Advancing Integrability for Strings in AdS$_3$/CFT$_2$

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Finally I would like to thank my family and my partner Jenny for the countless ways, both small and large, they have supported me throughout this project.

Declaration

The work contained in this thesis was carried out by the author while studying for the degree of Doctor of Philosophy at City University London. Parts of this work were previously published in the co-authored papers [1] and [2].

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Abstract

In this thesis we develop techniques of integrability in the study of dualities between two-dimensional conformal field theories and theories of closed strings on three-dimensional Anti-de Sitter background geometries. For several years after integrability was first applied to the 3d/2d dualities, it was an unanswered question how to incorporate the so-called "massless modes" of these theories into the integrability machinery. Here we tackle this problem in several contexts. We show that in the classical integrable description of closed strings the implementation of the string Virasoro constraints needs to be modified for geometries with multiple factors where massless modes are present. We show further that with the correct implementation of the Virasoro constraints, massless modes can be included in integrability techniques for obtaining quantum corrections to physical quantities such as the energies of string solutions. Lastly, we consider the scattering of fundamental string excitations and derive all-loop expressions for the scattering matrix that includes both massless and massive excitations.

Chapter 1

Introduction: gauge/gravity dualities

1.1 Elements of String Theory

String theory is a unified theory of quantum gauge and gravity interactions. The concept of a fundamental relativistic string can be understood by comparison with a point-particle. A single particle propagating in a spacetime with coordinates X^m can be described by its worldline $X^m(\tau)$ giving its position at each moment in proper time τ. A fundamental string extends along a spacelike coordinate σ in addition to a timelike coordinate τ, and is described by its worldsheet $X^m(\tau, \sigma)$. The coordinate σ may be identified periodically in which case we call the string closed, otherwise it is an open string.
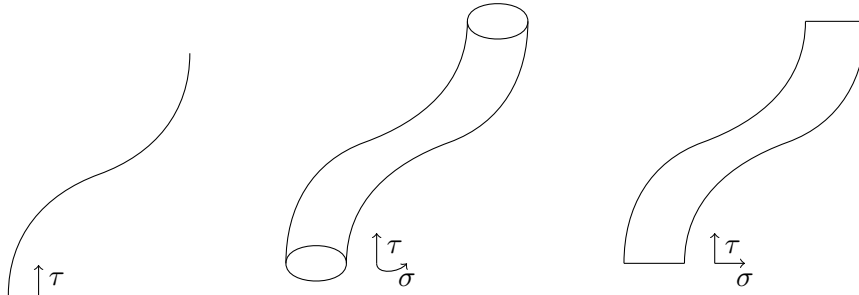
Figure 1: The worldline for a particle (left) and worldsheets for a closed string (centre) and open string (right).

There are two parameters that describe fundamental strings. One is the dimensionless string coupling constant g_s, which controls the strength of the string splitting and joining interactions. In perturbative calculations an expansion in g_s corresponds to a sum over different string topologies. The other fundamental parameter is the string scale l_s, or equivalently the string tension T, which are related by

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2} \ ,$$

(1.1)

where the parameter α' is called the Regge slope. The string can be excited along vibrational modes, and the tension determines the energy of these excitations. The low energy limit is $\alpha' \to 0$. The natural way to study string theory perturbatively is to consider a dual expansion in the parameters α' and g_s. If one first expands in g_s then each term in this expansion corresponds to a particular string topology as mentioned above. Next, given a fixed string
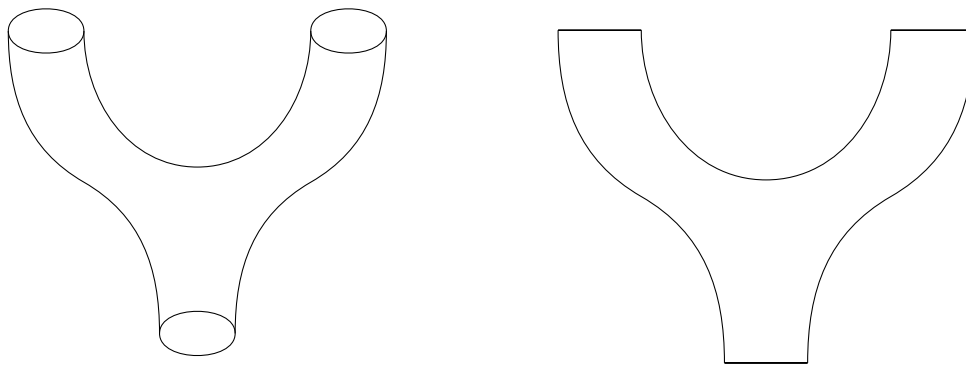
Figure 2: String interactions: splitting of a closed string (left) and an open string (right).

topology, there is an expansion in α'.

String theory places demands on the spacetime in which it is constructed in order to maintain physical consistency. In particular, superconformal invariance requires that the spacetime be ten-dimensional and obey the supergravity generalised Einstein equations [3]. The simplest possible background is ten-dimensional Minkowski spacetime. In the Green-Schwarz formalism [4] for implementing supersymmetry, the theory is described by the evolution of a string not just in spacetime, but in superspace. This means that in addition to the bosonic fields $X^m(\tau, \sigma)$ there are spacetime fermionic fields $\theta^I(\tau, \sigma)$. In the theory of type IIB superstrings that we will focus on, there are two such fermions, $I = 1, 2$, and each is a Majorana-Weyl spinor of the same chirality. This means that each has 16 independent real components and in total there are 32 supersymmetries.

In the spectrum of vibrational excitations of the closed superstring in the critical dimension, the lowest energy states are massless. The bosonic states transform as a spacetime tensor which can be decomposed into symmetric traceless, antisymmetric and trace parts. The first of these is a graviton field G_{mn}, the second B_{mn} is referred to as the B-field and the last Φ is called the dilaton. The B-field acts as a potential for a Neveu-Schwarz-Neveu-Schwarz (NS-NS) 3-form H_{mnp}. There are also Ramond-Ramond (R-R) fields arising from odd and even rank potentials in the type IIA and type IIB theories respectively. For the backgrounds of interest to us in this thesis the only non-zero R-R field is the R-R 3-form F_{mnp}. Consistency of the theory implies conditions on these fields which are identical to the equations of motion arising from a particular ten-dimensional action. This action is that of type IIB supergravity [5–8]. Since in the low energy limit $\alpha' \to 0$ other states above these massless states are suppressed, we conclude that type IIB supergravity is the low energy limit of type IIB superstring theory.

Open strings have endpoints, and so variation of the string action places boundary conditions at these endpoints in addition to equations of motion. The two simplest boundary conditions are Neumann boundary conditions,

where we set $\partial_\sigma X^m = 0$ at the endpoints, and Dirichlet boundary conditions, where we fix X^m to be constant at the endpoints. If we choose Neumann boundary conditions for endpoints in spatial dimensions $m = 0 \ldots p$, then the string endpoints are fixed along the remaining $(10 - p - 1)$ dimensions, i.e. they are constrained to lie in a $(p+1)$-dimensional hypersurface, which we call a Dp-brane [9, 10].

D-branes have a dual identity in string theory. As well as their role as endpoints for open strings, they also appear as particular solutions called p-branes in supergravity, meaning that they are dynamical objects of superstring theory in their own right [11]. As objects in string theory they couple to R-R fields. For type IIB supergravity, stable Dp-branes exist for p odd. In addition there are 5-branes that are also charged under the bosonic B-field, the so-called NS5-branes.

The low energy spectrum of open type II strings again starts with massless states. Consider the excitations of an open string ending on a Dp-brane from the point of view of the $(p + 1)$-dimensional space in which the brane propagates. From this perspective the bosonic massless excitations of the open string divide into those perpendicular to the brane, which appear as massless scalars, and those parallel to the brane, which appear as a gauge vector. In this way the excitations on a single Dp-brane describe a $U(1)$ gauge theory. This gauge symmetry can be enhanced to a non-abelian one if we consider open strings on not just a single Dp brane, but N branes coincident with one another. In this case we can introduce labels called Chan-Patton factors [12] which specify which of the N branes the endpoints of an open string are on. Counting these labels for both ends of the string, each state now comes with a multiplicity of N^2. In particular the gauge group is enhanced to $U(N)$ [13]. The explicit form of the resulting gauge theory is that of a Yang-Mills theory [14].

1.2 The Maldacena Conjecture

The Maldacena conjecture [15] relates strings on a background involving a factor of $(d + 1)$-dimensional Anti-de Sitter (AdS) space to a conformal field theory (CFT) in d dimensions. The canonical example of this conjecture states that the theory of type IIB strings propagating on a background of $AdS_5 \times S^5$ with constant R-R 5-form flux is dual to the gauge theory known as $\mathcal{N} = 4$ Super-Yang-Mills [16, 17]. This conjecture arises by considering first type IIB strings in flat space with a stack of N coincident D3-branes which will then backreact on the geometry. The system is examined in the low energy limit from two perspectives corresponding to the two guises of D-branes as endpoints for open strings and as p-brane supergravity solutions.

The supergravity metric arising from the stack of D3-branes is [18]

$$ds^2 = \frac{1}{\sqrt{f(r)}} \left(-dt^2 + \sum_{i=1}^{3} dx_i^2 \right) + \sqrt{f(r)} \left(dr^2 + r^2 d\Omega_5^2 \right) \ , \qquad (1.2)$$

where $d\Omega_n^2$ is the usual round metric in n-dimensions, and r is the associated radial coordinate. The function $f(r)$ is given by

$$f(r) = 1 + \frac{R^4}{r^4} \ , \qquad (1.3)$$

so that this solution has a horizon at $r = 0$. Finally the parameter R is related to the string parameters g_s and α' and the number N by

$$R^4 = 4\pi g_s \alpha'^2 N \ . \qquad (1.4)$$

The number N of D3-branes in the construction enters the low energy supergravity solution in the 5-form R-R flux F_5, with

$$\int_{S^5} F_5 = N \ . \qquad (1.5)$$

The Maldacena conjecture arises by looking at the low energy limit of this setup of D3-branes from two perspectives. In the first, we have a set of open strings propagating on the branes and closed strings in the bulk (the rest of spacetime away from the branes). In the low energy limit, the interaction between closed and open strings is subleading and so we have two decoupled systems. The dynamics of the open strings are described by a gauge theory on the four-dimensional space spanned by the D3-branes, with gauge group $SU(N)$. This gauge theory is $\mathcal{N} = 4$ Super-Yang-Mills (SYM). It has been shown that its beta function is exactly zero [19–22], and so it is a CFT.

In the second perspective, we replace the open strings by the backreaction of the D3-branes on the geometry, that is we have a system of closed strings propagating in the spacetime (1.2). One part of the low energy limit of this spacetime is again free supergravity coming from low energy excitations of strings in the bulk away from the branes. Another comes from the near-horizon geometry. The energy E of an excitation at distance r from the horizon is related to the energy E_∞ that an observer at infinity sees by the redshift factor

$$E_\infty = f(r)^{-\frac{1}{4}} E \ . \qquad (1.6)$$

Low energy excitations in the bulk are unable to be brought close to the horizon. Therefore, in the low energy limit of the second perspective we again have two decoupled systems, one of which is free supergravity. The other is closed strings on $AdS_5 \times S^5$, since in the near-horizon limit $r \ll R$, the metric (1.2) approaches that of $AdS_5 \times S^5$, with the metric for n-dimensional

Anti-DeSitter space in the form

$$ds^2_{AdS_n} = -dt^2 + \frac{1}{r^2}dr^2 + \sum_{i=1}^{n-2} dx_i^2 \ . \tag{1.7}$$

By comparing the two perspectives, we are led to infer that this is an equivalent system to $\mathcal{N} = 4$ SYM.

In the SYM gauge theory describing the dynamics of the open strings, the physical parameters are the Yang-Mills coupling g_{YM} and the number of colour charges N. The duality relates these to the string parameters g_s and α' via the following identifications:

$$g_s = \frac{g_{YM}^2}{4\pi} \ , \quad \frac{R^4}{\alpha'^2} = g_{YM}^2 N \ . \tag{1.8}$$

The supergravity description of the backreaction is appropriate when the radius of curvature R of the spacetime is much larger than the string scale, $R \gg l_s$. On the CFT side of the duality this is the region $g_{YM}^2 N \gg 1$. Hence the duality relates low energy strings to the strongly coupled sector of the gauge theory, and vice versa. This is an example of a strong-weak duality. It means that perturbative string quantization and perturbative field theory expansions cannot be compared, making the duality harder to test. On the other hand, it promises the possibility of solving problems of strong coupling by solving in the dual weakly coupled regime.

1.2.1 The planar limit

The "large N" or "planar" limit of the field theory corresponds to taking $g_{YM} \to 0$, $N \to \infty$ but keeping the 't Hooft coupling λ defined as

$$\lambda \equiv g_{YM}^2 N = \frac{R^4}{\alpha'^2} \tag{1.9}$$

fixed. It has been known for a long time [23] that in this limit of Yang-Mills theories there is a simplification of perturbative calculations. The only contributions that survive the limit come from Feynman diagrams that are planar, meaning they can be drawn in two dimensions without crossing, see figure 3. We can see from equation (1.8) relating the parameters that on the string side this limit corresponds to $g_s \to 0$, i.e. to a limit of free strings with no interactions. Therefore, the planar limit of the Maldacena conjecture is a duality between planar $\mathcal{N} = 4$ SYM and free superstrings on $AdS_5 \times S^5$.

It is in the planar limit that integrability appears. On the gauge side, integrability appears in terms of an integrable spin-chain [24]. We will discuss the integrability of free strings on particular backgrounds in the next chapter.
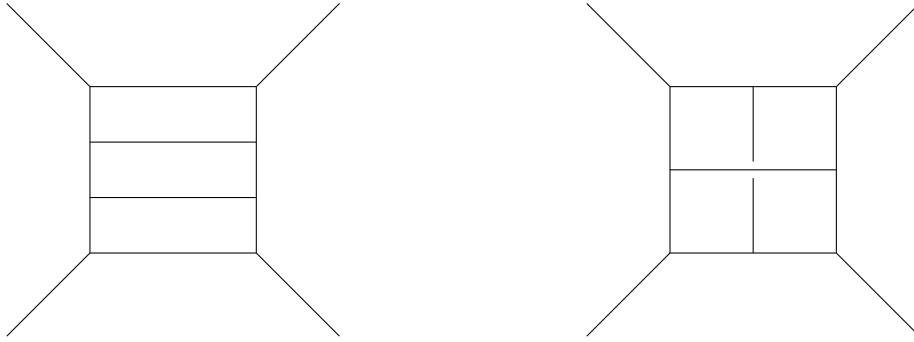
Figure 3: A planar (left) and non-planar (right) Feynman diagram.

