Scaling and Multi-scaling in Financial Markets

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Abstract.
This paper reviews some of the phenomenological models which have been introduced to incorporate the scaling properties of financial data. It also illustrates a microscopic model, based on heterogeneous interacting agents, which provides a possible explanation for the complex dynamics of markets' returns. Scaling and multi-scaling analysis performed on the simulated data is in good quantitative agreement with the empirical results.

I INTRODUCTION

The probability distribution of returns of many market indexes and exchange rates, at a given (but not too long) time scale seems consistent, with an asymptotic power law decay $P(r) \sim r^{-(1+\mu)}$. A crossover between a Levy stable regime with $\mu < 2$ [1], and a power law with an exponent $2 < \mu < 4$ [2] has been reported [3]. Power law distributions are self-similar, therefore the distribution of returns $r_\tau(t) = \log(P(t+\tau)/P(t))$ at different time scales τ, rescaled by a lag-dependent factor $\eta(\tau)$, satisfies

$$P(r_\tau) = \frac{1}{\eta(\tau)} \mathcal{F}(\frac{r_\tau}{\eta(\tau)}) \tag{1}$$

where $\mathcal{F}(u)$ is a time independent scaling function. For self-affine function $\eta(\tau) = \tau^H$ with $H = 1/2$ for a Gaussian process and $H = 1/\mu$ for a Levy process. If scale consistency holds the time-scale at which the process is observed becomes irrelevant. The situation in financial markets is more complicate than this and one observes that when τ increases over a certain value (a few days for market indexes and a few weeks for individual stocks) the scaling breaks down and the the shape of a Gaussian, as predicted by the random-walk hypothesis, is recovered. This implies that it is not possible to find a unique real number H such that the statistical properties of the rescaled variables $r_\tau(t)/\tau^H$ do not depend on τ. The notion of multiaffinity is introduced to characterize a stochastic process $X(t)$ which satisfies

$$E(|X(t+\tau) - X(t)|^q) = c(q)\tau^{\phi_q} \tag{2}$$

where ϕ_q is the scaling function. Self-affine processes are characterized by a ϕ_q which is linear and fully determined by its index H: $\phi_q = Hq$. Multi-affine processes instead are characterized by a non-linear scaling function ϕ_q. Anomalous scaling, or multiscaling, appear in a wide class of phenomena where global dilatation invariance fails. For example intermittent behaviour in dynamical system, i.e. strong time dependence in the degree of chaoticity, is accompanied by anomalous scaling with respect to time dilatations in the trajectory space. If the non linear shape of the scaling exponents ϕ_q is a consequence of intermittent behaviour multiaffinity also involves multifractality of an opportunely defined probability measure. A measure of the degree of intermittency could than be provided in terms of an infinite set of exponents associated to the geometrical structure of this probability measure [4].

Multiscaling in financial markets is an indication that returns, even if uncorrelated, are not independent stochastic variables and it reveals the presence of wild fluctuations. The wilder the fluctuations, the larger the difference of ϕ_q from a linear behaviour in q. Indeed it is well known that while stock market returns are uncorrelated, the autocorrelation function of a measure of volatilty, such as absolute value of returns, is positive and slowly decaying, indicating long memory effects. This phenomenon, known in the literature as volatility clustering implies temporal dependencies in the alternation of period of large price changes with period of smaller changes. Empirical analysis shows that the decay of the autocorrelations of absolute returns is hyperbolic over a large range of time lags (from one day to one year), with an exponent $0.1 < \gamma < 0.4$ [5].

Considerable attention has been devoted in detecting comovements of volatility with other economic variables in the attempt to interpret and capture the source of clustering effects in returns. In particular a big effort has been devoted to the analysis of correlations between volatility of returns and trading volume. Empirical evidence has been provided [2] of a positive cross correlation between these two quantities. Furthermore a lack of regularity has been observed in the trading or business time. Stochastic trading time models [6–8] have also been proposed as a possible explanation for the emergence of persistency in volatility.

II PHENOMENOLOGICAL MODELS

A simple model proposed by Mandelbrot and van Ness [9] which can generate volatilty persistence is the Fractional Brownian Motion. FBM is a random process with stationary, self-similar increments which grow locally at a rate τ^H, where H is the self-affinity exponent. Even thought this model account for correlated volatility fluctuations, these are generated through dependent, and hence predictable, price increments with negative autocorrelation when $0 < H < 1/2$ and positive correlation when $1/2 < H < 1$.

