circle centred at the origin. Basically, the conditions for quadratic stability of discrete time systems may be directly derived from those for quadratic $\mathcal{D}$-stability of continuous time systems when $\mathcal{D}$ is the unit disk centred at the origin. This statement can be formally expressed in the form of the following theorem.

Theorem 5.9.1 [100] Quadratic stability of a discrete time system is equivalent to quadratic $\mathcal{D}$-stability of its continuous time counterpart when $\mathcal{D}$ is the unit disk centred at the origin of the complex plane.

Let's again take the case of the system (5.9.1). Due to the reasons stated in section 5.4, the perturbation will be considered to be structured and norm-bounded. This leads to the subsequent description of the system:

$$x(k+1) = (A + \Delta A)x(k) = (A + D\Delta(I - L\Delta)^{-1}E)x(k) \qquad (5.9.4)$$

with real scaling matrices $D \in \mathbb{R}^{n \times p}$ and $E \in \mathbb{R}^{q \times n}$ characterizing the structure of the uncertainty, and the unknown matrix Δ designating the modelling uncertainty defined as:

$$\Delta \in \boldsymbol{\Delta} := \{\Delta \colon \|\Delta\|_2 \le I\} \qquad (5.9.5)$$

The direct feed-through matrix $L \in \mathbb{R}^{q \times p}$ (refer to (5.5.5)-(5.5.7)), is considered to be nonzero to account for larger class of uncertain systems. As before, for the sake of well-posedness, the constraint $\|L\|_2 < 1$ will be assumed throughout the section.

Theorem 5.9.2 [100] Necessary and sufficient condition for quadratic stability of the system (5.9.4) is the existence of a positive definite matrix P such that:

$$\begin{bmatrix} A^T P A - P + E^T E & A^T P D + E^T L \\ D^T P A + L^T E & D^T P D - (I - L^T L) \end{bmatrix} < 0 \qquad (5.9.6)$$

The LMI-based feasibility condition of (5.9.6) is readily achievable in view of theorems 5.7.5 and 5.9.1, and through substitution of:

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

where Θ is factorized as:

$$\Theta = \Theta_1^T \Theta_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Note that as the matrix Θ is of rank one, i.e. $\kappa = 1$, the matrix $V \in \mathbb{R}^{\kappa \times \kappa}$ in the inequality (5.7.9) will be scalar, which as mentioned before, without loss of generality may be equated to one. The scalar nature of the matrix V, makes the condition of (5.9.6) not just sufficient but also necessary.

The theorem 5.9.1 may also be exploited to attain the criteria for quadratic stability of more general form of a perturbed system, that is when the system is subject to both model parameter and input connection parameter uncertainties. Obviously, such a system is describable by the ensuing difference equation:

$$x(k + 1) = A(p)x(k) + B(p)u(k) \qquad (5.9.7)$$

with $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$.

In the case that the structured norm-bounded formulation for perturbations is used, (5.9.7) is replaced by:

$$x(k + 1) = (A + \Delta A)x(k) + (B + \Delta B)u(k) \qquad (5.9.8)$$

with the uncertainties parameterized as:

$$[\Delta A \quad \Delta B] = D\Delta(I - L\Delta)^{-1}[E_1 \quad E_2], \quad \Delta \in \mathbb{R}^{p \times q}, \|\Delta\|_2 \leq I, \text{ and } \|L\|_2 < 1 \qquad (5.9.9)$$

Now, the criteria for quadratic stability of the above system is expressible in the light of the theorems 5.9.1 and 5.8.1, and also the expression 5.8.4, as an LMI based feasibility condition as shown next:

Theorem 5.9.3 Necessary and sufficient condition for quadratic stabilizability of the system (5.9.8), (5.9.9) via static state feedback control is the existence of a positive definite matrix S and a matrix Y such that:

$$\begin{bmatrix} -S & AS + BY & D & 0 \\ SA^T + Y^T B^T & -S & 0 & SE_1^T + Y^T E_2^T \\ D^T & 0 & -I_p & L^T \\ 0 & E_1 S + E_2 Y & L & -I_q \end{bmatrix} < 0 \qquad (5.9.10)$$

The quadratically stabilizing state feedback control law is given by $u = Kx$, in which $K = YS^{-1}$.

By recalling the fact that a deadbeat controller locates the eigenvalues at the origin, it is easily deduced that theorem 5.9.3 could provide the means for synthesis of a robust deadbeat controller when the system is subject to uncertainty. This motivates the discussion of the next sections.

5.10 Quadratic $\mathcal{D}$-stability of discrete time systems with structured norm-bounded parametric uncertainty when $\mathcal{D}$ is a circular region:

In most practical applications we are interested in locating the eigenvalues in some specific region of the complex plane, which is considered as one of the major approaches for guaranteeing the dynamical behaviour of the system. In our case, i.e. deadbeat control, it was observed that achieving the deadbeat response necessitates the placement of all

closed-loop poles at the origin. According to the model-based design procedure of the compensator, and the fact that in practice all the mathematical models of real physical systems are subject to modelling errors, which can severely compromise the robustness and performance of the system, it seems natural to look for robust pole clustering schemes. In the case of the deadbeat controller, in order to maintain as far as possible the approximate deadbeat characteristic of the response, the eigenvalues should be robustly assigned as close as possible to the origin of the complex plane, despite the variation of the uncertain parameters within the uncertainty region. In this case, if the achievable worst-case spectral radius of the closed-loop matrix is sufficiently small, the tail of the impulse response of the system (although not optimal-time deadbeat any longer) will be guaranteed to exhibit the sufficiently high decay characteristics and thus will be a good approximation to an ideal deadbeat response. Note that for uncertain systems, despite the presence of the model of the perturbations, it is in general impossible to exactly locate the eigenvalues at a specific place of the complex plane. Therefore, the solution of a robust regional pole-placement problem is of great significance. It is important to note that, from the robust performance point of view, it is essential to locate the poles of the uncertain system as close as possible to those of the nominal system, because the performance of two systems with severely separated eigenvalues differ significantly from one another. This in turn, compels that the region in which the eigenvalues of the uncertain system are cluttered, is as small as possible. Hence, the aim will be quantitatively defining the smallest region which encompasses all the eigenvalues of the closed-loop system, and also designing a compensator which assigns the eigenvalues to such a region.

In the previous section, the criterion for quadratically stabilizing a discrete time system was investigated. In the current section though, the required modifications on the stated condition in theorem 5.9.3 will be examined, in order to synthesize a quadratically $\mathcal{D}$-stabilizing compensator, when $\mathcal{D}$ is a circular region of radius r and centred at $\alpha \in \mathbb{C}$. This region is denoted by $D(\alpha, r)$ and is such that $r \leq 1 - |\alpha|$. Notice that in the case of the robust deadbeat controller, it is desired that this circular region is centred at the origin and has the smallest possible radius.

Consider the case of an uncertain discrete time system described as:

$$x(k + 1) = (A + \Delta A)x(k) + (B + \Delta B)u(k) \qquad (5.10.1)$$

with the structured perturbations $\Delta A \in \mathbb{R}^{n \times n}$ and $\Delta B \in \mathbb{R}^{n \times m}$, assumed to belong to the compact bounding set Ω in $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$:

$$(\Delta A, \Delta B) \in \Omega \qquad (5.10.2)$$

Theorem 5.10.1 [102, 119] When $\mathcal{D}$ is $D(\alpha, r)$, the uncertain system (5.10.1), (5.10.2) is quadratically $\mathcal{D}$-stabilizable with the static state feedback controller $u(k) = Kx(k)$ if there exists a matrix $K \in \mathbb{R}^{m \times n}$ and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that:

$$r^2 P - (A_\alpha + BK + \Delta A + \Delta BK)^T P (A_\alpha + BK + \Delta A + \Delta BK) > 0 \qquad (5.10.3)$$

where $A_\alpha = A - \alpha I$. α is regarded as the design parameter which varies in accordance with the intended location of the poles.

In view of the theorem 5.10.1, the objective is rephrased as determining the minimum radius r^* of the disk $D(\alpha, r)$ enclosing all the poles of the system (5.10.1), when it is subject to the perturbations characterized in (5.10.2). In addition, the controller which renders the described system quadratically stable in $D(\alpha, r^*)$ needs to be computed.

In view of the reasons for assuming structured norm-bounded perturbations presented in section 5.5, let us again consider the model parameter and input connection parameter uncertainties to have the characterization defined in (5.9.9) as:

$$[\Delta A \quad \Delta B] = D\Delta[E_1 \quad E_2], \quad \Delta \in \mathbb{R}^{p \times q}, \|\Delta\|_2 \leq I \qquad (5.10.4)$$

Note that for the sake of simplicity here the feed-through matrix L is set to zero. Now, the condition for quadratic $\mathcal{D}$-stability of the uncertain system (5.10.1), (5.10.4), when $\mathcal{D}$ is the disk $D(\alpha, r)$, is stated in following theorem.