1.3 AdS_3/CFT_2 dualities

1.3.1 AdS_3 backgrounds from brane constructions

Alongside the AdS_5/CFT_4 correspondence outlined above, another duality was posited [15] which related strings on a background containing AdS_3 to a two-dimensional CFT. The brane construction leading to this duality was a system of N_1 D1-branes together with N_5 D5-branes.[1] Four spatial dimensions along which the D5-branes extend are compactified on a T^4. The D1-branes then extend along the same remaining non-compact spatial dimensions which the $D5$-branes span. The metric for this setup is

$$ds^2 = \frac{1}{\sqrt{f_1 f_5}}(-dt^2 + dx_1^2) + \sqrt{f_1 f_5}\left(dr^2 + r^2 d\Omega_3^2\right) + \sqrt{\frac{f_1}{f_5}}\sum_{i=6}^{9} dx_i^2 \quad (1.10)$$

where

$$f_1(r) = 1 + \frac{g_s \alpha' N_1}{v r^2} \ , \quad f_5(r) = 1 + \frac{g_s \alpha' N_5}{r^2} \ , \quad v = \frac{V_{T^4}}{(2\pi)^4 \alpha'^2} \ , \quad (1.11)$$

and V_{T^4} is the volume of the T^4. The low energy limit involving $\alpha' \to 0$ is taken together with the T^4 compactification in such a way that v remains finite. Just as in the D3-brane setup, there is a horizon at $r = 0$ and in the low energy limit the near-horizon geometry decouples from that of the bulk. In this case the near-horizon limit $r \to 0$ produces the background $AdS_3 \times S^3 \times T^4$. The radii of both AdS_3 and S^3 are equal and given by

$$R = \frac{g_s \alpha' \sqrt{N_1 N_5}}{\sqrt{v}} \ . \quad (1.12)$$

In both the D3-brane stack and the D1-D5 system, the branes couple to R-R fields, and a full description of the supergravity solution includes these fields. In the case of the D1-D5 system, there is a 3-form R-R flux $F^{(3)}$ which in the near-horizon limit is a constant, proportional to the volume forms on AdS_3

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[1]This D1-D5 system was already well-studied for its use in describing black-hole microstates in string theory [25].

and S^3. More generally, superstrings on this background can be supported by a mix of R-R and NS-NS fluxes. Supergravity backgrounds supported by these mixed fluxes are generated by NS5-branes and fundamental strings in addition to D1-branes and D5-branes. There is a one-parameter family of supersymmetric backgrounds involving this mix of branes, and so a one-parameter family of dualities with mixed fluxes. Namely, when we set the AdS_3 radius to 1, the fluxes can be written as

$$F = \tilde{q}(\text{Vol}(AdS_3) + \text{Vol}(S^3)) \ , \quad H = q(\text{Vol}(AdS_3) + \text{Vol}(S^3)) \qquad (1.13)$$

where

$$q^2 + \tilde{q}^2 = 1 \ . \qquad (1.14)$$

Another set of AdS_3/CFT_2 dualities is found involving the background $AdS_3 \times S^3 \times S^3 \times S^1$. The radii R_+ and R_- of the two three-spheres in this background are related to the radius R of AdS_3 as follows:

$$\frac{1}{R^2} = \frac{1}{R_+^2} + \frac{1}{R_-^2} \ , \qquad (1.15)$$

which is required to make the background a consistent supergravity solution. We introduce a parameter α defined by

$$\frac{1}{R_+^2} = \frac{\alpha}{R^2} \ , \quad \frac{1}{R_-^2} = \frac{1 - \alpha}{R^2} \ , \qquad (1.16)$$

and again we have a one-parameter family of such backgrounds. This parameter α appears in the symmetry superalgebra of these backgrounds as discussed in section 1.3.2. When giving worldsheet expressions on this background in this thesis we will generally use the alternative parameter ϕ defined by

$$\alpha = \sin^2 \phi \ . \qquad (1.17)$$

The brane construction used to construct this background [26–28] is that of a D1-D5-D5' system, which can be thought of as a D1-D5 system to which is added a second set of D5-branes, compactified on the directions transverse to the first set and vice versa. We denote this setup in the following diagram:

	1	2	3	4	5	6	7	8	9
$D1$	×								
$D5$	×	×	×	×	×				
$D5'$	×					×	×	×	×

where ×'s mark spatial directions spanned by the various branes, see also figure 4. One subtlety is that this construction actually leads to the geometry of $AdS_3 \times S^3 \times S^3 \times \mathbb{R}$ in the near-horizon limit, and so it is believed that to obtain a dual conformal theory there is some additional process of compactification
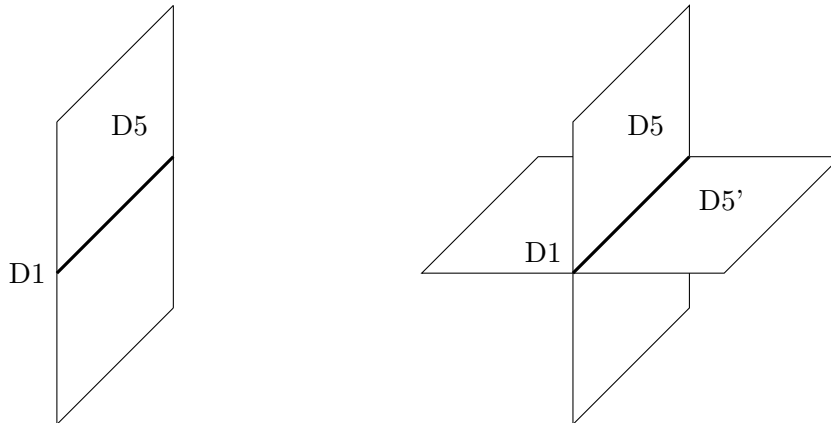
Figure 4: Three-dimensional representation of a D1-D5 system (left) and D1-D5-D5' system (right).

from $\mathbb{R}$ to S^1.

Early progress in studying strings on these AdS_3 backgrounds was achieved using worldsheet CFT techniques [29–36] to study the pure NS-NS backgrounds, which are related to the pure R-R backgrounds by S-duality. The use of integrability in AdS/CFT dualities, which we discuss in the following chapter, avoids the need for applying an S-duality and also makes it possible to study the mixed-flux backgrounds.

1.3.2 Dual CFT_2

Identifying the dual conformal gauge theory to the AdS_3 string background is a harder problem than in the case of AdS_5/CFT_4. In particular, the gauge theory describing the open string dynamics in the low energy limit is not conformally invariant. The dual CFT is instead conjectured to arise from the renormalization group flow to the IR of this gauge theory. Another important difference between this AdS_3/CFT_2 duality and the canonical AdS_5/CFT_4 duality is the presence of a large moduli space [37]. $\mathcal{N} = 4$ SYM is a theory with only two tuneable parameters: λ and N. However, there are various scalars which arise from the open string dynamics on the D1-D5 system which have non-zero expectation values. The field content of the D1-D5 systems contains vector multiplets and hypermultiplets. Both of these multiplets contain scalar fields. The two branches of the moduli space are the Higgs branch, where the hypermultiplet scalars have non-zero vacuum expectation values, and the Coulomb branch where the vector multiplet scalars have non-zero vacuum expectation values. The IR Higgs branch CFT is conjectured to be dual to AdS_3 strings [15], because it is this branch that corresponds to motion of the D1-branes inside the D5-branes and so is consistent with the near-horizon picture.

The CFT can also be understood by treating the D1-branes as an instanton on the D5-branes [38], and it has been proposed [39] that it is at the point

in the moduli space where this instanton shrinks to zero size that the spin-chain picture which integrability relies on emerges. It has been believed for some time that this moduli space can be obtained as a deformation of the space $Sym^N(T^4)$, the symmetric space of N copies of T^4, where $N = N_1 N_5$ is the product of the number N_1 of D1-branes and N_5 of D5-branes [40]. Early work on the CFT side of the AdS_3 dualities was undertaken by studying the $Sym^N(T^4)$ orbifold theories and their deformations in [41–44].

In identifying the CFT dual of $AdS_3 \times S^3 \times S^3 \times S^1$, there is the additional problem that in studying the open string dynamics on the brane construction, one has to deal with the difference between the factor of $\mathbb{R}$ in the near-horizon limit and the factor of S^1 in the final geometry. Partly owing to this, the dual conformal theory to this background has proved among the most elusive in all AdS/CFT dualities, with early work studying it in [45] and since then in [46, 47]. One thing that is known is the necessary superconformal symmetry algebra of the dual CFT. All the brane constructions discussed in this section preserve 16 real supersymmetries. In two dimensions these can be decomposed by chirality and we say that we have $\mathcal{N} = (4, 4)$ supersymmetry in two dimensions. Unlike for example $\mathcal{N} = 4$ supersymmetry in four dimensions, we can distinguish further between different supersymmetry algebras [48]. The CFT dual to $AdS_3 \times S^3 \times T^4$ preserves what is called the small $\mathcal{N} = (4, 4)$ algebra, while the CFT dual to $AdS_3 \times S^3 \times S^3 \times S^1$ preserves the large $\mathcal{N} = (4, 4)$ algebra. These algebras are infinite-dimensional. Their finite-dimensional subgroups which can be defined globally are the algebras of the groups of superisometries of the dual string backgrounds, namely $PSU(1, 1|2)^2$ for $AdS_3 \times S^3 \times T^4$ and $D(2, 1; \alpha)^2$ for $AdS_3 \times S^3 \times S^3 \times S^1$. The parameter α here is the same one appearing in the geometry that defines the relative radii of the two three-spheres as in equation (1.16).

In this thesis we will focus on studying the string side of the AdS_3/CFT_2 dualities introduced in this chapter. In the next chapter we introduce the ideas and techniques we will use to do so.

Chapter 2

Strings and Integrability

In this chapter we will outline the basics of the ideas and techniques used in this thesis. Full definitions of the ideas we introduce here will generally be given in the main text.

2.1 Strings in Anti-de Sitter Space

2.1.1 The String Sigma Model and its symmetries

Our starting point is the standard bosonic string action on a curved background, the non-linear sigma model. For a background with spacetime metric $G_{mn}(X)$ and supported by a non-zero two-form gauge field $B_{mn}(X)$, the action is

$$S_{bos} = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left(\sqrt{-\gamma}\gamma^{\alpha\beta} G_{mn}(X) + \epsilon^{\alpha\beta} B_{mn}(X) \right) \partial_\alpha X^m \partial_\beta X^n \ , \quad (2.1)$$

where X^m are the spacetime coordinates, $\gamma_{\alpha\beta}$ is the worldsheet metric, carrying indices associated to the worldsheet coordinates $\sigma^\alpha = (\tau, \sigma)$ and $\epsilon^{\alpha\beta}$ is the alternating tensor. Since we will be interested in critical superstrings of type IIB, the background spacetime will always be ten-dimensional. In particular we will be interested in the AdS backgrounds discussed in the previous chapter and in limits of these backgrounds.

As in flat space, this action possesses several symmetries at the classical level, namely invariance under reparametrisation of the worldsheet coordinates,

$$\sigma^\alpha \to f^\alpha(\sigma) \qquad (2.2)$$

and Weyl or scale invariance under transformation of the worldsheet metric

$$\gamma_{\alpha\beta} \to \Omega^2(\sigma)\gamma_{\alpha\beta} \ . \qquad (2.3)$$

As well as the equations of motion coming from varying the action with respect to the spacetime fields X^m, the action (2.1.1) also gives us the Virasoro constraints [49],

$$G_{mn}(X) \left(\dot{X}^m \pm X'^m \right) \left(\dot{X}^n \pm X'^n \right) = 0 \qquad (2.4)$$

arising, as in flat space, from variations with respect to the worldsheet metric.

The full superstring action is given by the bosonic action above together with a fermionic part. As in flat space, one convenient description of the

latter is given by a Green-Schwarz action [4, 50], but this now requires us to describe spinors on a curved background. We give the details of how this Green-Schwarz action is described in chapter 5 and an alternative formulation of the fermionic action in a group theoretic description in section 2.2.2. For now we will note several important points about the Green-Schwarz action.

For type IIB strings the fermionic fields are two 10-dimensional spinors θ_I, $I = 1, 2$. These are Majorana-Weyl spinors, meaning they satisfy first the reality condition that they are equal to their Majorana conjugate,

$$\theta_I^C = \theta_I \ , \tag{2.5}$$

and second that they have definite chirality,[1]

$$\Gamma^{0123456789}\theta_I = \theta_I \ . \tag{2.6}$$

In appendix B we give an explicit basis of ten-dimensional gamma matrices Γ which we use throughout this thesis whenever one is needed, and define Majorana conjugation in terms of this explicit basis. Each of the two conditions above reduces the number of independent components of the spinors by half, so having started with 32 complex components they end up with 16 independent real components.

As discussed in chapter 1, the backgrounds of interest to us arise in the near-horizon limit of particular brane constructions, and as such the resulting backgrounds are supported by the p-forms which are the charges of these branes. In particular, the AdS_3 backgrounds of interest to us can in general be supported by a combination of R-R and NS-NS 3-form fluxes. The fermionic fields θ^I couple to themselves and to the bosonic fields through these 3-form fluxes, just as the bosonic fields X^m couple amongst themselves through the NS-NS B-field in the bosonic action (2.1.1).

In flat space, the fermionic kappa symmetry which we discuss below can be fixed in such a way that the Green-Schwarz action becomes quadratic in fermionic fields. For a generic curved background this is no longer possible. However, the general type II action is known to quartic order in fermions [51] and for the backgrounds of interest here, kappa symmetry can be fixed in such a way that there are no higher order terms above this [52]. Regardless, for our purposes in this thesis it will suffice to use the action to quadratic order in fermions.

The full Green-Schwarz action is manifestly invariant under spacetime supersymmetry, which is a global fermionic symmetry. There is also another fermionic off-shell symmetry of the theory, called kappa-symmetry, which is a gauge symmetry. Fixing a particular gauge for this symmetry is neces-

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[1]Since we are interested in type IIB strings, both spinors θ_1 and θ_2 have the same chirality, which we have taken to be positive in equation (2.6).

sary to show explicitly that the Green-Schwarz action is also invariant under
worldsheet supersymmetry. We will introduce it here by the simplest theory
in which such a symmetry shows up [53]: the worldline action of a massless
superparticle given by

$$S = \int \mathrm{d}\tau \frac{1}{e} \left(\dot{X}^m - i\bar{\theta}\Gamma^m\dot{\theta} \right)^2 . \tag{2.7}$$

This action is invariant under the transformation

$$\delta\theta = i\Gamma^m p_m \kappa , \quad \delta X^m = i\bar{\theta}\Gamma^m \delta\theta , \quad \delta e = 4e\dot{\bar{\theta}}\kappa , \tag{2.8}$$

where p_m is the conjugate momentum to X^m and $\kappa = \kappa(\tau)$ is any spinor
function of proper time.

Gauge-fixing this symmetry has the effect of reducing the number of in-
dependent fermionic components by half. This last fact can be observed as
follows. The equation of motion for θ is $(\Gamma^m p_m)\dot{\theta} = 0$. Meanwhile the analo-
gous statement of the Virasoro constraints for the superparticle action is the
vanishing of the total momentum

$$p^2 = 0 , \quad p^m = \dot{X}^m - i\bar{\theta}\Gamma^m\dot{\theta} . \tag{2.9}$$

which in turn implies $(\Gamma^m p_m)^2 = 0$. Hence the rank of the matrix $\Gamma^m p_m$ is
at most half of its size (as follows for example from the rank-nullity theo-
rem). Indeed as there are no other constraints on the momentum, it is exactly
half. We conclude that half of the components of θ do not actually enter the
equations of motion and cannot be physical. This ensures that the number
of independent real components of the fermions is equal to the number of
transverse bosons, as is required for consistent worldsheet supersymmetry.