An earlier model introduced by Mandelbrot [10], which has become very popular in the physics literature, is the Levy Flight (LF) model. A LF is a random walk in which the step lenght is chosen from a Levy distribution. Since Levy distributions are stable under convolution the LF process exhibits exact self-similarity. The exact scaling of both FBM and LF is not consistent though with the scaling break down observed in financial market data. To overcome the many limitations of Levy flights (not last the difficulties to deal with a process with a diverging second moment) Truncated Levy flights (TLF) have been introduced. A TLF is a process based on a truncated Levy stable distribution with a cut-off in its power law tail. The truncation can be introduced as a sharp cut-off as in Mantegna and Stanley [1] or, to preserve the infinitely divisible property of the pdf through a smoother exponential decay in the tail as in Koponen [11]. Because of the cut-off the distribution is no longer self-similar when convoluted and has finite variance. Even though the TLF pdf belongs to the basin of attraction of a Gaussian, it converges to the Gaussian very slowly and the process exhibits Levy scaling in a wide range of sampling intervals. Nakao [12] has shown that TLFs a la Koponen exhibit the simplest form of multiscaling, i.e. bi-fractality. Nonetheless his results, i.e. $\phi_q = q/\alpha$ for $0 < q < \alpha$ and $\phi_q = 1$ for $q > \alpha$ are not consistent with the empirical findings. Similar results have been found by Chechkin [13] who analized a finite sample of a simulated ordinary LF. Indeed, even though Levy flights have stationary, statistically self-affine and stably distributed increments, the finiteness of the sample size violate sef-affinity giving rise to spurios multi-scaling. Consequently, while the moments of stable Levy distribution with $\alpha < 2$ diverge, together with the q-th order structure function, at $q > \alpha$ both quantities are finite for finite sample size. A rough estimate of finite sample effects gives [13] a linear τ-dependence of the q-th order structure function with an exponent which does not depend on q at $q > \alpha$. So both ordinary or truncated Levy flights can generate multiaffinity but the shape of the scaling function ϕ_q is not consistent with the empirical one.

Few aspects remain unexplained by the TLF: first is that the TLF describes well only the central part of distribution of return but not the far tails which decay with an exponent μ well outside the Levy regime. Second the crossover to the Gaussian regime occurs at much larger times than the one expected from the TLF and, finally, it does not account for correlated variance fluctuations.

In the economics literature the modelling of financial time series developed in a quite different direction and leptokurtotic distributions have been introduced through a variance conditionally dependent on its past (heteroschedasticity), as in the ARCH/GARCH models [14]. GARCH models nonetheless, even if fat-tailed, only achieve weak memory effetcs, manifest in an exponential (and not hyperbolic) decay of the volatility autocorrelation function. The recently developed FIGARCH [15] process achives long memory still mantaining the martingale property of asset returns. Nonetheless, unlike GARCH, FIGARCH does not converge to a Gaussian process over long sampling intervals and fails to describe the scaling properties of pdfs at different time horizons.

As a generalization of ARCH models Heteroskedastic Levy Flight (HLF) processes have been recently introduced. Podobnik et al. [16] show that due to the correlations in the variance the process generates power-law tails in the distribution of returns whose exponents can be controlled through the way the correlations in the variance are introduced. The crossover between the two power-law regimes ($\mu < 2$ and $2 < \mu < 4$) can also be generated in these models. Santini [17] moreover has shown that in the HLF the Levy scaling of the pdf persists for times of order of magnitude larger than for uncorrelated variance fluctuations. It would be interesting to analyze whether this model also reproduces the multiscaling behaviour of financial data.

An alternative exactly soluble model that mimics the long range volatility correlations has been introduced by Bouchaud et al. [18]. Although their model is uni-affine by construction, it shows apparent multiscaling, in good agreement with empirical data, as a result of very long transient effects, induced by the long range nature of the volatility correlations.

This scenario seems to indicate that multiscaling in financial markets is not a trivial effect of the truncature in the tails but is a consequence of correlated variance fluctuations.

Following a completely different approach, an altenative model, the Multifractal Model of Asset Return (MMAR), has been introduced by Mandelbrot [19]. Fluctuations in volatility are introduced in MMAR by a random trading time, generated as the cumulative ditribution function of a random multifractal measure. Trading time is assumed highly variable and contains long memory. Both these characteristics are passed on to the price process trough compounding. Subordinate stochastic process have been extensively used in the economic literature where either the trading volume [6] or the trading time [7,8] has been chosen as the directing process. Note that in general the distributions of subordinated stochastic processes do not possess scaling properties. In the MMAR model the return process is a compound process $X(t) = B_H[\theta(t)]$ where $B_H(t)$ is a fractional Brownian Motion with self-affinity index H, and $\theta(t)$ is a stochastistic trading time. In particular $\theta(t)$ is a multifractal process with continuous, non-drecreasing paths and stationary increments. If $B_H(t)$ is a Brownian motion ($H = 1/2$) without drift the MMAR generates, together with multiscaling, uncorrelated increments and persistence in volatility.

Aside the phenomelogical characterization of the scaling properties of financial data the microscopic origins of the complexity of financial markets needs to be investigated. From a physicist point a view the market is a perfect example of a complex system, with a large number of heterogenous agents interacting in an intricate way. It is than tempting to describe the behaviour of market's player using models developed in the context of the statistical mechanics of disordered systems, as it is proposed in the following section.

