A unifying E2-quasi exactly solvable model

Andreas Fring

Department of Mathematics, City University London,
Northampton Square, London EC1V 0HB, UK
E-mail: a.fring@city.ac.uk

Abstract: A new non-Hermitian E2-quasi-exactly solvable model is constructed containing two previously known models of this type as limits in one of its three parameters. We identify the optimal finite approximation to the double scaling limit to the complex Mathieu Hamiltonian. A detailed analysis of the vicinity of the exceptional points in the parameter space is provided by discussing the branch cut structures responsible for the chirality when exceptional points are surrounded and the structure of the corresponding energy eigenvalue loops stretching over several Riemann sheets. We compute the Stieltjes measure and momentum functionals for the coefficient functions that are univariate weakly orthogonal polynomials in the energy obeying three-term recurrence relations.

1. Introduction

In addition to the interesting mathematical aspect of enlarging the set of $sl_2(\mathbb{C})$ [1, 2] to E_2-quasi-exactly solvable models [3], the latter type also constitutes the natural framework for various physical applications in optics where the formal analogy between the Helmholtz equation and the Schrödinger equation is exploited [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Furthermore, a special case of these systems with a specific representation corresponds to the complex Mathieu equation that finds an interesting application in nonequilibrium statistical mechanics, where it corresponds to the eigenvalue equation for the collision operator in a two-dimensional classical Lorentz gas [14, 15].

Here we are mainly concerned with the extension of quasi-exactly solvable models [16, 17, 18, 19, 3] to non-Hermitian quantum mechanical systems [20, 21, 22, 23] within the above mentioned scheme. So far two different types of E_2-models have been constructed in [3, 24] and the main purpose of this manuscript is to investigate whether it is possible to construct a more general model that unifies the two. We show that this is indeed possible by combining the two models and introducing a new parameter into the system that interpolates between the two. In a similar fashion as the previously constructed models, also this one reduces in the double scaling limit to the complex Mathieu equation. As that equation is not fully explored analytically this limit provides an important option to obtain interesting information about the complex Mathieu system. On the other hand, for some applications it may also be sufficient to study an approximate behaviour for some finite values of the coupling constants. For that purpose we identify the parameter for which the general model is the optimal approximation for the complex Mathieu system.

Our manuscript is organized as follows: In section 2 we introduce the general unifying model involving three parameters. We determine the eigenfunctions by solving the standard three-term recurrence relations for the coefficient functions and determine the energy eigenfunction from the requirement that the three-term recurrence relations reduce to a two-term relation. We devote section three to the study of the exceptional points and their vicinities in the parameter space. The explicit branch cut structure is provided that explains the so-called energy eigenvalue loops. In section 4 we compute the central properties of the weakly orthogonal polynomials entering as coefficient functions in the Ansatz for the eigenfunctions, i.e. their norms, the corresponding Stieltjes measure and the momentum functionals. We state our conclusions in section 5.

2. A unifying E2-quasi-exactly solvable model

The general notion [1, 2] underlying solvable Hamiltonian systems is that its Hamiltonian operators $\mathcal{H}$ acting on some graded space V_n as $\mathcal{H} : V_n \mapsto V_n$ preserves the flag structure $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$ A distinction is usually made between exactly and quasi-exactly solvable, depending on whether the structure preservation holds for an infinite or a finite flag, respectively. Here we are concerned with the latter. Lie algebraic versions of Hamiltonians in this context are usually taken to be of $sl_2(\mathbb{C})$-type [1, 2], but as recently proposed [3, 24], they may also be taken to be of a Euclidean Lie algebraic type, thus giving rise to qualitatively new structures.

At present two different types of E_2-quasi-exactly solvable models were identified

$$\mathcal{H}_{E_2}^{(1)} = J^2 + \zeta^2(u^2 - v^2)^2 + 2i\zeta N(u^2 - v^2), \qquad \zeta, N \in \mathbb{R}, \tag{2.1}$$

$$\mathcal{H}_{E_2}^{(0)} = J^2 + \zeta uvJ + 2i\zeta N(u^2 - v^2), \tag{2.2}$$

in [3] and [24], respectively. Both Hamiltonians are expressed in terms of the E_2-basis operators u, v and J that obey the commutation relations

$$[u, J] = iv, \qquad [v, J] = -iu, \qquad [u, v] = 0. \tag{2.3}$$

Except for $\mathcal{H}_{E_2}^{(0)}$ at $N = 1/4$, both Hamiltonians are non-Hermitian, but respect the antilinear symmetry [25] $\mathcal{PT}_3 : J \to J,\ u \to v,\ v \to u,\ i \to -i$ as defined in [10]. For the particular representation $J := -i\partial_\theta$, $u := \sin\theta$ $v := \cos\theta$ the $\mathcal{PT}_3$-symmetry is simply $\mathcal{PT}_3 : \theta \to \pi/2 - \theta$, $i \to -i$, such that the invariant vector spaces over $\mathbb{R}$ were defined as

$$V_n^s(\phi_0) := \operatorname{span}\left\{ \phi_0 \left[\sin(2\theta), i\sin(4\theta), \ldots, i^{n+1}\sin(2n\theta)\right] \middle|\, \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}, \tag{2.4}$$

$$V_n^c(\phi_0) := \operatorname{span}\left\{ \phi_0 \left[1, i\cos(2\theta), \ldots, i^n\cos(2n\theta)\right] \middle|\, \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}. \tag{2.5}$$

In order to construct Hamiltonians that preserve the flag structure one needs to identify the action of the E_2-basis operators and its combinations on these spaces as explained in more detail in [3]. The behaviour found allowed to identify the Hamiltonians $\mathcal{H}_{E_2}^{(1)}$ and $\mathcal{H}_{E_2}^{(0)}$ in (2.1) and (2.2) as quasi-exactly solvable. The general structure suggests that there might be a master Hamiltonian that unifies the above Hamiltonians into one preserving the quasi-exact solvability. We demonstrate here that this is possible and study the properties of that model.