Theorem 5.10.2 [120, 118] The uncertain system (5.10.1), (5.10.4) is quadratically $\mathcal{D}$-stabilizable in $D(\alpha, r)$ if and only if there exists a symmetric positive definite matrix $S \in \mathbb{R}^{n \times n}$ and a matrix $Y \in \mathbb{R}^{m \times n}$ such that:

$$\begin{bmatrix} r^2S - DD^T & AS + BY - \alpha S & 0 \\ SA^T + Y^T B^T - \alpha S & S & SE_1^T + Y^T E_2^T \\ 0 & E_1 S + E_2 Y & I \end{bmatrix} > 0 \qquad (5.10.5)$$

Clearly, the LMI-based feasibility condition of (5.10.5) may be derived directly as a special case of theorem 5.9.3.

The stated condition of theorem 5.10.2 leads to the formulation of the problem of defining the minimum radius r^* of the disk $D(\alpha, r)$ in which the eigenvalues are clustered. In fact, the problem is expressible as the ensuing optimization problem [118]:

$\min \beta$

subject to:

$$\begin{bmatrix} \beta S - DD^T & AS + BY - \alpha S & 0 \\ SA^T + Y^T B^T - \alpha S & S & SE_1^T + Y^T E_2^T \\ 0 & E_1 S + E_2 Y & I \end{bmatrix} > 0 \qquad (5.10.6)$$

The problem has a solution β^* if and only if the perturbed system (5.10.1), (5.10.4) is quadratically $\mathcal{D}$-stabilizable. In this case, the disk with minimum radius r^* centred at α, containing all the closed-loop eigenvalues, is achievable as:

$$r^* = \sqrt{\beta^*} \le r \le 1 - |\alpha| \qquad (5.10.7)$$

Moreover, the state feedback compensator which assigns all the poles inside the circular region $D(\alpha, r^*)$ is $u(k) = Kx(k)$, where $K = YS^{-1}$ with Y and S being any feasible solution of the linear matrix inequality (5.10.5). It is apparent that if the optimization problem is infeasible, there is no quadratically $\mathcal{D}$-stabilizing controller.

It is interesting to note that by substituting for $r = 1$ and $\alpha = 0$, the theorem 5.10.2 yields the criterion for quadratic stability of a discrete time system.

5.11 Robust deadbeat controller:

In chapter 3, the deadbeat control synthesis problem was defined as that of designing a control law such that the closed-loop system matrix is nilpotent, i.e. it has all its eigenvalues located at the origin of the complex plane. The nilpotency property, enabled us to recast the controller synthesis as an eigenvalue assignment problem. However, due to the presence of uncertainties, the deadbeat characteristic of the system response may be adversely affected and lost. Therefore, it seems essential to robustly cluster all the closed-loop poles at the origin. Clearly, since the nature and structure of the perturbations are known to us only to a limited extent, it is almost impossible to exactly assign all the poles to the origin. This reveals the requisite to place the eigenvalues in the smallest possible region centred around the origin.

In the previous section, we studied a procedure that provided the smallest circular region in which all the poles could be located. In this section, the procedure will be employed to attain the minimum radius of the disk centred at the origin, encompassing all the closed-loop eigenvalues. This will obviously, yield the best achievable approximate deadbeat response. The controller which accomplish the assignment will also be defined.

In chapter 2, the framework based on which this thesis has been developed was illustrated in figure 2.3.1. The set of all internally stabilizing controllers were obtained as the combination of a stable observer and a stabilizing state feedback, depicted in figure 2.7.1. According to the mathematical description of the closed-loop system, characterized in (2.7.6), the set of closed-loop poles is achievable as the union of that of the state feedback and the observer, i.e. the eigenvalues of $A + B_2 F$ and $A + HC_2$, respectively. Hence, the requirement for having a deadbeat response compels us to assign the whole set of the modes to the origin, or in the presence of the uncertainty, to the smallest circular region centred at the origin.

Due to the structure of the closed-loop system matrix:

$$A_{cl} = \begin{bmatrix} A + B_2 F & -HC_2 \\ 0 & A + HC_2 \end{bmatrix}$$

where A, B_2, and C_2 matrices are all assumed to be subject to the structured norm-bounded parametric uncertainties, ΔA, ΔB_2, and ΔC_2, it may seem that the problem of determining the smallest circular region enclosing all the closed-loop modes is dependent on both of the state feedback and the observer, while they are coupled. However, in what follows we show that computation of the optimum radius could in fact be reduced to finding the maximum value of the two achievable minimum radii associated with each of the state feedback and the observer systems, whose perturbed state space matrices are designated by the pairs $(A + \Delta A, B_2 + \Delta B_2)$ and $(A + \Delta A, C_2 + \Delta C_2)$, respectively. Hence, the problem of robust deadbeat controller design may be formulated as that of determining the maximum radius of the two calculated circular regions of minimum radius, each encompassing the eigenvalues of the state feedback or the observer. To see this, take the perturbed state space description of the closed-loop system as:

$$x(k + 1) = [A + \Delta A]x(k) + B_1 w(k) + [B_2 + \Delta B_2]u(k)$$
$$z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k) \qquad\qquad (5.11.1)$$
$$y(k) = [C_2 + \Delta C_2]x(k) + D_{21} w(k) + [D_{22} + \Delta D_{22}]u(k)$$

in which the uncertainties are characterized as:

$$\begin{bmatrix} \Delta A & \Delta B_2 \\ \Delta C_2 & \Delta D_{22} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \Delta [E_1 \quad E_2] \qquad\qquad (5.11.2)$$

with D_1, D_2, E_1, and E_2 being known real matrices of compatible sizes, describing the structure of the perturbations, while the unknown matrix Δ designates the modelling uncertainty. The uncertainty is assumed to be bounded and normalized as follows:

$$\Delta \in \boldsymbol{\Delta} := \{\Delta: \|\Delta\|_2 \leq I\} \qquad\qquad (5.11.3)$$

Now, the robust deadbeat problem may be defined in the form of problem 5.11.1:

Problem 5.11.1 For $(\Delta A, \Delta B_2) \in \Omega_1$ and $(\Delta A, \Delta C_2) \in \Omega_2$ where Ω_1 and Ω_2 are compact sets, minimize the radius r of the disk $D(0, r)$, enclosing all the closed-loop eigenvalues of A_{cl}, such that the matrices:

$$A + \Delta A + (B_2 + \Delta B_2)F \quad \text{and} \quad A + \Delta A + H(C_2 + \Delta C_2)$$

are quadratically $\mathcal{D}$-stable.

In (5.11.2), for the sake of simplicity, and without loss of generality, it will be assumed that D_{22} and accordingly ΔD_{22} are zero. The procedure for eliminating the feed-through matrix has been elaborated in [6]. It will also be presumed that $D_1 = D_2 = E_1 = E_2 = 1$. The assumptions leads to the simplification of (5.11.2) to:

$$\begin{bmatrix} \Delta A & \Delta B_2 \\ \Delta C_2 & 0 \end{bmatrix} = \Delta \qquad (5.11.4)$$

Relative to (5.11.3), the uncertainty may now be described by:

$$\|\Delta\|_2 = \left\| \begin{bmatrix} \Delta A & \Delta B_2 \\ \Delta C_2 & 0 \end{bmatrix} \right\|_2 \leq I \qquad (5.11.5)$$

In [137], Parrott investigates minimization problem of the norm on the left hand side of the above inequality. The result of his work is given in the form of the following theorem.

Theorem 5.11.2 [137, 1] The infimum over all choices of the operator X of the norm of the operator matrix $\begin{bmatrix} A & B \\ C & X \end{bmatrix}$ whose entries are matrices of appropriate dimensions, is the minimum of the norms of the first row and of the first column:

$$\min_X \left\| \begin{bmatrix} A & B \\ C & X \end{bmatrix} \right\|_2 = \max \left\{ \left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\|_2, \|[A \quad B]\|_2 \right\}$$

Parrott's theorem, which plays an important role in many control related optimization problems, states that the minimization problem can be expressed as two decoupled minimization problems. In view of his theorem, (5.11.5) can be restated as two independent norm bounds as:

$$\|[\Delta A \quad \Delta B_2]\|_2 \leq I \quad \text{and} \quad \left\| \begin{bmatrix} \Delta A \\ \Delta C_2 \end{bmatrix} \right\|_2 \leq I \qquad (5.11.6)$$

From (5.11.6) it can be inferred that the robust deadbeat control design problem, which in fact is equivalent to minimizing the radius of the circular region enclosing all the closed-loop poles, is expressible as two decoupled minimization problems in which the radius of each of the disks enclosing the eigenvalues of the state feedback and the observer are minimized. The smallest possible circular region in which all the closed-loop poles can be located is then achieved as the disk with greater radius. The minimum radius associated to each of the state feedback and observer systems can be readily computed by means of the procedure presented in the theorem 5.10.2. The state feedback and observer gains, respectively denoted by F and H, are also achievable as the by-product of the theorem.

The proposed control design procedure has been clarified by means of the following example.