This argument we have presented to show how kappa symmetry arises for
a superparticle in flat space in fact extends to the full superstring action in flat
space [4] and further to any supergravity background [54]. Kappa symmetry
is as such a symmetry of the full string action.

2.1.2 Gauge-fixing

In this section we discuss the various gauge choices that can be made to fix
the symmetries discussed in the previous section. In looking at these gauge
choices we draw attention to the issues that arise on curved backgrounds in
general, and on the *AdS* spaces of interest to us in particular.

One gauge choice that can be made is to use reparametrisation of the
worldsheet coordinates and Weyl invariance to fix conformal gauge, where the
worldsheet metric is fixed to be everywhere equal to the 2d Minkowski metric,
$\gamma^{\alpha\beta} = \eta^{\alpha\beta}$. In flat space this gauge choice leaves enough symmetry left in the
bosonic sector to also fix either lightcone gauge or static gauge, however this

is no longer true for generic curved backgrounds as we discuss further below.

Lightcone gauge [55] is given by choosing lightcone spacetime coordinates

$$x^{\pm} = \psi \pm t \ , \tag{2.10}$$

then fixing

$$x^{+} = p^{+}\tau \tag{2.11}$$

and then using the Virasoro constraints to solve for x^{-} in terms of the remaining fields. On curved backgrounds, as in flat space, it is a convenient gauge to quantize the theory in because the Virasoro constraints are solved at the classical level before quantizing. For supersymmetric theories, the use of lightcone gauge is complicated by the need to combine it with a choice of kappa gauge, and the resulting choice for the latter is often referred to as lightcone kappa gauge.

As mentioned above, lightcone gauge cannot be consistently taken with conformal gauge for a generic background spacetime. The question of when these gauges are consistent was addressed in [56, 57], where it was found that these gauges can be simultaneously imposed for spacetimes which possess a covariantly constant null Killing vector, in addition to satisfying the usual conditions for Weyl invariance such as being in the critical dimension. The origin of this additional requirement is that if (2.11) can be imposed, then the vector

$$V = \frac{\partial}{\partial_{x^-}} \tag{2.12}$$

is such a constant null Killing vector. Note that the AdS spacetimes of interest to us do not satisfy this condition[2], and as such the fixing of lightcone gauge for these backgrounds [58, 59] does not use the conformal gauge. We will discuss in section 2.1.3 examples of non-flat spacetimes where this condition is satisfied.

Static gauge consists of the choice

$$t = \kappa\tau \tag{2.13}$$

where t represents the usual time coordinate of AdS. In some sense, it represents a half-way choice between lightcone quantization and conformal quantization, since the number of degrees of freedom is reduced by one below the critical dimension, but is still one higher than the number of physical degrees of freedom. However, at the classical level it is often a natural choice to study particular string solutions where the Virasoro constraints can be checked explicitly. These solutions can then be studied semiclassically in the static gauge by requiring the timelike bosonic field to not receive quantum corrections. Be-

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[2] Although they possess null Killing vectors, they are not covariantly constant ones. This is necessary so that, for example, $G_{--} = 0$, and thus the Virasoro conditions have a linear solution for x^-.

cause of its frequent use in classical and semiclassical string solutions, it is generally encoded into the classical description of strings in the integrability framework which we discuss below.

Different kappa gauges have been studied. One of these is known as the coset gauge [60], which for example on $AdS_3 \times S^3 \times T^4$ is [61]

$$\Gamma^{6789}\theta^I = 0 \ , \tag{2.14}$$

where 6789 are the T^4 directions. In this gauge the Green-Schwarz superstring action reduces to a coset action. Any such action as this admits an algebraic structure and is classically integrable [62]. However, the coset kappa gauge does not lead to a conventional quadratic kinetic term for the massless fermions, but rather a kinetic term which is higher order in fields. Such a higher order kinetic term is not convenient for the introduction of canonical Poisson brackets. As such we will find it convenient, following [63–65], to use a different kappa gauge,

$$\Gamma^+\eta_I = 0 \ , \quad \Gamma^+\chi_I = 0 \ , \tag{2.15}$$

where η_I and χ_I are suitable redefined fermions. We will discuss the exact field redefinitions required in chapter 5. Here we will simply mention that η, χ are defined so as to make them neutral under the $U(1)$'s associated with shifts in t and ψ.

2.1.3 Strings on plane-waves

We turn now to strings on backgrounds which are not Anti-de Sitter, but a class of spacetimes known as plane-wave spacetimes. These arise in particular limiting process from other geometries. Studying strings on these backgrounds is an interesting question in its own right because they provide a rare example where it is understood how to exactly quantize the theory even in the presence of curvature. We will be interested in them because of the guide they provide to understanding strings on AdS spaces. Once we fix a particular lightcone gauge, if we consider an expansion in transverse fields, the leading order terms are the same as those in the plane-wave theory, and higher order terms can be treated as corrections away from the plane-wave. In particular, considering string theory on the plane-wave limit of the backgrounds of interest to us will provide us with what we can think of as a set of elementary excitations of the theory with a set of associated masses.

The general metric for a plane-wave spacetime is

$$ds^2 = -4dx^+dx^- + \sum_{i=1}^{D-2} m_i x_i^2 (dx^+)^2 + \sum_{i=1}^{D-2} dx_i^2 \ . \tag{2.16}$$

where m_i is in general allowed to be a function of x^+. For our purposes as will shortly become clear we will be interested solely in considering m_i to be constants, and of course we are also considering only the critical dimension $D = 10$. Plane-wave metrics of the form (2.16) are defined within the larger class of pp-wave spacetimes as those which possess a covariantly constant null Killing vector, and hence as explained in section 2.1.2, strings on these backgrounds can be quantized similarly to strings on flat space. When this is done, the bosonic sigma model action reduces to that of 8 free bosons whose masses are given by the constants m_i [66, 67]. This can be compared with the action of free massless bosons which arises from string theory on flat space, which is of course exactly what the metric (2.16) reduces to when all $m_i = 0$.

It was observed in [66, 68] that one particular plane-wave background represents another maximally supersymmetric solution of type IIB string theory possessing 32 supercharges, alongside flat space and $AdS_5 \times S^5$. This case is when each of the masses m_i is the same, $m_i = m$, and the bosonic background is supplemented by a 5-form R-R flux given by

$$F_{+1234} = F_{+5678} = 2m \ .\tag{2.17}$$

This background is obtained by taking a Penrose limit of the $AdS_5 \times S^5$ background [69, 70]. A Penrose limit involves selecting a particular null geodesic in spacetime and considering a limit in the near-vicinity of the geodesic such that the resulting metric remains non-singular. In the case of the possible Penrose limits of $AdS_5 \times S^5$, the choice of null geodesic which preserves maximally supersymmetry in the IIB theory is a null geodesic whose spacelike part simply moves around the equator of the sphere.

The theory of IIB superstrings on the plane-wave background obtained by a Penrose limit of $AdS_5 \times S^5$ can be quantized in the lightcone gauge. The question of how this Penrose limit relates to the duality with $\mathcal{N} = 4$ SYM was first considered in [70]. As well as a geometric limit of the background spacetime, the plane-wave IIB theory can also be thought of as a limit where particular Noether charges are taken to be large. In particular, given the energy E and an angular momentum J associated to motion around the S^5 equator, we are considering a limit where

$$E \to \infty \ , \quad J \to \infty \tag{2.18}$$

This "large charge" limit can be carried over naturally to the gauge side of the duality, where we will be taking the dimensions of the relevant operators to be large. The limit is called the BMN limit (Berenstein, Maldecena, Nastau) [70] in the context of AdS/CFT, and perturbative calculations have been made on both sides of the duality in this limit.

An similar limit can be taken for the AdS_3 backgrounds, choosing a Pen-

rose limit which maximises supersymmetry, and consequently leading to a theory which is most likely to be solvable. For the case of $AdS_3 \times S^3$ [71–75] the appropriate limit is one where again the spacelike component of the null geodesic in the limit runs around the equator of the sphere, so the geodesic is given by

$$t = \psi = \tau \ . \tag{2.19}$$

For $AdS_3 \times S^3 \times S^3$ [76,77] we choose a null geodesic which runs around both equators of the spheres. In this case there is an additional parameter involved in choosing the limit according to which linear combination one choose the geodesic to take over the two spheres. The choice which turns out to maximise supersymmetry is [46,76]

$$t = \tau \ , \quad \psi_1 = \cos^2 \phi \ \tau \ , \quad \psi_2 = \sin^2 \phi \ \tau \ . \tag{2.20}$$

For these particular Penrose limits, the number of supersymmetries of the two theories are not just preserved, but are in fact enhanced, to 20 supersymmetries in the case of the $AdS_3 \times S^3 \times S^3$ theory [76] and 24 in the case of the $AdS_3 \times S^3$ theory. [78] After these limits, the metrics of the two backgrounds are given by equation (2.16) with masses

$$\{m_i\} = \{1,0\} \qquad\qquad (AdS_3 \times S^3 \times T^4)$$
$$\{m_i\} = \{1,\cos\phi,\sin\phi,0\} \qquad (AdS_3 \times S^3 \times S^3 \times S^1) \tag{2.21}$$

where each mass comes with a multiplicity of four bosons and four fermions in the former case, and two in the latter.

As can be seen, both backgrounds have massless fundamental excitations in their plane-wave limit. This is a new feature of the AdS_3 backgrounds which does not show up in higher dimensions. At the level of the Penrose limit of the geometry, the factors of T^4 and S^1 are essentially decoupled in the limiting procedure and so remain flat subspaces in the plane-wave spacetime. These flat subspaces give rise to four and one massless boson(s) respectively. The S^1 background has another massless boson coming from a transverse direction around both S^3 equators. Each of the massless bosons has a superpartner massless fermion. Despite their simple origin from the Penrose limit perspective, the massless modes have proven difficult to implement in the integrability descriptions of the full theories, as we will discuss further, and developments on how to incorporate them will constitute a major recurring aspect of the work in this thesis.

2.2 Integrability of classical strings

We now turn to the subject of integrability. We will begin by introducing the ideas of classical integrable systems, in particular how the idea of Liouville

integrability as defined for finite-dimensional systems can be extended to field theories with infinitely many degrees of freedom, using the so-called Lax formulation. Then we will discuss how these ideas apply to classical strings, and discuss how, for particular backgrounds, the string sigma model can be cast in an integrable form.

2.2.1 Classical Integrability

The original notion of integrability is that of a Liouville integrable classical system. This concept applies to finite-dimensional dynamical systems, usually Hamiltonian systems with $2n$ degrees of freedom given by positions q_i and momenta p_i depending on a time parameter τ, with evolution in τ governed by a Hamiltonian H. In particular, for any function $f(q_i, p_i)$, its evolution in time is given by

$$\dot{f} = \{f, H\} .$$

(2.22)

where the Poisson bracket of two functions f, g, $\{f, g\}$ is defined to be

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) .$$

(2.23)

The system is defined to be Liouville integrable if there are n functions f_i that are in involution,

$$\{f_i, f_j\} = 0 ,$$

(2.24)

one of which is the Hamiltonian, and hence all are conserved. For such systems, the Arnold-Liouville theorem guarantees that there exists a coordinate system in which all momenta are constant, and the dynamical evolution of the positions is uniform motion around a torus.

The key idea in this original notion of integrability is that there be as many conserved quantities as degrees of freedom in the theory. For the extension to field theories with infinitely many degrees of freedom, it is natural to assume integrability must mean there are infinitely many conserved quantities, but it is unclear whether this should be a sufficient condition by itself. One way to proceed is to use the ideas of the Lax formalism [79], which can be defined for finite-dimensional integrable systems and which then generalise naturally to field theories. This works as follows: suppose square matrices L and M of size n can be built out of the variables of the theory such that the dynamical equations of the theory can be written as

$$\dot{L} = [M, L] .$$

(2.25)

Then it follows that

$$\frac{d}{d\tau} \text{tr}(L^k) = 0$$

(2.26)

for any integer k. Hence the n eigenvalues of L are all conserved quantities.

The obvious generalisation of the Lax formalism to infinite-dimensional systems is simply to take the Lax pair (L, M) to be infinite-dimensional matrices. We achieve the same result however by introducing the so-called spectral parameter z and considering a one-parameter family of matrices $(L(z), M(z))$. The advantage of this approach is that we can deduce properties of a particular solution, or obtain equations that characterise different solutions, by considering the analytic structure of the matrices on z.

The generalisation of equation (2.25) to field theories relies on something which is unique to two-dimensional field theories, and hence these are the only ones for which we can use integrability.[3] When $d = 2$ we can regard the Lax pair as being made up of the two components of a connection on a vector bundle, $(L(z), M(z)) \to (L_\tau(z), L_\sigma(z))$ and then the condition that this connection L_α be flat,

$$\partial_\alpha L_\beta(z) - \partial_\beta L_\alpha(z) - [L_\alpha(z), L_\beta(z)] = 0 \ , \tag{2.27}$$

is the generalisation of equation (2.25).

In the finite-dimensional case we saw the conserved charges arising from the Lax formulation as the eigenvalues of L. Now in the case of field theory, we expect to have an infinite set of conserved charges arising in a similar way. We need now to define the monodromy matrix as the path-ordered exponential of the Lax connection,

$$M(z, \tau) = \text{Pexp} \int_0^{2\pi} d\sigma L_\sigma \ . \tag{2.28}$$

From the flatness of L_α, this obeys the evolution equation

$$\partial_\tau M(z, \tau) = [L_\tau(2\pi, \tau, z), M(z, \tau)] \tag{2.29}$$

and so

$$\partial_\tau \text{tr}(M^k) = 0 \ , \tag{2.30}$$

and now the eigenvalues of M are conserved quantities. By expanding in the spectral parameter z, typically ub oiwers if $\frac{1}{z}$ around $z = \infty$, we obtain an infinite set of conserved charges.

2.2.2 Classical strings

In this section we will give an overview of how classical integrability shows up in particular string backgrounds. The key idea is that for these backgrounds,

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[3]Two-dimensional field theories can be regarded as uniquely capable of admitting integrability for a different reason. The theorem of Coleman and Mandula [80] restricts the possible symmetries of a QFT whose S-matrix obeys certain general conditions. One of these conditions is analyticity of the S-matrix in the scattering angle. This is generically true of physically interesting QFT's in $d > 2$ but in two dimensions the only possible "angles" for scattering are forwards or backwards. As such two-dimensional field theories can possess higher conserved charges, as is fundamental for integrability, even in the absence of supersymmetry.

the classical string action can be formulated as a coset action for a supergroup whose bosonic subgroup represents the isometries of the relevant background. An action of this form was first given for strings on $AdS_5 \times S^5$ in [81], following the earlier use [82] of a coset action to describe strings on flat space. A key feature of many examples of cosets describing strings on AdS spaces is that they admit a $\mathbb{Z}_4$ automorphism [83].[4] This allows a flat Lax connection to be written down for the theory [62] and hence the theory is classically integrable in the manner discussed in the previous section. Superspaces admitting a $\mathbb{Z}_4$ automorphism are called semisymmetric spaces [85]. Following [61, 86, 87], we will find it convenient to describe the setup of strings on semisymmetric spaces in a general group-theoretic form, and this can then be specialised to the particular cases of interest to us by specifying the relevant supergroups.