Thus we introduce the new Hamiltonian

$$\mathcal{H}(N,\zeta,\lambda) = J^2 + 2(1-\lambda)\zeta uvJ + \lambda\zeta^2(u^2-v^2)^2 + 2i\zeta N(u^2-v^2), \qquad \lambda,\zeta,N \in \mathbb{R}, \tag{2.6}$$

and demonstrate explicitly that it is indeed E_2-quasi-exactly solvable. First we observe that $\mathcal{H}(N,\zeta,\lambda)$ interpolates between the two models in (2.1) and (2.2) by varying λ, since

$$\lim_{\lambda\to 1}\mathcal{H}(N,\zeta,\lambda) = \mathcal{H}_{E_2}^{(1)} \qquad \text{and} \qquad \lim_{\lambda\to 0}\mathcal{H}(2N,\zeta/2,\lambda) = \mathcal{H}_{E_2}^{(0)}. \tag{2.7}$$

Furthermore, $\mathcal{H}(N,\zeta,\lambda)$ reduces to the complex Mathieu Hamiltonian in the double scaling limit $\lim_{N\to\infty,\zeta\to 0}\mathcal{H}(N,\zeta,\lambda) = \mathcal{H}_{\mathrm{Mat}} = J^2 + 2ig(u^2-v^2)$ for $g := N\zeta < \infty$. We also note that $\mathcal{H}^\dagger(N,\zeta,\lambda) = \mathcal{H}(1-\lambda-N,\zeta,\lambda)$, which implies that $\mathcal{H}(N,\zeta,\lambda)$ is non-Hermitian unless $2N = 1-\lambda$, with free coupling constant $\zeta \in \mathbb{R}$.

Given the structure for the vector spaces in (2.4) and (2.5) we now make the following Ansätze for the two fundamental solutions of the corresponding Schrödinger equation $\mathcal{H}_N\psi_N = E\psi_N$

$$\psi_N^c(\theta) = \phi_0 \sum_{n=0}^{\infty} i^n c_n P_n(E)\cos(2n\theta), \quad \text{and} \quad \psi_N^s(\theta) = \phi_0 \sum_{n=0}^{\infty} i^{n+1} c_n Q_n(E)\sin(2n\theta), \tag{2.8}$$

where the $\mathcal{PT}_3$-symmetric ground state is taken to be $\phi_0 = e^{\frac{i}{2}\zeta\cos(2\theta)}$ and the constant c_n is $c_n = 1/\zeta^n(N+\lambda)(1+\lambda)^{n-1}\left[(1+N+2\lambda)/(1+\lambda)\right]_{n-1}$ with $(a)_n := \Gamma(a+n)/\Gamma(a)$ denoting the Pochhammer symbol. The constants are chosen conveniently in order to ensure the simplicity of the to be determined n-th and $(n-1)$-th order polynomials $P_n(E)$, $Q_n(E)$ in the energies E, respectively. Upon substitution into the Schrödinger equation we obtain the three-term recurrence relations

$$\begin{aligned}
P_2 &= (E-\lambda\zeta^2-4)P_1 + \mathbf{2}\zeta^2\left[N-1\right]\left[N+\lambda\right]P_0, && (2.9)\\
P_{n+1} &= (E-\lambda\zeta^2-4n^2)P_n + \zeta^2\left[N+n\lambda+(n-1)\right]\left[N-(n-1)\lambda-n\right]P_{n-1}, && (2.10)\\
Q_2 &= (E-4-\lambda\zeta^2)Q_1, && (2.11)\\
Q_{m+1} &= (E-\lambda\zeta^2-4m^2)Q_m + \zeta^2\left[N+m\lambda+(m-1)\right]\left[N-(m-1)\lambda-m\right]Q_{m-1}, && (2.12)
\end{aligned}$$

for $n = 0,2,\ldots$ and for $m = 2,3,4,\ldots$ Note that a more generic Ansatz for the unifying model involving two independent coupling constants μ, λ in the terms $\mu\zeta uvJ + \lambda\zeta^2(u^2-v^2)^2$ leads to a four term recurrence relation in which the highest term is always proportional to $\mu+2\lambda-2$. Thus taking this term to zero with the appropriate choice for μ reduces this to the desired three term relations that may be solved in complete generality as outlined in [3]. The lowest order polynomials are easily computed in a recursive way. Taking $P_0 = 1$

we obtain

$$
\begin{aligned}
P_1 &= E - \lambda\zeta^2, \qquad (2.13)\\
P_2 &= \lambda^2\zeta^4 + 2\zeta^2\left[\lambda - \lambda E + N(\lambda + N - 1)\right] + (E-4)E,\\
P_3 &= -\lambda^3\zeta^6 + \lambda\zeta^4\left(\lambda(2\lambda + 3E - 13) - 3N^2 - 3(\lambda-1)N + 2\right) + (E-16)(E-4)E\\
&\quad -\zeta^2\left[3\lambda E^2 + E\left(2\lambda^2 - 3N^2 - 3\lambda(N+11) + 3N + 2\right) + 32(\lambda + N(\lambda + N - 1))\right],
\end{aligned}
$$

and likewise with $Q_1 = 1$ we compute

$$
\begin{aligned}
Q_2 &= E - 4 - \lambda\zeta^2, \qquad (2.14)\\
Q_3 &= \lambda^2\zeta^4 + \zeta^2\left[\lambda(15 - 2\lambda - 2E) + N^2 + (\lambda-1)N - 2\right] + (E-16)(E-4),\\
Q_4 &= -\lambda^3\zeta^6 + \lambda\zeta^4\left[8 + \lambda(8\lambda + 3E - 38) - 2N^2 - 2(\lambda-1)N\right] + (E-36)(E-16)(E-4)\\
&\quad +\zeta^2\left[-8\left(-12\lambda^2 + 69\lambda + 5\lambda N + 5(N-1)N - 12\right)\right]\\
&\quad +\zeta^2\left[-3\lambda E^2 + 2E\left((47 - 4\lambda)\lambda + N^2 + (\lambda-1)N - 4\right)\right].
\end{aligned}
$$

In both cases we observe the typical feature for quasi-exactly solvable systems that the three term relation can be reset to a two-term relation at a certain level. This is due to the fact that in (2.10) and (2.12) the last term vanishes when $m = n = \hat{n} = -(1+N)/(1+\lambda)$ or $m = n = \tilde{n} = (\lambda + N)/(1+\lambda)$. Thus when taking $N = \tilde{n} + (\tilde{n}-1)\lambda$ we find the typical factorization

$$
P_{\tilde{n}+\ell} = P_{\tilde{n}} R_\ell \qquad \text{and} \qquad Q_{\tilde{n}+\ell} = Q_{\tilde{n}} R_\ell. \qquad (2.15)
$$

The first solutions for the factor R_ℓ are easily found from (2.10) and (2.12) to

$$
R_1 = E - 4\tilde{n}^2 - \lambda\zeta^2, \qquad (2.16)
$$

$$
R_2 = (E - 4\tilde{n}^2 - \lambda\zeta^2)(E - 4(\tilde{n}+1)^2 - \lambda\zeta^2) - 2\tilde{n}(1+\lambda)^2\zeta^2. \qquad (2.17)
$$

