Mixture and Continuous 'Discontinuity' Hypotheses:

An Earnings Management Model with Auditor-Required Adjustment[†]

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A parsimonious model is developed to understand two perplexing, salient features of the distributions of earnings, earnings change, and earnings surprise. The model provides guidance for empirical work to uncover the unmanaged earnings important to firm valuation and public scrutiny, yet unobserved by outside parties. Simulation results based on the model show that the puzzling volcano shape of the distributions can arise from the mixture of a spiky distribution of managed earnings with a bell-shaped distribution of unmanaged earnings. The spiky distribution is due to cookie-jar earnings management that compresses unmanaged earnings from both sides toward an earnings benchmark, leading to a concentration of density around there. The mixture is due to the auditor's adjustment decision, which seems stochastic from the public's or client firm's perspective. Additional simulation results suggest that the widely documented discontinuity in the distributions can be partly due to a steep increase in density appearing like a discontinuity when a continuous distribution is plotted in terms of frequency counts in histogram bins. The main analytical results are a general characterization of the optimal earnings management strategy and the derivation of closed-form solutions for particular functional form assumptions. Potential applications include structurally estimating the model for policy analysis to assess the impact on earnings manipulation. (*JEL* M43/M49/K42)

Key words: Misreporting, Earnings Manipulation, Cookie-jar Accounting, Benchmark Reference, Auditor-client Interaction

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1. Introduction

This paper develops a parsimonious model to understand within a unified framework two puzzling phenomena. They are: (i) a discontinuity at zero in the distributions of the earnings triplet (i.e., earnings, earnings change, and earnings surprise); (ii) the volcano shape of the distributions. Hayn (1995) was the first to point out a discontinuity at zero in the distribution of earnings. Burgstahler and Dichev (1997) systematically showed that the discontinuity occurs in the distribution of earnings change as well as earnings, suggesting that it could be caused by earnings management to avoid earnings decreases and losses. Since then, a number of studies have found similar phenomena for particular types of institutions and for different countries (e.g., Beatty et al. 2002, Beaver et al. 2003, Coulton et al. 2005, Gore et al. 2007, and Parte Esteban and Such Devesa 2011). Degeorge et al. (1999) found a similar discontinuity in the distribution of earnings surprise (also known as unexpected earnings or negative forecast error) using forecast data in 1974-1996. Bhojraj et al. (2009) documented such a discontinuity using forecast data in 1988-2006.

Besides the discontinuity, the distributions of the earnings triplet usually have a volcano shape. "The common mental image of a volcano is that of a steep symmetrical cone sweeping upward in a [convex] curve to a sharp summit peak" (Rafferty 2010, p. 169).[1] The sharp peak of the volcano shape is markedly different from the smoother, rounder top of the bell shape of a

[1] In reality, not many volcanoes have this ideal shape. Those qualified as good examples of this image usually are categorized as *stratovolcanoes* (also known as *composite volcanoes*). According to Wicander and Monroe (2006), composite volcanoes are "steep sided near their summits … but the slope decreases toward the base" (p. 107). A drawing of a composite volcano on the website of the United States Geological Survey (http://pubs.usgs.gov/gip/volc/types.html), or a photo of Mayon Volcano in the Philippines on the website of San Diego State University's Department of Geological Sciences (http://www.geology.sdsu.edu/how_volcanoes_work/Thumblinks/Mayon_page.html), illustrates what I refer to as a volcano shape in this paper. However, the crater of a volcano, which is very small in the classic stratovolcanoe form, is a feature to deemphasize.

normal distribution, or that of similar alternatives such as Student, logistic, or extreme value type I. The volcano shape of the distributions of the earnings triplet can be noticed in prior studies (e.g., figure 1 of Burgstahler and Dichev 1997, figure 6 of Degeorge et al. 1999, and figure 2 of Frankel et al. 2010). To my knowledge, no study has highlighted the volcano shape of the distributions, let alone explain it.

Using simulation results based on the model developed, I show that the widely documented discontinuity can be partly due to a steep increase in density appearing like a discontinuity when a continuous distribution is plotted in terms of frequency counts in histogram bins. The steep increase in density is a consequence of the compression of unmanaged earnings into reported earnings toward an earnings benchmark as a result of cookie-jar earnings management. This provides a novel alternative perspective to interpret the discontinuity phenomenon, supplementing the prevalent explanation based on upward earnings management by “just-missed” firms. A number of studies have examined whether the prevalent explanation is indeed the main cause of the discontinuity or even questioned the existence of the discontinuity (Dechow et al. 2003, Durtschi and Easton 2005 and 2009, Beaver et al. 2007, Jacob and Jorgensen 2007, and Kerstein and Rai 2007). The alternative perspective provided here adds to the debate by setting a different null hypothesis for testing the existence of a discontinuity distinct from what could simply be interpreted as a continuous increase in density.

Together with the auditor’s strategic response, the compression of earnings arising from cookie-jar earnings management attempts can also generate a volcano-shaped distribution of reported earnings. Auditors sometimes require adjustments to remove earnings management attempts (Nelson et al. 2002). Consequently, the distribution of reported earnings observed by the public is a mixture of the distributions of unmanaged and managed earnings. The mixture can retain the volcano-shape characteristics of the managed earnings distribution. I illustrate this explanation using simulation results based on the model that takes into account auditor-required adjustment. Prior models of misreporting have omitted this aspect in order to focus on other issues, such as the interaction between a firm's disclosure and investors' rational expectation.

Many of the studies investigating earnings management and fraud are empirical-based, with only a few of them (e.g., Kedia and Philippon 2009) closely guided by formal theoretical models. Models of misreporting exist in the literature (e.g., Kumar and Langberg 2009, Guttman et al. 2006, Ewert and Wagenhofer 2005, Kirschenheiter and Melumad 2002, Sankar and Subramanyam 2001, and Fischer and Verrecchia 2000). Yet, except Caskey et al. (2010) and Newman et al. (2001), few have explicitly modeled the auditor, which plays an important role in the corporate reporting process (Bollen and Pool 2009, Caramanis and Lennox 2008, Brown and Pinello 2007, and Liang 2003). Some of the models above have provided explanations to the discontinuity phenomenon. However, no prior study has provided a single theoretical framework to understand the two salient features of the earnings triplet distributions.

The sort of earnings manipulation discussed in this paper is better understood as borderline misreporting rather than outright fraud. Proving the intention of borderline misreporting is often difficult. Usually what auditors can do is simply requiring adjustments to remove such earnings management attempts. In the unusual circumstances where the extent of misreporting is so outrageous to strongly suggest outright fraud, auditors may notify regulators. Otherwise, rarely would client firms bear any significant consequences of borderline misreporting disallowed by auditors. Therefore, the main consequence of unsuccessful borderline misreporting is the reversal of the attempted manipulations before announcing the earnings to the public.

The distinction between borderline misreporting and outright fraud suggests a novel assumption on a firm's cost of misreporting for downward manipulations, sometimes referred to as cookie-jar accounting. In contrast to the conventional quadratic assumption, I assume that the cost of misreporting is an increasing function of earnings manipulation, even for the negative values representing downward manipulations. In other words, a larger downward manipulation results in a more negative misreporting cost standing for the opportunity benefit of "saving for the future." An auditor-required adjustment reversing a downward manipulation attempt means a reversal of the negative misreporting cost as well.

Downward manipulations look like conservative accounting when viewed with respect to the current period. However, they can be used to build up cookie-jar reserves for subsequent upward manipulations (Cohen et al. 2011, Jackson and Liu 2010, and Moehrle 2002). Cookie-jar accounting has caused serious concern to regulators. In his famous 1998 speech entitled *The 'Numbers Game'*, the then Chairman of the U.S. Securities and Exchange Commission (SEC) Arthur Levitt said: "[U]sing unrealistic assumptions to estimate liabilities for such items as sales returns, loan losses or warranty costs … [some companies] stash accruals in cookie jars during the good times and reach into them when needed in the bad times." [2] The model of this paper accommodates cookie-jar accounting as equilibrium behavior under a wide range of circumstances. In contrast, models focusing on upward manipulations usually admit downward manipulations as possible behavior occurring infrequently in equilibrium.

In the model, a firm manipulates the earnings before providing the figure to an auditor for audit. The audit allows the auditor to separate the unmanaged and manipulated components of the

[2] Recently, the Public Company Accounting Oversight Board (PCAOB) fined Ernst & Young (E&Y) $2 million for failing to properly evaluate in the 2005 to 2007 audits of Medicis Pharmaceutical Corp. the amount set aside to account for the cost of product returns. In addition, it sanctioned three current partners of the audit firm plus one retired, barring some of them from auditing public companies for one to two years or more and imposing fines from $25,000 to $50,000. Medicis corrected the misreporting when it was discovered in 2008 and has restated the financial statements for the years affected. The company said: "It actually revealed that we were more profitable across the overall six-year restatement period." The company also emphasized that "[s]everal independent reviews found that the errors didn't stem from any improper efforts to inflate earnings" (Gordon 2012). This example suggests that regulators disapprove non-inflationary manipulations as much as inflationary ones, and companies do use cookie-jar accounting.

pre-audit (managed) earnings. He can require the firm to make an adjustment to remove the manipulation before announcing the post-audit (reported) earnings to the public. However, the auditor must incur a cost, privately known to him, to convince the firm to follow the requirement.[3] Depending on the magnitude of the cost and that of the expected liability cost arising from tolerating the manipulation, the auditor decides whether to require an adjustment or not. From the firm's perspective, this outcome of the auditor's adjustment decision is stochastic. Considering the chance of a required adjustment, the firm chooses the extent of an upward or downward manipulation to balance the benefit and cost of misreporting. If the auditor tolerates the manipulation, the managed earnings are reported to the public. If he requires an adjustment, the manipulation is reversed. So is the firm's cost of misreporting because it captures the opportunity cost (or benefit) of "borrowing from (or saving for) the future." The earnings reported to the public are then the unmanaged earnings.

In reality, an auditor influences the earnings reported to the public through a complicated auditor-client negotiation process (see, e.g., Perreault and Kida 2011, Bame-Aldred and Kida 2007, Beattie et al. 2004, and Gibbins et al. 2001). Modeling this process in an elaborated manner requires a separate paper. The moderate goal here is to use a tractable "ban-or-tolerate" game to capture only the first-order impact of the auditor-client negotiation.

A key element of the model is a cutoff point that divides the model into two separate regions, with upward manipulation being optimal in one and downward manipulation in the other. Whether the model can generate a "discontinuity" resembling the one observed in reality depends

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[3] See section 2 and related discussions in McCracken et al. (2008), Gibbins et al. (2001), and Beattie et al. (2004) for explanations about the adjustment requirement cost borne by the auditor. The consideration of materiality is implicitly captured in the specification of the distribution function of this cost. When the manipulation is immaterial, it is highly likely that the auditor's adjustment requirement cost will exceed the expected liability cost arising from the manipulation, leading to the tolerance for the immaterial manipulation.

critically on the location of the cutoff point, which is determined by the underlying parameters of the model. The fact that the model cannot explain the discontinuity location with a wide range of parameter values is a strength rather than weakness of the model. It is the alignment of the discontinuity location with the location of the cutoff point, together with the change in the density around that point implied by the model, that provides the identifiability of the underlying parameters. Without identifiability, the model cannot be used for structural estimation and counterfactual policy analysis (see below for further discussion).

The paper contributes to the literature by offering a single framework to understand two puzzling features of the earnings triplet distributions. In particular, the predicted change in the density around the cutoff point provides a theoretical prediction for what one should expect when reported earnings are affected by cookie-jar earnings management only. This prediction constitutes a null hypothesis for testing the existence of a discontinuity due to other reasons over and above cookie-jar earnings management.

My alternative interpretation of the discontinuity phenomenon is based on earnings compression from both sides toward the earnings benchmark. This is distinct from the prevalent explanation based on moving earnings across the benchmark to turn small losses into small profits (TSLISP). The alternative interpretation concerns the behaviors of all firms with opposite directions of earnings management. The latter is a local explanation concerning upward manipulation by only a small group of "just missed" firms, without saying anything about the behaviors of other firms. If a more general perspective can demystify both the discontinuity and the volcano shape, it is not necessary to invoke a second explanation specifically for the discontinuity. Therefore, testing the existence of a discontinuity against the null hypothesis based on the prediction of a structurally estimated model of this paper can provide evidence on the necessity of the TSLISP hypothesis. A reasonably good alignment of the data with the prediction of the model rejects the TSLISP hypothesis as a necessary explanation.

Second, the model of this paper provides a framework for structural estimation of the underlying parameters, allowing inferring the unmanaged earnings not observed by outside parties. This is important to firm valuation, investment decisions, and public scrutiny. The framework also allows policy or event analysis to be conducted based on the hypothesized impacts of a policy or event on the parameters (see Heckman 2010 for a discussion on counterfactual policy analysis). By hypothesizing how regulation changes might affect certain model parameters (or the distribution of unmanaged earnings), one can use the model of this paper to predict the impacts on earnings manipulation. This provides an assessment useful for comparing the anticipated impact to the implementation cost of a regulation change, helping regulators to make informed decisions.

The rest of the paper is organized as follows. The next section introduces the model setup. In section 3, I provide a general characterization of the optimal misreporting strategy, which is unique under some mild conditions. In section 4, two (effectively four) closed-form solutions are derived for particular functional form assumptions. Section 5 presents the results of two simulation exercises demonstrating the model's capability of generating the two salient features of the earnings triplet distributions. Further details of why the volcano shape is a puzzle are discussed in section 6. The difference between a sharp peak and the widely documented leptokurtic property of stock return distributions are also discussed there, followed by concluding remarks on various issues, including potential applications of the model.

Maximum likelihood and nonlinear least squares methods to estimate the model parameters are discussed in appendix A. Appendix B suggests a few ways to extend the model, such as allowing analyst forecast dispersion to play a role. Technical proofs are relegated to appendix C. The appendices are available upon request.

2. Model Setup

In the model, a firm needs to prepare the earnings figure for audit before announcing it to the public. The firm generally has an interest to manipulate the figure away from the *unmanaged earnings* denoted by y. The unmanaged earnings are defined as the cash flows plus the *unmanaged accruals*, i.e., the accruals most correctly determined according to the auditor's understanding of the financial reporting standards. Therefore, the auditor knows the value of y, given that the audit engagement allows unrestricted access to the firm's accounting system. It is assumed that the firm also knows the y in the auditor's mind. This way in defining the unmanaged earnings means that any real activities earning management, if exists, is part of y.