We consider a coset G/H consisting of a supergroup G equipped with a $\mathbb{Z}_4$ automorphism $\Omega : G \to G$, and where $H \in G$ is the invariant subspace of Ω. To form a string action on such a space, the elements in G are functions over the worldsheet, $g(\tau, \sigma) \in G$ and we can relate these group elements to the usual bosonic and fermionic fields of the Green-Schwarz action in some (non-unique) way. We form the standard Maurer-Cartan one-forms j_α in the Lie algebra $\mathfrak{g}$ of G,

$$j_\alpha = g^{-1} \partial_\alpha g \; , \tag{2.31}$$

and then the automorphism Ω also acts on these one-form currents, giving a decomposition of the algebra which we denote

$$j_\alpha = \sum_{n=0}^{3} j_\alpha^{(n)} \; , \tag{2.32}$$

where the automorphism Ω acts as

$$\Omega \left(j_\alpha^{(n)} \right) = i^n j_\alpha^{(n)} \; . \tag{2.33}$$

In this decomposition $j^{(0)}$ and $j^{(2)}$ represent the bosonic parts of the algebra and $j^{(1)}$ and $j^{(3)}$ the fermionic parts.

The string action is given by

$$S = \frac{\sqrt{\lambda}}{8\pi} \int d^2\sigma \operatorname{Str} \left(\sqrt{-\gamma} \gamma^{\alpha\beta} j_\alpha^{(2)} j_\beta^{(2)} + \epsilon^{\alpha\beta} j_\alpha^{(1)} j_\beta^{(3)} \right) \; , \tag{2.34}$$

where Str denotes the unique invariant bilinear form on $\mathfrak{g}$ which all superspaces of interest to us possess.[56] It does not depend on $j_\alpha^{(0)}$ which is invariant under

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[4]This was first shown for the case of $AdS_5 \times S^5$ explicitly in [84].

[5]See e.g. [88]

[6]If G is realised as a group of supermatrices, this bilinear form is the supertrace:

$$\operatorname{Str} \begin{pmatrix} A & B \\ C & B \end{pmatrix} = \operatorname{tr}(A) - \operatorname{tr}(D)$$

.

Ω, and as such is an action on the coset G/H. We should note that that in general the coset action (2.34) does *not* describe a consistent string action. It is necessary to check, given a particular group G, that the background associated to it is a consistent string theory background (for example, that it is a supergravity solution etc.).

A Lax connection for the action (2.34) is given by

$$L_\alpha = j_\alpha^{(0)} + \frac{z^2 + 1}{z^2 - 1} j_\alpha^{(2)} - \frac{2z}{z^2 - 1} \frac{1}{-\gamma} \gamma_{\alpha\beta} \epsilon^{\beta\gamma} j_\gamma^{(2)}$$
$$+ \sqrt{\frac{z+1}{z-1}} j_\alpha^{(1)} + \sqrt{\frac{z-1}{z+1}} j_\alpha^{(3)} \ . \tag{2.35}$$

The flatness condition for this Lax connection is equivalent to the equations of motion arising from the action (2.34), together with the Maurer-Cartan equations for the flatness of the current (2.31). The Virasoro constraints are an additional requirement which needs to be imposed in addition to the flatness of the Lax connection. They arise, just as in the usual sigma model, from variations of the worldsheet metric. In conformal gauge they are given explicitly by

$$\mathrm{Str}\left[\left(j_\tau^{(2)} \right)^2 + \left(j_\sigma^{(2)} \right)^2 \right] = \mathrm{Str}\left[j_\tau^{(2)} j_\sigma^{(2)} \right] = 0 \ . \tag{2.36}$$

As discussed in the previous section, given a monodromy matrix M arising from a flat Lax connection we know that the eigenvalues of M are time-independent, so they are functions only of the spectral parameter z. We diagonalise the monodromy matrix arising from the Lax connection (2.35) in terms of a particular Cartan basis H_l for G as

$$M(z) = U^{-1}(z) e^{p_l(z) H_l} U(z) \ . \tag{2.37}$$

We refer to the functions $p_l(z)$ as the quasimomenta of the system. The infinite set of conserved charges that characterise the theory as being integrable show up in the quasimomenta in a convenient way. We find that expanding at large z, the behaviour of any quasimomentum p_l is

$$p(z) = -\frac{2}{z} Q_l + \dots \tag{2.38}$$

where Q_l is a conserved charge of the system, in fact it is a Noether charge associated to a global symmetry. Higher terms in the expansion give an infinite set of conserved charges.

The quasimomenta $p_l(z)$ have a more interesting analytic structure than $M(z)$. The Lax connection is defined as a z-dependent function of the currents in such a way that its only singularities in z are at $z = \pm 1$, and for a solution just in the bosonic sector these are simple poles. The monodromy matrix inherits singularities at $z = \pm 1$, but like the Lax connection is elsewhere analytic. On the other hand the quasimomenta can possess branch cuts, and

indeed we can use these branch cuts to characterise classical string solutions. On each branch cut, the quasimomenta are required to satisfy the monodromy condition

$$A_{lm} p\!\!\!/_m(z) = 2\pi n_l \ , \quad z \in C_{l,i} \tag{2.39}$$

where $A_{lm} = \text{Str}(H_l H_m)$ is the Cartan matrix of the group, $C_{l,i}$ denotes the set of cuts on the sheet for the corresponding quasimomentum p_l, and $p\!\!\!/_l$ is the continuous part of the quasimomentum on the cut.

The monodromy condition (2.39) can be recast as a set of integral equations, called the finite-gap equations. We will see in detail in the following chapter how this is done. Solutions to these equations are often called the algebraic curve. Finite-gap equations have been written down for strings on the $AdS_5 \times S^5$ background, first in subsectors [89–91] and then for the full background [92]. Following this, finite-gap equations were written down for strings on $AdS_4 \times \mathbb{CP}^3$ [93], for the pure R-R AdS_3 backgrounds of interest to us in this thesis [61], for backgrounds involving AdS_2 [87] and most recently for the mixed-flux $AdS_3 \times S^3 \times T^4$ background [94]. One important result of the work in this thesis is to show how the finite-gap equations need to be modified in the presence of massless excitations.

2.2.3 Coset model on $\mathbb{R} \times S^2$ and a single-cut algebraic curve

We will now illustrate the ideas of coset models and classical string solutions described as algebraic curves by considering a simple example: bosonic strings on $\mathbb{R} \times S^2$. We will take as the coset for this model $SU(2)/U(1)$, so the timelike coordinate is supplementary. We parametrise elements $g \in G = SU(2)$ in terms of $\mathbb{C}^2$ coordinates as

$$g = \begin{pmatrix} Z_1 & -Z^2 \\ Z_2 & Z^1 \end{pmatrix} \tag{2.40}$$

where $Z^i \equiv Z_i^*$ and $|Z_1|^2 + |Z_2|^2 = 1$. In terms of this parametrisation the currents (2.32) are given by

$$j = \begin{pmatrix} Z^1 dZ_1 + Z^2 dZ_2 & Z^2 dZ^1 - Z^1 dZ^1 \\ Z_1 dZ_2 - Z_2 dZ_1 & Z_1 dZ^1 + Z_2 dZ^2 \end{pmatrix} \ . \tag{2.41}$$

For a sigma model describing bosonic strings only, the $\mathbb{Z}_4$ automorphism of a semisymmetric superspace is reduced to a $\mathbb{Z}_2$ automorphism, that is G is now a symmetric space. We can define a particular choice of this automorphism Ω acting on $\mathfrak{su}(2)$ in terms of its action on the Pauli matrices. A useful choice is

$$\Omega(\sigma_1) = -\sigma_1 \ , \quad \Omega(\sigma_2) = -\sigma_2 \ , \quad \Omega(\sigma_3) = \sigma_3 \ . \tag{2.42}$$

This is clearly a $\mathbb{Z}_2$ automorphism, and the current j is split into $j^{(0)}$ and $j^{(2)}$

by its diagonal and off-diagonal parts respectively. The invariant subalgebra of $\mathfrak{su}(2)$ under this automorphism is clearly a $U(1)$ subalgebra, so this is indeed giving us the coset we want.

The other simplification of the general picture when considering bosonic strings only is that we can replace the "supertrace" in the action by the usual trace. The action is therefore

$$S = \frac{\sqrt{\lambda}}{8\pi} \int \mathrm{d}^2\sigma \sqrt{-\gamma}\gamma^{\alpha\beta}\left[\mathrm{tr}\left(j_\alpha^{(2)}j_\beta^{(2)}\right) - \partial_\alpha t\partial_\beta t\right] \tag{2.43}$$

We can see that this action is indeed equivalent to the usual action for bosonic strings on $\mathbb{R} \times S^2$, and in this process see also how the action of the coset works explicitly in this case, by changing from complex coordinates Z_i to angular coordinates on S^3. If we take

$$Z_1 = \cos\theta e^{i\varphi} , \quad Z_2 = \sin\theta e^{i\psi} \tag{2.44}$$

then we find

$$-\frac{1}{2}\mathrm{tr}\left(\left[j^{(2)}\right]^2\right) = d\theta^2 + \sin^2\theta\cos^2\theta\,(d\varphi - d\psi)^2 \ . \tag{2.45}$$

The automorphism Ω selecting only the off-diagonal parts of the current is therefore giving us the S^2 subsector of S^3 given in these coordinates by $\psi + \varphi = 0$.

We will now look briefly at how particular solutions to the classical evolution of strings on $\mathbb{R} \times S^2$ can be readily seen to give rise to quasimomenta possessing branch cuts. We will consider strings in conformal gauge, and solutions of the form

$$t = \kappa\tau , \quad \varphi = -\psi = n\sigma , \quad \theta = \theta(\tau) \tag{2.46}$$

in the coordinates (2.44). The equation of motion for θ can be solved with the Virasoro constraints to give a general elliptic integral solution, but we can ignore the exact form of this solution. The resulting Lax connection for this solution has a component L_σ given by

$$L_\sigma = in\cos(2\theta)\sigma_3 - in\sin(2\theta)\frac{z^2+1}{z^2-1}\sigma_1 - i\dot{\theta}\frac{2z}{z^2-1}\sigma_2 \ . \tag{2.47}$$

Now we see that for this solution the Lax connection is independent of σ, so the path-ordered exponential required to give us the monodromy matrix simplifies greatly to an ordinary matrix exponential. When we then diagonalise this we can see that for this simple case of $\mathfrak{su}(2)$ we have just one quasimomentum $p(z)$ and the monodromy matrix diagonalises to $e^{p(z)\sigma_3}$. $p(z)$ is given

by

$$p(z) = \frac{2\pi}{z^2 - 1} \sqrt{n^2(z^2 - 1)^2 \cos^2 \theta + 4n^2 z^2 \sin^2 \theta + (z^2 + 1)^2 \dot{\theta}^2} \ . \qquad (2.48)$$

In this form it appears that $p(z)$ depends on τ through θ, but the path-ordered exponential is to be taken at any fixed τ, so given a particular solution $\theta(\tau)$ we could simply substitute its values at say $\tau = 0$. We know that the flatness of the Lax connection means that for any $\theta(\tau)$ which obeys the equations of motion, the expression in the square root here is indeed τ-independent. In fact for this solution, once we make use of the Virasoro constraints we can explicitly rewrite it as

$$p(z) = \frac{2\pi}{z^2 - 1} \sqrt{\kappa^2 z^2 + n^2 (z^2 - 1)^2} \ . \qquad (2.49)$$

We can see from the form of $p(z)$ in (2.49) that it indeed has simple poles at $z = \pm 1$ as we always expect, and in general has four branch points. Thus we have shown that in general solutions of the form (2.46) are described by an algebraic curve with two branch cuts. In fact these two branch cuts are related, so that we regard this solution as having one *physical* branch cut. This arises through the action of the $\mathbb{Z}_4$ automorphism on the quasimomenta.[7] Given its action on the currents, we can find that the action of the automorphism Ω on the general Lax connection (2.35) is

$$\Omega(L_\alpha(z)) = L_\alpha\left(\frac{1}{z}\right) \ . \qquad (2.50)$$

Also, given a particular Cartan basis, we can write the action of Ω on the elements of this basis via a symmetry matrix S as

$$\Omega(H_l) = H_m S_{lm} \qquad (2.51)$$

and then we have a symmetry in the quasimomenta of inversion in the spectral parameter z,

$$p_l\left(\frac{1}{z}\right) = S_{lm} p_m(z) \ . \qquad (2.52)$$

In our example above, we have a single Cartan element and S is the 1×1 identity matrix (see equation (2.42)) and so our single quasimomentum $p(z)$ must always have an even number of branch cuts. We can classify solutions by the number of branch cuts possessed in the physical region $|z| > 1$. The solution (2.49) has, for $\kappa^2 > 2n^2$, a single branch cut in the physical region connecting two branch points on the imaginary axis, as we show in figure 5.

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[7]Here with quasimomenta for the bosonic sector only, the $\mathbb{Z}_4$ automorphism in fact reduces to a $\mathbb{Z}_2$ automorphism.
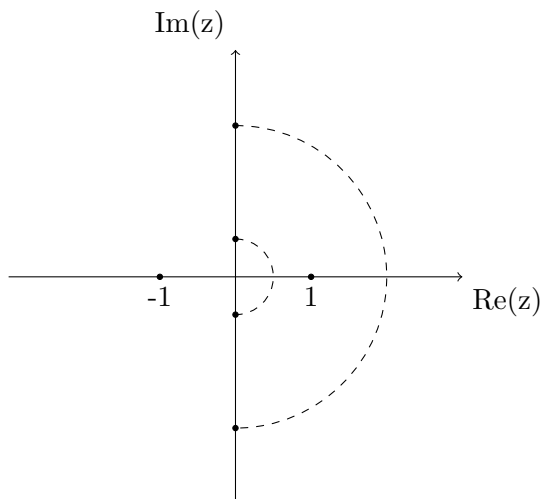
Figure 5: The analytic structure of the quasimomentum (2.49). There is a single branch cut in the physical region $|z| > 1$, which is reflected in the region $|z| < 1$ by inversion symmetry. There are also the usual simple poles at $z = \pm 1$.

2.3 Outline of following chapters

A common theme throughout the work of this thesis will be developing the understanding of how the massless modes that are seen in the plane-wave limits of the AdS_3 backgrounds should be incorporated into the machinery of integrability. It was demonstrated initially when the coset model describing classical dynamics of strings on these backgrounds was written down [61] that it did not include these massless modes. However, this did not prevent progress in understanding these theories through integrability. In particular, there was a successful programme of applying the techniques of quantum integrability to the massive-only sector of the theories. The all-loop Bethe ansatz for the massive sector of both the $D(2,1;\alpha)$ and $PSU(1,1|2)$ theories (with pure R-R flux) was calculated by directly reverse-engineering from the finite-gap equations and associated spin-chains [61, 95][8], and also by means of first obtaining the exact S-matrices from the symmetries of the theories [97–100]. In addition, integrability played a role in direct worldsheet calculations [101–108], where the problem of missing massless modes as in the coset model did not arise.