Next we compute the energy eigenvalues $E_{\tilde{n}}$ from the constraints $P_{\tilde{n}}(E) = 0$ and $Q_{\tilde{n}}(E) = 0$ for the lowest values of N. For the solutions related to the even fundamental solution in (2.8) we find

$$
N = 1: \qquad E_1^c = \lambda\zeta^2, \qquad (2.18)
$$

$$
N = 2 + \lambda: \qquad E_2^{c,\pm} = 2 + \lambda\zeta^2 \pm 2\sqrt{1 - (1+\lambda)^2\zeta^2}, \qquad (2.19)
$$

$$
N = 3 + 2\lambda: \qquad E_3^{c,\ell} = \frac{20}{3} + \lambda\zeta^2 + \frac{4\hat{\Omega}}{3}e^{\frac{i\pi\ell}{3}} + \frac{1}{3}\left[52 - 12(1+\lambda)^2\zeta^2\right]e^{-\frac{i\pi\ell}{3}}\hat{\Omega}^{-1}, \qquad (2.20)
$$

with $\hat{\Omega}^3 := 35 + 18(\lambda+1)^2\zeta^2 + \sqrt{\left[3(\lambda+1)^2\zeta^2 - 13\right]^3 + \left[18(\lambda+1)^2\zeta^2 + 35\right]^2}$, $\ell = 0, \pm 2$.

For the solutions related to the odd fundamental solution in (2.8) we obtain

$$
N = 2 + \lambda: \qquad E_2^s = 4 + \lambda\zeta^2, \qquad (2.21)
$$

$$
N = 3 + 2\lambda: \qquad E_3^{s,\pm} = 10 + \zeta^2\lambda \pm 2\sqrt{9 - (\lambda+1)^2\zeta^2}, \qquad (2.22)
$$

$$
N = 4 + 3\lambda: \qquad E_4^{s,\ell} = \frac{56}{3} + \lambda\zeta^2 + \frac{4\Omega}{3}e^{\frac{i\pi\ell}{3}} + \frac{1}{3}\left[196 - 12(1+\lambda)^2\zeta^2\right]e^{-\frac{i\pi\ell}{3}}\Omega^{-1}, \qquad (2.23)
$$

with $\Omega^3 := 143 + 18\zeta^2(\lambda+1)^2 + \sqrt{\left(3\zeta^2(\lambda+1)^2 - 49\right)^3 + \left(18\zeta^2(\lambda+1)^2 + 143\right)^2}$, $\ell = 0, \pm 2$. Solutions for higher order may of course also be obtained, but are rather lengthy and therefore not reported here.

3. Exceptional points and their vicinities

The special point in parameter space where two real energy eigenvalues viewed as functions of the coupling constants merge and subsequently split into a complex conjugate pair is usually referred to as exceptional point [26, 27, 28, 29]. In our system these points can be computed in an explicit simple and straightforward manner. Using that by definition the discriminant Δ equals the product of the squares of the differences of all energy eigenvalues E_i for $1 \leq i \leq n$, i.e. $\Delta = \prod_{1\leq i<j\leq n}(E_i - E_j)^2$ one obtains the exceptional points from the real zeros of $\Delta(E)$. For practical purposes one may also exploit the fact [3], that the discriminant equals the determinant of the Sylvester matrix. This viewpoint has the advantage that it does not require the computation of all the eigenvalues and is more efficient when the sole purpose is to find the exceptional points. Thus in our case we have to find the real zeros of the discriminants $\Delta_{\tilde{n}}^c$ and $\Delta_{\tilde{n}}^s$ for the polynomials $P_{\tilde{n}}(E)$ and $Q_{\tilde{n}}(E)$, respectively. Extracting overall constant factors κ as $\Delta = \kappa\tilde{\Delta}$, that do not contribute to the zeros, we obtain for the lowest values of $\tilde{n}$

$$\begin{aligned}
\tilde{\Delta}_2^c &= \hat{\zeta}^2 - 1, \qquad (3.1)\\
\tilde{\Delta}_3^s &= \hat{\zeta}^2 - 9,\\
\tilde{\Delta}_3^c &= \hat{\zeta}^6 - \hat{\zeta}^4 + 103\hat{\zeta}^2 - 36,\\
\tilde{\Delta}_4^s &= \hat{\zeta}^6 - 37\hat{\zeta}^4 + 991\hat{\zeta}^2 - 3600,\\
\tilde{\Delta}_4^c &= \hat{\zeta}^{12} + 2\hat{\zeta}^{10} + 385\hat{\zeta}^8 - 33120\hat{\zeta}^6 + 16128\hat{\zeta}^4 - 732276\hat{\zeta}^2 + 129600,\\
\tilde{\Delta}_5^s &= \hat{\zeta}^{12} - 94\hat{\zeta}^{10} + 7041\hat{\zeta}^8 - 381600\hat{\zeta}^6 + 6645600\hat{\zeta}^4 - 78318900\hat{\zeta}^2 + 158760000,
\end{aligned}$$

where we abbreviated $\hat{\zeta} := \zeta(1+\lambda)$.

There exist many detailed studies about the structures in the coupling constant space in the vicinity of the exceptional points [30, 31, 32, 33, 34]. It is evident that when tracing a complex energy eigenvalue E as functions of the coupling constants, λ or ζ in our case, the corresponding path in the energy plane will inevitably pass through various Riemann sheets due to the branch cut structure. As a consequence one naturally generates eigenvalue loops that stretch over several Riemann sheets. This phenomenon is well studied for a large number of models and we demonstrate here that it also occurs in quasi-exactly solvable models. The basic principle can be demonstrated with the square root singularity occurring in $E_2^{c,\pm}$ with branch cuts from $(-\infty, -1-1/\zeta)$ and $(1/\zeta - 1, \infty)$. The energy loops are generated by computing $E_2^{c,\pm}(\lambda = \tilde{\lambda} + \rho e^{i\pi\phi}, \zeta)$ for some fixed values of ζ, center $\tilde{\lambda}$ and the radius ρ in the λ-plane as functions of ϕ as illustrated in figure 1(a) and (b). In panel (a) we simply trace the energy around a point in parameter space that leads to two real eigenvalues. For a small radius ones reaches the starting point by encircling $\tilde{\lambda}$ just once. However, when the radius is increased one needs to surround $\tilde{\lambda}$ twice to reach the starting point and when the radius is increased even further one only needs to surround $\tilde{\lambda}$ once switching, however, between both energy eigenvalues.