The interest to manipulate the earnings figure is affected by some *earnings benchmark* denoted by z, which is common knowledge in the model. Real-world examples of earnings benchmarks include the profit/loss cutoff at zero, last year's earnings highlighting earnings increase/decrease, and the analyst consensus forecast reflecting market expectation (see section 3.1.5 of Dechow et al. 2010). In circumstances where a firm's accounting choices are heavily influenced by certain executives, earnings benchmarks can be some internal yardsticks used for performance evaluation related to salary increases, bonuses, promotion, etc.

One of the key insights of this paper is that the widely documented discontinuity phenomenon could be partly due to a steep but *continuous* increase in the density of the earnings distribution. Therefore, I will specify a payoff function of the firm that is smooth even at the earnings benchmark. This specification sets the model apart from others (e.g., Degeorge et al. 1999) that rely on a jump (e.g., a lump-sum bonus) to generate a discontinuity in the distribution of reported earnings.

The earnings figure prepared for audit, hereafter the *pre-audit earnings* (or *managed earnings*), is denoted by m. In general, it may include manipulation that nonetheless is not outright fraud. Let $a \in (-\infty,\infty)$ denote such manipulation. By comparing m to the y in its mind, the auditor knows the value of a.[4]

In the very beginning of the model, the auditor explains the audit plan to the firm. The quality level $q \in [0,1]$ of the audit is known to the firm at that time before the firm knows the unmanaged earnings y and selects the manipulation a accordingly. Denote by $\varepsilon_q + \varepsilon_{1-q}$ the total amount of unintentional errors contained in y. By definition, the firm is unaware of these errors. The components $\varepsilon_q \sim \text{Normal}(0,\ q\sigma_{\text{u}}^2)$ and $\varepsilon_{1-q} \sim \text{Normal}(0,\ (1-q)\sigma_{\text{u}}^2)$ are independent random variables, with the parameter $\sigma_{\text{u}} > 0$. The part of unintentional errors discovered and removed by the auditor is ε_q, with ε_{1-q} remaining in y even after the audit. For ease of exposition, q is treated as an exogenous parameter in this paper. Endogenizing it does not critically affect the main results. (I discuss how it can be endogenized in appendix B.) Excluding unintentional errors from the model does not qualitatively change the results either. However, their inclusion helps to understand the model's implications; for example, it allows one to see the effect of the quality parameter q, which is interesting.

Let $x \in \{0,1\}$ denote the auditor's adjustment decision, with 0 standing for no adjustment required and 1 otherwise. Requiring an adjustment to remove the manipulation is not as simple as just saying no. To achieve the objective, the auditor must bear a cost, $X \in [0,\infty)$, privately known to him. Field studies (e.g., McCracken et al. 2008) have documented the stress auditors face and the effort they make during the negotiation process to convince client firms to make adjustments.

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[4] The simplifying assumption of the auditor being capable of separating manipulation from earnings seems to be consistent with practitioners' observation (e.g., Bagshaw 2013, p. 266): "[T]he evidence of the relevant fraud or accounting irregularity was not hidden from auditors. Reports note that such evidence was sometimes recorded in the audit file, but that auditors did not appreciate the significance of the issue, follow it up or make the right linkages. Reports by professional bodies and independent regulators on audit firms highlight the same problem. Audit failure is more likely to involve errors of judgement and a failure to evaluate audit findings properly than deficient audit methodologies, or sampling risk, for example." In this model, the "errors of judgment" or "failure to evaluate audit findings properly" is modeled in a stylized way as a willful decision to tolerate manipulation based on expected liability and adjustment requirement cost comparison.

Accomplishing this without jeopardizing the relationship with the clients and the likelihood of being retained as the auditors is a big challenge. Such a process can take weeks, or even months, to finish (Gibbins et al. 2001), potentially distracting the auditors from concentrating on other work like the audits of other clients. Preparation work such as producing sufficient literature research to back up the adjustment requirement, gathering evidence of similar practices by other companies, securing the support from the national office, or even obtaining a second opinion from another audit firm also contributes to the cost of requiring an adjustment. Alternatively, the cost may be viewed as a parsimonious way to model whether the auditor's personality is closer to a crusader type (who never tolerates any manipulation), as opposed to an accommodator type (who is prepared to bend the rules to interpret a manipulation as complying with GAAP) (Beattie et al. 2004).

The cost X is modeled as a random variable independent of ε_q and ε_{1-q}. It follows a probability distribution $G(l) = \Pr\{X \leq l\}$ with a differentiable probability density $g(l) = G'(l) > 0$ for all $l > 0$. The hazard rate function $h(l) = [1-G(l)]/g(l)$ is therefore also differentiable. To ensure that the firm's objective function (to be specified shortly) is twice-differentiable in a, I assume the existence of bounded limits $\lim_{l\downarrow 0} l^{1/2}g(l)$ and $\lim_{l\downarrow 0} [g(l) + 2lg'(l)]$. This is a mild condition satisfied by a number of distributions including those discussed in section 4. The auditor learns the realized value of X just before making the adjustment decision. The firm knows only the distributional properties of X without observing the realized value.

The expected liability cost to the auditor is assumed to be a quadratic function of the extent of the manipulation: $L = ka^2/2$, where $k > 0$. (Any expected liability cost that might arise from failing to remove all the unintentional errors is irrelevant to the auditor's adjustment decision and need not be specified.) Assuming a quadratic form is common in the literature to maintain analytical tractability. The symmetric loss represented by the quadratic form is not critical to the model. At the expense of expositional simplicity, alternative specifications such as an asymmetric expected liability cost like $L = [k_2\mathbf{1}\{a > 0\} + k_1\mathbf{1}\{a \leq 0\}]a^2/2$, where $k_2 > k_1 > 0$, can be used, without critically changing the nature of the results.

The optimal adjustment decision, $x^*(a)$, is determined by minimizing the sum of the expected liability cost and the cost of requiring an adjustment:

$$\text{Min}_{x\in\{0,1\}} \quad (1-x)L + x(X). \tag{1}$$

Clearly, $x^*(a) = \mathbf{1}\{X \leq L\}$, i.e., equal to 1 when the event $\{X \leq L\}$ holds and 0 otherwise. In other words, the auditor will require an adjustment if and only if the cost of doing so is not higher than the expected liability cost from tolerating the manipulation.

The firm's manipulation incentive is driven by the benefit and cost of misreporting (see Jiambalvo 1996 and Marquardt and Wiedman 2004 for discussions on the benefits and costs). Because I focus on cookie-jar earnings management, a dollar of downward manipulation today helps build up the cookie-jar reserve, facilitating future upward manipulations. Similarly, every dollar of upward manipulation today is essentially borrowed from the future. Hence, if in the absence of distortions and discounting, the cost of misreporting can be as simple as $c(a) = a$. Allowing for discounting and potential distortions (e.g., a small psychic cost of misreporting), I assume a more general $c(a)$ with the requirements that $c(0) = 0$, $\lim_{a\uparrow\infty} c(a) = \infty$, $c'(a) > 0$ with $c'(0) = c_0 < \infty$, and $c''(a) \geq 0$.[5] The monotonicity assumption, $c'(a) > 0$, is a departure from the

[5] Although the model technically is a single-period model, it should be viewed as modeling a representative period of an infinite-period setting, rather than as a one-shot model. The misreporting cost in this reduced-form representation of an infinite-period setting plays a role analogous to the next-period value function in a Bellman equation for dynamic programming. Such a value function is endogenously determined in a Bellman equation but the misreporting cost in the reduced-form representation here is exogenously specified. The reduced-form representation could be formulated as a two-period model like some others in the literature. Such two-period models typically assume that the manipulation in the first period must be reversed in the second period, effectively turning the two-period choices of manipulation into a one-period choice. Therefore, the difference between the two formulations is not as big as they appear. The representative-period formulation, however, seems to be more consistent with the going-concern perspective of accounting.

usual specification of a quadratic misreporting cost function (e.g., Guttman et al. 2006 and Fischer and Verrecchia 2000). The assumption is essential for deriving downward manipulations in the model. I will elaborate on this further shortly.

The firm benefits more from reporting earnings that exceed the earnings benchmark to a greater extent and bears more negative consequences from missing the benchmark more.[6] This is captured by a negative exponential benefit function. If the auditor does not require an adjustment to remove the manipulation a, the net benefit to the firm is

$$\left[\frac{1-\exp\left(-\alpha[y+a-\varepsilon_q-z]\right)}{\alpha}\right]-c(a), \tag{2}$$

where $\alpha > 0$, with the limit case of $\alpha = 0$ accommodating a linear benefit function. If an adjustment is required, a is reversed. So is $c(a)$, which captures the opportunity cost (or benefit) of "borrowing from (or saving for)" the future. Accordingly, the net benefit to the firm is simply $[1-\exp(-\alpha[y-\varepsilon_q-z])]/\alpha$, as though there was no manipulation attempt.

To reduce notation complexity, I will simply write $x^*(a)$ as x to denote the binary random variable induced by the auditor's optimal adjustment decision. The earnings announced to the public after the audit are referred to as the *post-audit earnings* (or *reported earnings*) defined as follows:

$$r = xy + (1-x)m - \varepsilon_q. \tag{3}$$

The firm's manipulation decision is made before ε_q is known. Anticipating the auditor's response $x^*(a)$, the firm chooses a to maximize the expected net benefit given below:

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[6] A strand of literature in economics assumes that utility is reference-dependent. See discussions in Rablen (2010) and Koszegi and Rabin (2006).

$$\mathrm{E}\left\{(1-\mathbf{1}\{X \le L\})\left[\frac{1-\exp\left(-\alpha[y+a-\varepsilon_q-z]\right)}{\alpha}-c(a)\right]+\mathbf{1}\{X \le L\}\left[\frac{1-\exp\left(-\alpha[y-\varepsilon_q-z]\right)}{\alpha}\right]\right\}, \quad (4)$$

which can be simplified as

$$v=\frac{(1-b)}{\alpha}+[1-G(L)]\left[\frac{b[1-\exp(-\alpha a)]}{\alpha}-c(a)\right], \quad (5)$$

where

$$b=\exp\left[-\alpha(y-z)+\frac{\alpha^2\sigma_{\mathrm{u}}^2 q}{2}\right].^{7} \quad (6)$$

Note that the parameter b defined above summarizes the impacts of the quality parameter q and the deviation of y from z on the firm's misreporting incentive. For ease of reference, I simply call b the *marginal expected benefit of manipulation*, although strictly speaking it is only one of the contributing factors.

The firm can control a but not ε_q. When ε_q is more uncertain as a result of a higher quality level q, it raises b for any given levels of $y-z$ and a. Consequently, the slope of the expected net benefit function changes, providing a different incentive to manipulate earnings. Likewise, a different benchmark z can also change the incentive.

Whether the optimal manipulation is upward or downward also depends on the marginal cost of manipulation, $c'(a)$, relative to the marginal benefit affected by b. If $c(a)$ were assumed to be quadratic as in other earnings management models, $c'(a)$ would be negative for $a < 0$. Downward manipulation would never constitute an equilibrium. Therefore, the assumption of $c'(a) > 0$ is critical for accommodating both upward and downward manipulations. Given that cookie-jar

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[7] $\mathrm{E}[\,[1-\mathbf{1}\{X \le L\}][[1-\exp(-\alpha[y+a-\varepsilon_q-z])]/\alpha-c(a)]+\mathbf{1}\{X \le L\}[1-\exp(-\alpha[y-\varepsilon_q-z])]/\alpha\,]$
$= \mathrm{E}[1-\exp(-\alpha[y-\varepsilon_q-z])]/\alpha+\mathrm{E}[1-\mathbf{1}\{X \le L\}]\;\mathrm{E}[\,\exp(-\alpha[y-\varepsilon_q-z])[1-\exp(-\alpha a)]/\alpha-c(a)\,]$
$= (1-b)/\alpha+[1-G(L)][\,b[1-\exp(-\alpha a)]/\alpha-c(a)\,],$
where $b = \mathrm{E}[\exp(-\alpha[y-z-\varepsilon_q])] = \exp[-\alpha(y-z)+\alpha^2\sigma_{\mathrm{u}}^2 q/2]$. The first equality above uses the assumption that X and ε_q are independent, and the second utilizes the normality assumption on ε_q.

accounting is an important regulatory concern, it is interesting to explore the implications of this alternative assumption that captures the idea of "saving for the future."

I end this section with table 1 that summarizes the notations used and the timeline in figure 1 that summarizes the sequence of events in the model.

3. Optimal Misreporting Strategy

This section gives a general characterization of the optimal manipulation. In the next section, closed-form solutions for particular adjustment requirement cost distribution and misreporting cost functions are derived. Then in section 5 the solutions are used to conduct two simulation exercises to see how well, or not so well, the model is able to accommodate the two salient features of the earnings triplet distributions.

Central to the characterization of the optimal manipulation is a cutoff point y_0 given by the equation $\exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2] = c_0$. That is to say, $y_0 \equiv z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$. This cutoff point of the unmanaged earnings determines whether the optimal manipulation is upward or downward. The proposition stated below proves the existence of the optimal misreporting strategy and characterizes it with several equality and inequality conditions.

Proposition 1 (Existence and Characterization of Optimal Misreporting Strategy): *Let* $b = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2]$ *and* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$. *An optimal manipulation of the firm, denoted by* a^**, exists and satisfies*

(POS) $$b[1 - \exp(-\alpha a)]/\alpha - c(a) \geq 0,$$

which defines a convex set of a, denoted by Φ*. Moreover,*

(i) a^* *is a solution of the following equation:*

(FOC) $$[1-G(L)][\, b\exp(-\alpha a) - c'(a)\,] = kag(L)[\, b[1 - \exp(-\alpha a)]/\alpha - c(a)\,],$$

where $L \equiv ka^2/2$*;*

(ii) a^* *satisfies the following inequality:*

(SOC) $$[1 - G(L)][\ \alpha b\exp(-\alpha a) + c''(a)\ ]$$

$$+\ k[\ g(L) + 2Lg'(L) + 4Lg(L)^2/(1 - G(L))\ ][\ b[1 - \exp(-\alpha a)]/\alpha - c(a)\ ] \geq 0;$$

(iii) unless $y = y_0$, *zero manipulation (i.e.,* $a = 0$*) is suboptimal;*

(iv) for $y > y_0$, *any optimal manipulation is downward (i.e.,* $a^* < 0$); *for* $y < y_0$, *any optimal manipulation is upward (i.e.,* $a^* > 0$).