Chapter 3, which is based on [1], deals directly with the issue of the massless modes in the coset model. It begins with a review of the Penrose limit of $AdS_3 \times S^3 \times S^3 \times S^1$ and the bosonic sector of the spectrum of strings on the resulting plane-wave background. In this spectrum there are two massless bosons. One arises on the factor of S^1, and its absence in the coset model is unsurprising based on the symmetries of the coset. The other, which we will sometimes refer to as the "coset" massless boson, emerges on the transverse

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[8]There was also progress in understanding the massless modes in the spin-chain by considering the limit in which extra massless modes appear [96].

direction along the equators of the two three-spheres, and as such is within the part of the geometry which the coset describes. The key result of chapter 3 is to show how, through a correction to the way the Virasoro constraints had previously been implemented in the coset model, this mode can be incorporated. Once this correction is made, the remaining massless boson, and all four in the T^4 background, can be incorporated through $U(1)$ quasimomenta which interact with the coset only through the corrected Virasoro constraints.

Chapter 4 applies the results of chapter 3 to the semiclassical methods of quasimomenta fluctuations, again primarily in order to show how massless modes can appear explicitly in calculations where they were previously absent. Fluctuations of the algebraic curve had been studied previously for a variety of solutions in the AdS_3 backgrounds [104,109–111], where the massless modes were absent and had to be put back in by hand. Progress in understanding the massless fermions in this context was made in [112], where it was shown how to include additional modes that appeared with non-zero mass in a class of background solutions containing the BMN vacuum, with energy corrections which smoothly approached that of a massless mode in the limit to the BMN background. Chapter 4 contains work which has not been previously published. In this chapter, we combine the prescriptions of [112] with the results of chapter 3 to show how the semiclassical algebraic curve can produce a complete spectrum of energy corrections for semiclassical fluctuations around classical string solutions, including both massless and massive mode contributions.

Chapter 5 is based on work in [2] which combines the extension of integrability techniques into the massless sector of the backgrounds with another extension of integrability into the $AdS_3 \times S^3 \times T^4$ backgrounds with mixed R-R and NS-NS flux. These mixed-flux backgrounds have been studied through a generalisation of the coset model [94, 113], in particular the action (2.34) requires an additional term to incorporate the Wess-Zumino (WZ) term of the GS action. The S-matrix of the massive sector of the mixed-flux backgrounds was studied in [114,115]. An important difference from the pure R-R backgrounds arises in the exact dispersion relation of fundamental excitations which was further studied in [116]. The work of chapter 5 follows on from a project initiated in the pure R-R background in [64, 65] which constructs the exact S-matrix using symmetry arguments but working from the Green-Schwarz action rather than the coset action. In this way massless excitations can be included and the resulting S-matrix consists of massive-massive scattering which matches results obtained from the coset action, together with sectors for massless-massless and massive-massless scattering. Chapter 5 of this thesis describes the application of this project to the mixed-flux background.

Chapter 3

Finite-gap equations and massless modes

In this chapter we will study the finite-gap equations describing classical string solutions in AdS_3 backgrounds. In particular our key result will be a modification of the finite-gap equations written down for these backgrounds in [61]. There it was already observed that the integrability methods used were not capturing the full spectrum of the theories. The AdS_3 backgrounds contain "massless modes" which are a novel feature in relation to the higher-dimensional AdS backgrounds. These massless modes are seen in the plane-wave limit, as we saw already in section 2.1.3.

We will see that to incorporate massless modes into the classical integrability machinery, we need to look carefully at how this machinery implements the Virasoro conditions. It is shown in this chapter that the way the constraints had been imposed previously in the literature (for example in [61]) is, in general, too strict. We will identify the precise condition placed on the finite-gap equations by the Virasoro constraints. This condition will be referred to as the generalised residue condition (GRC).

Having identified the need to use a more general condition to correctly implement the Virasoro conditions in the finite-gap equations, we will see how this relates to the issue of including massless modes. We do this by focusing our attention on one particular massless boson in the $AdS_3 \times S^3 \times S^3 \times S^1$ spectrum, the "coset boson" not associated to the factor of S^1 but to a transverse direction along both S^3 radii which we define precisely in equation (3.2). We consider bosonic solutions in the $\mathbb{R} \times S^1 \times S^1$ subsector of the full theory where this massless boson is the only transverse mode, and explicitly construct the quasimomenta associated to these solutions. We find that their residues do *not* satisfy the old residue conditions, but *do* satisfy the GRC, thereby showing that the GRC is necessary if this massless mode is to be included.

3.1 The Penrose Limit of $AdS_3 \times S^3 \times S^3 \times S^1$ and the spectrum of bosonic strings on plane-waves

As we will be studying solutions corresponding to excitations of a particular massless bosonic mode, we will begin by reviewing the Penrose limit of $AdS_3 \times S^3 \times S^3 \times S^1$ to make clear the origin of this particular mode. We will

also review the derivation of the spectrum of bosonic strings on plane-wave backgrounds. In particular we will be interested in the dispersion relation between the energy E and the angular momentum J associated to these backgrounds. This has a characteristic dependence on the mass of the possible excitations.

We use the following form of the metric for $AdS_3 \times S^3 \times S^3 \times S^1$:

$$
ds^2 = R^2 \bigg[d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\gamma^2
$$
$$
+ \frac{1}{\cos^2 \phi} \left(d\theta_1^2 + \cos^2 \theta_1 d\psi_1^2 + \sin^2 \theta_1 d\varphi_1^2 \right)
$$
$$
+ \frac{1}{\sin^2 \phi} \left(d\theta_2 + \cos^2 \theta_2 d\psi_2^2 + \sin^2 \theta_2 d\varphi_2^2 \right) + du_9^2 \bigg] , \qquad (3.1)
$$

we change coordinates as follows (with ζ being any real constant for now):

$$
t = x^+ + \frac{x^-}{R^2} , \quad \rho = \frac{\tilde{x}_2}{R} , \quad \theta_1 = \cos \phi \frac{\tilde{x}_4}{R} , \quad \theta_2 = \sin \phi \frac{\tilde{x}_6}{R} , \quad u_9 = \frac{x_8}{R} ,
$$
$$
\psi_1 = \cos \zeta \cos \phi \left(x^+ - \frac{x^-}{R^2} \right) - \sin \zeta \cos \phi \frac{x_1}{R} ,
$$
$$
\psi_2 = \sin \zeta \sin \phi \left(x^+ - \frac{x^-}{R^2} \right) + \cos \zeta \sin \phi \frac{x_1}{R} \qquad (3.2)
$$

and keep only the leading term in the limit $R \to \infty$. The metric reduces to

$$
ds^2 = -4dx^+ dx^- + \sum_{i=1}^{8} m_i^2 x_i^2 (dx^+)^2 + \sum_{i=1}^{8} dx_i^2 , \qquad (3.3)
$$

with

$$
(x_2, x_3) = (\tilde{x}_2 \cos \gamma, \tilde{x}_2 \sin \gamma) , \quad (x_4, x_5) = (\tilde{x}_4 \cos \varphi_1, \tilde{x}_4 \sin \varphi_1) ,
$$
$$
(x_6, x_7) = (\tilde{x}_6 \cos \varphi_2, \tilde{x}_6 \sin \varphi_2) \qquad (3.4)
$$

and masses m_i, given by

$$
m_2 = m_3 = 1 , \quad m_4 = m_5 = \cos \zeta \cos \phi ,
$$
$$
m_1 = m_8 = 0 , \quad m_6 = m_7 = \sin \zeta \sin \phi . \qquad (3.5)
$$

The parameter ζ defines a 1-parameter family of metrics obtained from $AdS_3 \times S^3 \times S^3 \times S^1$ via Penrose limits. This freedom comes from the choice of a relative angle between the geodesics in the two S^3 factors. Type II string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ preserves 16 supersymmetries. These remain symmetries of the plane wave limit metric (3.3); in addition for special values of ζ there are extra supersymmetries. If we choose $\zeta = \phi$, string theory on (3.3) preserves 20 supersymmetries [76,77]. From now on, it will be assumed

that we are making this choice, and that the BMN limit has masses

$$m_2 = m_3 = 1 , \quad m_4 = m_5 = \cos^2 \phi ,$$
$$m_1 = m_8 = 0 , \quad m_6 = m_7 = \sin^2 \phi . \tag{3.6}$$

To find the bosonic spectrum of string theory, we impose conformal gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ and lightcone gauge $x^+ = \kappa\tau$. The equation of motion for x_i then becomes

$$(-\partial_\tau^2 + \partial_\sigma^2)x_i = \kappa^2 m_i^2 x_i \tag{3.7}$$

and x^- is determined uniquely from the Virasoro constraints, which in this gauge are

$$\partial_\tau x^- = \frac{1}{4\kappa} \sum_i ((\partial_\tau x_i)^2 + (\partial_\sigma x_i)^2 - \kappa^2 m_i^2 x_i^2),$$
$$\partial_\sigma x^- = \frac{1}{2\kappa} \sum_i (\partial_\tau x_i)(\partial_\sigma x_i) . \tag{3.8}$$

In lightcone gauge x^+ and x^- become non-dynamical variables and the gauge-fixed Hamiltonian is

$$H = \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \sum_{i=1}^{8} \left[(2\pi\alpha')^2 p_i^2 + (\partial_\sigma x_i)^2 + \kappa^2 m_i^2 x_i^2 \right] . \tag{3.9}$$

Solving the equations of motion (3.7), the x^i have the following mode expansion:

$$x^i = X_0^i + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n^i}} \left(a_n^i e^{-i(\omega_n^i \tau + n\sigma)} + a_n^{i\,\dagger} e^{i(\omega_n^i \tau + n\sigma)} \right.$$
$$\left. + \tilde{a}_n^i e^{-i(\omega_n^i \tau - n\sigma)} + \tilde{a}_n^{i\,\dagger} e^{i(\omega_n^i \tau - n\sigma)} \right) , \tag{3.10}$$

where

$$\omega_n^i = \sqrt{n^2 + \kappa^2 m_i^2} , \tag{3.11}$$

and

$$X_0^i = x_0^i \cos \kappa m\tau + \frac{\alpha'}{\kappa m} p_0^i \sin \kappa m\tau \tag{3.12}$$

for massive modes and

$$X_0^i = x_0^i + \alpha' p_0^i \tau + w^i \sigma \tag{3.13}$$

in the massless case $m_i = 0$.[1]

We can insert this mode expansion into the lightcone Hamiltonian (3.9).

---

[1] The winding w in the massless mode is only present if the direction associated to the massless mode in the metric is compact.

Define the zero modes, for the massive case, as

$$a_0^i = \tilde{a}_0^i = \frac{1}{2}\sqrt{\frac{\alpha'}{\kappa m_i}}p_0^i + \frac{i}{2}\sqrt{\frac{\kappa m_i}{\alpha'}}x_0^i \ , \tag{3.14}$$

then we have

$$H = \sum_{i=1}^{8}\sum_{n=0}^{\infty}\omega_n^i N_n^i + \frac{1}{2\alpha'}\sum_{i=1,8}\left[(\alpha' p_0^i)^2 + (w^i)^2\right] \ , \tag{3.15}$$

with N_n^i the number operator defined as

$$N_n^i = a_n^{i\,\dagger}a_n^i + \tilde{a}_n^{i\,\dagger}\tilde{a}_n^i \ . \tag{3.16}$$

Now we consider conserved Noether charges. From the independence of the metric on the coordinates x^+ and x^- we get conserved charges P_+ and P_- upon integrating the conjugate momenta p_+ and p_-. These are related to more natural charges: the energy $E = i\partial_\tau$, and an angular momentum $J = -i\partial_\eta$ coming from the spatial coordinate

$$\eta = x^+ - \frac{x^-}{R^2} \ . \tag{3.17}$$

Then we have

$$P_+ = i\partial_+ = i(\partial_t + \partial_\eta) = E - J,$$
$$P_- = i\partial_- = \frac{i}{R^2}(\partial_t - \partial_\eta) = \frac{E+J}{R^2} \ , \tag{3.18}$$

and

$$P_+ = \frac{H}{\kappa} = E - J$$
$$= \frac{1}{\kappa}\sum_{i=1}^{8}\sum_{n=0}^{\infty}\omega_n^i N_n^i + \frac{1}{2\alpha'\kappa}\sum_{i=1,8}\left[(\alpha' p_0^i)^2 + (w^i)^2\right] \ . \tag{3.19}$$

Since

$$P_- = \int_0^{2\pi}\mathrm{d}\sigma\, p_- = \frac{1}{\pi\alpha'}\int_0^{2\pi}\mathrm{d}\sigma\,\partial_\tau x^+ = \frac{2\kappa}{\alpha'} \ , \tag{3.20}$$

we find $E+J = 2\sqrt{\lambda}\kappa$, with $\sqrt{\lambda} = \frac{R^2}{\alpha'}$. To leading order in a large J expansion, $E + J \approx 2J$. So writing the right-hand side of (3.19) in terms of J instead of κ, to leading order we have $\kappa = \frac{J}{\sqrt{\lambda}}$ and so

$$E - J = \sum_{i=1}^{8}\sum_{n=0}^{\infty}\sqrt{m_i^2 + \frac{\lambda n^2}{J^2}}\ N_n^i + \frac{\sqrt{\lambda}}{2\alpha' J}\sum_{i=1,8}\left[(\alpha' p_0^i)^2 + (w^i)^2\right] \ . \tag{3.21}$$

3.2 Quasimomenta and finite-gap equations

In this section we will describe in more detail the classical integrability of strings on symmetric space cosets and finite-gap equations [89, 92, 117].[2] that we introduced in section 2.2.2. There we saw that on integrable coset backgrounds the fundamental objects describing classical string solutions are the currents $j_\alpha(\sigma, \tau)$, and that we can equivalently use the so-called quasimomenta $p_l(z)$ defined as functions of the complex spectral parameter z. The quasimomenta determine classical solutions in terms of their analytic structure. All quasimomenta possess simple poles at $z = \pm 1$, and in addition may possess a number of branch cuts. On these cuts they are required to satisfy a monodromy condition, and this ensures that the classical string solution they describe obeys the equations of motion. The Virasoro constraints do not translate to a condition on the cuts of the quasimomenta, but rather a restriction on the possible residues of the simple poles.

In this section we will examine these two different sets of restrictions on the quasimomenta in more detail. First we will review briefly how the monodromy condition on the cuts can be rewritten in terms of a set of integral equations, called the finite-gap equations. This will serve mostly to establish some of the notation we will use in the rest of the chapter. Next we examine the implementation of the Virasoro constraints on the residues in some detail. In particular, we will introduce the generalised residue condition that will play the crucial role in incorporating massless modes, and explain how it differs from the previously-used residue condition.