Essentially this structure survives when the two eigenvalues merge into an exceptional point. However, since the exceptional point is a branch point we no longer have the option for a closed loop around it produced from only one energy eigenvalue as seen in figure 1(b).

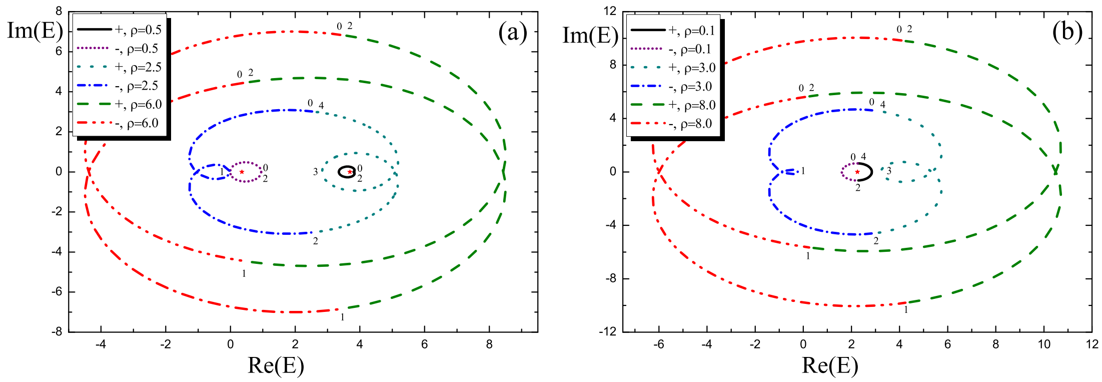

Figure 1: Energy eigenvalue loops $E_2^{c,\pm}(\tilde{\lambda}+\rho e^{i\pi\phi},\zeta)$ around two real eigenvalues panel (a) and around an exceptional point panel (b) as functions of ϕ, indicated by the numbers on the loops, for fixed value of $\zeta = 1/2$ at $\tilde{\lambda} = 1/10$ in (a) and $\tilde{\lambda} = 1$ in (b). The energy eigenvalues for $\rho = 0$ are distinct in panel (a) as $E_2^{c,-} = 0.35$, $E_2^{c,+} = 3.70$ and coalesce to an exceptional point in panel (b) as $E_2^{c,-} = E_2^{c,+} = 9/4$.

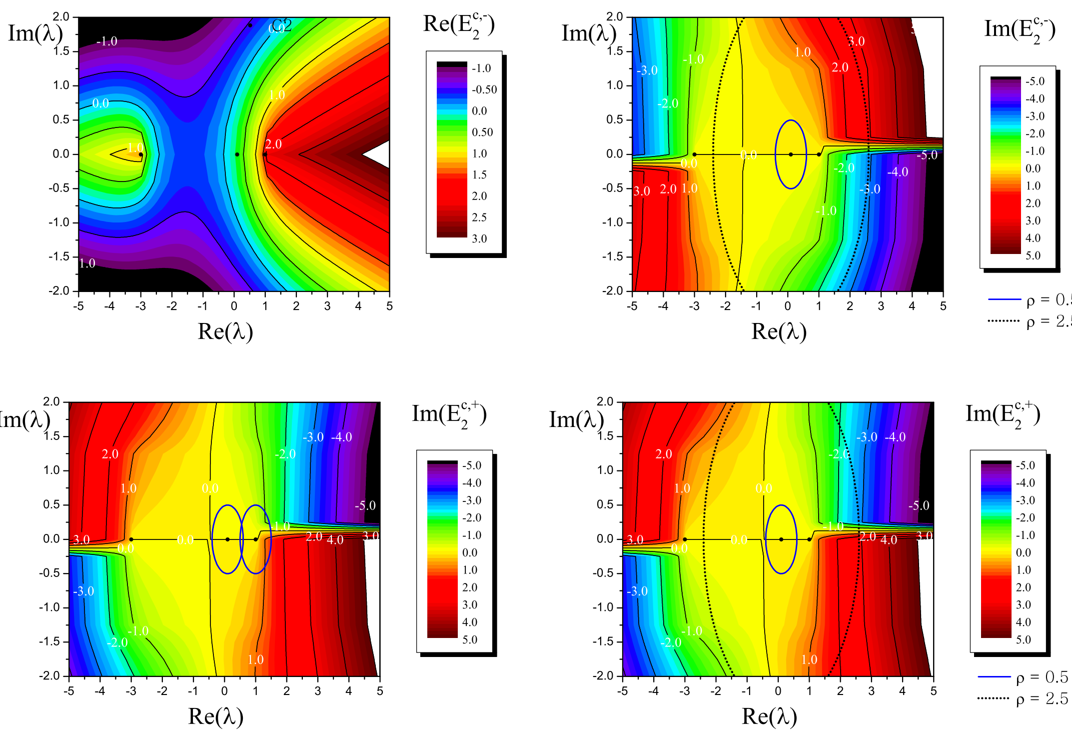

Figure 2: Energy levels and branch cut structure for $E_2^{c,\pm}$ for fixed $\zeta = 1/2$ as functions of λ. The branch cuts extend to the left and right from the exceptional points $(-\infty,-3)$ and $(1,\infty)$.

This behaviour is easily understood from the structure of the branch cuts as depicted in figure 2. Whereas for small radii it is possible to encircle for instance the point $\tilde{\lambda} = 1/10$ without crossing any branch cut, this is not possible when encircling the exceptional point at $\tilde{\lambda} = 1$ where we have to analytically continue from $E_2^{c,-}$ to $E_2^{c,+}$ when crossing a cut.

This structure is the same for intermediate radii. For large radii we cross the first cut already at a half circle turn, such that one returns back to the original value already after one complete turn.

When more eigenvalues are present the structure will be more intricate. Considering for instance a scenario with four eigenvalues in the form of two complex conjugate eigenvalues and an exceptional point, see figure 3(a), we need to perform again at least two turns in the λ-plane in order to return to the initial position for the energy loops when surrounding an exceptional point. The two complex conjugate eigenvalues may be enclosed with just one turn, albeit we require again different energy eigenvalues for this. When enlarging the radius the loops will eventually merge as depicted in figure 3(b) for a situation with a degenerate complex eigenvalue and two complex eigenvalues. We observe that for the given values we have to surround the chosen point at least three times to obtain a closed energy loop surrounding the indicated centers.