For a manipulation to be optimal, the firm's expected net benefit v for $a > 0$ minus the v for $a = 0$ must be non-negative, i.e., the manipulation must bring a non-negative gain to the firm. Since $[1 - G(L)] > 0$, the non-negativity requirement is simply condition POS. The equality first-order condition FOC can pin down candidates of optimal manipulation, of which some are actually suboptimal. The second-order condition SOC can differentiate between them but oftentimes condition POS can do so more conveniently. The last two parts of the proposition highlight a very simple structure of the optimal manipulation: Downward manipulation is optimal when the unmanaged earnings are sufficiently high (i.e., $y > y_0$); otherwise, upward manipulation is optimal. This result can provide the force to turn normally distributed unmanaged earnings into a volcano-shaped distribution of post-audit earnings. I will explain this further in section 5.

The next result stated below establishes the uniqueness of the optimal manipulation. Other misreporting models often use a signaling approach, leading to multiple equilibria. This limits the potential of using such models to guide *structural* estimation in empirical studies because it is not clear which of the equilibria is observed in data.[8] In contrast, the uniqueness result in the next

[8] One can choose to focus only on the linear equilibrium when linking signaling-based misreporting models to data. In fact, this is typically the only equilibrium analyzed for such models, as a way to get around the intractability of analyzing all the multiple equilibria. The linear equilibrium specifies a rational price conjecture under which the share price is an affine function of the reported earnings. Although this equilibrium appears to be simplest and most intuitive, taking it to data causes some technical complications. For example, the reported earnings in principle can take an arbitrarily large positive or negative value, e.g., as a realization from a normal distribution. However, the share price cannot be negative. Therefore, the linear rational price conjecture cannot be well-defined for all possible values of the reported earnings if the distribution of earnings has the real line as the support, which seems very reasonable. In contrast, the model of this paper does not run

proposition removes such ambiguity when linking the model of this paper to data.

Proposition 2 (Uniqueness of Optimal Misreporting Strategy): *Suppose* $1 + 2l[d\ln g(l)/dl] - 4l[d\ln(1 - G(l))/dl] \geq 0$ *(e.g., if* $d\ln g(l)/dl \geq -1$*), or equivalently,* $d\ln[h(l)(1 - G(l))]/dl \leq 1/2l$ *(e.g., if* $h'(l) \leq 0$*). Then any manipulation satisfying conditions FOC and POS also satisfies condition SOC with strict inequality. Consequently, the firm has a unique optimal manipulation* a^* *fully characterized by conditions FOC and POS.*

The proof of this proposition shows that under the conditions on the adjustment requirement cost distribution specified in the proposition, conditions FOC and POS can pin down the unique optimal manipulation. This approach offers the way to derive the closed-form solutions given in the next section.

4. Closed-Form Solutions

I begin with the following lemma showing that several families of the adjustment requirement cost distribution can ensure the uniqueness of the optimal manipulation.

Lemma 1 (Distributions Ensuring Unique Optimal Misreporting Strategy): *Suppose that the adjustment requirement cost distribution belongs to the following families:*

(i) Weibull distribution with $\lambda > 0$ *and* $\theta > 0$*, i.e.,* $g(l) = \theta\lambda^\theta l^{\theta-1}\exp[-(\lambda l)^\theta]$ *with* $1 - G(l) = \exp[-(\lambda l)^\theta]$ *and* $h(l) = l^{1-\theta}/\theta\lambda^\theta$*, provided that* $\theta \geq ½$ *(including Exponential when* $\theta = 1$*);*

(ii) Gompertz distribution with $\lambda > 0$ *and* $\theta > 0$*, i.e.,* $g(l) = \lambda\theta\exp(\lambda l)\exp(-\theta[\exp(\lambda l) - 1])$ *with* $1 - G(l) = \exp(-\theta[\exp(\lambda l) - 1])$ *and* $h(l) = \exp(-\lambda l)/\lambda\theta$*;*

(iii) Pareto Type II distribution with $\lambda > 0$ *and* $\theta > 0$*, i.e.,* $g(l) = (\theta/\lambda)[\lambda/(\lambda + l)]^{\theta+1}$ *with* $1 - G(l) = [\lambda/(\lambda + l)]^\theta$ *and* $h(l) = (\lambda + l)/\theta$*, provided that* $\theta \geq ½$.

Then the firm has a unique optimal manipulation a^*.

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into such a complication.

I derive two closed-form solutions under the assumptions of an exponential and a linear misreporting cost function, respectively. This effectively provides two more, with the additional ones corresponding to hybid misreporting cost functions combined from the exponential and linear functions. Before elaborating on this further, let me first present the more specific form of condition FOC under the assumption of a Weibull adjustment requirement cost distribution with $\theta = ½$. This proves to be a useful assumption that greatly simplifies the first-order condition and allows the solution to be expressed in closed form.

Corollary 1 (First-Order Condition with Weibull Adjustment Requirement Cost Distribution): *Suppose the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$. *Let* $\eta = (k\lambda/2)^{½}$, *which is a parameter capturing the relative importance of the expected liability cost and adjustment requirement cost to the auditor. Moreover, let* $b = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2]$, $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$, *and* Φ *be the convex set of a defined by condition POS. Then*

(i) for $y > y_0$ *(i.e.,* $b < c_0$*), the optimal manipulation is downward (i.e.,* $a^* < 0$*) and uniquely determined by*

$$\exp(\alpha a)[\, c'(a) + \eta c(a) - (\eta/\alpha)b \,] = (1 - \eta/\alpha)b \qquad \textit{for } a \in \Phi;$$

(ii) for $y < y_0$ *(i.e.,* $b > c_0$*), the optimal manipulation is upward (i.e.,* $a^* > 0$*) and uniquely determined by*

$$\exp(\alpha a)[\, c'(a) - \eta c(a) + (\eta/\alpha)b \,] = (1 + \eta/\alpha)b \qquad \textit{for } a \in \Phi;$$

(iii) for $y = y_0$ *(i.e.,* $b = c_0$*), the optimal manipulation is zero manipulation (i.e.,* $a^* = 0$*), which solves both of the conditions above regardless of* η.

As mentioned, the cutoff point y_0 plays an important role in dividing the model into two separate regions. This is reflected in the two parts of the simplified condition FOC stated in the corollary above, with the dividing point at y_0 fitting either part. An important implication of this structure is that if two misreporting cost functions are identical on one side of the cutoff point, the optimal manipulation must be identical as well for that side. So when the closed-form solutions for an exponential and a linear misreporting cost function are derived, a mix-and-match of them yields the closed-form solutions for some hybrid misreporting cost functions. These functions are referred to as "left-linear, right-exponential" (LLRE) and "left-exponential, right-linear" (LERL) functions, with $a = 0$ that corresponds to $y = y_0$ as the dividing point:

(LLRE) $c(a) = c_0 a$ for $a < 0$; $c(a) = (c_0/\gamma)[\exp(\gamma a) - 1]$, with $\gamma > 0$, for $a \geq 0$

(LERL) $c(a) = (c_0/\gamma)[\exp(\gamma a) - 1]$, with $\gamma > 0$, for $a < 0$; $c(a) = c_0 a$ for $a \geq 0$.

Using the simplified condition FOC in Corollary 1, I am able to derive the closed-form solutions given in the next two propositions.

Proposition 3 (Solution with Exponential Misreporting Cost and Weibull Adjustment Requirement Cost Distribution): *Suppose the firm's cost of misreporting is exponential, i.e.,* $c(a) = (c_0/\gamma)[\exp(\gamma a) - 1]$ *with* $\gamma > 0$*, and the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$*. Let* $\eta = (k\lambda/2)^{½}$*. Moreover, let* $b = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2]$ *and* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$*. Then the optimal manipulation* a^* *is given as follows:*

(i) For $\eta = \alpha$*, if* $y \geq y_0$ *(i.e.,* $b \leq c_0$*),* $a^* = (1/\alpha)\ln(\ [(b/c_0) + (\alpha/\gamma)]/[1 + (\alpha/\gamma)]\ )$.

(ii) For $\eta = \gamma$*, if* $y \leq y_0$ *(i.e.,* $b \geq c_0$*),* $a^* = (1/\alpha)\ln(\ [1 + (\alpha/\gamma)]/[1 + (c_0/b)(\alpha/\gamma)]\ )$.

(iii) For $\alpha = \gamma$*, if* $y \geq y_0$ *(i.e.,* $b \leq c_0$*),*

$$a^* = (1/\alpha)\ln(\ [\ (1 + b/c_0) + \{(1 - b/c_0)^2 + 4(\alpha/\eta)^2(b/c_0)\}^{½}\ ] / 2(\alpha/\eta + 1)\ );$$

if $y \leq y_0$ *(i.e.,* $b \geq c_0$*),*

$$a^* = (1/\alpha)\ln(\ [\ \{(1 - b/c_0)^2 + 4(\alpha/\eta)^2(b/c_0)\}^{½} - (1 + b/c_0)\ ] / 2(\alpha/\eta - 1)\ )$$

for $\eta \neq \alpha$*, and* $a^* = (1/\alpha)\ln[2b/(c_0 + b)]$ *for* $\eta = \alpha$*.*

Parts (i) and (ii) of the proposition consider the optimal manipulation for specific cases with $\eta = \alpha$ or $\eta = \gamma$. Under these cases, the first-order conditions for the relevant regions take a linear form. Solving out a^* in closed form for those regions is thus trivial.

The closed-form solution given in part (iii), with $\alpha = \gamma$, comes from a quadratic equation that conveniently arises under this case for an exponential misreporting cost function assumed. The proof involves considering multiple cases to ensure that the optimal manipulation is well-defined even for certain parameter values that appear to result in an ill-defined first-order condition.

Proposition 4 (Solution with Linear Misreporting Cost and Weibull Adjustment Requirement Cost Distribution): *Suppose the firm's cost of misreporting is linear, i.e.,* $c(a) = c_0 a$, *and the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$. *Let* $\eta = (k\lambda/2)^{½}$. *Moreover, let* $b = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2]$ *and* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$. *Then the optimal manipulation* a^* *is given as follows:*

(i) For $y \geq y_0$ *(i.e.,* $b \leq c_0$*),*

$$a^* = (1/\alpha)\{ (b/c_0) - (\alpha/\eta) + W_0((\alpha/\eta - 1)\exp(\alpha/\eta)(b/c_0)\exp(-b/c_0)) \};$$

(ii) for $y \leq y_0$ *(i.e.,* $b \geq c_0$*),*

$$a^* = (1/\alpha)\{ (b/c_0) + (\alpha/\eta) + W_{-1}(-(\alpha/\eta + 1)\exp(-\alpha/\eta)(b/c_0)\exp(-b/c_0)) \},$$

where W_0 *and* W_{-1}, *with* $W_0 \geq -1 \geq W_{-1}$, *are the single-valued upper and lower segments of the real branch of the Lambert W function defined on the domains* $[-\exp(-1),\infty)$ *and* $[-\exp(-1),0]$, *respectively.*

The linear misreporting cost function considered in this proposition leads to a simple form of the first-order condition. It has a closed-form solution that can be expressed in terms of the Lambert W function. This special function is defined as the (multi-valued) inverse of the function $f(W) = W\exp(W)$. The real branch of the function has an upper and a lower (single-valued) segment, denoted by W_0 and W_{-1}, defined on the domains $[-\exp(-1),\infty)$ and $[-\exp(-1),0]$, respectively, with $W_0 \geq -1 \geq W_{-1}$. Figure 2 shows the shape of the function. The figure is adopted from Corless et al. (1996) that reviews the function and further develops its properties, making it more widely known than before. However, work related to the function dates back to Johann Lambert (1728-1777) and Leonhard Euler (1707-1783) (Brito, Fabiao, and Staubyn 2008).[9]

The shape of the Lambert W function gives a special touch to the optimal manipulation. It can result in a rather drastic change in the density around the cutoff point y_0 in the earnings triplet distributions. This provides a continuous alternative interpretation to the discontinuity in histogram documented in the literature (e.g., Burgstahler and Dichev 1997 and Degeorge et al. 1999). I will elaborate on this further in section 5 where two simulation exercises based on solutions involving the Lambert W function are discussed.

Figure 3a illustrates the closed-form solution of the optimal manipulation (a^*) in Proposition 4 based on the Lambert W function, alongside with the marginal expected benefit of manipulation (b), both as functions of the unmanaged earnings y. In figure 3b, the optimal manipulation is added to the unmanaged earnings to depict the pre-audit earnings $m = y + a^*$ as a function of the unmanaged earnings. The shaded areas in the figure indicate the optimal manipulations that constitute the pre-audit earnings.

I end this section with the following result that shows the one-to-one mapping from the pre-audit earnings m back into the unmanaged earnings y. Holding for the two (effectively four) closed-form solutions, this result provides a hope to derive conditions under which such a one-to-one inverse mapping exists generally. The challenge is left for future research.

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[9] The Lambert W function finds its role in many fields including mathematics (e.g., combinatorial number theory), computer science (e.g., algorithm and data structures), statistics (e.g., generalized skewed distributions and risk estimation), engineering (e.g., combustion, fuel consumption, and time-delayed systems), geology (e.g., earthquake forecasting), population ecology (e.g., Lotka-Volterra equations for population growth), health science (e.g., epidemic models), chemistry (e.g., enzyme kinetics), and especially physics (e.g., electrostatics, statistical mechanics, general relativity, inflationary cosmology, radiative transfer, and quantum chromodynamics) (Goerg 2011, Brito et al. 2008, Scott et al. 2006, and Corless et al. 1996). A decade ago, an "editorial in *Focus*, the newsletter of the Mathematical Association of America, asked: 'Time for a new elementary function?'," suggesting that the usefulness of the function in diverse fields qualifies it to be considered a candidate member of elementary functions, like the familiar sin, cosine, logarithm, exponential, etc (Hayes 2005).