Example 5.11.1 This example considers the design of a robust deadbeat controller when the system is subject to structured norm-bounded parametric uncertainties. The system under consideration is a DC motor with the electrical model depicted in figure 5.11.1.
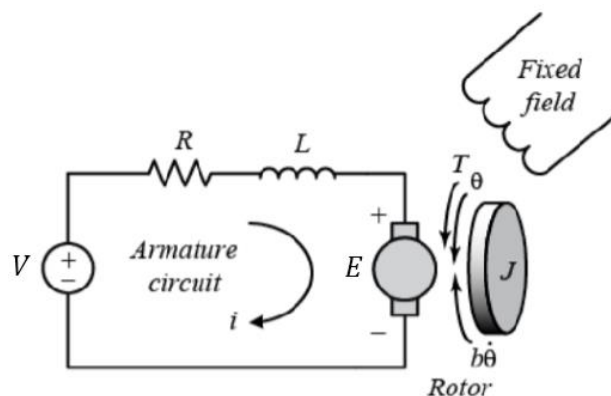
Figure 5.11.1 The electric equivalent circuit of the armature and the free-body diagram of the rotor

It is assumed that the input of the system is the voltage source V applied to the motor's armature, while the output is the rotational speed of the shaft $\dot{\theta}$. A viscous friction model is further assumed. The friction torque is proportional to the shaft angular velocity. The physical parameters of the system are as follows:

J = moment of inertia of the rotor = $0.02 \ kg.m^2$

b = motor viscous friction constant = $0.2 \ N.m.s$

k_e = electromotive force constant = $0.1 \ v/rad/s$

k_t = motor torque constant = $0.1 \ N.m/Amp$

R = electric resistance = $2 \ \Omega$

L = electric resistance = $0.5 \ H$

In figure 5.11.1, the back-emf is designated by E; this is proportional to the angular velocity of the shaft, the constant of proportionality being k_e:

$$E = k_e \dot{\theta}$$

For a fixed external magnetic field, the motor torque, denoted by T, is proportional to the armature current i with constant of proportionality k_t:

$$T = k_t i$$

Based on the above description, the system state space model when the state vector is defined as $x = [\dot{\theta} \quad i]$, is:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

in which:

$$A = \begin{bmatrix} -\dfrac{b}{J} & \dfrac{K}{J} \\ -\dfrac{K}{L} & -\dfrac{R}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix}, \qquad C = [1 \quad 0]$$

Note that in the state space model it is assumed that $k_e = k_t = K$. The continuous time model is discretized with sampling interval of $T_s = 0.1 \ s$, and the zero-order-hold equivalent state space discrete-time model is:

$$x(k + 1) = A_d x(k) + B_d u(k)$$
$$y(k) = C_d x(k)$$

In order to design a robust deadbeat compensator, it is assumed that all the matrices A_d, B_d, and C_d in the discrete state space model are uncertain, with the uncertainty models ΔA_d, ΔB_d, and ΔC_d, respectively, which are shown below:

$$[\Delta A_d \quad \Delta B_d] = D_{AB} \Delta_{AB} [E_{AB_1} \quad E_{AB_2}], \quad \|\Delta_{AB}\|_2 \leq 1$$
in which

$$D_{AB} = [0.2113 \quad 0.1207]^T$$
$$E_{AB_1} = [0.1143 \quad 0.1091]$$
$$E_{AB_2} = 0.35$$

and

$$[\Delta A_d \quad \Delta C_d]^T = D_{AC} \Delta_{AC} [E_{AC_1} \quad E_{AC_2}], \quad \|\Delta_{AC}\|_2 \leq 1$$

in which:

$$D_{AC} = [0.1115 \quad 0.0108]^T$$
$$E_{AC_1} = [0.1258 \quad 0.1979]$$
$$E_{AC_2} = 0.18$$

In view of the discussion in this section, the robust deadbeat control design problem reduces to that of finding the maximum value of the two achievable minimum radii associated with the state feedback and the observer eigenvalues, respectively.

The example is simulated in MATLAB. The minimum radii of the two circular regions containing the eigenvalues of the pairs $A_d + B_d F$ and $A_d + H C_d$, for the defined structured model uncertainty in the pairs $(A_d + \Delta A_d , B_d + \Delta B_d)$ and $(A_d + \Delta A_d , C_d + \Delta C_d)$ are obtained as 0.34249 and 0.037804, respectively.

The locus of the eigenvalues of $A_d + B_d F$ and $A_d + HC_d$ for 100 equally spaced values of Δ_{AB} and Δ_{AC} between -1 and 1 are depicted in figures 5.11.2 and 5.11.3. Note that this entails no conservativeness as at least one eigenvalue is located at the boundary in each case. Clearly, the smallest possible circular region in which the closed-loop eigenvalues of the uncertain system are clustered is a disk of radius 0.34249. The controller gain which assigns the eigenvalues to this disk was achieved as $[-1.302 \quad -4.5066]$.
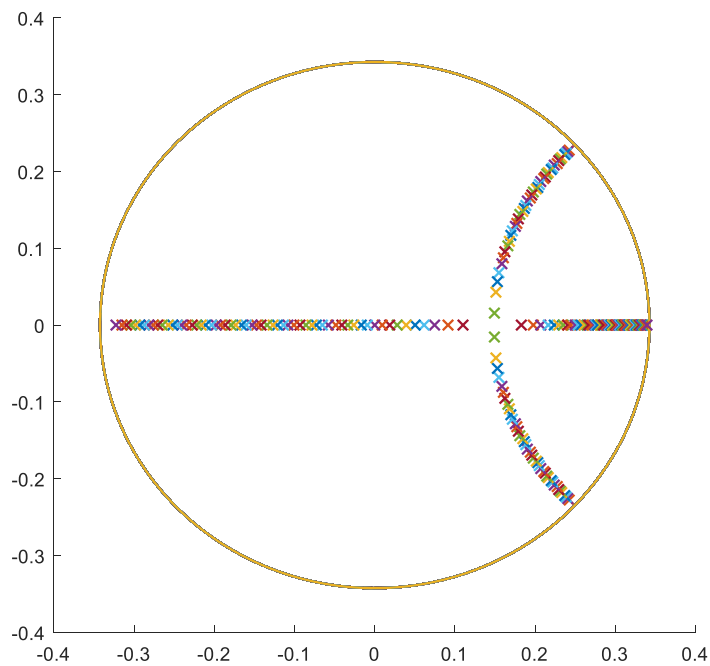

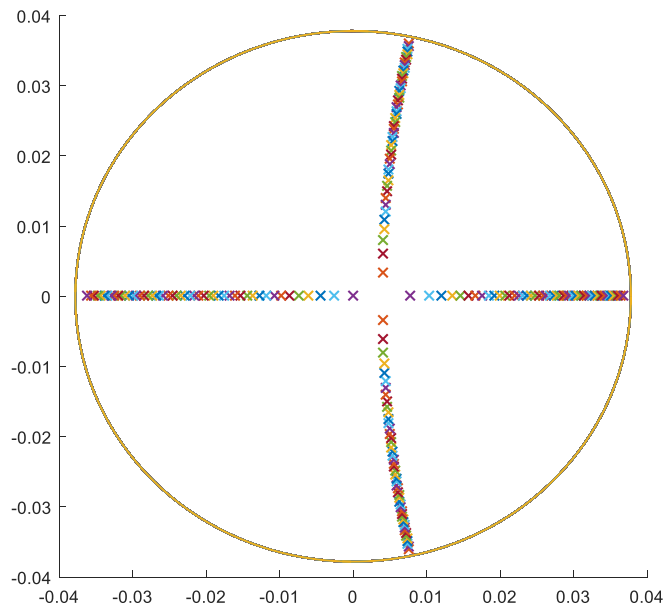Figure 5.11.2 The eigenvalue locus of $A_d + B_d F$

Figure 5.11.3 The eigenvalue locus of $A_d + HC_d$

5.12 Conclusion:

This chapter considered the synthesis problem of robust deadbeat controller when the system is subject to the parametric uncertainty. First, the two major classes of perturbations, namely the parametric uncertainty and the neglected or unmodelled dynamic uncertainties, were briefly reviewed. Next, the sensitivity of an eigenvalue λ_j, designated by s_j, to parametric uncertainties was examined. It was shown that s_j depends critically on the magnitude of the condition number associated with λ_j. It was also observed that the condition number of the eigenvector matrix may provide an upper bound on the sensitivities of the eigenvalues, hence establishing a measure for the robustness of the eigenvalues to model perturbations. This was then followed by a revision of the Lyapunov theorem, which is known to be one of the major tools in stability analysis.

General descriptions of uncertain models were then given. By assuming structured norm-bounded parametric uncertainties for characterization of the perturbations, the notion of quadratic stability was generalized to uncertain systems subject to perturbations of this type entering either the state, or both the state and input matrices, respectively, known as

model parameter uncertainty and input connection parameter uncertainty. Necessary and sufficient conditions for quadratic stability of these systems were obtained in the form of the LMI-based feasibility conditions. As was discussed, by imposing structure for the perturbations, quadratic stability is achievable through application of a linear time-invariant compensator.

The demand to have robust performance in the face of the uncertainty, which in turn necessitates robust assignment of the closed-loop poles to generalized stability regions of the complex plane, denoted by $\mathcal{D}$, led us to the problem of quadratic $\mathcal{D}$-stability. As a matter of fact, quadratic $\mathcal{D}$-stability extended $\mathcal{D}$-stability to uncertain systems in a similar way that quadratic stability extends stability to uncertain systems. Generalization of the Lyapunov theorem to the sub-regions of the complex plane was accomplished through introduction of LMI-defined regions. Again, the necessary and sufficient conditions for quadratic $\mathcal{D}$-stability of a system when it was subject to only the model parameter uncertainty, or both model and input connection parametric uncertainties, were derived.