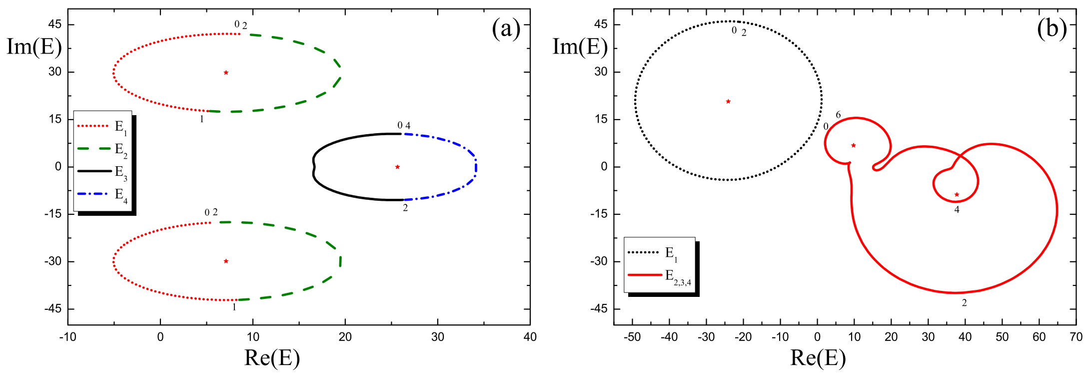

Figure 3: Energy eigenvalues $E_4^c(\tilde{\lambda} + \rho e^{i\pi\phi}, \zeta)$ as functions of ϕ, indicated by the numbers on the loops, for fixed value $\zeta = 1/2$ at $\tilde{\lambda} = 9.5284$ in (a) and $\tilde{\lambda} = 5.2562 + i9.9526$ in (b). The energy eigenvalues for $\rho = 0$ in panel (a) are $E_4^{c,1} = E_4^{c,2} = 25.6613$, $E_4^{c,3} = (E_4^{c,4})^* = 7.1029 + i29.8106$ and $E_4^{c,1} = E_4^{c,2} = 37.7449 - i8.7611$, $E_4^{c,3} = 9.8103 + i6.7668$, $E_4^{c,4} = -24.0439 + i20.7081$ in panel (b). The radii are $\rho = 4.0$ and $\rho = 8.5$ in panel (a) and (b), respectively.

In the same manner as for the simpler scenario one may understand the nature of these loops from an analysis of the branch cut structure of the energy as seen in figure 4. Tracing the indicated radii at $\rho = 4.0$ and $\rho = 8.5$ in figure 4 produces the energy loops in figure 3 when properly taking care of the analytic continuation at the branch cuts.

As discussed earlier the Hamiltonian $\mathcal{H}(N, \zeta, \lambda)$ has the interesting property that in the double scaling limit it reduces to the complex Mathieu equation for which only incomplete information is available, especially concerning the locations of the exceptional points. In comparison with the previously analyzed models $\mathcal{H}_{E_2}^{(1)}$ in [3] and $\mathcal{H}_{E_2}^{(0)}$ in [24] we have now the additional parameter λ at our disposal and we may investigate how the complex Mathieu system is approached. In particular we may address the question of whether there exists a value λ for which this is optimal. Our numerical results are depicted in figure 5. We find a similar qualitative behaviour for the other exceptional points, which we do not report here.

Comparing the rate of the approach for different values of λ we conclude that $\mathcal{H}(N, \zeta, \lambda = 1)$ is the best approximation to the complex Mathieu system for some finite values of N.