Proposition 5 (Invertibility of Pre-audit Earnings to Unmanaged Earnings): *Suppose the distribution of the auditor's adjustment requirement cost is Weibull with* $\lambda > 0$ *and* $\theta = ½$. *Let* $\eta = (k\lambda/2)^{½}$ *and* $\mu = \exp(\ -\alpha(m - z) + \alpha^2\sigma_u^2 q/2\ )/c_0$. *Then the unmanaged earnings y are below (above)* $y_0 = z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$ *if and only if the pre-audit earnings* $m = y + a^*$ *are below (above)* y_0. *Moreover, the auditor can infer y from m using the relation* $y = Y(\mu)$, *with Y defined as follows:*

(i) If the firm's cost of misreporting is exponential with $\gamma = \alpha$, *i.e.,* $c(a) = (c_0/\alpha)[\exp(\alpha a) - 1]$,

$$Y(\mu) \equiv z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\ c_0[\mu + (\alpha/\eta - 1)\mu^2]/(\alpha/\eta + 1 - \mu)\ ] \qquad \text{for } m \geq y_0 \text{ (i.e., } \mu \leq 1)$$

$$Y(\mu) \equiv z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\ c_0[(\alpha/\eta + 1)\mu^2 - \mu]/(\alpha/\eta - 1 + \mu)\ ] \qquad \text{for } m \leq y_0 \text{ (i.e., } \mu \geq 1);$$

(ii) if the firm's cost of misreporting is linear, i.e., $c(a) = c_0 a$,

$$Y(\mu) = z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\ -c_0 W_0(\ -\exp(-\alpha/\eta)\mu\exp[(\alpha/\eta - 1)\mu]\ )\ ] \text{ for } m \geq y_0 \text{ (i.e., } \mu \leq 1)$$

$$Y(\mu) = z + \alpha\sigma_u^2 q/2 - (1/\alpha)\ln[\ -c_0 W_{-1}(\ -\exp(\alpha/\eta)\mu\exp[-(\alpha/\eta + 1)\mu]\ )\ ] \text{ for } m \leq y_0 \text{ (i.e., } \mu \geq 1).$$

The unmanaged earnings $y = Y(\mu)$ *are continuous and strictly increasing in the pre-audit earnings* $m \in (-\infty,\infty)$.

In the next section, I present the results of two simulation exercises based on the model. The analysis helps to assess the limit and the potential of the model as a framework for guiding empirical research.

5. Empirical Contents of the Model

Before presenting the simulation results in subsections 5.2 and 5.3, I first briefly explain why the model should be capable of accommodating the two salient features of the earnings triplet distributions.

5.1 Why the Model Can Accommodate the Puzzling Features

To elucidate how the model can lead to a phenomenon similar to the discontinuity documented in the literature, I plot the distribution of unmanaged earnings, *y*, in figure 4a and superimpose on it the distribution of pre-audit earnings, m. In the figure, the unmanaged earnings are assumed to follow a normal distribution. Only the left side of the density line (in gray) appears in the plot given the scale chosen. The earnings benchmark z is marked by the vertical solid line (in black). It is assumed to be 0.02, rather than zero, to highlight that the model does not require the earnings benchmark to be equal to zero.

The vertical dash line (in green) indicates the location of the cutoff point y_0 (assumed to be 0.06), which divides the two directions of earnings manipulation. Unmanaged earnings below y_0 are manipulated upward, pushing the density line to the right toward the cutoff point. Similarly, the density line on the right of y_0 is pushed to the left toward the point. Mathematically, this one-to-one remapping of the density of unmanaged earnings into the "density" of pre-audit earnings is straightforward and represented by the thin solid curve (in black). However, the area under this curve need not add up to one. To ensure that the remapped density meets the "sum to one" requirement of probability theory, the thin solid curve is multiplied by a normalizing factor to obtain the density line of pre-audit earnings. This is represented by the thick solid curve (in blue).

Figure 4a assumes that the firm's misreporting cost function is linear. Consequently, the optimal manipulation can be solved in terms of the Lambert W function. The shape of this function leads to a particularly steep slope around y_0. A zoom-in view of that part of the density lines (surrounded by red dotted rectangles) is provided on the left side of the figure.

Imagine that the probability distribution of pre-audit earnings described by the model is sampled, with the distribution of observations plotted in histogram form. Because of the steep slope around y_0, it is likely to see a sharp difference between the frequencies observed in the immediate left and right bins next to the cutoff point. A "discontinuity" similar to those documented in the literature can thus arise in the continuous model of this paper. Several issues however should be kept in mind before concluding that the model can accommodate the discontinuity phenomenon.

First, the documented discontinuity occurs at zero in the distributions of earnings, earnings change, and earnings surprise. For the model to predict the phenomenon, y_0 must be close to the earnings benchmark in concern, be it the profit/loss cutoff, earnings increase/decrease, or beating/missing an earnings forecast. This proximity between y_0 and z is assumed in figure 4a and in the simulation exercises. Empirically, it is the alignment of the predicted location of the "discontinuity" with the actual location observed from the data that gives the identifiability of the underlying parameters y_0 and z. I discuss the estimation of the model parameters in appendix A.

Second, figure 4a only shows a "discontinuity" in the distribution of pre-audit earnings, which are not exactly the same as the reported earnings in audited financial statements. I therefore conduct the two simulation exercises to fill the gap between these earnings concepts.

Third, the documented discontinuity is not about the distribution of earnings from the same firm observed multiple times. Instead, the earnings concerned in the literature come from different firms. Thus, they are likely to be drawn from different distributions, rather than from a single distribution assumed in figure 4a. The simulation exercises make an attempt to address this issue.

Figure 4b illustrates that even when the unmanaged earnings are distributed normally, the distribution of pre-audit earnings may have a volcano shape with the peak sharper than that of a bell-shaped normal distribution. This may occur if y_0 is located around the peak of the unmanaged earnings distribution, with the left and right sides of the density line pushed toward the peak strongly. For example, imagine that the expected liability cost is low and the adjustment requirement cost tends to be high. So the auditor is reluctant to require an adjustment. The firm will have a strong incentive to manipulate earnings, resulting in a big push of the density line toward the middle and hence a volcano-shaped distribution of pre-audit earnings.

Before presenting the simulation results in the next two subsections, let me first introduce some terminology. Recall that the audit will remove some of the unintentional errors, namely ε_q, before the firm publicly announces the post-audit earnings $r = xy + (1 - x)m - \varepsilon_q$. Without an auditor-required adjustment (i.e., $x = 0$), r is simply the pre-audit earnings m corrected for the discovered unintentional errors. If an adjustment is required (i.e., $x = 1$), r is the unmanaged earnings y corrected for the discovered unintentional errors.

The first simulation exercise focuses on the distribution of excess earnings, defined as the part of earnings exceeding the earnings benchmark z. (Negative excess earnings mean the part of earnings falling short of the benchmark.) Based on this definition, the *post-audit excess earnings* are

$$\delta r = [xy + (1 - x)m - \varepsilon_q] - z. \tag{7}$$

Similarly, the pre-audit and unmanaged excess earnings are $m - z$ and $y - z$, respectively. If the analyst consensus forecast is taken as the earnings benchmark, post-audit excess earnings coincide with the concept of earnings surprise.

To simulate the distribution of earnings change, I consider a simple repetition of the model for two periods. This yields some interesting results. However, one needs to bear in mind a caveat. Repeating the model for multiple periods does not give a truly dynamic model. In a dynamic model, care must be taken to consider the accumulation of past earnings manipulations in the total assets (see, e.g., Barton & Simko 2002 and Baber et al. 2011), which might enhance the auditor's incentive to require an adjustment in the future. Owing to the limited space here, the analysis of a truly dynamic version of the model is left for future research.

Let r_1 denote the *lagged post-audit earnings*, i.e., the post-audit earnings in the earlier period of a two-period repetition of the model. The second simulation exercise assumes that in the current period of the two-period repetition some firms use the profit/loss cutoff as the earnings benchmark (i.e., $z = 0$) while others use the earnings increase/decrease cutoff as the benchmark (i.e., $z = r_1$). Because of this diversity in the benchmarks assumed, a discontinuity can occur in the distributions of earnings and earnings change/difference simultaneously. The *post-audit earnings change* is defined as

$$\Delta r = [xy + (1 - x)m - \varepsilon_q] - r_1. \tag{8}$$

Similarly, the pre-audit and unmanaged earnings differences are defined as $m - r_1$ and $y - r_1$, respectively. For firms using lagged post-audit earnings as the benchmark, the post-audit earnings change is simply their excess earnings. However, for firms using zero earnings as the benchmark, the post-audit earnings change is not the same as excess earnings but the post-audit earnings are.

5.2 Simulating a Volcano-shaped Distribution of Excess Earnings with a Sharp Peak

Figure 5a visualizes the results of the first simulation with $y_0 - z$ = \$0.011 and a linear misreporting cost function. The simulation assumes a population of firms each with possibly a different distribution of unmanaged earnings (per share in cents). For simplicity, the distributions are all normal with the same standard deviation but possibly different means. The means are themselves drawn from a normal distribution.

I am interested in simulating the situation where earnings forecasts taken as the earnings benchmarks are equal to the means of the unmanaged earnings distribution. The purpose is to see how far the distribution of the simulated post-audit excess earnings can get close to its counterpart reported in figure 2 of Bhojraj et al. (2009), which is included in figure 5b for ease of reference. Because this simulation is meant to be a first look at the empirical contents of the model, I do not consider the more complicated situation with the forecasts set to the post-audit earnings of an earlier period. The second simulation presented later will include such dynamic considerations.

In figure 5a, the distributions of the pre-audit, post-audit, and unmanaged excess earnings are plotted in histogram form, with the latter overlaid on the former one after another. The bin width of the histogram is \$0.01, to be consistent with the choice in Bhojraj et al. (2009). To provide a better angle in viewing the three distributions, a three-dimensional plot of the distributions is given in figure 5c. The shape of the Lambert W function that characterizes the optimal manipulation behind this simulation induces a huge spike in the frequency distribution of the pre-audit excess earnings (in yellow) in the back. This starkly differs from the nearly flat distribution of the unmanaged excess earnings (in gray) in the front. The stochastic nature of the adjustment decision, together with the correction for the discovered unintentional errors, mixes the two publicly unobservable distributions into the distribution of the post-audit excess earnings (in blue) in the middle, which is observable to the public.

The especially sharp peak of the distribution of earnings surprise is a feature easily noticed in related studies (e.g., figure 6 of Degeorge et al. 1999 and figure 2 of Frankel et al. 2010). The simulation illustrates how the model turns a rather flat distribution of unmanaged excess earnings (originating from normally distributed unmanaged earnings) into a dramatically different distribution of post-audit excess earnings. The assumption of $y_0 - z = \$0.011$ ensures that the sharp peak of the distribution occurs in the two right bins next to zero excess earnings. The solid stair-step line (in blue) in figure 5a outlines the distribution of the post-audit excess earnings partially hidden behind the distribution of unmanaged excess earnings in the front. The overall shape of the post-audit excess earnings distributions looks quite similar to its counterpart in figure 2 of Bhojraj et al. 2009, which however is fatter at the mid-level and has thinner tails. Given the simple structure of the model, compared to the complex reality it tries to approximate, such mismatches seem unsurprising.

The following hypothesis summarizes the key insight from the first simulation.

The Mixture Hypothesis: *The volcano-shaped distribution of earning surprise with a sharp peak documented in the literature is due to a mixture of a relatively flat distribution of unmanaged excess earnings with a spiky distribution of pre-audit excess earnings.*

Although cookie-jar earnings management alone can give rise to a volcano-shaped distribution, recognizing the auditor's role in the model is necessary to the discovery of the mixture hypothesis. Most important, the auditor's role cannot be ignored, or the empirical content of the model would be adversely impacted. Ignoring the role is equivalent to imposing the a priori restriction that the expected adjustment requirement cost is infinitely large (so that any misreporting attempt will be tolerated). This will introduce a bias in the estimated coefficients when the model parameters are structurally estimated. Whether the expected adjustment requirement cost is large or not is an empirical question. However, what appears to be certain, even without any estimation, is that the cost should be neither infinitely large nor negligibly small.

5.3 Simulating a 'Discontinuity' in the Distributions of Earnings and Earnings Change

Throughout the second simulation, I assume $y_0 - z$ = \$0.015 and a "left-exponential, right-linear" (LERL) misreporting cost function (see section 4 for the definition). Assuming a linear misreporting cost function as in the first simulation would not change the results critically. However, the overall shapes of the simulated distributions would look less similar to their counterparts based on actual data.

The first simulation directly assumes a distribution of unmanaged earnings per share (in cents). In contrast, the second simulation uses a more complicated procedure to simulate the distribution of unmanaged earnings based on the actual data of total assets in 1988-2006, assuming a relation between the unmanaged earnings and total assets. The distribution of unmanaged earnings per share is then computed using the actual shares outstanding data associated with the total assets data. I use this unmanaged earnings distribution as the "seed" for simulating the distribution of post-audit earnings for an earlier period, assuming zero earnings as the benchmark for manipulation. The earnings increase/decrease cutoff is not a choice at this point because there are no lagged post-audit earnings yet.

After obtaining the post-audit earnings for the earlier period, they become the lagged post-audit earnings for the current period of the simulation. The beginning value of the total assets for the current period is updated from that of the earlier period using the simulated post-audit earnings, assuming a 75% payout ratio. The updated total assets are used as a basis to simulate the current period's unmanaged earnings, assuming a relation augmented by some "natural growth."

There are three main sources for such "natural growth." First, it can come from technological advancement that improves the productivity of any given asset base. Second, it can arise from an expansion of the asset base due to new investment opportunities discovered. (But this is not captured in the simple clean-surplus updating of the total assets assumed above.) Third, the earnings concerned in the literature are in nominal value. Inflation can contribute to the "natural growth" of earnings even when the productivity is fixed and the asset base is constant.

Allowing for some "natural growth" is important. Otherwise, the peak of the distribution of the simulated unmanaged earnings difference would not be so far to the right as in figure 7a. Consequently, the peak of the distribution of the simulated post-audit earnings change would be much sharper, unlike its counterpart in figure 7b based on actual earnings data in 1988-2006. It is important to keep in mind that the model of this paper does not explain the shape of the (unobservable) unmanaged earnings distributions. It only explains how the distribution may be transformed into the (observable) post-audit earnings with a distinctly different look. Therefore, the purpose of assuming some "natural growth" is to come up with something reasonably close to the reality and let the model explain the remaining difference, which otherwise is perplexing.