Up to this point, all the stability criteria obtained were based on the description of continuous time systems. All the results were next translated into the discrete time case. This was achieved by using the fact that quadratic stability of a discrete time system is equivalent to quadratic $\mathcal{D}$-stability of its continuous time counterpart, when $\mathcal{D}$ is the unit disk centred at the origin of the complex plane.

In order to design a robust deadbeat controller, the results were employed to robustly locate all the closed-loop poles in the smallest possible circular region, centred at the origin of the complex plane. The radius of the disk, was computed as the solution to an optimization problem with LMI constraints. It was observed that the problem of robust deadbeat controller synthesis is expressible as that of determining the maximum radius of the two circular regions of minimum radius, each encompassing the eigenvalues of the state feedback or the observer parts of the closed-loop state matrix. So, in spite of what could be inferred from the structure of the uncertainty, the two problems were decoupled. Therefore, the main contribution of this chapter was proposing a new procedure for designing robust deadbeat controller when the system is subject to structured norm-bounded parametric uncertainties. The procedure was in terms of LMIs.

Chapter 6

Deadbeat controller design with H_∞ norm minimization constraint; the LMI approach

6.1 Introduction:

This chapter deals with the synthesis problem of deadbeat controller subject to the H_∞ norm optimization constraint, which is regarded as a frequency domain constraint.
First, a brief introduction to the notion of the H_∞ norm and its interpretation is given. This is followed by the formulation of the optimal and suboptimal H_∞ control problem and its motivations. Three major schemes to solve the H_∞ problem, namely the model-matching, Riccati equation-based, and the LMI approaches, are then reviewed.

Based on the Q-parameterization of the closed-loop system discussed in chapter 2, it is shown that the Markov parameters of the design parameter Q appear affinely in only the C and D matrices of the state space realization of the closed-loop system. This feature will be exploited to formulate the constrained design problem of deadbeat controller in the LMI framework. The resulting LMI will be a linear function of the matrix variables, which is due to the above property in the description of the closed-loop system.
As in earlier chapters, $\mathbb{R}H_\infty$ will denote the space of real-rational proper and stable transfer matrices.

6.2 The H_∞ norm:

In any control system, further to providing the internal stability, the objective is to fulfil certain design specifications. A direct criterion corresponding to the description of performance specifications, is offered by the notion of the signal norms. Quantitative

treatment of the performance of a control system requires introduction of appropriate norms for certain signals of interest, which are directly dependent on the situation at hand. Another concept closely related to the size of a signal is the size of a system, which is also of great importance. As it is well known, a system is a mapping from one signal space, the input space, to another signal space, the output space, respectively designated by S_1 and S_2. To facilitate accommodating the discussion in this chapter to the work in this thesis, take the general framework of the figure 2.3.1, which resulted in the equivalent construction of the figure 6.2.1, and its associated input-output mathematical characterization as:

$$H_{zw} = \mathcal{F}_l\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix}, Q\right) = T_{11} + T_{12}QT_{21} \qquad (6.2.1)$$

The closed-loop mapping H_{zw} may be described by:

$$H_{zw} : S_1 \rightarrow S_2$$
$$: w \rightarrow z = H_{zw}w \qquad (6.2.2)$$

Regarding H_{zw} as an operator from input space to the output space, a norm is induced by the normed-spaces S_1 and S_2 on H_{zw}, which loosely speaking quantifies the amplification (or attenuation) applied by the system on a given input signal w. Hence, the system norm which is also known as the system gain, gives an implication of the achievable performance of a system for various classes of input signals.
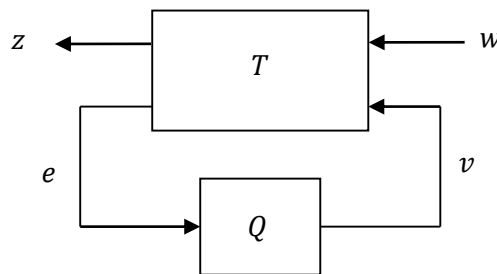

Figure 6.2.1 The equivalent representation of the general framework of the figure 2.3.1

One of the most fundamental norms defined for characterizing the performance of a system in analysis and design, is the so-called H_∞ norm. The H_∞ norm is an operator norm which is induced by the l_2 norm, i.e. when both the input and output signals belong to the l_2 space. The set of square-summable signals (sequences), i.e. the signals with finite energy, forms the l_2 space. The l_2 norm of a signal $u = (u_0, u_1, u_2, \dots) \in l_2$ in the time domain, denoted by $\|u\|_2$, is given by [121]:

$$\|u\|_2 = [\textstyle\sum_{k=0}^{\infty} u_k^T u_k]^{1/2} \qquad (6.2.3)$$

The l_2 space in the time domain may be related to the l_2 space in the frequency domain, respectively denoted by $l_2(-\infty, +\infty)$ and $l_2(j\mathbb{R})$. This is accomplished according to the fact that a function in the l_2 space in the time domain admits a bilateral Fourier transform, which yields an isometric isomorphism between $l_2(-\infty, +\infty)$ and $l_2(j\mathbb{R})$ [1]. It can be shown that discrete Fourier transform of the signal $u \in l_2(-\infty, +\infty)$, designated by $\hat{u}(e^{j\omega})$, belongs to the space of square-integrable functions on the unit circle [121].

Having reviewed the l_2 norm of signals, the H_∞ norm of the system H_{zw}, designated by $\|H_{zw}\|_\infty$, may be defined as the norm induced by the l_2 norms of the input signal w and the output signal z, as:

$$\|H_{zw}\|_\infty = \sup \frac{\|H_{zw}w\|_2}{\|w\|_2}, \quad w \in l_2 \text{ and } \|w\|_2 \neq 0 \qquad (6.2.4)$$

For obvious reasons, the H_∞ norm is also referred to as the l_2 gain of the system. Clearly, this quantity represents the largest possible l_2 gain provided by the system over the set of all square-summable signals.

The H_∞ norm may also be computed in the frequency domain, and is expressible as:

$$\|H_{zw}\|_\infty = \sup_\omega \bar{\sigma}\left(H_{zw}(e^{j\omega})\right) \qquad (6.2.5)$$

in which $\bar{\sigma}$ denotes the largest singular value of the system, with the supremum being over all frequencies ω [1].

In the case of H_{zw} being a scalar transfer function, its infinity norm can be interpreted as the distance between the origin and the farthest point on the Nyquist plot of H_{zw}, or as the peak value on the Bode magnitude plot of $|H_{zw}(j\omega)|$ [6]. However, in general it is an indicator of the worst-case energy of the output for energy bounded inputs, which accordingly could come naturally as a measure of the worst case performance for many classes of input signals.

6.3 The H_∞ control problem formulation and its motivations:

As was mentioned in the previous section, many of the control objectives can be formulated as requiring a certain closed-loop transfer matrix be small in some sense. One of the measures to express the smallness of the transfer matrix, was defined as the H_∞ norm or equivalently the l_2 gain of the system.

Basically, the standard form for the control problem in which the objective is H_∞ norm minimization of the transfer matrix of interest, which in our case is H_{zw}, can be formulated as what follows.

Definition 6.3.1 [122] Given the closed-loop mapping $H_{zw} = \mathcal{F}_l(P, K)$, find a stabilizing proper compensator K that minimizes the H_∞ norm of the transfer matrix from w to z:

$$\min \ \|H_{zw}\|_\infty = \min \ \|\mathcal{F}_l(P, K)\|_\infty \qquad (6.3.1)$$

As can be seen, the H_∞ problem is stated based on the scheme of the figure 2.3.1. However, in view of the Q-parameterization of the closed-loop map, illustrated in figure 6.2.1, and its associated mathematical description of (6.2.1), the problem may be recast as:

Definition 6.3.2 Find the stable (otherwise arbitrary) parameter Q, such that the following H_∞ norm in minimized:

$$\min_{Q\in\mathbb{R}H_\infty} \|\mathcal{F}_l(T, Q)\|_\infty = \min_{Q\in\mathbb{R}H_\infty} \|T_{11} + T_{12}QT_{21}\|_\infty \qquad (6.3.2)$$

Evidently, two above descriptions of the problem leads to the optimum value. Yet, very often in practice, finding the optimal solution is both theoretically and numerically involved. In general, such solutions may be unattainable, ill-conditioned of high McMillan degree [123]. This reveals the need for seeking suboptimal controllers, achievable as the solution to the following problem definition:

Definition 6.3.3 [1] Given $\gamma > 0$, find all admissible controllers K, if any, such that $\|T_{zw}\|_\infty < \gamma$.

Originally, the H_∞ optimization problem initiated from the requirement to reduce the sensitivity of a feedback system against disturbances. It first appeared in the seminal work of Zames [125] and Doyle, Stein [126]. Some of the most celebrated examples of control objectives expressible as H_∞ norm constraints are disturbance attenuation, robust control, and the mixed sensitivity problem. These problems and more other classical synthesis problems which can be recast as an H_∞ optimization problem, have been discussed in [124, 83, 19]. For an extensive list of references we refer the reader to [122].

6.4 Approaches to solve the H_∞ optimization problem:

Having reviewed the description of H_∞ minimization problem in previous section, we aim now to concentrate on the solution of the problem. Various approaches for tackling the H_∞ problem have been introduced in the literature, some of which are developed in the frequency domain and some in the state space framework. In this section, the focus will be on the three major procedures, which will be discussed briefly as we proceed. The discussion pertains to the figure 6.2.1, and its associated mathematical characterization in (6.2.1).