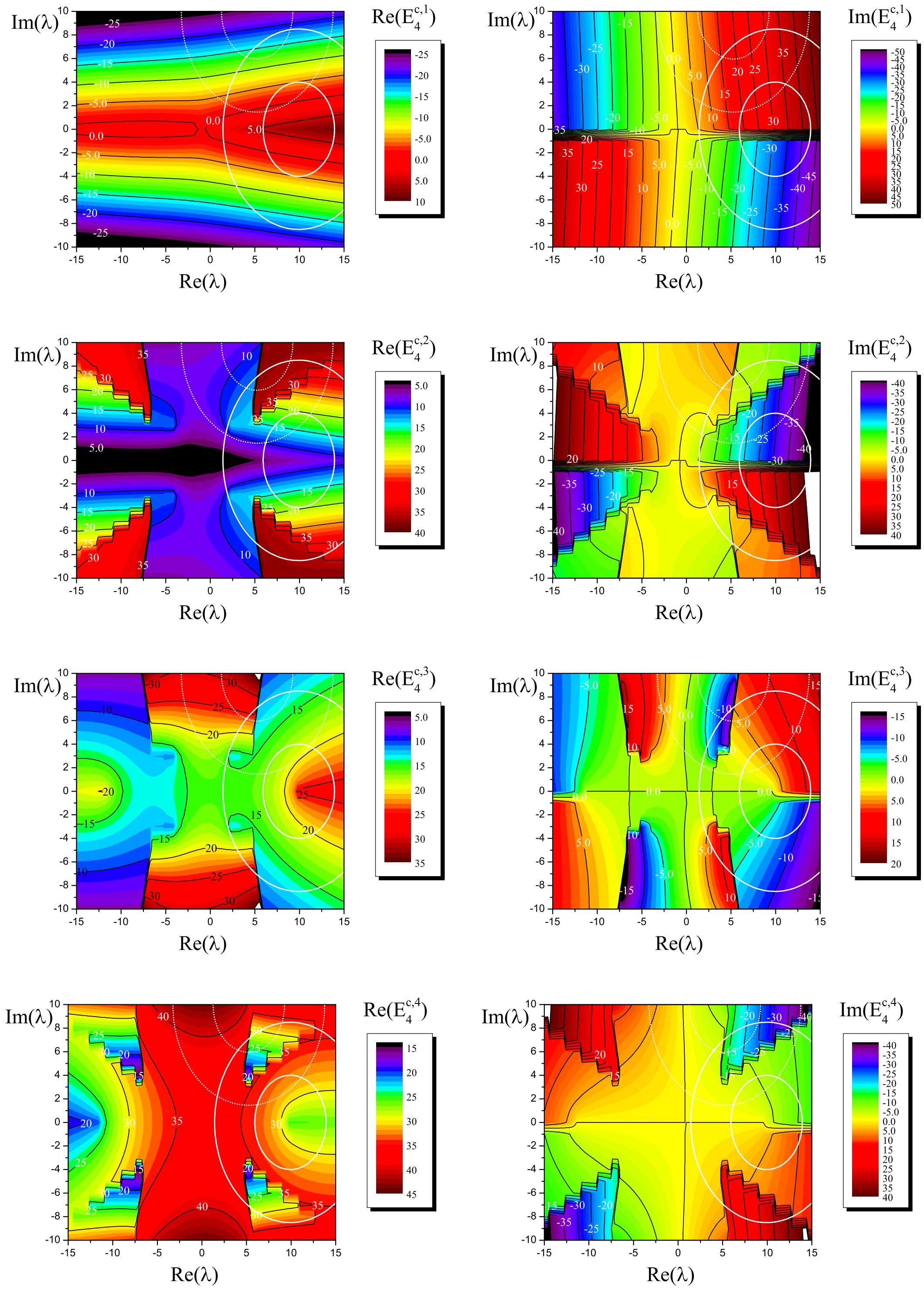

Figure 4: Energy levels and branch cut structure for $E_2^{c,1,2,3,4}$ for fixed $\zeta = 1/2$ as functions of λ.

If one is exclusively interested in the computation of the exceptional point it is most efficient to carry out the double scaling limit already for the three-term relation (2.10) and (2.12) as explained in [3, 24].

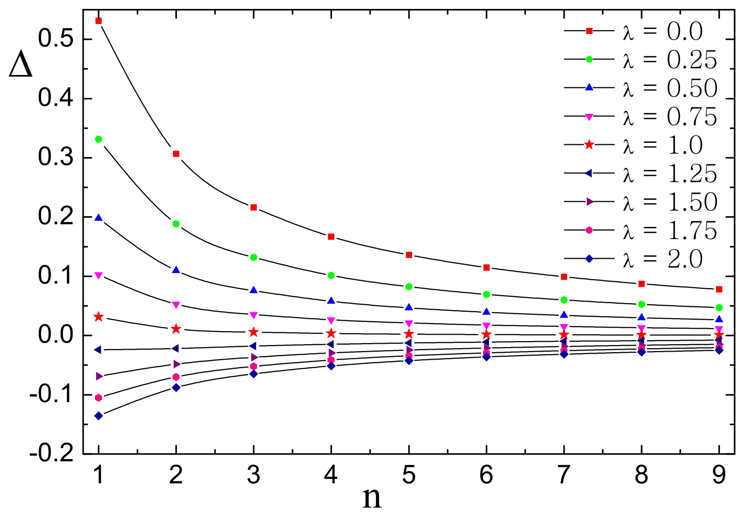

Figure 5: Double scaling limit of $\lim_{N\to\infty,\zeta\to 0}\mathcal{H}(N,\zeta,\lambda)=\mathcal{H}_{\mathrm{Mat}}$ to the smallest exceptional point at $\zeta_M = 1.46877$ with $\Delta(n)=\zeta_0 N(n)-\zeta_M$, $N(n)=(n+1)+n\lambda$ for $n=1,2,3,\ldots$.

4. Weakly orthogonal polynomials

It is well known from Favard's theorem [35, 36] that polynomials $\Phi_n(E)$ constructed from three-term relations in the way mentioned above possess a norm N_n^Φ

$$\mathcal{L}(\Phi_n\Phi_m)=N_n^\Phi\delta_{nm}. \tag{4.1}$$

defined by the action of a linear functional $\mathcal{L}$ acting on arbitrary polynomials p in E as

$$\mathcal{L}(p)=\int_{-\infty}^{\infty}p(E)\omega(E)dE, \qquad \mathcal{L}(1)=1. \tag{4.2}$$

This norm may be computed in two alternative ways. The simplest way is to multiply the three-term relation by Φ_{n-1} and act subsequently on the resulting equation with $\mathcal{L}$. Using the property $N_n^\Phi=\mathcal{L}(\Phi_n^2)=\mathcal{L}(E\Phi_{n-1}\Phi_n)$ together with (4.1) then simply yields $N_n^\Phi=\prod_{k=1}^n b_k$, where the b_k are the negative coefficients in front of Φ_{n-1}. Whereas the first method simply assumes that the functional exist the second method goes further and actually provides an explicit expressions for the measure. As argued in [37] the concrete formulae for $\omega(E)$ may be computed from

$$\omega(E)=\sum_{k=1}^{\ell}\omega_k\delta(E-E_k), \tag{4.3}$$

where the energies E_k are the ℓ roots of the polynomial $\Phi(E)$. The ℓ constants ω_k can be determined by the ℓ equations

$$\sum_{k=1}^{\ell} \omega_k \Phi_n(E_k) = \delta_{n0}, \qquad \text{for } n \in \mathbb{N}_0. \tag{4.4}$$

In our case the integer ℓ are determined from $N = \ell + (\ell - 1)\lambda$ and $N = (\ell + 1) + \ell\lambda$ for the $P_\ell(E)$ and $Q_{\ell+1}(E)$, respectively.

Using the first method we obtain

$$N_n^P = 2\zeta^{2n}(1+\lambda)^{2n} \left(\frac{1-N}{1+\lambda}\right)_n \left(\frac{\lambda+N}{1+\lambda}\right)_n, \qquad n = 1, 2, 3, ... \tag{4.5}$$

$$N_n^Q = \frac{1}{2(N+\lambda)(1-N)} N_n^P, \qquad n = 2, 3, 4, ... \tag{4.6}$$

with $N_0^P = N_1^Q = 1$. Due to the non-Hermitian nature of the Hamiltonian this norm is in general not positive definite. For instance for $N = 4 + 3\lambda$ we have

$$N_0^P = 1, \quad N_1^P = -24\zeta^2(1+\lambda)^2, \quad N_2^P = 240\zeta^4(1+\lambda)^4, \quad N_3^P = -1440\zeta^6(1+\lambda)^6. \tag{4.7}$$

The exception is the class of models where the Hamiltonian becomes Hermitian, i.e. when $\lambda = 1 - 2N$ holds. For this value of λ the expressions in (4.5) and (4.6) become positive definite

$$N_n^P = 2^{1+2n}\zeta^{2n}(N-1)^{2n} \left(\frac{1}{2}\right)_n^2 = 2\zeta^2(N-1)^2 N_n^Q. \tag{4.8}$$