With the unmanaged earnings for the current period simulated, the model then converts them into pre-audit and post-audit earnings. Unlike in the first simulation, now firms can differ in the earnings benchmarks assumed: some use $z = 0$, while others use $z = r_1$. For simplicity, I assign these two benchmarks randomly with 40% of the chance setting $z = 0$ and 60% setting $z = r_1$. The distributions of the pre-audit, post-audit, and unmanaged earnings are plotted in figure 6a, with the latter overlaid on the former one after another. Again, the bin width of the histogram is $0.01. I use small solid circles (in blue) to outline the distribution of the post-audit earnings partially hidden behind the distribution of unmanaged earnings (in gray) in the front. Whenever the frequency of the post-audit earnings exceeds that of the unmanaged earnings, the exceeding part can be clearly seen as a blue bar segment on top of a gray histogram bar. The key difference between the distributions of the post-audit and unmanaged earnings is the higher frequencies in the several right bins next to zero earnings. A close-up view of that part (surrounded by red dotted rectangles) is provided on the left side of the figure. The distribution of the actual earnings in 1998-2006 is given in figure 6b for comparison.

Figure 7a shows the distributions of the pre-audit earnings difference (in yellow) in the back, the post-audit earnings change (in blue) in the middle, and the unmanaged earnings difference (in gray) in the front. Again, small solid circles (in blue) are used to outline the distribution of the post-audit earnings change partially hidden behind the distribution of the unmanaged earnings difference. Note that 40% of the simulated observations in the figure use $z = 0$ (rather than $z = r_1$) as the benchmark for earnings manipulation. Still the remaining 60% are sufficient to induce the noticeably higher frequencies in the several right bins next to zero earnings change indicated by the solid vertical line (in black). The left side of the figure provides a zoom-in view for that part of the post-audit earnings change distribution (surrounded by red dotted rectangles).

The key insight from the second simulation is summarized as the following hypothesis:

The Continuous 'Discontinuity' Hypothesis: *The simultaneous existence of a "discontinuity" in the distributions of earnings and earnings change documented in the literature is due to a continuous but drastic increase in the density of the distributions around the respective* *earnings benchmarks.*

6. Discussions and Concluding Remarks

6.1 Why is the Volcano Shape a Puzzle?

To my knowledge, there is no known reason suggesting that the earnings triplet distributions should have a volcano shape. Below I draw on the insights from the literature on stock return distributions to argue that there are reasons to believe that the earnings distribution may have a normal-like shape. It is important to recognize that the model in this paper does not impose any normality requirement on the unmanaged earnings distribution. The model simply transforms whatever smooth, single-peaked distribution of unmanaged earnings into a reported earnings distribution with a shaper peak.

Similar to the classical model of return distributions (Osborne 1959 and Fama 1965),[10] it seems reasonable to assume that earnings from transaction to transaction in a firm approximately are independent, identically distributed random variables with a finite variance. This can occur, for example, when the price of a product is marked up from its cost by a fixed percentage and the quantity purchased by a customer and that of the next are independent, identically distributed random variables with a finite variance. With numerous transactions accumulated in a quarter or a year, the central limit theorem ensures that the quarterly or annual earnings of a firm should have a normal distribution.[11] This holds regardless of the distributional form of the earnings from an

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[10] Another model of return distributions assumes that the returns from individual transactions follow a distribution in the *stable paretian* class, with α–characteristic < 2 (Mandelbrot 1963). This *stable* model fits the behavior of daily stock returns better, compared to the classic model. Blattberg and Gonedes (1974), however, show that their Student model fits the daily returns data even better. Focusing on monthly, quarterly, and annual returns, Upton and Shannon (1979) reject the stable model in favor of their lognormality model. Lau et al. (1990) provide additional evidence further rejecting the empirical validity of the stable model.

[11] It should be emphasized that although the i.i.d. condition is invoked here for the sake of easy understanding by typical readers, recent development in probability theory has proved that exact independence is not necessary for achieving some notion of normality, unlike the classical central limit theorem. See Chen, Goldstein, and Shao (2011), Barbour and Xia (2006), Chen and Shao (2004), Romano and Wolf (2000), Szeidl and Zolotarev (1998), de Jong (1997), Zolotarev (1997), Hall and Heyde (1980), and Ibragimov (1975).

individual transaction. So it is a very powerful argument. In the context of stock returns, even though daily returns are leptokurtic, Fama (1976) finds that "monthly returns are close enough to normal for the normal model to be a good working approximation." Assuming a Student distribution to capture the fat tails of stock returns, Blattberg and Gonedes (1974) also find that for monthly returns the estimated distribution is "almost indistinguishable from a normal distribution."

Given that the annual or quarterly earnings of a firm should be normally distributed, the earnings change is the difference between two normally distributed variates. Thus, the change is also normally distributed. If the earnings forecast is largely based on the earnings of a previous period, the earnings surprise is like an earnings change and should follow a distribution close to normal. Alternatively, if the earnings forecast is a close enough prediction of the mean of the earnings, the earnings surprise should be normally distributed too. Even though the earnings triplet distributions might not be exactly normal, there seems to be no obvious reason why they should have a volcano shape with a sharper peak, instead of a rounder top in bell-shaped distributions such as logistic, Student, and normal.

The arguments for expecting a normal distribution have been so far based on the earnings of an individual firm. The earnings triplet distributions discussed in the literature, however, are constructed using a cross section of firms in a period, whose earnings might not be identically distributed. By imposing more structure on the earnings from different firms, it is not unreasonable to expect a distribution close to normal. One such structure was considered by Praetz (1972) and Blattberg and Gonedes (1974) in the context of stock returns. They prove that if the variance of otherwise identically normally distributed stock returns follows an inverted gamma distribution, the posterior distribution of the returns is Student. Analogously, the earnings from a cross section of firms can follow a Student distribution if the earnings of each firm are identically normally distributed with the variance inverted-gamma distributed. Moreover, if the value of the degrees of freedom parameter is high (e.g., over 25), the distribution can be close to normal. Again, even when normality is not a good approximation, there has been no theory suggesting a volcano-shaped distribution. This shape goes against intuition and is perplexing.

Some authors argue that normalizing earnings by market value or other size deflators contributes to the discontinuity phenomenon (Durtschi and Easton 2005). One might suspect that the volcano shape could be caused by deflation as well. It should be stressed that the volcano shape, or the discontinuity, appears not only in studies examining scaled earnings (e.g., Burgstahler and Dichev 1997) but also in others focusing on unscaled earnings per share (e.g., Degeorge et al. 1999 and Bhojraj et al. 2009). Hence, normalization by size deflators cannot explain the volcano-shape phenomenon. Providing a unified framework to understand this puzzle as well as the discontinuity phenomenon therefore is interesting.

6.2 Shape Peak versus Fat Tails

The earnings compression explanation of the volcano-shape feature suggests that the feature is closer to the statistical concept of peakedness than tailweight (see Birnbaum 1948 and Groeneveld and Meeden 1984 for formal definitions of *more peaked* and *heavier-tailed*). The feature is not the same as the *leptokurtosis* found in the empirical distributions of daily stock returns (Fama 1976, p. 24).

Kurtosis is often defined as the fourth central moment normalized by the squared variance. It is an inverse measure for the concentration of probability density around the “shoulders” of the distribution, i.e., the two points at one standard deviation above and below the mean (Moors 1986). If kurtosis is high, i.e., greater than 3, the distribution is said to be *leptokurtic*; a normal distribution has a kurtosis equal to 3 (Abell, Braselton, and Rafter 1999). High kurtosis (or leptokurtosis) can be due to a concentration of probability density near the mean, or in the tails, or both, relative to a normal distribution. So there is no logical connection between leptokurtosis and the high concentration of density near the mean compared to a normal distribution with the same mean and variance (Balanda and MacGillivray 1988 and Kaplansky 1945).

Finance professionals and researchers' interest in the leptokurtic behavior of asset returns is often due to the importance of extreme rare events in risk management (e.g., Sheikh and Qiao 2009 and Lucas 2000). Therefore, leptokurtosis in finance tends to mean fat tails (e.g., Blattberg and Gonedes 1974 and Kon 1984). This feature is crucially different from the sharper peak emphasized with the volcano shape description of a distribution.

6.3 Concluding Remarks

The model of this paper can accommodate the two salient features of the earnings triplet distributions, despite the simple representation of the auditor-client negotiation process as a "ban-or-tolerate" game. Aside from some mismatches that seem unsurprising given the simple model structure, the simulation results demonstrate the model's potential to capture the driving forces behind the two salient features.

Two caveats for the simulation results are worth noting. In the second simulation, I assume for simplicity that firms stochastically choose the profit/loss or the earnings increase/decrease cutoffs as the benchmarks for manipulation. In unreported analysis, I also consider a behavioral model of benchmark selection based on the unmanaged earnings' proximity to different benchmarks. The results are broadly similar. Because constructing such a selection model is not the focus of this paper, it is not reported here. However, endogenizing the benchmark selection is an interesting extension of the model.

The second simulation also tries to be dynamic. But the model here is only "pseudo-dynamic," i.e., it uses an exogenously specified misreporting cost function to represent in reduced form any future benefits and costs of the current-period manipulation. A truly dynamic version of the model requires carefully considering the connection between periods through a misreporting cost function endogenously determined in equilibrium.

Appendix A discusses how the parameters of the model can be estimated structurally. Alternatively, methodologies such as the geophysical inverse theory and nonparametric mixture of regression models can be borrowed from geophysics and statistics to uncover the underlying unmanaged earnings using the theoretical framework developed here (Ganse 2008 and Huang, Li, and Wang 2013).

Besides providing a single framework to understand two puzzling phenomena, the model also sheds light on an interesting finding of Dichev and Tang (2009): earnings with low volatility have remarkably high persistence. Ignoring any earnings management consideration, they simply see "earnings volatility as arising from two factors, volatility due to economic shocks and volatility due to problems in the accounting determination of income, and both of these factors reduce the predictability of earnings." This paper provides a different perspective to understand their finding. Suppose many firms use lagged earnings with a constant growth rate as their earnings benchmarks. Cookie-jar earnings management smoothens earnings over time for the purpose of not deviating too much from the earnings benchmark. This can result in earnings being more persistent – in the sense that the reported earnings always follow closely the path of the lagged earnings with a constant growth rate. Connecting persistence to volatility (variance) is an incomplete story, if viewed through the model here. A tighter connection is with the shape of the distribution of earnings, which are compressed by cook-jar earnings management and hence have a lower variance. An empirical research design to measure the connection to the shape would provide evidence on whether cookie-jar earnings management is the more complete story.

There are other potential applications of the model. Besides the policy analysis application already mentioned in the introduction, I will discuss two more here. The first is to use the model as a framework for examining the effectiveness of certain corporate governance mechanisms in curbing earnings manipulation. For such mechanisms to be useful, they must change the model parameters related to the benefit and cost of misreporting (e.g., c_0). By including corporate governance measures as explanatory variables in the estimation procedures, one can test whether the relevant parameters are sensitive to these variables. Conclusions can then be drawn on the effectiveness of the variables in discouraging earnings manipulation.

Besides viewing the model from a positive perspective, it may be used as a framework for developing decision aids to help auditors improve adjustment decisions. In the model, the auditor is assumed to know all the parameters without doubt. In reality, this seems unlikely. Instead, auditors might have a subconscious assessment of the factors corresponding to the model parameters and make decisions based on "professional judgment." Less experienced auditors might misjudge and make mistakes against their own interest. With the decision made in a judgmental manner, even experienced auditors might make occasional mistakes due to subconscious psychological biases. Systematically collecting information useful for estimating the model parameters and developing a formal decision aid based on the model can raise the awareness of the auditors making adjustment decisions. By highlighting the strategic considerations in the model, the auditors can carefully balance such considerations with other unmodeled factors. The decision quality can thus be improved.

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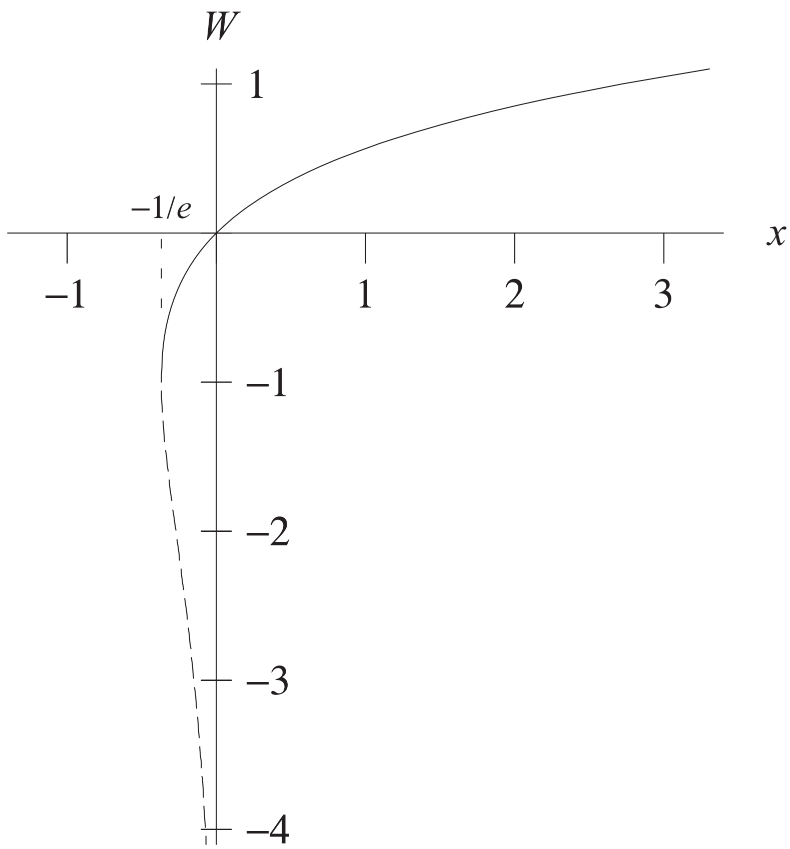
Table 1: Notations