One of the earliest approaches to treat the H_∞ problem, is the so-called model-matching problem. As the name suggests, in this scheme the H_∞ problem is considered to be equivalent to that of matching two models. With regard to the closed-loop description of the figure 6.1.1 as $H_{zw} = T_{11} + T_{12}QT_{21}$, it is readily observed that the H_∞ minimization problem may be interpreted as matching the given transfer matrix $-T_{11} \in \mathbb{R}H_\infty$, with the

cascade of the three transfer matrices $T_{12}QT_{21}$, in which T_{12} and T_{21} are known transfer matrices in $\mathbb{R}H_\infty$, whereas the $Q \in \mathbb{R}H_\infty$ is the design parameter. This interpretation has been depicted in figure 6.4.1 [127].
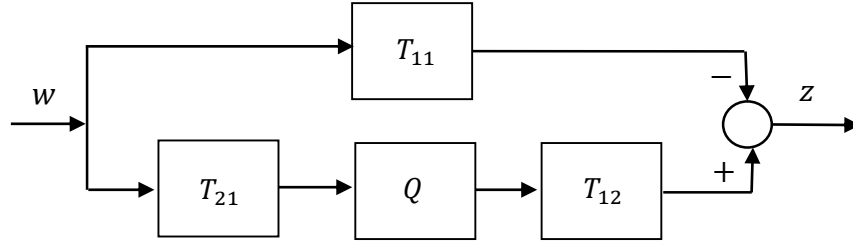

Figure 6.4.1 Model-matching problem illustration

Based on the above interpretation, the main problem is now expressible as finding $Q \in \mathbb{R}H_\infty$ such that the model-matching error $\|T_{11} + T_{12}QT_{21}\|_\infty$ is minimized (or kept below a specified level) [127]. According to [128], a sufficient condition to achieve the minimum is that the two matrices T_{12} and T_{21} have constant ranks for all the frequencies $0 \leq \omega < \infty$. In practice, this condition for well-defined problems is fulfilled.

In order to compute a solution to the model-matching problem, it has been shown in [128] that the problem is equivalent to yet another problem, the so-called Hankel-norm approximation problem, also known as the Nehari extension problem. To see this, let us assume that T_{12} and T_{21} are square and inner (or all-pass), i.e. [1, 133]:

$$T_{12}T_{\widetilde{12}} = I \quad \text{and} \quad T_{\widetilde{21}}T_{21} = I \qquad (6.4.1)$$

in which the tilde designates the parahermitian transpose of the transfer matrix. Now, the model-matching error, owing to the norm-preserving property of the inner matrices [1], may be reformulated as:

$$\|T_{11} + T_{12}QT_{21}\|_\infty = \|T_{12}(T_{\widetilde{12}}T_{11}T_{\widetilde{21}} + Q)T_{21}\|_\infty = \|T_{\widetilde{12}}T_{11}T_{\widetilde{21}} + Q\|_\infty$$
$$= \|T_{21}T_{\widetilde{11}}T_{12} + Q^\sim\|_\infty \qquad (6.4.2)$$

In (6.4.2), while $Q^\sim$ is unstable, $T_{21}T_{11}^\sim T_{12}$ has only stable eigenvalues [129]. By denoting $T_{21}T_{11}^\sim T_{12}$ as R, the model-matching problem transforms into that of approximating a stable transfer matrix R by the unstable one $-Q^\sim$: [122]

$$\min_{Q\in\mathbb{R}H_\infty} \|T_{11} + T_{12}QT_{21}\|_\infty = \min_{Q\in\mathbb{R}H_\infty} \|R + Q^\sim\|_\infty \qquad (6.4.3)$$

The above results rely on the assumptions that T_{12} and T_{21} are inner. As it is stated in [127, 130], this may be accomplished through appropriate selection of the state feedback gain F and the observer gain H, in the Q parameterization of the system. However, the first assumption made on T_{12} and T_{21}, i.e. being square, is violated by some important classes of problems, e.g. the mixed performance and robustness problem [83]. In the general case that T_{12} and T_{21} are neither square nor inner, it is always possible to find orthogonal complements of the transfer matrices T_{12} and T_{21}, respectively denoted by $T_{12\perp}$ and $T_{21\perp}$, such that $[T_{12} \quad T_{12\perp}]$ and $[T_{21}^T \quad T_{21\perp}^T]^T$ are square and inner [1]. Following the same procedure in (6.4.2), we will have [133, 127]:

$$\|T_{11} + T_{12}QT_{21}\|_\infty = \left\|\begin{bmatrix} R_{11} + Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\right\|_\infty \qquad (6.4.4)$$

in which:

$$R_{11} = T_{12}^\sim T_{11}T_{21}^\sim \quad , \quad R_{12} = T_{12}^\sim T_{11}T_{21\perp}^\sim \qquad (6.4.5)$$
$$R_{21} = T_{12\perp}^\sim T_{11}T_{21}^\sim \quad , \quad R_{22} = T_{12\perp}^\sim T_{11}T_{21\perp}^\sim$$

Therefore, (6.4.4) converts the problem of minimizing the model-matching error to that of the H_∞ norm minimization of the quantity on the right hand side of the equality as:

$$\min_{Q\in\mathbb{R}H_\infty} \|T_{11} + T_{12}QT_{21}\|_\infty = \min_{Q\in\mathbb{R}H_\infty} \left\|\begin{bmatrix} R_{11} + Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\right\|_\infty \qquad (6.4.6)$$

This is known as the four-block problem, compared to the special case of (6.4.3), which consists of just one block, hence the name one-block problem.

The right hand side identity in the equality (6.4.6) may also be inferred as the distance between a transfer matrices $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ and $-Q \in \mathbb{R}H_\infty$, hence the name distance problem as an alternative [133]:

$$\min_{Q \in \mathbb{R}H_\infty} \left\| \begin{bmatrix} R_{11} + Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty = \mathrm{dist}\left(R, \begin{bmatrix} \mathbb{R}H_\infty & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad (6.4.7)$$

The solution to such problems is extensively elaborated in [130] for the case of continuous time systems. It is developed based on the notion of the norm of a certain operator, the so-called Hankel operator [132], usually designated by Γ. It is shown that the norm of Γ equals the spectral norm of the square root of the product of the controllability and observability Gramians of the transfer matrix R, and that the minimal model-matching error equals the norm of the Hankel operator [133, 131, 130]. For a discrete time treatment of the distance problem the reader is referred to [134].

According to the fact that in the Hankel approximation problem, the procedure to attain the solution is both theoretically and computationally very involved, in [136] Glover et al. propose a new approach which relies on the solution to two algebraic Riccati equations with the same order as the system. Here, we will briefly describe their approach. The results are given in terms of the description of systems in the continuous time framework, which is regarded as the more standard framework.

It is well-known that associated with the continuous time algebraic Riccati equation:

$$A^*X + XA + XRX + S = 0 \qquad (6.4.8)$$

in which A, S and R are real $n \times n$ matrices with S and R symmetric, there exists a $2n \times 2n$ Hamiltonian matrix:

$$H = \begin{bmatrix} A & R \\ -S & -A^* \end{bmatrix} \qquad (6.4.9)$$

Assuming that H has no eigenvalues on the imaginary axis, the spectrum of H, i.e. $\sigma(H)$, will be symmetric about the imaginary axis. It is then possible to construct two invariant subspaces of dimension n, corresponding to the stable and unstable modes of H,

respectively denoted by $\mathcal{X}_-(H)$ and $\mathcal{X}_+(H)$. If we could find a basis for $\mathcal{X}_-(H)$ and partition it as:

$$\mathcal{X}_-(H) = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad X_1, X_2 \in \mathbb{C}^{n \times n} \qquad (6.4.10)$$

such that X_1 is nonsingular, or equivalently, the two following subspaces are complement:

$$\mathcal{X}_-(H) , \quad \text{Im} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (6.4.11)$$

we can then define X as $X = X_2 X_1^{-1}$. X is uniquely determined by H. In other words, $H \rightarrow X$ serves as a function, known as Ric, with the domain designated by $\text{dom}(\text{Ric})$. Therefore, $\text{dom}(\text{Ric})$ encompasses Hamiltonian matrices with no purely imaginary eigenvalues, and those for which the two subspaces in (3.6.11) are complementary. These two features of the elements of $\text{dom}(\text{Ric})$ are usually recognized as the stability property and the complementarity, respectively. [1]

Theorem 6.4.1 [1] Suppose $H \in \text{dom}(\text{Ric})$ and $X = \text{Ric}(H)$. Then:

(i) X is real symmetric.
(ii) X satisfies the algebraic Riccati equation of (6.4.8).
(iii) $A + RX$ is stable.

In the above theorem, X is called a stabilizing solution to the Riccati equation of (6.4.8), i.e. the set of spectrum of $A + RX$ is in the open LHP.

The proposed approach in [136] to tackle the sub-optimal H_∞ problem and the conditions for solvability of the problem, is based on the above way of constructing stabilizing solutions to the Riccati equation in terms of invariant subspaces of H. In [135], Doyle et al. consider a simplified version of the problem stated in [136], by equating the D_{11} and D_{22} matrices in the plant state space description (expression (2.3.2)) to zero. For the sake of brevity, in here we just represent the results stated in [135] in the form of the theorem 6.4.2. It should be pointed out that the results are based on further assumptions on the

plant, which will not be restated here. These assumptions are quite standard and may be found in many references, e.g. [135, 136, 1, 85].