3.2.1 Finite-gap equations

We have a set of quasimomenta $p_l(z)$ associated with a Cartan basis H_l with Cartan matrix $A_{lm} = \mathrm{Str}(H_l H_m)$. These have some number of branch cuts, and we denote the set of branch cuts on the sheet of p_l by $\{C_{l,i}\}$. Then the monodromy condition on these cuts is given by

$$A_{lm}\slashed{p}_m(z) = 2\pi i n_{l,i}, \quad z \in C_{l,i}, \quad n_{l,i} \in \mathbb{Z}, \tag{3.22}$$

where $\slashed{p}_l(z)$ is the continuous part of the quasimomenta across the cuts:

$$\slashed{p}_l(z) = \lim_{\epsilon \to 0}(p_l(z + \epsilon) + p_l(z - \epsilon)), \quad z \in C_{l,i}, \tag{3.23}$$

with ϵ a complex number normal to the branch cut.

We will use a so-called *spectral representation* to write $p_l(z)$ in terms of integral along its branch cuts. To do so, we introduce the notation for the

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density function as the difference across the cuts

$$\rho_l(z) = \lim_{\epsilon \to 0} \left(p_l(z + \epsilon) - p_l(z - \epsilon) \right), \quad z \in C_{l,i} , \tag{3.24}$$

and we choose to parametrize the residues at the poles by their sum and difference, defining constants κ_l and m_l so that as $z \to \pm 1$

$$p_l = \frac{1}{2} \frac{\kappa_l z + 2\pi m_l}{z \mp 1} + \dots . \tag{3.25}$$

Then the spectral representation of p_l is $[119, 120]^3$

$$p_l(z) = \frac{\kappa_l z + 2\pi m_l}{z^2 - 1} + p_l(\infty) + \int_{\bigcup_i C_{l,i}} \mathrm{d}w \frac{\rho_l(w)}{z - w} . \tag{3.26}$$

We saw in section 2.2.2 that the quasimomenta posses an inversion symmetry arising from the $\mathbb{Z}_4$ automorphism. This symmetry is

$$p_l \left(\frac{1}{z} \right) = S_{lm} p_m(z) \tag{3.27}$$

where the symmetry matrix S_{lm} is determined from the action of the $\mathbb{Z}_4$ automorphism Ω on the Cartan basis via

$$\Omega(H_l) = S_{lm} H_m . \tag{3.28}$$

The inversion symmetry allows us to write the spectral representation in terms of integrals over only those cuts in the physical region $|z| > 1$:

$$p_l(z) = \frac{\kappa_l z + 2\pi m_l}{z^2 - 1} + p_l(\infty) + \int_{\bigcup_i C_{l,i}} \mathrm{d}w \left(\frac{\rho_l(w)}{z - w} - \frac{S_{lm}}{w^2} \frac{\rho_m(w)}{z - \frac{1}{w}} \right) . \tag{3.29}$$

It also places restrictions on κ_l, m_l and $p_l(\infty)$:

$$S_{lm}\kappa_m = -\kappa_l , \qquad S_{lm}m_m = -m_l ,$$
$$S_{lm}p_m(\infty) = p_l(\infty) - 2\pi m_l . \tag{3.30}$$

With $p(z)$ defined for z away from the branch cuts as in (3.29), we can apply the Sochocki-Plemelj formula $[119, 120]$ to evaluate $p(z)$ when z lies on a branch cut. This is essentially solving the associated Riemann-Hilbert problem. With the monodromy of the quasimomentum given by equation

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[3]This result comes from applying the Cauchy integral formula on an infinite domain to the function obtained by subtracting the poles from p_l, which is analytic outside this contour surrounding all the cuts. (3.26) then follows by shrinking the contour down onto the cuts. In the case that p_l is meromorphic, (3.26) still holds with $\rho_l = 0$, since in this case subtracting the poles from the quasimomentum gives an entire function, and the only entire function satisfying the inversion symmetry is a constant.

(3.22), we get from the Sochocki-Plemelj formula

$$A_{lm} \!\!\!\fint dw \frac{\rho_m(w)}{z-w} - A_{lk}S_{km} \int \frac{dw}{w^2} \frac{\rho_m(w)}{z-\frac{1}{w}} = -A_{lm}\frac{\kappa_m z + 2\pi m_m}{z^2 - 1} - \pi A_{lm}m_m$$

$$+ 2\pi n_{l,i}, \quad z \in C_{l,i} . \tag{3.31}$$

These are the finite-gap equations of the system. In the next subsection we see how the Virasoro constraints place further restrictions on κ_l and m_l in addition to the inversion symmetry restrictions (3.30).

3.2.2 The Generalised Residue Conditions

There is an equivalent setting [79] in which to define the monodromy matrix and quasimomenta from a flat Lax connection. In this section we introduce this setting and show one use for it: considering how the Virasoro constraints appear at the level of the quasimomenta.

In the so-called auxiliary linear problem, we study the first order differential equation

$$\partial_\sigma \Psi(\sigma, \tau, z) = L_\sigma \Psi(\sigma, \tau, z) . \tag{3.32}$$

where Ψ is an element of the representation space of $\mathfrak{g}$. In this setting the monodromy matrix is defined by

$$\Psi(\sigma + 2\pi, \tau, z) = M(z)\Psi(\sigma, \tau, z) \tag{3.33}$$

This definition is equivalent to (2.28). We use a basis where M is diagonalised as in equation (2.37), so that

$$\Psi(\sigma + 2\pi, \tau, z) = e^{ip_l(z)H_l}\Psi(\sigma, \tau, z) . \tag{3.34}$$

We know that the quasimomenta have poles at $z = \pm 1$. Let us determine the residues of these poles by solving the auxiliary linear problem (3.32) in the limit $z \to \pm 1$. We denote $h = z \mp 1$ in this limit, so that h is a small parameter we can expand in, and define

$$V = hL_\sigma = \left(j_\tau^{(2)} \pm j_\sigma^{(2)} \right) + \mathcal{O}(h), \quad h = z \mp 1 . \tag{3.35}$$

Since L has simple poles at $z = \pm 1$, V is a regular function of h. We make the Wentzel-Kramers-Brillouin (WKB) ansatz

$$\Psi(\sigma, \tau, z) = \exp\left(\frac{S_l(\sigma, \tau, h)H_l}{h} \right) \xi(h) , \tag{3.36}$$

where S_l are some complex-valued functions. Then the defining equation (3.32) of the system becomes

$$V\Psi = (\partial_\sigma S_l)H_l\Psi . \tag{3.37}$$

This implies

$$\text{Str}\left(V^2\right) = \text{Str}\left((\partial_\sigma S_l)(\partial_\sigma S_m)H_l H_m\right) = A_{lm}(\partial_\sigma S_l)(\partial_\sigma S_m) . \tag{3.38}$$

Since
$$\text{Str}(V^2) = \text{Str}\left(j_\tau^{(2)} \pm j_\sigma^{(2)}\right)^2 + \mathcal{O}(h), \quad h = z \mp 1 , \tag{3.39}$$

the Virasoro constraints, (2.36), can be written as

$$\lim_{h \to 0} \text{Str}(V^2) = 0. \tag{3.40}$$

With the ansatz (3.36), equation (3.34) is solved by

$$p_l(z) = \frac{1}{h}\left(S_l(\sigma + 2\pi, h) - S_l(\sigma, h)\right) = \frac{1}{h}\int_0^{2\pi} d\sigma \partial_\sigma S_l(\sigma, h) . \tag{3.41}$$

If we define functions $f_l^\pm(\sigma, \tau)$ by

$$f_l^\pm(\sigma, \tau) = \lim_{h \to 0} \partial_\sigma S_l(\sigma, \tau, h) , \tag{3.42}$$

then we can see that the residues of the quasimomenta at $z = \pm 1$ are precisely the integrals of these functions,

$$\frac{1}{2}(\kappa_l \pm 2\pi m_l) = \int_0^{2\pi} f_l^\pm d\sigma . \tag{3.43}$$

At the same time, we have now seen that the Virasoro constraints can be written in terms of the same functions. Using the relation (3.38), the Virasoro constraints (3.40) can be written in terms of $f_l^\pm$ as

$$A_{lm} f_l^\pm f_m^\pm = 0 . \tag{3.44}$$

Thus, the condition that the Virasoro constraints place upon the residues of the quasimomenta can be stated as follows: the residues can be written as integrals in the form (3.43), such that the integrands satisfy equation (3.44). To clarify this further: there are obviously many different functions of σ which give the same result upon integration from 0 to 2π, and so many choices of $f_l^\pm$ such that (3.43) holds. The condition placed on the residues by the Virasoro constraints is that for at least one of these choices, equation (3.44) holds. We will refer the condition as the generalised residue condition (GRC).

If we knew the residues, and wanted to write down functions to represent them via (3.43), the most obvious and simple choice would be to choose the constant functions
$$f_l^\pm(\sigma) = \frac{1}{4\pi}(\kappa_l \pm 2\pi m_l) . \tag{3.45}$$

Although we can always make this choice to satisfy equation (3.43), it is not in general guaranteed that this choice for $f_l^\pm$ will satisfy the condition (3.44).

The Virasoro constraints imply only that one of the many possible choices for $f_l^\pm$ in equation (3.43) satisfies equation (3.44), not that all possible choices do, or that one particular simple choice does. When the constant functions given by equation (3.45) do satisfy equation (3.44), then the condition on the residues can be written as

$$A_{lm}(\kappa_l \pm 2\pi m_l)(\kappa_m \pm 2\pi m_m) = 0 \; . \tag{3.46}$$

In much of the literature (see [87] for example), it is the condition of equation (3.46) that has been taken to hold. In the next section we consider explicit sigma model solutions for strings on $AdS_3 \times S^3 \times S^3$ and their associated quasimomenta. For each solution we will discuss whether the residues satisfy the old condition (3.46) or the GRC (3.43) and (3.44). We will see that solutions containing massless modes do not satisfy the old condition, but do satisfy the GRC. This will show explicitly that the GRC must be used in the finite-gap equations in order to capture the dynamics of the massless modes.

3.3   Strings on $\mathbb{R} \times S^1 \times S^1 \subset AdS_3 \times S^3 \times S^3$

In this section we consider solutions on the subspace $\mathrm{R} \times S^1 \times S^1 \subset AdS_3 \times S^3 \times S^3$, with the metric

$$ds^2 = R^2 \left[- dt^2 + \frac{1}{\cos^2 \phi} d\psi_1^2 + \frac{1}{\sin^2 \phi} d\psi_2^2 \right] \; . \tag{3.47}$$

This subspace contains the coset massless mode of the spectrum in the plane-wave limit.[4] If we choose to consider solutions in lightcone gauge in this space with the Virasoro constraints solved before quantization, then we are looking at precisely the same BMN massless mode quantization that we considered as part of the full space in section 3.1. We will look first at general solutions in lightcone gauge, and then at particular solutions in static gauge, since this latter gauge features prominently in the finite-gap analysis. As we will see, the choice of gauge will not affect the dynamics of the general solution. Indeed we will check very explicitly that we have the same form of expression for $E - J$ for each.

We will see presently that the quasimomenta on this subspace have a very simple analytic structure; they have no branch points or cuts, only simple poles at $z = \pm 1$. This makes it straightforward to write down the most general quasimomenta for any solution on this space and will serve as a guide for how to incorporate this massless mode into the finite-gap equations.

---

[4]Not the one which appears simply as the dynamics of the isolated S^1.

3.3.1 Coset representatives and quasimomenta

To begin we will take an explicit coset representation for solutions on the $\mathbb{R} \times S^1 \times S^1$ subspace, chosen in such a way that the quasimomenta are particularly simple to compute. We show that the quasimomenta have no branch points or cuts, and so can be written completely in terms of the residues. In particular, we will write down the most general quasimomenta for any solution on this subspace in terms of the numbers κ_l and m_l, and what κ_l and m_l are in terms of a particular coordinate solution $t(\sigma, \tau)$, $\psi_1(\sigma, \tau)$ and $\psi_2(\sigma, \tau)$. We show how the generalised residue conditions (3.43) and (3.44) are clearly equivalent to the Virasoro conditions expressed in terms of the coordinates. Lastly we write down an expression for $E - J$ in terms of κ_l and m_l, which we will use later when we consider particular solutions to show that the correct massless dispersion relation appears from the quasimomenta of those solutions.

In the bosonic case the most natural choice of group representative g is a direct sum $g = g_0 \oplus g_1 \oplus g_2$ with $g_0 \in SU(1,1) \times SU(1,1)$ and $g_i \in (SU(2)_i)^2$, where $SU(2)_1$, respectively $SU(2)_2$, is the group manifold for the sphere of radius $\frac{1}{\cos^2 \phi}$, respectively $\frac{1}{\sin^2 \phi}$. In particular, we choose the coset representatives as follows:

$$g_1 = \frac{1}{\cos \phi} \mathrm{diag} \left(e^{i \frac{\psi_1}{2}}, e^{-i \frac{\psi_1}{2}}, e^{i \frac{\psi_1}{2}}, e^{-i \frac{\psi_1}{2}} \right),$$
$$g_2 = \frac{1}{\sin \phi} \mathrm{diag} \left(e^{i \frac{\psi_2}{2}}, e^{-i \frac{\psi_2}{2}}, e^{i \frac{\psi_2}{2}}, e^{-i \frac{\psi_2}{2}} \right), \tag{3.48}$$

and

$$g_0 = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \oplus \begin{pmatrix} \cosh \frac{t}{2} & -\sinh \frac{t}{2} \\ -\sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}. \tag{3.49}$$

Then the current $j = g^{-1} dg$ is

$$j = \frac{dt}{2} \left[\sigma_1 \oplus (-\sigma_1) \right] \oplus \frac{i}{\cos \phi} \frac{d\psi_1}{2} \left[\sigma_3 \oplus \sigma_3 \right] \oplus \frac{i}{\sin \phi} \frac{d\psi_1}{2} \left[\sigma_3 \oplus \sigma_3 \right]. \tag{3.50}$$

The $\mathbb{Z}_2$ automorphism on the space is defined here as $\Omega(j) = K j^t K$, where

$$K = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{\oplus 3}. \tag{3.51}$$

For all j's given here, this acts as $\Omega(j) = -j$, so $j^{(0)} = \frac{1}{2}(j + \Omega(j)) = 0$, $j^{(2)} = \frac{1}{2}(j - \Omega(j)) = j$.