y	$\in$	$(-\infty,\infty)$ is the unmanaged earnings of the firm
z	$\in$	$(-\infty,\infty)$ is the earnings benchmark affecting the firm's incentive to manipulate earnings
a	$\in$	$(-\infty,\infty)$ is the earnings manipulation chosen by the firm
m	$=$	$y + a$ is the pre-audit earnings (or managed earnings) provided to the auditor for audit.
q	$\in$	$[0,1]$ is the quality level of the audit.
ε_q	$\sim$	Normal$(0, q\sigma_u^2)$, with $\sigma_u > 0$, is the part of the unintentional errors contained in y that is discovered and removed by the auditor.
ε_{1-q}	$\sim$	Normal$(0, (1-q)\sigma_u^2)$ is the part of the unintentional errors remaining in y even after the audit.
L	$=$	$ka^2/2$, where $k > 0$, is the expected liability cost to the auditor arising from tolerating the earnings manipulation.
x	$\in$	$\{0,1\}$ is the auditor's adjustment decision.
X	$\in$	$[0,\infty)$ is the auditor's cost of requiring an adjustment, which follows a probability distribution $G(l) = \Pr\{X \le l\}$, with a differentiable probability density $g(l) = G'(l) > 0$ for all $l > 0$ and a differentiable hazard rate function $h(l) = [1-G(l)]/g(l)$. The existence of bounded limits $\lim_{l\downarrow 0} l^{1/2}g(l)$ and $\lim_{l\downarrow 0} [g(l) + 2lg'(l)]$ is assumed.
$x^*(a)$	$=$	$\mathbf{1}\{X \le L\}$ is the optimal adjustment decision
$c(a)$	$=$	the firm's misreporting cost, where $c(0) = 0$, $\lim_{a\uparrow\infty} c(a) = \infty$, $c'(a) > 0$ with $c'(0) = c_0 < \infty$, and $c''(a) \ge 0$.
b	$=$	$\mathrm{E}[\exp(-\alpha[y - z - \varepsilon_q])] = \exp[-\alpha(y - z) + \alpha^2\sigma_u^2 q/2]$ is the "marginal expected benefit of manipulation" that summarizes the impacts of the quality parameter q and the deviation of y from z on the firm's misreporting incentive.
v	$=$	$(1 - b)/\alpha + [1 - G(L)][\, b[1 - \exp(-\alpha a)]/\alpha - c(a)\,]$, with $\alpha > 0$, is the firm's expected net benefit from earnings manipulation.
y_0	$=$	$z - (\ln c_0)/\alpha + \alpha\sigma_u^2 q/2$ is the cutoff of the unmanaged earnings that determines whether the optimal manipulation is upward or downward.
λ	$>$	0 is the reciprocal of the scale parameter of a Weibull adjustment requirement cost distribution.
θ	$=$	½ is the shape parameter of a Weibull adjustment requirement cost distribution that simplifies the first-order condition for the optimal manipulation, allowing the solution to be expressed in closed form.
η	$=$	$(k\lambda/2)^{1/2}$ is a parameter capturing the relative importance of the expected liability cost and adjustment requirement cost to the auditor.
W	$=$	the Lambert W function, which is the (multi-valued) inverse of the function $f(W) = W\exp(W)$. The real branch of the function has an upper and a lower (single-valued) segment defined on the domains $[-\exp(-1),\infty)$ and $[-\exp(-1),0]$, respectively. The two segments are denoted by W_0 and W_{-1}, respectively, with $W_0 \ge -1 \ge W_{-1}$.
μ	$=$	$\exp(\, -\alpha(m - z) + \alpha^2\sigma_u^2 q/2\,)/c_0$ is the counterpart of b for $Y(\mu)$, the inverse mapping from m back to y.
r	$=$	$xy + (1 - x)m - \varepsilon_q$ is the post-audit earnings (or reported earnings) announced to the public.
δr	$=$	$[xy + (1 - x)m - \varepsilon_q] - z$ is the post-audit excess earnings.
r_1	$=$	the lagged post-audit earnings.
Δr	$=$	$[xy + (1 - x)m - \varepsilon_q] - r_1$ is the post-audit earnings change.

Figure 1. Timeline of the model

1. The auditor explains the audit plan to the firm, letting it know the quality level q of the audit.
2. The firm assumes an earnings benchmark z (e.g., analyst consensus forecast), which is common knowledge in the model.
3. The firm learns the unmanaged earnings y, chooses the manipulation a, and provides the pre-audit earnings $m = y + a$ to the auditor for audit.
4. The auditor conducts the audit and removes the discovered part ε_q of the unintentional errors in y, with the undiscovered part ε_{1-q} remaining in y.
5. After the audit, the auditor knows the components y and a of m.
6. The auditor decides whether to incur a cost X, privately known to him, in order to require an adjustment ($x = 1$) to remove the manipulation a, or not ($x = 0$).
7. The firm announces to the public the post-audit earnings $r = xy + (1 - x)m - \varepsilon_q = y + (1 - x)a - \varepsilon_q$.

Figure 2. The two real branches of the Lambert W function
(Source: Figure 1 of Corless et al 1996)



The two real branches of $W(x)$. ——, $W_0(x)$; − − −, $W_{-1}(x)$.

Figure 3a. Optimal manipulation and marginal expected benefit of manipulation as functions of unmanaged earnings

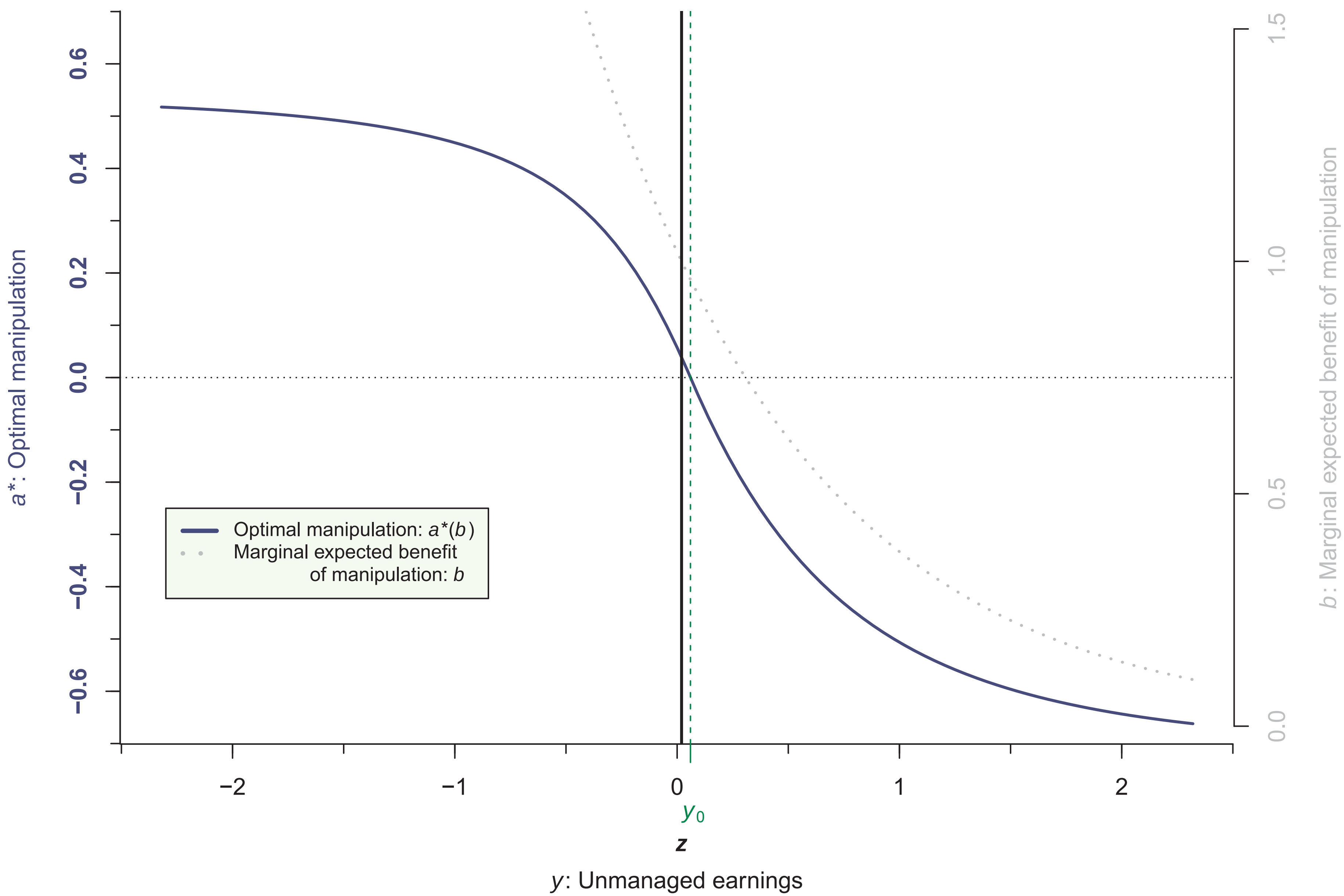


Figure 3b. Pre−audit earnings as a function of unmanaged earnings

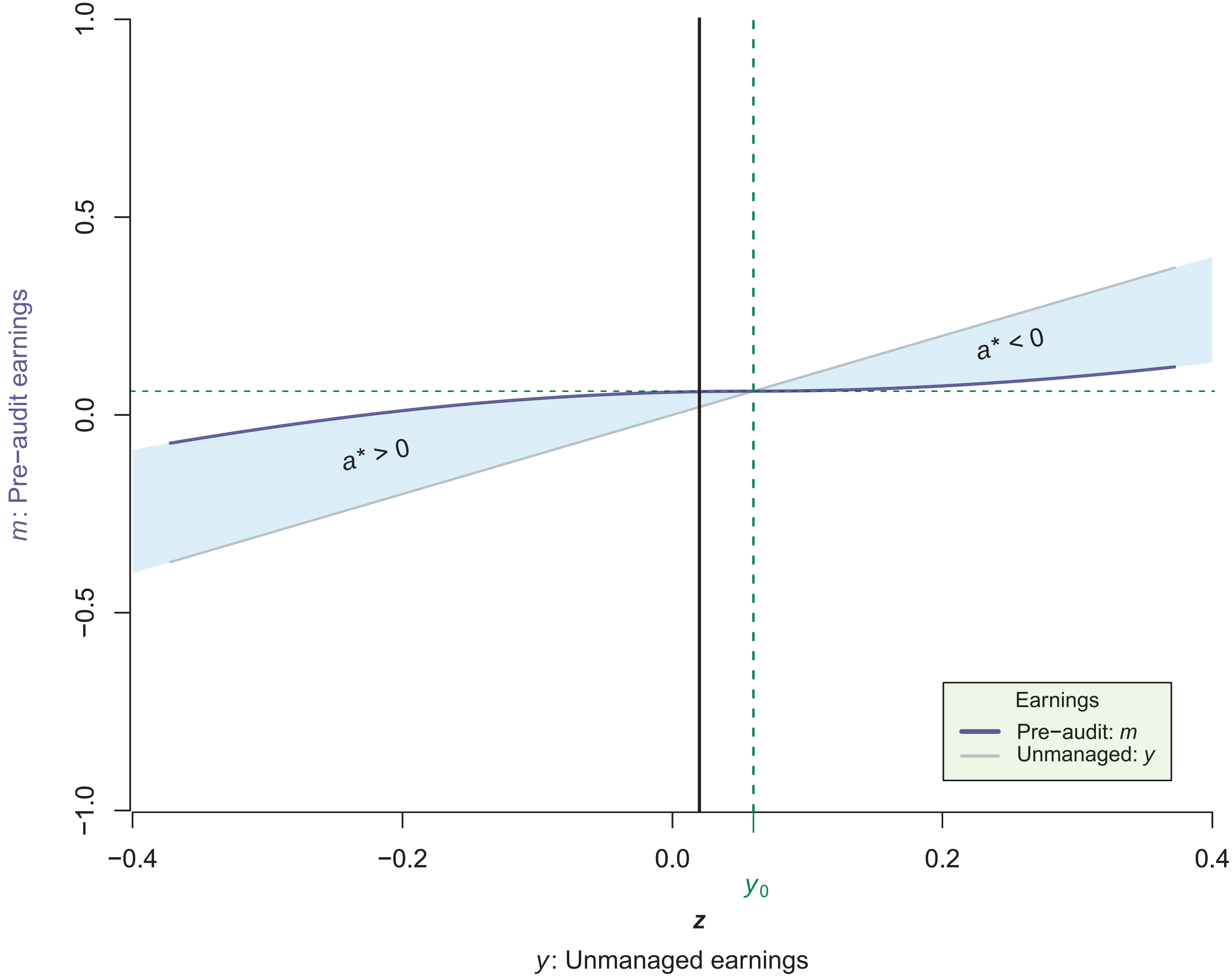


Figure 4a. Distributions of pre−audit and unmanaged earnings

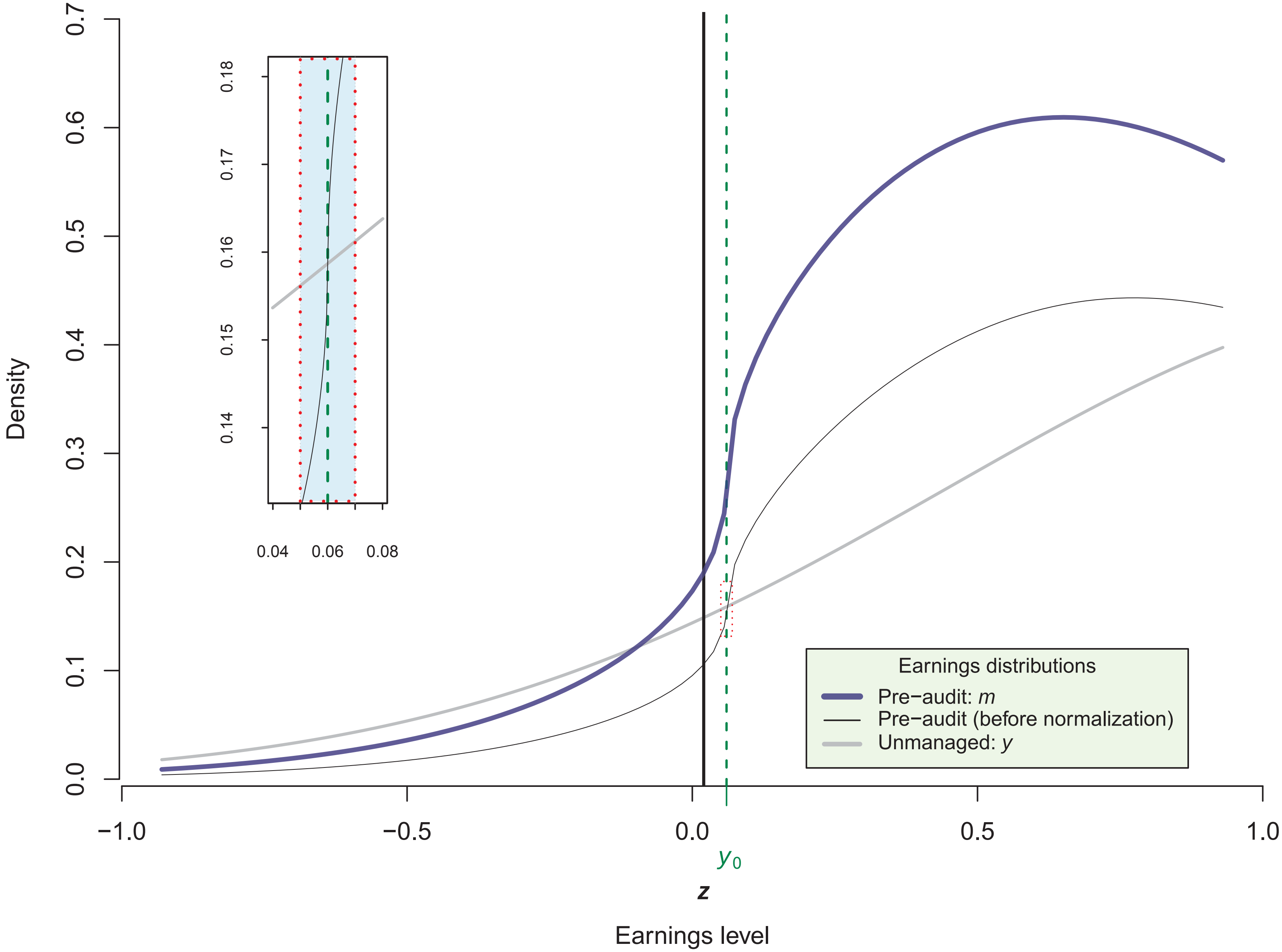


Figure 4b. Normal distribution of unmanaged earnings transformed into volcano−shaped distribution of pre−audit earnings