III A MICROSCOPIC MODEL

It is not settled yet whether the emergence of power law fluctuations and volatility clustering observed in financial data is due to external factors, like the arrival of new information, or to the inherent interaction among market players and the trading process itself.

Iori [20] proposed a model where large fluctuations in returns arise purely from communication and imitation among traders. The key element in the model is the introduction of a trade friction (representing transactions costs) which, by responding to price movements, creates a feedback mechanism on future trading and generates volatility clustering. In [20] (to which the reader is refered for more details and for references of alternative agents based models), the market consists of a market maker plus a number of noise traders. Traders buy from or sell to the market maker and respond to a signal which incorporates idiosyncratic preferences and the influence of the traders nearest to them. Only one kind of stock is traded, whose price is set by the market maker on the basis of the observed order flow and the overall trading volume.

The model is studied through numerical simulations. It repoduces correctly the scaling of the distribution of returns, with a power law decay with $\mu \sim 3$, at short time and a slow convergences to a Gaussian distribution at larger time scales. Moreover the model generates volatility clustering and positive cross-correlation between volatility and trading volume. The autocorelation function of the volatility decays hyperbolically with an exponent $\gamma \sim 0.3$ consistent with empirical observations.

The analysis of the moments of the distribution of the simulated data also reveals multiscaling (fig.(1a)). By using a step-wise linear regression, the slopes in the structure function are estimated as 0.47 for $q < 3$ and 0.14 for $q > 3$. These results are in good agreement with the empirical analysis performed by Baviera et al. [21]

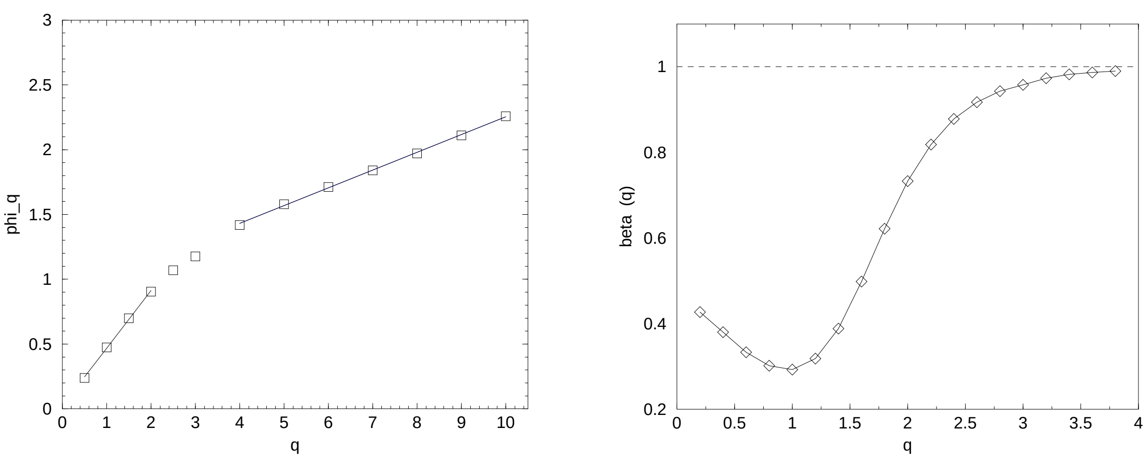

FIGURE 1. Exponent ϕ_q versus q (left). The slope of the two regions are 0.47 at $q < 3$ and 0.14 at $q > 3$. Scaling exponent $\beta(q)$ as a function of q (right). Anomalous scaling with $\beta(q) < 1$ is shown.

on the DM/US exchange rate quotes.

Baviera et al. [21] also detected multiscaling in the volatility autocorrelations by analyzing the generalized correlation functions:

$$C_q(L) =< |r(t)|^q |r(t+L)|^q > - < |r(t)|^q >< |r(t+L)|^q > . \qquad (3)$$

If the absolute returns' serie $|r(t)|^q$ has long term memory we would find $C_q(L) \sim L^{-\beta(q)}$ with $\beta(q) < 1$ while if $|r(t)|^q$ is an uncorrelated process $\beta(q) = 1$. Multiscaling would be signaled by a non linear shape of $\beta(q)$. The scaling exponent $\beta(q)$ measured from simulated data is shown in fig.(1b) for $0 < q < 4$. In analogy with the NYSE index and the USD-DM exchange rate [21,22], $\beta(q)$ is not a constant function of q revealing the presence of different anomalous scales. The convergence of $\beta(q)$ to one reveals that large fluctuations are practically independent. This observation might justify the convergence of the distribution of returns to a Gaussian even though returns are not independent random variables.

This simple model can reproduce many of the stylized facts of stock market returns and has outlined a mechanism which can explain the emergence of the observed power-law fluctuations.

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