We can check explicitly that the bosonic part coset action (2.34) with j given as above is identical to the sigma model action on the metric (3.47),

noting that

$$\mathrm{tr}((j^{(2)})^2) = -2\left(-dt^2 + \frac{1}{\cos^2\phi}d\psi_1^2 + \frac{1}{\sin^2\phi}d\psi_2^2\right) . \tag{3.52}$$

Since $j^{(0)} = 0$, the Lax connection is (*cf.* equation (2.35))

$$L_\sigma = \frac{1}{z^2-1}\left((z^2+1)j_\sigma + 2zj_\tau\right) . \tag{3.53}$$

The Lax connection is given by a direct sum of three matrices, each of which takes the form of a constant matrix multiplied by a function.[5] In this case, the path-ordered exponential taking us from the Lax connection to the monodromy matrix, given in equation (2.28), reduces to an ordinary matrix exponential of the integrals of the scalar functions. It is then straightforward to read off the quasimomenta

$$p_1(z) = -\frac{1}{2\cos\phi}\frac{1}{z^2-1}\int_0^{2\pi}\mathrm{d}\sigma\left[(z^2+1)\psi_1' + 2z\dot{\psi}_1\right] , \tag{3.54}$$

$$p_2(z) = -\frac{1}{2\sin\phi}\frac{1}{z^2-1}\int_0^{2\pi}\mathrm{d}\sigma\left[(z^2+1)\psi_2' + 2z\dot{\psi}_2\right] \tag{3.55}$$

and

$$p_0(z) = \frac{i}{2}\frac{1}{z^2-1}\int_0^{2\pi}\mathrm{d}\sigma\left[(z^2+1)t' + 2z\dot{t}\right] . \tag{3.56}$$

The quasimomenta can be written in the form of the spectral representation (3.26), but with no cuts

$$p_l(z) = \frac{\kappa_l z + 2\pi m_l}{z^2-1} + \pi m_l \tag{3.57}$$

where

$$\begin{aligned}
\kappa_0 &= i\int_0^{2\pi}\dot{t}\,\mathrm{d}\sigma , & 2\pi m_0 &= i\int_0^{2\pi}t'\,\mathrm{d}\sigma , \\
\kappa_1 &= -\frac{1}{\cos\phi}\int_0^{2\pi}\dot{\psi}_1\,\mathrm{d}\sigma , & 2\pi m_1 &= -\frac{1}{\cos\phi}\int_0^{2\pi}\psi_1'\,\mathrm{d}\sigma , \\
\kappa_2 &= -\frac{1}{\sin\phi}\int_0^{2\pi}\dot{\psi}_2\,\mathrm{d}\sigma , & 2\pi m_2 &= -\frac{1}{\sin\phi}\int_0^{2\pi}\psi_2'\,\mathrm{d}\sigma .
\end{aligned} \tag{3.58}$$

Since t must be periodic in σ, we have $m_0 = 0$. We also get conditions for integer winding modes on ψ_1 and ψ_2, namely $m_1\cos\phi \in \mathbb{Z}$ and $m_2\sin\phi \in \mathbb{Z}$.

The Noether charges of these solutions are the energy E and angular mo-

---

[5]Classical solutions studied in [121] have a similarly simple Lax connection.

menta J_1 and J_2 given by:

$$J_1 = \frac{R^2}{2\pi\alpha' \cos^2 \phi} \int_0^{2\pi} \dot{\psi}_1 \, d\sigma \ ,$$

$$J_2 = \frac{R^2}{2\pi\alpha' \sin^2 \phi} \int_0^{2\pi} \dot{\psi}_2 \, d\sigma \ ,$$

$$E = \frac{R^2}{2\pi\alpha'} \int_0^{2\pi} \dot{t} \, d\sigma \ . \tag{3.59}$$

These are related to the residues κ_l as

$$\kappa_0 = i \frac{2\pi\alpha'}{R^2} E \ , \quad \kappa_1 = -\frac{2\pi \cos \phi \alpha'}{R^2} J_1 \ , \quad \kappa_2 = -\frac{2\pi \sin \phi \alpha'}{R^2} J_2 \ . \tag{3.60}$$

This shows explicitly in this case that the Noether charges do indeed arise from the quasimomenta via an expansion at $z \to \infty$ as

$$p_l(z) = -\frac{2}{z} Q_l + \dots \ . \tag{3.61}$$

The coefficients of higher order terms in this expansion give higher conserved charges. For these simple solutions in flat space we can easily see what these terms are. At $\mathcal{O}(z^{-n})$, the quasimomentum p_l is either proportional to κ_l or m_l, depending on whether n is odd or even.

We can see for these simple solutions how the Virasoro constraints restrict the residues of the quasimomenta, as discussed in section 3.2.2. Using equation (3.58), we can read off the functions f_l whose σ-integrals are related to the κ_l through (3.43)

$$f_0 = \frac{i}{2}(\dot{t} \pm t') \ ,$$

$$f_1 = -\frac{1}{2 \cos \phi}(\dot{\psi}_1 \pm \psi_1') \ ,$$

$$f_2 = -\frac{1}{2 \sin \phi}(\dot{\psi}_2 \pm \psi_2') \ . \tag{3.62}$$

A straightforward check then confirms how, for $\mathbb{R} \times S^1 \times S^1$, the generalised residue conditions (3.43) and (3.44) are equivalent to the Virasoro condition expressed on the coordinates,

$$(\dot{t} \pm t')^2 = \frac{1}{\cos^2 \phi}(\dot{\psi}_1 \pm \psi_1')^2 + \frac{1}{\sin^2 \phi}(\dot{\psi}_2 \pm \psi_2')^2 \ . \tag{3.63}$$

In appendix C we show that for quasimomenta describing bosonic strings on the full curved space, we can write their residues in terms of fields using the WKB analysis and show explicitly that the GRC does indeed reproduce the full usual Virasoro constraints there too.

We noted at the end of section 3.2.2 that the GRC reduces to the previously used condition (3.46) when the functions $f_l(\sigma)$ are constants. For these

solutions on $\mathbb{R} \times S^1 \times S^1$, we can see this occurs only when t, ψ_1 and ψ_2 are all linear functions of τ and σ (i.e. when the zero mode and winding mode are excited but all other excitations are absent).

It is useful at this point to write down a general expression for $E - J$ in terms of the κ_l. Recall that J was defined as the Noether charge associated with the angle η given in (3.17), so in the $\mathbb{R} \times S^1 \times S^1$ subspace it is given by

$$J = \frac{R^2}{\alpha'} \int_0^{2\pi} \dot{\eta} \, \mathrm{d}\sigma = \cos^2 \phi J_1 + \sin^2 \phi J_2 \tag{3.64}$$

and therefore

$$E - J = \frac{\sqrt{\lambda}}{2\pi} (-i\kappa_0 + \cos\phi \kappa_1 + \sin\phi \kappa_2) . \tag{3.65}$$

3.3.2 Solutions in lightcone gauge

We will now look at solutions in lightcone gauge $x^+ = \kappa\tau$. In this gauge, it is most natural to write down a solution in the coordinates (x^+, x^-, x_1) and then switch to the coordinates (t, ψ_1, ψ_2). We will look first at a simple example, which will be useful for comparing to the same mode in static gauge, and then consider the most general mode expansion for x_1. When we do so, we will see that imposing the condition (3.46) on the residues of the quasimomenta would remove every excitation of this massless mode.[6]

Consider first the solution for x_1 given by

$$x_1 = \sqrt{\frac{2\alpha'}{n}} \left(a \cos n(\sigma + \tau) + \tilde{a} \cos \tilde{n}(\tau - \sigma) \right) , \tag{3.66}$$

with a a real constant and n an integer. Then the Virasoro constraints determine x^- to be

$$x^- = \frac{\alpha'}{2\kappa} \left[na(\tau + \sigma) - \frac{a}{4} \sin 2n(\tau + \sigma) + \tilde{n}\tilde{a}(\tau - \sigma) - \frac{\tilde{a}}{4} \sin 2\tilde{n}(\tau - \sigma) \right] . \tag{3.67}$$

In terms of t, ψ_1 and ψ_2 the solution is

$$t = \kappa\tau + \frac{\alpha'}{2\kappa R^2} \left[na(\tau + \sigma) - \frac{a}{4} \sin 2n(\tau + \sigma) \right.$$
$$\left. + \tilde{n}\tilde{a}(\tau - \sigma) - \frac{\tilde{a}}{4} \sin 2\tilde{n}(\tau - \sigma) \right] , \tag{3.68}$$

$$\psi_1 = \kappa\tau \cos^2 \phi - \cos^2 \phi \frac{\alpha'}{2\kappa R^2} \left[na(\tau + \sigma) - \frac{a}{4} \sin 2n(\tau + \sigma) \right.$$
$$\left. + \tilde{n}\tilde{a}(\tau - \sigma) - \frac{\tilde{a}}{4} \sin 2\tilde{n}(\tau - \sigma) \right]$$
$$- \sin\phi \cos\phi \sqrt{\frac{2\alpha'}{n}} \left(a \cos n(\sigma + \tau) + \tilde{a} \cos \tilde{n}(\tau - \sigma) \right) , \tag{3.69}$$

---

[6]With the exception of the zero-mode and winding which we will discuss later.

$$\psi_2 = \kappa\tau\sin^2\phi - \cos^2\phi\frac{\alpha'}{2\kappa R^2}\left[na(\tau+\sigma) - \frac{a}{4}\sin 2n(\tau+\sigma)\right.$$

$$\left. +\tilde{n}\tilde{a}(\tau-\sigma) - \frac{\tilde{a}}{4}\sin 2\tilde{n}(\tau-\sigma)\right]$$

$$+\sin\phi\cos\phi\sqrt{\frac{2\alpha'}{n}}\Big(a\cos n(\sigma+\tau) + \tilde{a}\cos\tilde{n}(\tau-\sigma)\Big) . \qquad (3.70)$$

The quasimomenta for this solution are given in the standard form (3.57), with κ_l and m_l found by inserting the above expression for t, ψ_1 and ψ_2 into (3.58) to get

$$\kappa_1 + 2\pi m_1 = -2\pi\cos\phi\left(\kappa - \frac{\alpha'na}{\kappa R^2}\right) , \qquad \kappa_1 - 2\pi m_1 = -2\pi\cos\phi\left(\kappa - \frac{\alpha'\tilde{n}\tilde{a}}{\kappa R^2}\right)$$

$$\kappa_2 + 2\pi m_2 = -2\pi\sin\phi\left(\kappa - \frac{\alpha'na}{\kappa R^2}\right) , \qquad \kappa_2 - 2\pi m_2 = -2\pi\sin\phi\left(\kappa - \frac{\alpha'\tilde{n}\tilde{a}}{\kappa R^2}\right)$$

$$\kappa_0 + 2\pi m_0 = 2\pi i\left(\kappa + \frac{\alpha'na}{\kappa R^2}\right) , \qquad \kappa_0 - 2\pi m_0 = 2\pi i\left(\kappa + \frac{\alpha'\tilde{n}\tilde{a}}{\kappa R^2}\right) .$$
$$(3.71)$$

We can see explicitly that these do not satisfy the condition (3.46) that has been previously taken to hold for the residues of the quasimomenta, indeed we have

$$\sum_{l=0}^{2}(\kappa_l + 2\pi m_l)^2 = -\frac{16\pi^2\alpha'na}{R^2} , \qquad (3.72)$$

and

$$\sum_{l=0}^{2}(\kappa_l - 2\pi m_l)^2 = -\frac{16\pi^2\alpha'\tilde{n}\tilde{a}}{R^2} . \qquad (3.73)$$

We note that in order to have $m_0 = 0$ here (the condition that t is periodic in σ), we must have $na = \tilde{n}\tilde{a}$ and hence also $m_1 = m_2 = 0$. From (3.65) we have for this solution:

$$E - J = \frac{\sqrt{\lambda}\alpha'(na + \tilde{n}\tilde{a})}{\kappa R^2} = (na + \tilde{n}\tilde{a})\frac{\sqrt{\lambda}}{J} . \qquad (3.74)$$

This matches up with the expression (3.21) for dispersion relation when we have just a single massless excitation, so this solution does indeed correspond to a massless mode as we expected. This is our first example of a massless mode solution which satisfies the generalised residue conditions (3.43) and (3.44) but not the conditions (3.46).

Now we consider the most general mode expansion for the massless mode x_1, as in (3.10). We take

$$x_1 = \sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\Big(a_n e^{-in(\tau+\sigma)} + a_n^{\dagger}e^{in(\tau+\sigma)} + \tilde{a}_n e^{-in(\tau-\sigma)} + \tilde{a}_n^{\dagger}e^{in(\tau-\sigma)}\Big)$$

$$+x_0 + \alpha'p_0\tau + w\sigma . \qquad (3.75)$$

From x_1, x^- is determined via the Virasoro constraints, see equation (3.8). We

can then find t, ψ_1 and ψ_2 from x_1 and x^- via equation (3.2). The expressions are easily obtained but as they are long and we do not need them we will not write them down explicitly. The quasimomenta have the general form given by equation (3.57) so we only need to find κ_l and m_l, which (*cf.* equation (3.58)) requires only the τ and σ derivatives of t, ψ_1 and ψ_2. These derivatives will have a double sum in the mode expansion[7] coming from x^- and a single sum coming from x_1. When we integrate over σ in (3.58) the double sum reduces to a single sum and we pick up only the zero mode contribution from x_1. The conclusion is that the quasimomenta for these solutions are given in the simple form (3.57), with κ_l and m_l given by

$$
\kappa_0 = 2\pi i \kappa + \frac{i\pi\alpha'}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger + \tilde{a}_n \tilde{a}_n^\dagger) + \frac{i\pi(\alpha'^2 p_0^2 + w^2)}{2\kappa R^2} \; ,
$$

$$
\kappa_1 = -2\pi\kappa\cos\phi + \frac{\pi\alpha'\cos\phi}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger + \tilde{a}_n \tilde{a}_n^\dagger)
$$
$$
+ \frac{\pi(\alpha'^2 p_0^2 + w^2)\cos\phi}{2\kappa R^2} + \frac{2\pi\alpha' p_0 \sin\phi}{R} \; ,
$$

$$
\kappa_2 = -2\pi\kappa\sin\phi + \frac{\pi\alpha'\sin\phi}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger + \tilde{a}_n \tilde{a}_n^\dagger)
$$
$$
+ \frac{\pi(\alpha'^2 p_0^2 + w^2)\sin\phi}{2\kappa R^2} - \frac{2\pi\alpha' p_0 \cos\phi}{R} \; , \tag{3.76}
$$

$$
2\pi m_0 = \frac{i\pi\alpha'}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger - \tilde{a}_n \tilde{a}_n^\dagger) + \frac{i\pi\alpha' p_0 w}{\kappa R^2} \; ,
$$

$$
2\pi m_1 = \frac{\pi\alpha'\cos\phi}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger - \tilde{a}_n \tilde{a}_n^\dagger) + \frac{\pi\alpha' p_0 w\cos\phi}{\kappa R^2} + \frac{2\pi w \sin\phi}{R} \; ,
$$

$$
2\pi m_2 = \frac{\pi\alpha'\sin\phi}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger - \tilde{a}_n \tilde{a}_n^\dagger) + \frac{\pi\alpha' p_0 w\sin\phi}{\kappa R^2} - \frac{2\pi w \cos\phi}{R} \; .
$$

We note that the σ-periodicity of t, $m_0 = 0$, implies the level matching condition

$$
\sum_{n=1}^{\infty} n(a_n a_n^\dagger - \tilde{a}_n \tilde{a}_n^\dagger) + p_0 w = 0 \tag{3.77}
$$

and so

$$
m_1 = \frac{w\sin\phi}{R} \; , \quad m_2 = -\frac{w\cos\phi}{R} \; . \tag{3.78}
$$

Hence, the winding modes in ψ_1 and ψ_2 come from a winding mode in x_1, and the conditions $m_1 \cos\phi \in \mathbb{Z}$ and $m_2 \sin\phi \in \mathbb{Z}$ are both satisfied if

$$
\frac{w\sin\phi\cos\phi}{R} \in \mathbb{Z} \; . \tag{3.79}
$$

From (3.65) we get for $E - J$ for this general solution (approximating

---

[7]This follows since the terms in (3.8) are squares of derivatives of x_1.