The solution and solvability conditions for the sub-optimal H_∞ problem, i.e. $\|H_{zw}\|_\infty < \gamma$ for some $\gamma > 0$, involves two Hamiltonian matrices:

$$H_\infty = \begin{bmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix} \qquad (6.4.12)$$

$$J_\infty = \begin{bmatrix} A^T & \gamma^{-2}C_1^TC_1 - C_2^TC_2 \\ -B_1B_1^T & -A \end{bmatrix}$$

Theorem 6.4.2 [135] There exists an admissible controller such that $\|H_{zw}\|_\infty < \gamma$ if and only if the following three conditions are satisfied:

 i) $H_\infty \in \text{dom}(\text{Ric})$ and $X_\infty = \text{Ric}(H_\infty) \geq 0$

 ii) $J_\infty \in \text{dom}(\text{Ric})$ and $Y_\infty = \text{Ric}(J_\infty) \geq 0$

 iii) $\rho(X_\infty Y_\infty) < \gamma^2$

As can be seen, the feasibility condition is expressed in terms of the existence of unique positive definite stabilizing solutions to two algebraic Riccati equations, such that the spectral radius ρ, of their product is less than γ^2.

Conditions for the general case i.e. when D_{11} and D_{22} matrices are nonzero, are studied in [136]. The sub-optimal controller is parameterized in both [135] and [136], when the problem is feasible.

Having surveyed two of the major methods for tackling the H_∞ problem, we conclude this section by introducing yet another scheme for treating the problem. In this approach, the H_∞ norm minimization problem is transferred into a standard linear matrix inequality (LMI) feasibility problem. [142] The LMI characterization of the H_∞ problem, is the so-called bounded real lemma, which is stated in the form of the following theorem:

Theorem 6.4.3 [106, 6] For an asymptotically stable discrete time system with the state space realization $G = C(zI - A)^{-1}B + D$, $\|G\|_\infty < \gamma$ if and only if there exists a symmetric positive definite matrix P such that:

$$\begin{bmatrix} P - A^T P A & A^T P B & C^T \\ B^T P A & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \leq 0 \qquad (6.4.13)$$

It is well-known that LMIs arise in many control analysis and synthesis problems and are reformulable as convex optimization problems, which correspondingly makes them readily amenable to computer solutions. [141] Moreover, LMI problems can be solved via efficient tractable numerical algorithms, e.g. interior-point method. [9, 103, 104, 105, 106] LMIs are especially beneficial for solving problems lacking analytical solution. Due to these peculiar attributes of LMIs, they have always been of special interest to many researchers and engineers.

In the next section, this approach will be exploited to synthesize a deadbeat compensator with H_∞ norm constraint.

6.5 Synthesis of deadbeat controller subject to H_∞ norm constraint:

In chapter 4, we looked upon the synthesis problem of the deadbeat controller under time domain constraints. In this section though, we consider the compensator design problem subject to the H_∞ norm minimization. According to the brief introduction of the H_∞ norm and its interpretation given in section 6.2, by minimizing the H_∞ norm of the system we in fact minimize the largest possible l_2 gain provided by the system over the set of all square-summable input signals, i.e. the signals with finite energy. In other words, the peak in the magnitude frequency response of the closed-loop system is minimized for that specific class of the input signals.

The current section is again based on the closed-loop interconnection of the figure 6.2.1, and its associated characterization:

$$H_{zw} = T_{11} + T_{12}QT_{21} \qquad (6.5.1)$$

in which T_{11}, T_{12}, and T_{21} are all FIR and belong to the set of proper and stable rational transfer matrices $\mathbb{R}H_\infty$. The design parameter Q is also considered to be FIR and stable. The requirement of the transfer matrices being deadbeat, obviously renders the closed-loop system as deadbeat.

Now, consider that Q is an $(r + 1)$-tap FIR system as:

$$Q(z) = \sum_{p=0}^{r} Q_p z^{-p} \qquad (6.5.2)$$

with Q_p's being matrices of dimension $m \times n$:

$$Q_p = \begin{bmatrix} q_{11_p} & \cdots & q_{1n_p} \\ \vdots & \ddots & \\ q_{m1_p} & & q_{mn_p} \end{bmatrix} \qquad (6.5.3)$$

Each Markov parameter Q_p may alternatively be characterized in terms of the selection matrices E_{st} of size $m \times n$, with all elements equal to zero except the st-th element being one, which corresponds to the st-th element in Q_p. Parameterizing Q_p's in this fashion will accordingly yield to the following description of the closed-loop system:

$$H_{zw} = T_{11} + \sum_{i=1,j=1}^{m,n} q_{ij_0} T_{12} E_{ij} T_{21} + \sum_{i=1,j=1}^{m,n} (q_{ij_1} T_{12} E_{ij} T_{21}) z^{-1} + \cdots$$
$$+ \sum_{i=1,j=1}^{m,n} (q_{ij_r} T_{12} E_{ij} T_{21}) z^{-r} \qquad (6.5.4)$$

or equivalently:

$$H_{zw} = T_{11} + \sum_{i=1,j=1,k=0}^{m,n,r} (q_{ij_k} T_{12} E_{ij} T_{21}) z^{-k} \qquad (6.5.5)$$

The expression (6.5.5) reveals a substantial property of the closed-loop system, that is all the design parameters q_{ij_k}, $i = 1, \ldots, m, j = 1, \ldots, n, k = 0, \ldots, r$ appear affinely in only the C and D matrices of the state space realization of the closed-loop system H_{zw}. Hence, the realization of H_{zw} can be represented as:

$$H_{zw_q}(z) = C_q(zI - A)^{-1}B + D_q \qquad (6.5.6)$$

emphasizing the fact that the C and D matrices are affine functions of the design parameters q_{ij_k}.

Based on the above characterization of the closed-loop system which is clearly deadbeat, the design problem of the deadbeat compensator with H_∞ norm constraint may now be easily formulated in the framework of the theorem 6.4.3 as an LMI. It should be pointed out that what makes the description of (6.5.5) peculiar is that it imposes the LMI in (6.4.13) to be linear in terms of the matrix variables, which are the symmetric positive definite matrix P, C_q, and D_q.

In order to represent how the above procedure works, we have considered as an example the synthesis problem of a deadbeat controller with the requirement of the H_∞ norm of the regulated output being optimized, for the case of a random SISO system.

Example 6.5.1 Consider the unity feedback configuration of figure 6.5.1 in which the plant G is the DC motor with the state space model described in the example 5.11.1. The regulated variable y is the rotational speed of the motor shaft. In this example the aim is to design a deadbeat controller such that the H_∞ norm of the closed-loop system is minimized.
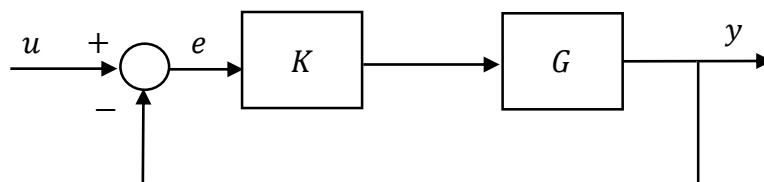

Figure 6.5.1 Unity feedback configuration

The feedback configuration of figure 6.5.1 can be reconstructed in the equivalent form of a lower LFT configuration, as illustrated in figure 6.5.2.
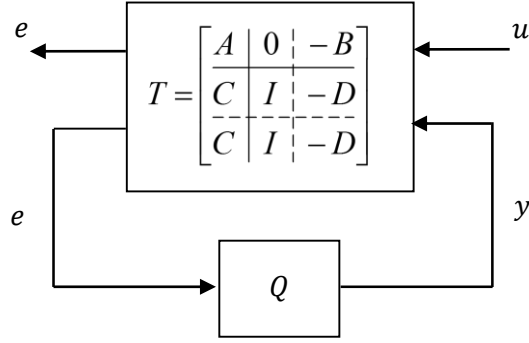
Figure 6.5.2 The equivalent lower LFT configuration of figure 6.5.1

In the new framework, the objective transforms into designing the deadbeat controller Q such that $\|H_{eu}\|_\infty$ is minimized.

Regarding the discussion in chapter 2, the closed-loop transfer matrix is parameterized as:

$$H_{eu} = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21}$$

in which the design parameter Q is considered to be FIR.

By designing the state feedback and observer gain matrices in the observer-based controller such that all the eigenvalues of the closed-loop system H_{eu} are assigned to the origin of the complex plane (expression (2.7.6)), all T_{11}, T_{12}, and T_{21} transfer functions are achieved respectively as 5-, 3-, and 3-tap FIR systems. Since in this example all the sub-systems are SISO, the series connection of T_{12}, and T_{21}, designated by T_{121}, can be obtained as the convolution of two polynomials. To start with, Q is assumed to be a 3-tap FIR system.