Let us now consider the second method and compute explicitly the measure for a few examples. For $N = 2 + \lambda$ and $N = 3 + 2\lambda$ we solve (4.4) for the even and odd solutions, respectively, to

$$\omega_\pm^c = \frac{1}{2} \pm \frac{1}{2\sqrt{1-(1+\lambda)^2\zeta^2}}, \quad \text{and} \quad \omega_\pm^s = \frac{1}{2} \pm \frac{3}{2\sqrt{9-(1+\lambda)^2\zeta^2}}. \tag{4.9}$$

Computing now (4.1) with (4.2) agrees with (4.5) and (4.6)

$$N_0^P = \mathcal{L}(P_0^2) = \omega_+^c + \omega_-^c = 1 \tag{4.10}$$

$$N_1^P = \mathcal{L}(P_1^2) = \omega_+^c \left(E_2^{c,+} - \lambda\zeta^2\right)^2 + \omega_-^c \left(E_2^{c,-} - \lambda\zeta^2\right)^2 = -4\hat{\zeta}^2, \tag{4.11}$$

$$N_2^Q = \mathcal{L}(Q_2^2) = \omega_+^s \left(E_3^{s,+} - 4 - \lambda\zeta^2\right)^2 + \omega_-^s \left(E_3^{s,-} - 4 - \lambda\zeta^2\right)^2 = -4\hat{\zeta}^2. \tag{4.12}$$

Similarly we compute for $N = 3 + 2\lambda$

$$\omega_1^c = \frac{1}{3} - \frac{\left(260 - 60\hat{\zeta}^2\right)\Omega + \left(3\hat{\zeta}^2 + 4\right)\Omega^2 + 20\Omega^3}{12\left[\left(13 - 3\hat{\zeta}^2\right)^2 + \left(13 - 3\hat{\zeta}^2\right)\Omega^2 + \Omega^4\right]}, \qquad \omega_2^c = \chi_{-2}, \qquad \omega_3^c = \chi_2, \tag{4.13}$$

$$\chi_\ell = \frac{1}{3} + \frac{\left(3\hat{\zeta}^2 - 20\Omega + 4\right)\left(1 + 2e^{\frac{i\pi\ell}{3}}\right)}{36(3\hat{\zeta}^2 + \Omega^2 - 13)} + \frac{4 + 3\hat{\zeta}^2 - 20e^{\frac{i\pi\ell}{3}}\Omega}{12\left(1 + 2e^{\frac{i\pi\ell}{3}}\right)\left(3\hat{\zeta}^2 - 13\right) + \left(1 - e^{\frac{i\pi\ell}{3}}\right)\Omega^2}$$

and confirm that

$$
\begin{aligned}
N_0^P &= \mathcal{L}(P_0^2) = \omega_1^c + \omega_2^c + \omega_3^c = 1, \\
N_1^P &= \mathcal{L}(P_1^2) = \omega_1^c P_1^2(E_3^{c,0}) + \omega_2^c P_1^2(E_3^{c,-2}) + \omega_3^c P_1^2(E_3^{c,2}) = -12\hat{\zeta}^2, \\
N_2^P &= \mathcal{L}(P_2^2) = \omega_1^c P_2^2(E_3^{c,0}) + \omega_2^c P_2^2(E_3^{c,-2}) + \omega_3^c P_2^2(E_3^{c,2}) = 48\hat{\zeta}^4 \\
\mathcal{L}(P_1 P_2) &= \omega_1^c P_1(E_3^{c,0}) P_2(E_3^{c,0}) + \omega_2^c P_1(E_3^{c,-2}) P_2(E_3^{c,-2}) + \omega_3^c P_1(E_3^{c,2}) P_2(E_3^{c,2}) = 0.
\end{aligned}
\tag{4.14}
$$

Note that the last relation in (4.14) does not follow from the first method.

As the final quantity we also compute the moment functionals defined in [35, 36] as

$$
\mu_n := \mathcal{L}(E^n) = \sum_{k=1}^{\ell} \omega_k E_k^n = \sum_{k=0}^{n-1} \nu_k^{(n)} \mu_k, \tag{4.15}
$$

Once again also these quantities can be obtained in two alternative ways, that is either from the computation of the integrals or directly from the original polynomials P_n and Q_n without the knowledge of the constants ω_k. In the last equation the coefficients $\nu_k^{(n)}$ are defined through the expansion $P_n(E) = 2^{n-1}E^n - \sum_{k=0}^{n-1} \nu_k^{(n)} E^k$ and $Q_n(E) = 2^{n-1}E^{n-1} - \sum_{k=0}^{n-2} \nu_k^{(n)} E^k$ for our even and odd solutions, respectively. For the even solutions with $N = 2 + \lambda$ we obtain

$$
\mu_0^P = 1, \tag{4.16}
$$

$$
\mu_1^P = \lambda\zeta^2, \tag{4.17}
$$

$$
\mu_2^P = \lambda^2\zeta^4 - 4\hat{\zeta}^2, \tag{4.18}
$$

$$
\mu_3^P = \lambda^3\zeta^6 - 12\lambda\zeta^2\hat{\zeta}^2 - 16\hat{\zeta}^2, \tag{4.19}
$$

$$
\mu_4^P = \lambda^4\zeta^8 - 24\lambda^2\zeta^4\hat{\zeta}^2 + 16\left(\zeta^2 - 1\right)^2 \zeta^4 - 64\hat{\zeta}^2, \tag{4.20}
$$

and similarly for the odd solutions with $N = 3 + 2\lambda$ we compute for instance

$$
\mu_0^Q = 1, \tag{4.21}
$$

$$
\mu_1^Q = 4 + \lambda\zeta^2, \tag{4.22}
$$

$$
\mu_2^Q = 16 - 4\hat{\zeta}^2 + \lambda^2\zeta^4, \tag{4.23}
$$

$$
\mu_3^Q = \lambda^3\zeta^6 - 12\left(\lambda^3 + \lambda^2 + \lambda\right)\zeta^4 - 48(2\lambda^2 + 3\lambda + 2)\zeta^2 + 64. \tag{4.24}
$$

Thus $\mathcal{H}(N, \zeta, \lambda)$ possesses indeed all the standard features of a quasi-exactly solvable model of E_2-type.

5. Conclusions

Following the principles outlined in [3] we have constructed a new three-parameter quasi-exactly solvable model of E_2-type. One of the parameters can be employed to interpolate between two previously constructed models. With regard to one of the original motivations that triggered the investigation of these models, that is the double scaling limit towards the complex Mathieu equation, we found that for $\lambda = 1$, i.e. $\mathcal{H}_{E_2}^{(1)}$, finite values for N best approximate the complex Mathieu system and mimic its qualitative behaviour. We provided a detailed discussion of the determination of the exceptional points and the energy branch cut structure responsible for the intricate energy loop structure stretching over several Riemann sheets. The coefficient functions are shown to possess the standard properties of weakly orthogonal polynomials.