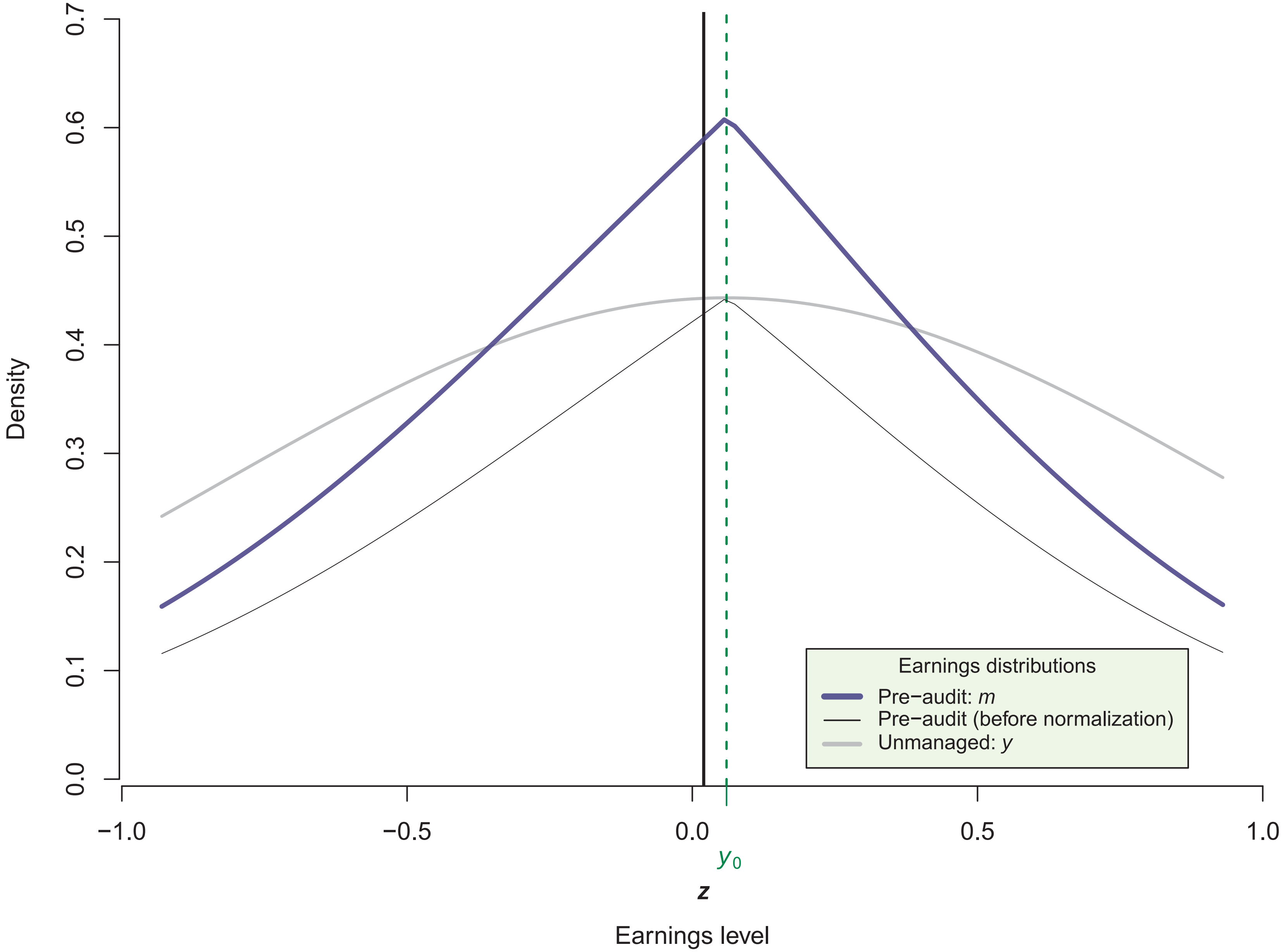


Figure 5a. Frequency distributions of simulated excess earnings

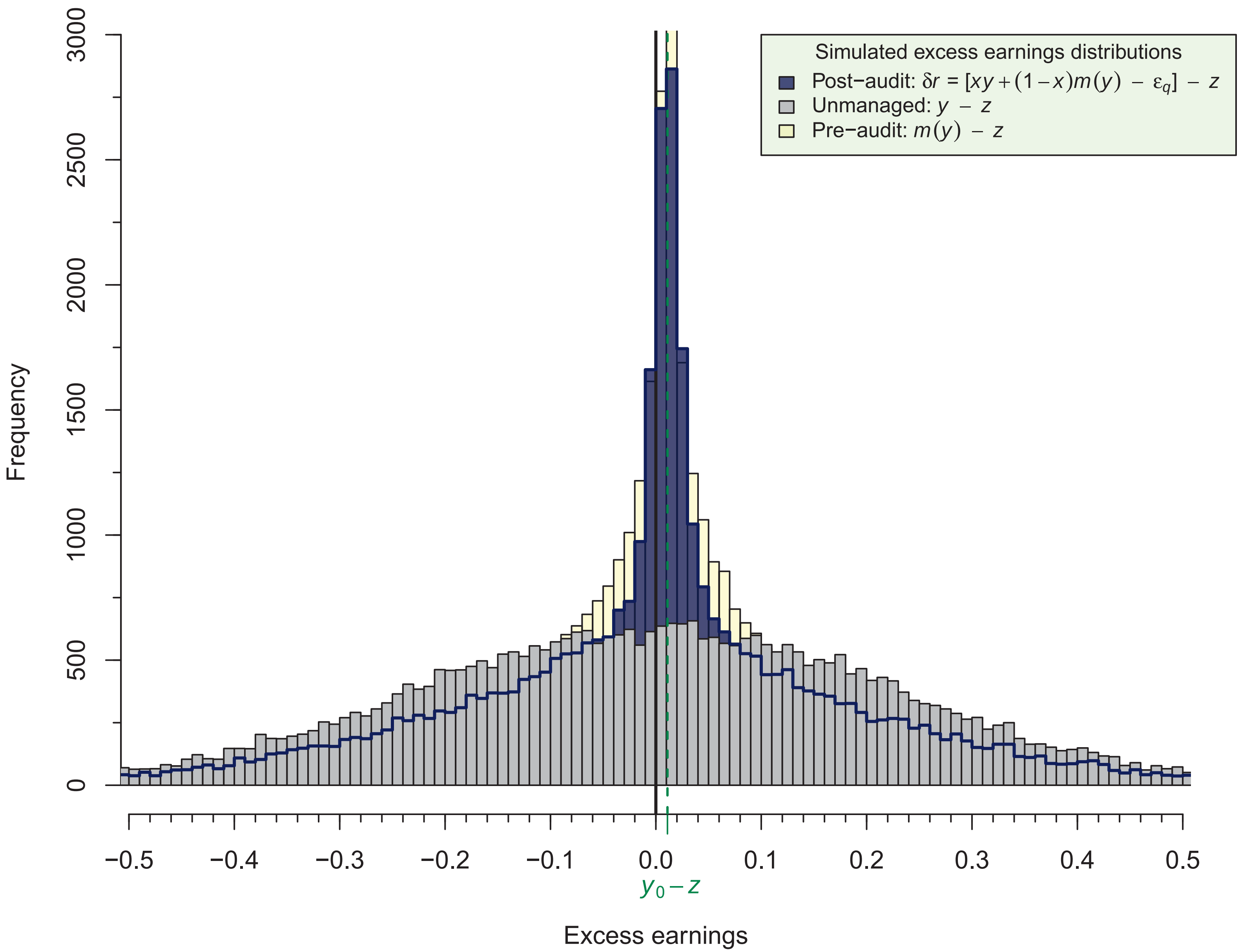
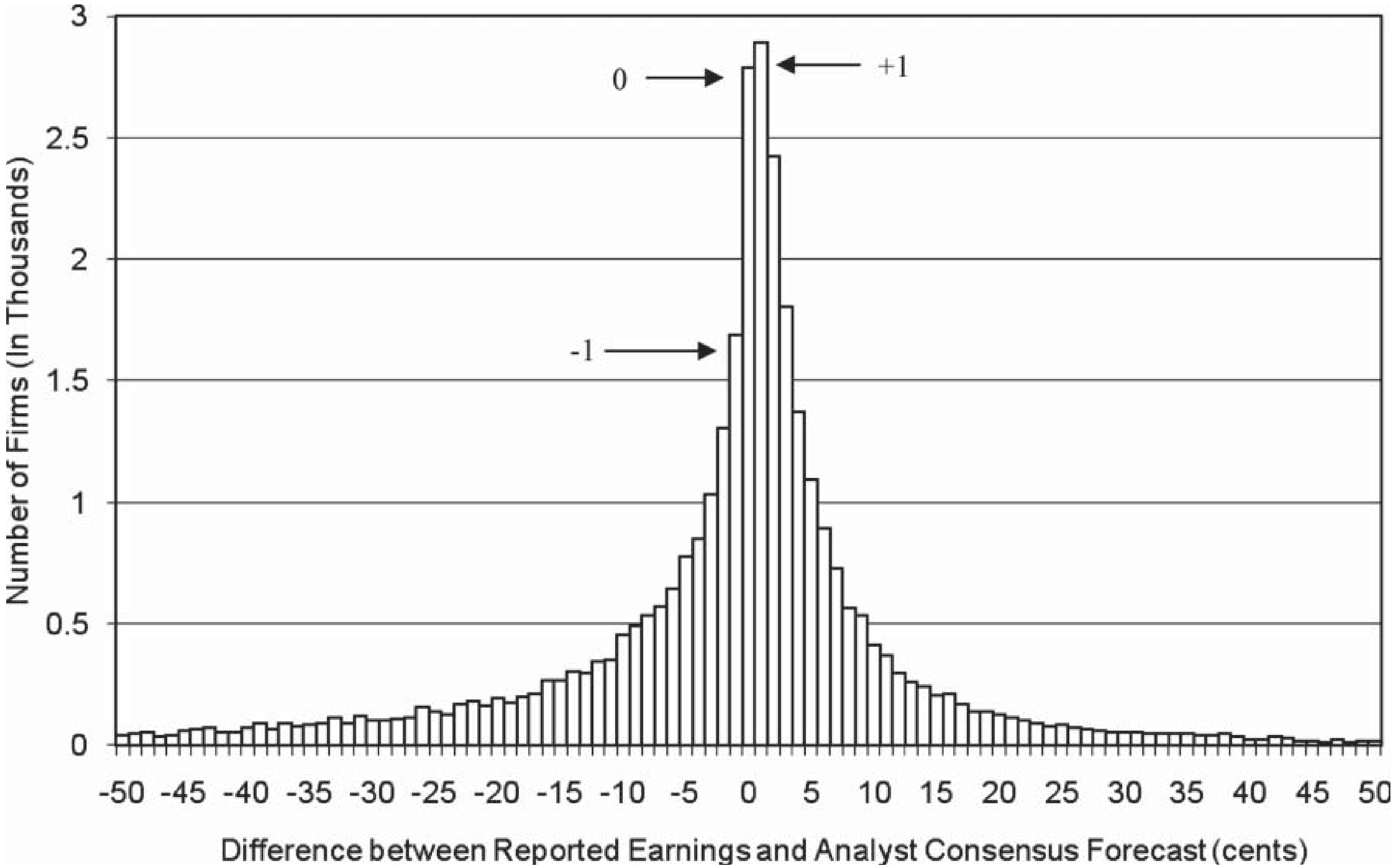


Figure 5b. Frequency distributions of excess earnings in 1988−2006
(Source: Figure 2 of Bhojraj et al 2009)



Excess earnings defined as earnings surprises relative to analysts' consensus forecast (in cents)