Having represented T_{11}, T_{12}, and T_{21} in terms of their Markov parameters as:

$$T_{11} = T_{11_0} + T_{11_1}z^{-1} + T_{11_2}z^{-2} + T_{11_3}z^{-3} + T_{11_4}z^{-4}$$
$$T_{121} = T_{121_0} + T_{121_1}z^{-1} + T_{121_2}z^{-2} + T_{121_3}z^{-3} + T_{121_4}z^{-4}$$
$$Q = q_0 + q_1z^{-1} + q_2z^{-2}$$

the closed-loop system is realizable as $H_{eu}(z) = C_q(zI - A)^{-1}B + D_q$ in which $A \in \mathbb{R}^{6 \times 6}$ is a shift matrix made up of all zeros except for ones on the first super-diagonal, $B = e_6$ denoting the last column of the identity matrix of size six. Also, the C_q and D_q matrices are as follows:

$$C_q = qC + C_0 =$$

$$[q_0 \quad q_1 \quad q_2] \begin{bmatrix} 0 & 0 & T_{121_4} & T_{121_3} & T_{121_2} & T_{121_1} \\ 0 & T_{121_4} & T_{121_3} & T_{121_2} & T_{121_1} & T_{121_0} \\ T_{121_4} & T_{121_3} & T_{121_2} & T_{121_1} & T_{121_0} & 0 \end{bmatrix}$$

$$+ [0 \quad 0 \quad T_{11_4} \quad T_{11_3} \quad T_{11_2} \quad T_{11_1}]$$

$$D_q = qD + D_0 = [q_0 \quad q_1 \quad q_2] \begin{bmatrix} T_{121_0} \\ 0 \\ 0 \end{bmatrix} + T_{11_0}$$

Now, the design problem is expressible as:

$$\min_{\gamma,P,q} \gamma$$

$$\text{S.t.} \begin{bmatrix} P - A^T P A & A^T P B & C_q^T \\ B^T P A & -\gamma I & D_q^T \\ C_q & D_q & -\gamma I \end{bmatrix} \leq 0$$

where $\gamma = \|H_{eu}\|_\infty$.

This example was simulated in MATLAB, and minimum value of $\|H_{eu}\|_\infty$ for various orders of the design parameter Q were achieved and represented in table 6.5.1. The third column in the table shows the number of variables involved in solving the minimization problem. Clearly, by increasing the order of the controller the number of decision variables drastically increases.

Order of Q	Value of the minimized norm	Number of variables
1	8.6404	11
2	5.5105	17
3	3.8628	24
5	1.9656	41
7	1.2424	62
9	1.0661	87

Table 6.5.1

The frequency response magnitude of the closed-loop system was also plotted for different orders of Q, and illustrated in figure 6.5.3. As can be seen, by increasing the order of Q, the l_2 gain of the system uniformly reduces towards zero and become more flat all the time.
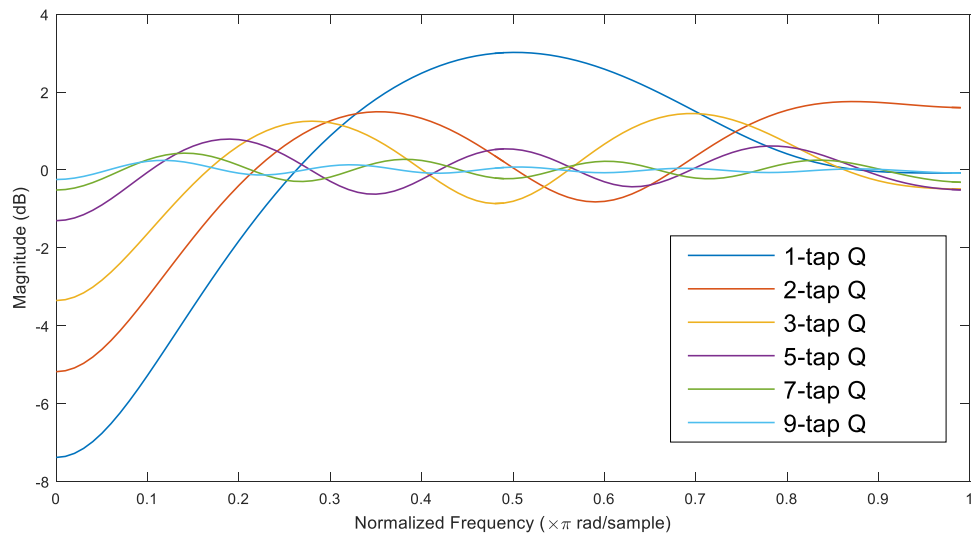

Figure 6.5.3 Frequency response magnitude for different orders of Q

6.6 Conclusion:

In this chapter, the design problem of deadbeat controller subject to the H_∞ norm minimization was investigated. As it is known, the H_∞ norm is an operator norm induced by the l_2 norm of the input and output signals. In the frequency domain though, it is interpreted as the largest singular value of the system over the whole set of frequencies. Therefore, it is an indicator of the worst-case energy of the output in response to energy bounded inputs, which accordingly could come naturally as a measure of the worst case performance for many classes of input signals.

Based on the considered framework in the form of a lower LFT, the H_∞ problem was formulated as finding the stable parameter Q such that the H_∞ norm of the closed-loop system $\|H_{zw}\|_\infty$ is minimized.

Among various approaches for tackling the H_∞ norm minimization problem, a brief overview on three major methods, namely the model-matching, Riccati equation-based, and the LMI approaches was given. In the model-matching scheme, the main problem is transferred to designing $Q \in \mathbb{R}H_\infty$ such that the given transfer matrix $T_{11} \in \mathbb{R}H_\infty$ matches the cascade of the three transfer matrices $T_{12}QT_{21}$. This in turn, was equivalent to yet another problem, the so-called Hankel approximation problem, or the Nehari extension problem, which was also discussed briefly. We then, looked into the Riccati equation-based approach, in which the solvability of the H_∞ problem was expressed in terms of the existence of unique positive definite stabilizing solutions to two algebraic Riccati equations and the spectral radius of their product.

In this work, the applied method for designing the deadbeat compensator with H_∞ norm constraint was the LMI approach, in that the problem is recast in the form of an LMI. As it is proved in section 6.5, the elements of the design parameter Q enter affinely in only the C and D matrices of the state space realization of the closed-loop system $H_{zw} = \mathcal{F}_l(T, Q)$. In view of this feature of the closed-loop system and the fact that T_{11}, T_{12}, and T_{21} transfer matrices are all designed to be FIR systems, we were able to formulate the design problem in terms of an LMI, that was a linear function of the matrix variables. The design procedure was then represented by means of an example.

Chapter 7

Conclusion

This thesis has addressed the synthesis problem of linear state deadbeat controller subject to time and frequency domain performance specifications. Moreover, the robust design problem was investigated when the plant is subject to structured norm-bounded parametric uncertainty.

Basically, the objective in a state deadbeat regulator is to drive a discrete time system from any arbitrary initial state to the desired final state (which without loss of generality can be considered to be the origin of the complex plane) in finite number of time steps. As was argued, deadbeat is a characteristic exclusive to discrete time systems.

The first part of the thesis presented a formal framework in which the design problem is treated. The framework is a natural extension of YJBK algebraic theory and is in the form of a lower LFT of a fixed deadbeat system ("generator" of all deadbeat controllers) with a free deadbeat system of arbitrary high McMillan degree.

The conditions for the fundamental requirement of internal stability were investigated. This was first accomplished using the state space realization of the closed-loop system. However, internal stability was alternatively treated in terms of the matrix fractional description (MFD) of the plant and the compensator. It was observed that expressing the plant and the controller as an irreducible quotient of elements from the set of proper and stable matrices, i.e. coprime factors in $\mathbb{R}H_\infty$, not only captures the usual notion of instability as the result of the presence of unstable closed-loop poles, but also excludes the possibility of unstable pole-zero cancellations between the plant and controller required by the notion of internal stability. This enabled us to characterize the whole family of admissible controllers in terms of the elements of a doubly coprime factorization of the system to be stabilized, and the proper and stable but arbitrary design parameter Q. Characterizing the controller in this fashion, reduced the linear fractional description of the closed-loop system $H_{zw} = \mathcal{F}_l(P, K) = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})$ to an affine parameterization in terms of the free parameter Q as $H_{zw} = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21}$.

This, in turn, simplified the design procedure by reducing the problem of search or optimization over the set of admissible controllers K to a search or unconstrained optimization over Q. It was finally shown that all stabilizing controllers are realizable in the form of observer-based state feedback.

In chapter 3 two major approaches to tackle the deadbeat controller design problem were discussed, namely the state space approach and the algebraic approach. The former method was based on the concepts of controllability and controllable subspaces. It was observed that the number of steps after which the states settle to their final value, and also the controller, are dependent on how n (the order of the system) linearly independent columns of the controllability matrix are selected. The minimum possible number of steps was given by the controllability index of the system.

The properties of the closed-loop system were investigated. It was observed that the system matrix is nilpotent with the index of nilpotency equal to the system controllability index. This attribute led to interpreting the problem of deadbeat controller design as an eigenvalue assignment problem, when all poles are located at the origin. The fact that the structure of the Jordan matrix assigned to the closed-loop system matrix is non-unique, again implies non-uniqueness of the set of deadbeat controllers.

The design problem was also examined via the algebraic approach, developed in view of the isomorphism between certain classes of formal series in one indeterminate over $\mathbb{R}$, and series expansion of functions over $\mathbb{R}$.