Acknowledgments: I am grateful to Kazuki Kanki for making reference [15] available to me.

References

[1] A. V. Turbiner, Quasi-Exactly-Solvable problems and sl(2) Algebra, Commun. Math. Phys. **118**, 467–474 (1988).

[2] A. Turbiner, Lie algebras and linear operators with invariant subspaces, Lie Algebras, Cohomologies and New Findings in Quantum Mechanics, Contemp. Math. AMS, (eds N. Kamran and P.J. Olver) **160**, 263–310 (1994).

[3] A. Fring, E2-quasi-exact solvability for non-Hermitian models, J. Phys. **A48**, 145301(19) (2015).

[4] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Optical Solitons in PT Periodic Potentials, Phys. Rev. Lett. **100**, 030402 (2008).

[5] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, PT-symmetric optical lattices, Phys. Rev. **A81**, 063807(10) (2010).

[6] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. Christodoulides, Observation of PT-Symmetry Breaking in Complex Optical Potentials, Phys. Rev. Lett. **103**, 093902(4) (2009).

[7] B. Midya, B. Roy, and R. Roychoudhury, A note on the PT invariant potential $4cos^2x + 4iV_0sin2x$, Phys. Lett. **A374**, 2605–2607 (2010).

[8] H. Jones, Use of equivalent Hermitian Hamiltonian for PT-symmetric sinusoidal optical lattices, J. Phys. **A44**, 345302 (2011).

[9] E. Graefe and H. Jones, PT-symmetric sinusoidal optical lattices at the symmetry-breaking threshold, Phys. Rev. **A84**, 013818(8) (2011).

[10] S. Dey, A. Fring, and T. Mathanaranjan, Non-Hermitian systems of Euclidean Lie algebraic type with real eigenvalue spectra, Annals of Physics **346**, 28–41 (2014).

[11] S. Dey, A. Fring, and T. Mathanaranjan, Spontaneous PT-symmetry breaking for systems of noncommutative Euclidean Lie algebraic type, arXiv:1407.8097, to appear in Int. J. Theor. Phys.

[12] S. Longhi and G. Della Valle, Invisible defects in complex crystals, Annals of Physics **334**, 35–46 (2013).

[13] K. Makris, C. D. Musslimani, Z.H., and S. Rotter, Constant-intensity waves and their modulation instability in nonHermitian potentials, arXiv:1503.08986.

[14] K. Kanki, Spontaneous breaking of a PT-symmetry in the Liouvillian dynamics at a nonhermitian degeneracy point, talk at the 15th International Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics, May 18-23, University of Palermo, Italy (2015).

[15] Z. Zhang, Irreversibility and extended formulation of classical and quantum nonintegrable dynamics, PhD Thesis, The University of Texas at Austin (1995).

[16] A. Khare and B. P. Mandal, A PT-invariant potential with complex QES eigenvalues, Phys. Lett. A **272**, 53–56 (2000).

[17] B. Bagchi, S. Mallik, C. Quesne, and R. Roychoudhury, A PT-symmetric QES partner to the Khare–Mandal potential with real eigenvalues, Phys. Lett. A **289**, 34–38 (2001).

[18] C. M. Bender and M. Monou, New quasi-exactly solvable sextic polynomial potentials, J. Phys. A **38**, 2179–2187 (2005).

[19] B. Bagchi, C. Quesne, and R. Roychoudhury, A complex periodic QES potential and exceptional points, J. Phys. A **41**, 022001 (2008).

[20] F. G. Scholtz, H. B. Geyer, and F. Hahne, Quasi-Hermitian Operators in Quantum Mechanics and the Variational Principle, Ann. Phys. **213**, 74–101 (1992).

[21] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, Phys. Rev. Lett. **80**, 5243–5246 (1998).

[22] C. M. Bender, Making sense of non-Hermitian Hamiltonians, Rept. Prog. Phys. **70**, 947–1018 (2007).

[23] A. Mostafazadeh, Pseudo-Hermitian Representation of Quantum Mechanics, Int. J. Geom. Meth. Mod. Phys. **7**, 1191–1306 (2010).

[24] A. Fring, A new non-Hermitian E2-quasi-exactly solvable model, Phys. Lett. **379**, 873–876 (2015).

[25] E. Wigner, Normal form of antiunitary operators, J. Math. Phys. **1**, 409–413 (1960).

[26] T. Kato, Perturbation Theory for Linear Operators, (Springer, Berlin) (1966).

[27] W. D. Heiss, Repulsion of resonance states and exceptional points, Phys. Rev. E **61**, 929–932 (2000).

[28] I. Rotter, Exceptional points and double poles of the S matrix, Phys. Rev. E **67**, 026204 (2003).

[29] U. Günther, I. Rotter, and B. F. Samsonov, Projective Hilbert space structures at exceptional points, Journal of Physics A: Mathematical and Theoretical **40**, 8815 (2007).

[30] W. D. Heiss and H. Harney, The chirality of exceptional points, The European Physical Journal D - Atomic, Molecular, Optical and Plasma Physics **17**, 149–151 (2001).

[31] H. Mehri-Dehnavi and A. Mostafazadeh, Geometric phase for non-Hermitian Hamiltonians and its holonomy interpretation, Journal of Mathematical Physics **49**, 082105 (2008).

[32] I. Rotter, A non-Hermitian Hamilton operator and the physics of open quantum systems, Journal of Physics A: Mathematical and Theoretical **42**, 153001 (2009).

[33] W. D. Heiss, The physics of exceptional points, Journal of Physics A: Mathematical and Theoretical **45**, 444016 (2012).

[34] W. D. Heiss and G. Wunner, Fano-Feshbach resonances in two-channel scattering around exceptional points, The European Physical Journal D **68** (2014) 284.

[35] J. Favard, Sur les polynomes de Tchebicheff., C. R. Acad. Sci., Paris **200**, 2052–2053 (1935).

[36] F. Finkel, A. Gonzalez-Lopez, and M. A. Rodriguez, Quasiexactly solvable potentials on the line and orthogonal polynomials, J. Math. Phys. **37**, 3954–3972 (1996).

[37] A. Krajewska, A. Ushveridze, and Z. Walczak, Bender–Dunne Orthogonal Polynomials General Theory, Mod. Phys. Lett. A **12**, 1131–1144 (1997).