Figure 5c. 3D plot of frequency distributions of simulated excess earnings

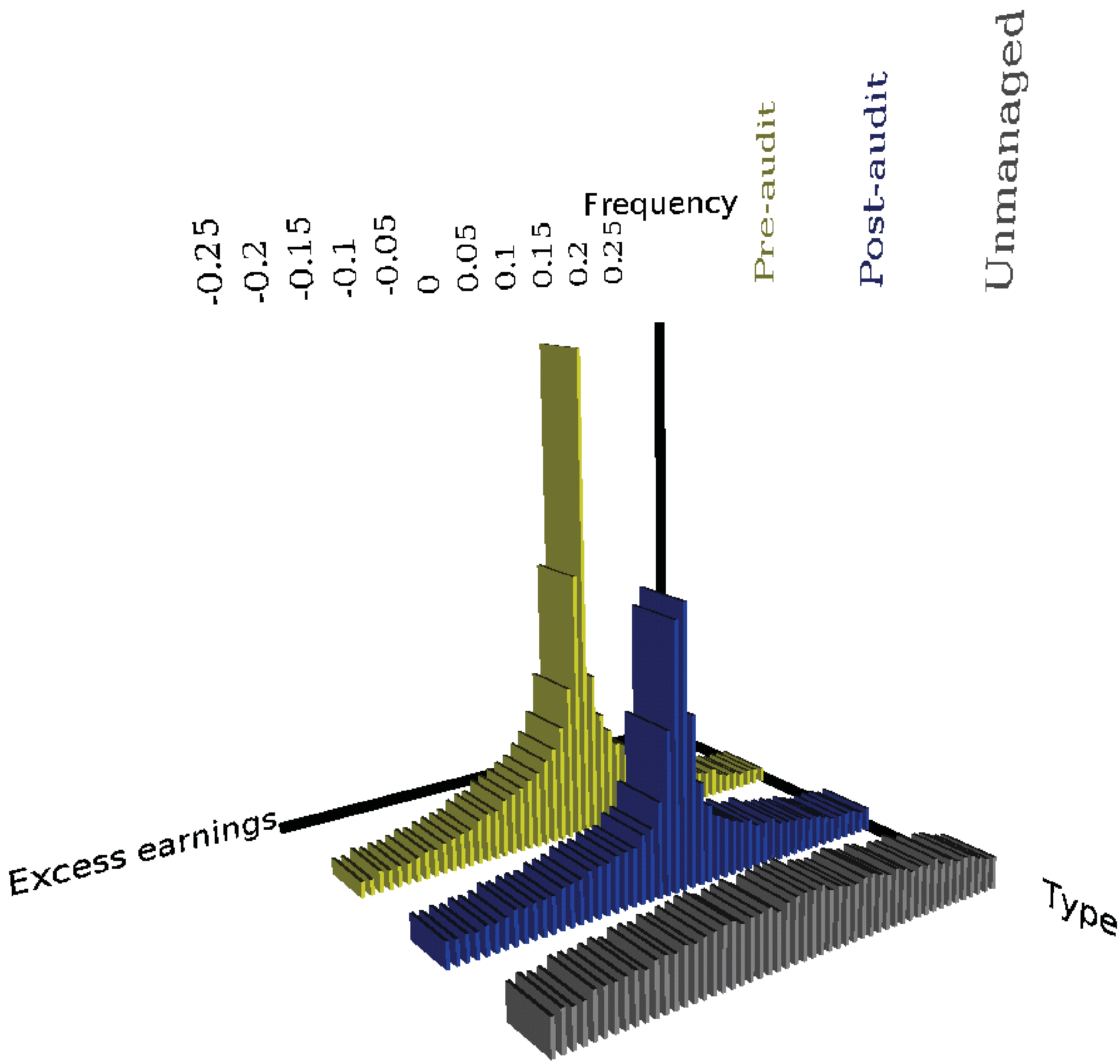


Figure 6a. Frequency distributions of simulated earnings

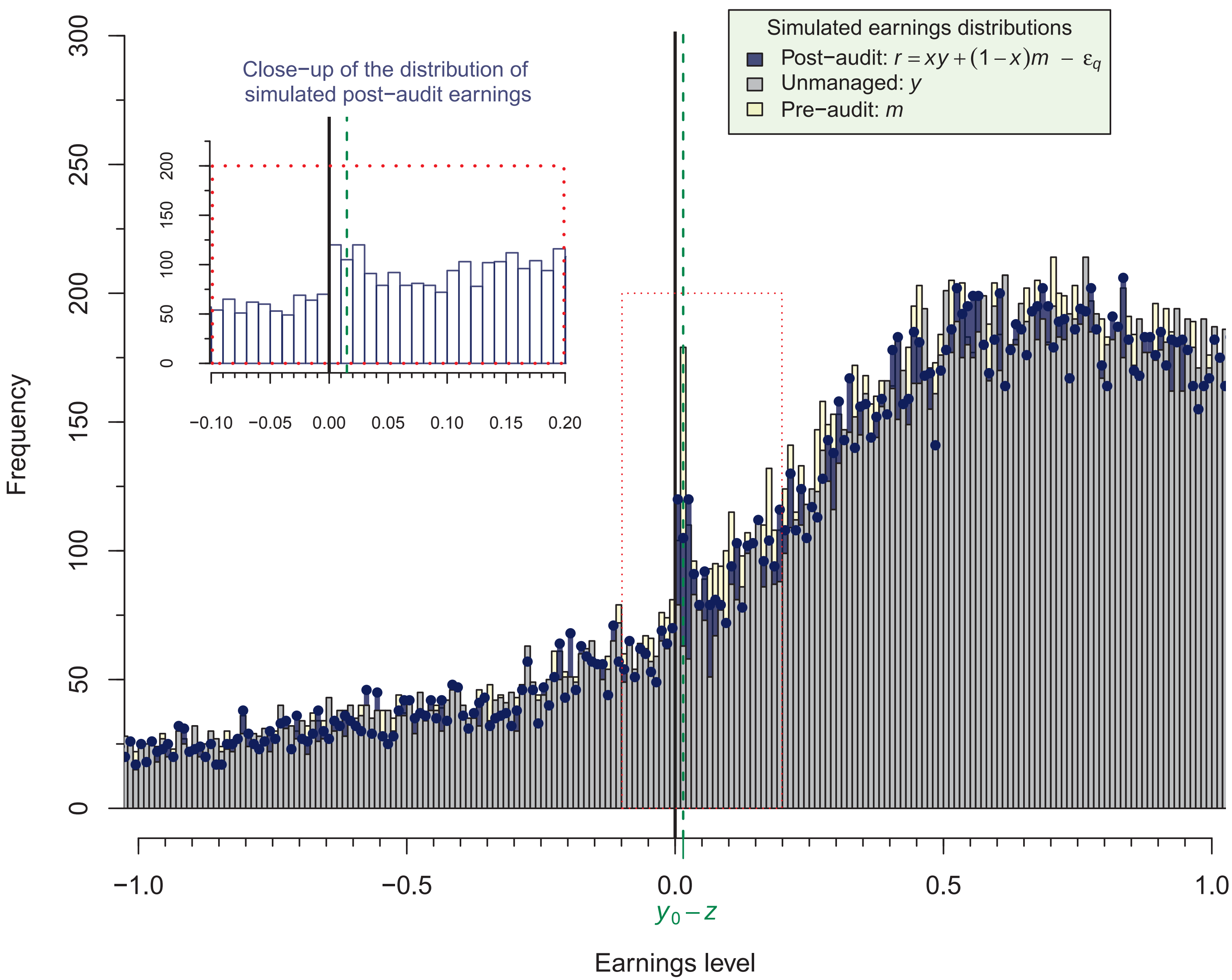


Figure 6b. Frequency distributions of earnings in 1988−2006

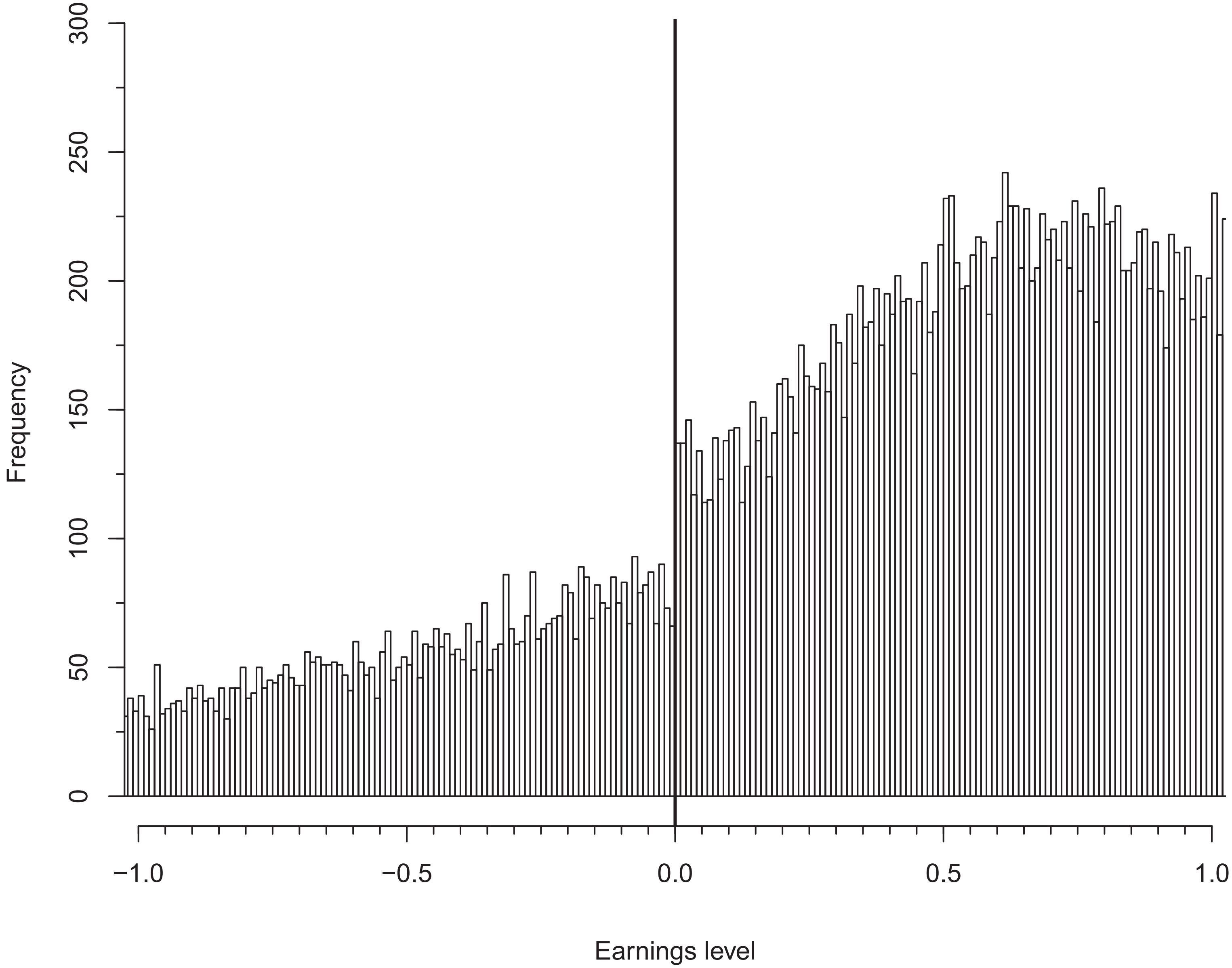


Figure 7a. Frequency distributions of simulated earnings differences

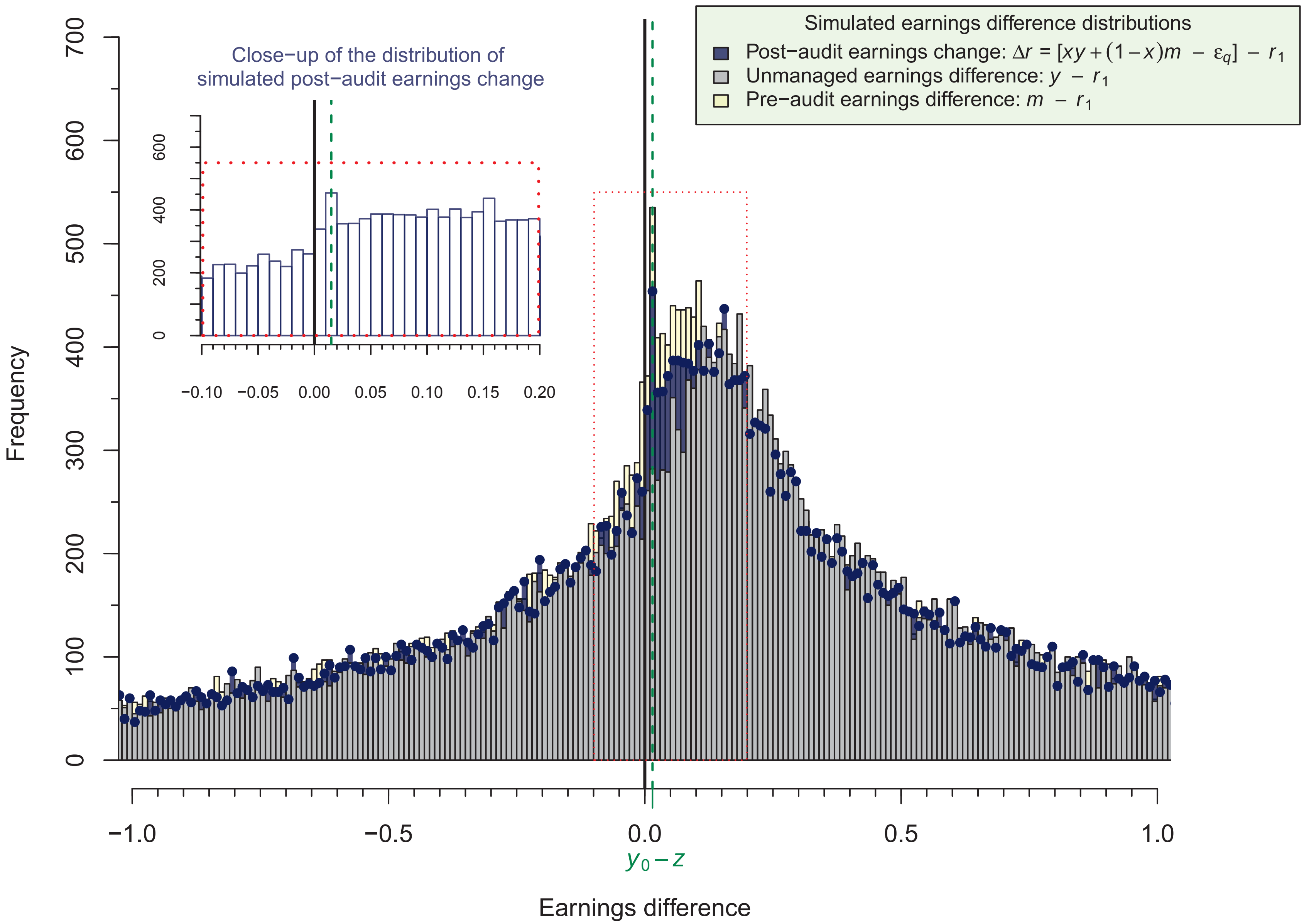


Figure 7b. Frequency distributions of earnings change in 1988−2006