The deadbeat regulator was derived as the solution to a polynomial matrix Diophantine equation. As in the state apace approach, the set of stabilizing controllers was parameterized in terms of a free parameter in a YJBK fomat.

In the final part of the chapter, a numerical algorithm was presented to compute the state feedback gain for assigning all the closed-loop controllable modes to the origin. This was developed based on the recursive construction of unitary transformations, resulting into a coordinate system in which the gain was computed by merely solving a set of linear equations. In order to split the controllable and uncontrollable parts, the system was first transformed into the staircase model. Provided that the uncontrollable subsystem was nilpotent, the problem was feasible and the algorithm was applied to the controllable

subsystem. The observer gain was also achieved as the solution to the dual problem of the state feedback. The numerical procedure was programmed in MATLAB, and illustrated with few examples.

In chapter 4, the design problem of deadbeat regulator subject to time domain constraints was addressed. First the input-output mathematical description of a system with deadbeat response was studied. It was observed that the impulse response of such systems are polynomials in the unit delay operator z^{-1}. Therefore, they are alternatively known as FIR (Finite Impulse Response) systems. With respect to the input-output characterization of the closed-loop map $H_{zw} = T_{11} + T_{12}QT_{21}$, having deadbeat response compelled T_{11}, T_{12}, T_{22}, and Q all to be FIR. This was equivalent to assigning all the eigenvalues of the observer-based state feedback to the origin, and selecting the design parameter Q such that its impulse response is of finite length.

Next, the attention of the work was turned to the constrained design problem of the deadbeat compensator. Due to the affine dependence of the closed-loop map on the design parameter Q, the performance specifications can be recast as linear constraints on the elements of Q. This is in contrast to the case when the closed-loop system is characterized in terms of the LFT interconnection of the plant and the controller K, where simple design specifications translate into complicated constraints on K. The achieved simplification in expressing the performance specifications accordingly enabled us to reformulate the constrained deadbeat regulator synthesis problem as a linear program.

More sophisticated design requirements in the form of LQG were also examined. First, the relation between the LQG and H_2 optimization problem was established. As was discussed, the LQG performance index may be stated as the system H_2 norm when it is excited by white noise input disturbance signals. The equivalence was then exploited to show that the design problem of deadbeat controller with LQG constraints reduces to quadratic programming, with constraints exerted on the design parameter Q.

Chapter 5 considered the robust design problem of deadbeat compensator, when the plant is subject to structured norm-bounded model parameter and input connection parameter uncertainties. The synthesis method was based on the Lyapunov approach.

First, the conditions for quadratic stability and the particular form of the quadratic $\mathcal{D}$-stability were discussed for the case of perturbed continuous time systems. The conditions were then readily accommodated to discrete time systems. The significant attribute of the attained stability criteria was that all were formulated as LMI feasibility problems. This property provided the means to treat the robust deadbeat design problem through existing efficient tractable numerical algorithms for solving LMIs.

When the plant is uncertain it is clearly impossible to place all closed-loop poles at the origin for all possible combinations of the uncertain parameters. In an approximately deadbeat system, the most natural region to confine the poles of the closed-loop system in, is a circle centred at the origin. Evidently, this is equivalent to the quadratic $\mathcal{D}$-stability of the system when $\mathcal{D}$ is a disk centred at the origin. To approximate the deadbeat characteristic as closely as possible, the circle is required to have minimum radius. This problem was recast as an optimization problem with the associated LMI condition being modified accordingly as its constraint. Due to the fact that the eigenvalues of a deadbeat system are the union of those of the state feedback and the observer, as the constituent elements of the observer-based controller, it was necessary to minimize the largest radius corresponding to the two subsystems. Using Parrott's lemma it was shown that the problem of finding the minimum radius could be split into finding the smallest circular regions associated with each of the subsystems, and then selecting the greater disk as the smallest possible circular region which encloses whole set of the closed-loop poles. The design procedure was illustrated by means of an example.

In the final chapter of this thesis, chapter 6, we studied the synthesis problem of the deadbeat regulator subject to H_∞ norm specifications. First, the interpretation of the H_∞ norm in both the time and frequency domains was discussed. As it is well-known, the H_∞ norm is an indicator of the worst-case energy of the output for energy bounded inputs. Hence, it may be regarded as a measure of the worst case performance for many classes of input signals.

Three major schemes to tackle the H_∞ norm optimization problem were briefly reviewed, namely model-matching, Riccati equation-based, and the LMI approaches. In this case, the last method was deemed to be the most appropriate one due to ease of implementation. The LMI characterization of the H_∞ problem, the so-called bounded real lemma, translates the H_∞ norm optimization problem into an LMI feasibility problem, expressed

in terms of the state space matrices of the closed-loop system. As , the Markov parameters of the design parameter Q appear affinely only in the C and D matrices of the state space realization of the closed-loop map H_{zw}. This makes the linear matrix inequality a linear function of the matrix variables, and so it can be easily solved. The chapter concluded by considering an example of the design problem.

The main achievements of the thesis are summarized as follows:

- It was shown that the Q parameterization of all stabilizing controllers extends naturally to the deadbeat case. The "central" deadbeat (FST) controller, obtained when the free parameter is set to zero, corresponds to minimum-time deadbeat control, which in a sense is the simplest controller in this framework, although not necessarily the most appropriate one when performance and robustness specifications are taken into account. By confining the design parameter Q to be FIR, the family of deadbeat regulators was parameterized affinely in terms of Q.

- The robust deadbeat control design problem was addressed. This was accomplished based on the notion of quadratic stability through application of the LMI approach and extends the results presented in the previous paragraph when the effect of numerical or model errors is significant. It was observed that the problem may be reduced to two decoupled robust eigenvalue assignment problems, in which the systems to be considered are the constituent elements of the controller, i.e. the observer and the state feedback. The circular regions with minimal radius centred at the origin of the complex plane were computed, corresponding to the worst-case eigenvalue assignment problem of the two subsystems. It was shown that the circular region with greater radius is the smallest region to which whole set of the closed-loop poles may be assigned.

- The synthesis problem of the deadbeat regulator subject to classical time domain constraints was reformulated as a linear program. This was accomplished in view of the affine dependence of the closed-loop map on the design parameter Q.

- The achieved parameterization of the deadbeat compensators was exploited to recast the design problem when the system is to satisfy more stringent time domain specifications in the form of LQG as a quadratic programming. The procedure is based on the equivalence between the LQG and H_2 problem.

- Shaping the frequency response of deadbeat system in terms of the worst case performance, quantified by its H_∞ norm, was formulated as a convex program. It was shown that in the state space characterization of the closed-loop map it is only the C and D matrices of the system which are dependent on the Markov parameters of the design parameter Q in an affine manner. This feature yields to the involved LMI constraint be a linear function of the matrix variables.

Suggestions for future work:

The work which has been done in this thesis may be developed and extended as what follows.

- FST-based multiple objective control: The thesis has considered individual optimization settings, such as H_2/LQG and H_∞ optimal control. It has been shown that the FIR framework can be used to formulate and solve problems of these two types by converting them to standard quadratic programming (QP) and convex programming (LMI-type) problems, respectively. Since QP can be effectively combined with LP, mixed LQG problems with time-domain specifications (e.g. hard constraints on state variables, slew-rates or transient response characteristics) can be easily be formulated and solved. In future work, the optimization of additional norms can be considered (e.g. generalized H_2, l_1-norm) which are appropriate for a variety of closed-loop specifications and models of disturbance and noise signals. In addition, multiple objective optimization problems using (simultaneously) different norms and mixed time-domain and frequency-domain specifications can be considered.

- Approximation Error analysis: The thesis has developed a version of Youla's parameterization of the set of all closed-loop transfer functions corresponding to

all internally stabilizing controllers, in the form of an affine map of an FIR system with the set of all H_∞ functions (Q parameterization). It has been argued that replacing the later set by the set of all matrix FIR filters is not too restrictive, provided a bound on the order of these filters is chosen sufficiently large. Although the validity of this method is intuitively clear, a more formal error analysis would be beneficial for quantifying the effects of the approximation on the achievable performance and stability properties. Such an analysis could also be used to provide a-priori estimates of the required degree of the FIR matrix parameter in terms of bandwidth or transient decay-rate specifications.

- Controller model reduction: The approach described in the thesis often results in controllers of high McMillan degree, obtained in the form of an LFT of a (typically low-degree) controller-generator with an FIR filter which has (typically) high McMillan degree. Thus, for reasons related to practical implementation, some form of model reduction should be applied. One way of doing this is to model reduce the FIR parameter directly, using for example the balanced-truncation or Hankel-norm based techniques of [169]. Alternatively, model reduction can be applied to the controller after the LFT of the generator with the FIR matrix filter has been computed. In either case, the effects of the controller approximation on the stability margins and performance of the closed-loop system should be carefully considered.

- Robust LP: In recent years, significant progress has been made in the area of robust optimization, especially in the area of Linear Programming. These methods typically assume that both the objective function which is optimized and the constraints of the problem are uncertain and worst-case or probabilistic techniques are employed to formulate and solve the problem in a robust setting. It seems that this optimization framework is directly applicable to our case when the model of the plant is assumed to be uncertain. The investigation of robust LP techniques in the context of the present work would be interesting and could provide valuable links between the fields of robust optimization and robust control.

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