$\kappa = \frac{J}{\sqrt{\lambda}}$ again)

$$E - J = \frac{\sqrt{\lambda}}{J} \sum_{n=1}^{\infty} n(a_n^{\dagger} a_n + \tilde{a}_n^{\dagger} \tilde{a}_n) + \frac{(\alpha' p_0^2 + \frac{w^2}{\alpha'})\sqrt{\lambda}}{2J} + \mathcal{O}\left(\frac{1}{J^2}\right) \qquad (3.80)$$

As expected this is precisely the same as the massless part of the BMN expression (3.21).

The above solutions give a clear indication for why we need to generalise the condition on the residues of the quasimomenta from the conventional one given in (3.46) to the one proposed in (3.43) and (3.44). To see this, we note that for these solutions, the generalised residue condition is explicitly satisfied.[8] On the other hand, when we compute the sums of squares of residues as in equation (3.46) we find

$$\sum_{l=0}^{2} (\kappa_l + 2\pi m_l)^2 = -\frac{16\pi^2 \alpha'}{R^2} \sum_{n=1}^{\infty} n a_n^{\dagger} a_n , \qquad (3.81)$$

$$\sum_{l=0}^{2} (\kappa_l - 2\pi m_l)^2 = -\frac{16\pi^2 \alpha'}{R^2} \sum_{n=1}^{\infty} n \tilde{a}_n^{\dagger} \tilde{a}_n . \qquad (3.82)$$

Imposing the conditions (3.46) would force us to set all of the massless excitations to zero, with the exception of the zero-mode p_0 and winding w.[9] Ignoring this single exception for now, the above equation demonstrates explicitly why in previous finite-gap analysis [61], the massless mode was not present. On the other hand, the conditions (3.43) and (3.44) are sufficiently general to incorporate all of the massless modes.

3.3.3 Solutions in static gauge

In static gauge, $t = \kappa\tau$, we cannot take the same approach to writing down a general massless mode solution as in the last sub-section. It has been noted previously [122], that quantization of string theory in static gauge is in a certain manner half-way between quantization in lightcone gauge and covariant

---

[8]We saw from the general expressions (3.62) for $f_l^{\pm}$ for any solution on $\mathbb{R} \times S^1 \times S^1$ in our coset parametrisation how equations (3.43) and (3.44) are equivalent to the Virasoro constraints. Hence our solutions satisfies the residue conditions (3.43) and (3.44) by construction. It can also be confirmed explicitly that the functions $f_l^{\pm}$ for this solution satisfy equation (3.44).

[9]We noted in section 3.2.2 that the generalised residue conditions (3.43) and (3.44) reduce to the condition (3.46) precisely when the functions $f_l^{\pm}$ are constant. In section 3.3.1 we saw that for our solutions on $\mathbb{R} \times S^1 \times S^1$, the functions $f_l^{\pm}$ are constant whenever the solution is linear in τ and σ, see equation (3.62). We will also see this linear solution in static gauge in the next section, but there is one difference between the two gauges. In lightcone gauge, suppose we set $a_n = \tilde{a}_n = 0$ for all $n > 1$, as is required if the condition (3.46) holds. Then the condition that t is periodic in σ, equation (3.77), becomes $p_0 w = 0$. Hence in lightcone gauge, we can have a solution for x_1 with the condition (3.46) holding on the residues of the quasimomenta if we have either only an excited zero-mode, $x_1 = \alpha' p_0 \tau$, or a winding mode, $x_1 = w\sigma$, but not both. In static gauge, t is already periodic in σ by the gauge choice, so we don't have this additional restriction.

quantization: in D dimensions gauge-fixing in static gauge reduces the degrees of freedom to $D - 1$, but it is most natural to impose Virasoro after quantization, so there still remains one spurious degree of freedom.

However, for particularly simple solutions in static gauge, it is possible to solve the Virasoro constraints at the classical level fairly simply. If we work in the coordinates (t, η, x_1),[10] then we can write down a solution for x_1, and write down the Virasoro constraints as

$$
(\partial_\tau \pm \partial_\sigma)\eta = \sqrt{((\partial_\tau \pm \partial_\sigma)t)^2 - \frac{1}{R^2}((\partial_\tau \pm \partial_\sigma)x_1)^2}
$$
$$
= \sqrt{\kappa^2 - \frac{1}{R^2}((\partial_\tau \pm \partial_\sigma)x_1)^2} \ . \tag{3.83}
$$

We can integrate this in principle to find η, but for a general x_1 the resulting η will be given as an integral not expressible in terms of standard functions.

We note that for all solutions in $\mathbb{R} \times S^1 \times S^1$ in static gauge, we can immediately give the component p_0 of the quasimomentum from (3.56) as

$$
p_0 = \frac{2i\pi\kappa z}{z^2 - 1} \ , \tag{3.84}
$$

which has the general form (3.57) with $\kappa_0 = 2\pi i \kappa$ and $m_0 = 0$.

Consider first a simple solution linear in τ and σ,

$$
x_1 = \alpha' p_0 \tau + w\sigma \ . \tag{3.85}
$$

In this case one can solve the Virasoro constraints (3.83) explicitly to get

$$
\eta = \frac{1}{2}\sqrt{\kappa^2 - \frac{(\alpha' p_0 + w)^2}{R^2}}(\tau + \sigma)
$$
$$
+ \frac{1}{2}\sqrt{\kappa^2 - \frac{(\alpha' p_0 - w)^2}{R^2}}(\tau - \sigma) \ . \tag{3.86}
$$

In terms of ψ_1 and ψ_2 we have

$$
\psi_1 = \cos\phi\left[\psi_1^+(\tau + \sigma) + \psi_1^-(\tau - \sigma)\right],
$$
$$
\psi_2 = \sin\phi\left[\psi_2^+(\tau + \sigma) + \psi_2^-(\tau - \sigma)\right] \ , \tag{3.87}
$$

with $\psi_1^\pm$ and $\psi_2^\pm$ constants given by

$$
\psi_1^\pm = \frac{1}{2}\cos\phi\left(\sqrt{\kappa^2 - \frac{(\alpha' p_0 \pm w)^2}{R^2}}\right) - \sin\phi\frac{(\alpha' p_0 \pm w)}{R})
$$
$$
\psi_2^\pm = \frac{1}{2}\sin\phi\left(\sqrt{\kappa^2 - \frac{(\alpha' p_0 \pm w)^2}{R^2}}\right) + \cos\phi\frac{(\alpha' p_0 \pm w)}{R}) \ . \tag{3.88}
$$

---

[10]Recall η was defined in (3.17).

The quasimomenta p_1 and p_2 are again in the form (3.57) with

$$\kappa_i = -2\pi(\psi_i^+ + \psi_i^-), \quad m_i = -(\psi_i^+ - \psi_i^-) \tag{3.89}$$

for $i = 1, 2$. The condition for integer winding on ψ_1 and ψ_2 is that $m_1 \cos\phi$ and $m_2 \sin\phi$ must be integers (*cf.* equation (3.58)).

Inserting this into (3.65) gives

$$E - J = \sqrt{\lambda}\left(\kappa - \frac{1}{2}\sqrt{\kappa^2 - \frac{(\alpha' p_0 + w)^2}{R^2}}\right.$$
$$\left. - \frac{1}{2}\sqrt{\kappa^2 - \frac{(\alpha' p_0 - w)^2}{R^2}}\right) . \tag{3.90}$$

Making again the approximation $J = \sqrt{\lambda}\kappa$ to eliminate J and taking only the leading term in a large J expansion gives

$$E - J = \frac{(\alpha' p_0^2 + \frac{w^2}{\alpha'})\sqrt{\lambda}}{2J} + \mathcal{O}\left(\frac{1}{J^2}\right) , \tag{3.91}$$

and we can compare this with (3.74) to see we have the same form for this expression as we did in lightcone gauge.

Now we consider the same solution for x_1 as we looked at in section 3.3.2, but this time in static gauge,

$$t = \kappa\tau, \quad x_1 = \sqrt{\frac{2\alpha'}{n}}\left(a\cos n(\sigma + \tau) + \tilde{a}\cos\tilde{n}(\tau - \sigma)\right) . \tag{3.92}$$

η is fixed by the Virasoro constraints:

$$(\partial_\tau + \partial_\sigma)\eta = \sqrt{\kappa^2 - \frac{8\alpha' n a^2}{R^2}\sin^2 n(\tau + \sigma)} ,$$
$$(\partial_\tau - \partial_\sigma)\eta = \sqrt{\kappa^2 - \frac{8\alpha' \tilde{n}\tilde{a}^2}{R^2}\sin^2 \tilde{n}(\tau - \sigma)} . \tag{3.93}$$

To integrate this we use the following definition of the incomplete elliptic integral of the second kind:[11]

$$\mathbb{E}(\phi, k) = \int_0^\phi d\theta\sqrt{1 - k^2 \sin^2\theta} , \tag{3.94}$$

so that

$$\int d\sigma^+ \partial_+\eta = \frac{\kappa}{2n}\mathbb{E}\left(n\sigma^+, \frac{2\sqrt{2\alpha' n}a}{\kappa R}\right) ,$$
$$\int d\sigma^- \partial_-\eta = \frac{\kappa}{2\tilde{n}}\mathbb{E}\left(\tilde{n}\sigma^-, \frac{2\sqrt{2\alpha' \tilde{n}}\tilde{a}}{\kappa R}\right) \tag{3.95}$$

---

[11]We use the non-standard notation $\mathbb{E}$ rather than E to avoid confusion with the energy E.

for $\sigma^\pm = \tau \pm \sigma$, and hence

$$\eta = \frac{\kappa}{2n} \mathbb{E}\left(n(\tau + \sigma), \frac{2\sqrt{2\alpha' n}a}{\kappa R}\right) + \frac{\kappa}{2\tilde{n}} \mathbb{E}\left(\tilde{n}(\tau - \sigma), \frac{2\sqrt{2\alpha' \tilde{n}}\tilde{a}}{\kappa R}\right) . \qquad (3.96)$$

From η and x_1 we have ψ_1 and ψ_2 (*cf.* equation (3.2)), and can take derivatives and then integrate again in order to determine κ_i and m_i (*cf.* (3.58)). We get

$$\kappa_1 = -2\kappa \cos\phi \left[\mathbb{E}\left(\frac{2\sqrt{2\alpha' n}a}{\kappa R}\right) + \mathbb{E}\left(\frac{2\sqrt{2\alpha' \tilde{n}}\tilde{a}}{\kappa R}\right)\right],$$

$$\kappa_2 = -2\kappa \sin\phi \left[\mathbb{E}\left(\frac{2\sqrt{2\alpha' n}a}{\kappa R}\right) + \mathbb{E}\left(\frac{2\sqrt{2\alpha' \tilde{n}}\tilde{a}}{\kappa R}\right)\right],$$

$$2\pi m_1 = -2\kappa \cos\phi \left[\mathbb{E}\left(\frac{2\sqrt{2\alpha' n}a}{\kappa R}\right) - \mathbb{E}\left(\frac{2\sqrt{2\alpha' \tilde{n}}\tilde{a}}{\kappa R}\right)\right],$$

$$2\pi m_1 = -2\kappa \sin\phi \left[\mathbb{E}\left(\frac{2\sqrt{2\alpha' n}a}{\kappa R}\right) - \mathbb{E}\left(\frac{2\sqrt{2\alpha' \tilde{n}}\tilde{a}}{\kappa R}\right)\right], \qquad (3.97)$$

written using the complete elliptic integral of the second kind

$$\mathbb{E}(k) = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \sqrt{1 - k^2 \sin^2\theta} . \qquad (3.98)$$

From (3.65) we have

$$E - J = \sqrt{\lambda}\kappa \left[1 - \frac{1}{\pi}\mathbb{E}\left(\frac{2\sqrt{2\alpha' n}a}{\kappa R}\right) - \frac{1}{\pi}\mathbb{E}\left(\frac{2\sqrt{2\alpha' \tilde{n}}\tilde{a}}{\kappa R}\right)\right] . \qquad (3.99)$$

We make again the approximation $J = \sqrt{\lambda}\kappa$ and expand to leading order in J, using the expansion for the elliptic integral

$$\mathbb{E}(k) = \frac{\pi}{2} - \frac{\pi}{8}k^2 + \mathcal{O}(k^4) \qquad (3.100)$$

for k small. From this we get

$$E - J = (na^2 + \tilde{n}\tilde{a}^2)\frac{\sqrt{\lambda}}{J} + \mathcal{O}\left(\frac{1}{J^2}\right) . \qquad (3.101)$$

Comparing this to both the lightcone gauge result (3.74) and the previous static gauge result for a linear solution (3.91) we see again the same form for the expression, confirming that this solutions corresponds to a massless mode in static gauge.

For this solution we have

$$\sum_{l=0}^{2}(\kappa_l + 2\pi m_l)^2 = -4\pi^2\kappa^2 + 16\kappa^2 \left[\mathbb{E}\left(\frac{2\sqrt{2\alpha' n}a}{\kappa R}\right)\right]^2 , \qquad (3.102)$$

and

$$\sum_{l=0}^{2} (\kappa_l - 2\pi m_l)^2 = -4\pi^2 \kappa^2 + 16\kappa^2 \left[\mathbb{E} \left(\frac{2\sqrt{2\alpha'}\tilde{n}\tilde{a}}{\kappa R} \right) \right]^2 , \qquad (3.103)$$

and these expressions are not zero unless $na = \tilde{n}\tilde{a} = 0$.[12] We conclude that these solutions do not satisfy the residue condition (3.46) and so would not have been part of the conventional finite-gap analysis. They do however satisfy the generalised conditions (3.43) and (3.44) proposed here.[13]

3.4 Finite-gap equations with Generalised Residue Conditions

The quasimomenta studied in section 3.3 described only a bosonic subsector of the full theory. We now turn our attention to the quasimomenta for the full supergroup and look at the implications of the GRC for quasimomenta on AdS_3 backgrounds. In particular we present versions of the finite-gap equations that account for the GRC and thus should include the massless bosonic modes. In appendices D and E we show that for strings on $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$, the old residue conditions are equivalent to the GRC. This was to be expected since for those backgrounds the conventional finite-gap equations are well known to capture the complete string spectrum. In appendix F we show that the GRC for $D(2,1;\alpha)^2$ in a mixed grading, as in [98], is equivalent to the GRC for $D(2,1;\alpha)^2$ in the grading used here.

We start with quasimomenta for the coset $D(2,1;\alpha)^2/(SU(1,1)\times SU(2)^2)$. We denote the Cartan basis for one factor of $D(2,1;\alpha)$ by H_l and the other by $H_{\bar{l}}$. We use the same grading for each, with Cartan matrix given by

$$A_{lm} = A_{\bar{l}\bar{m}} = \begin{pmatrix} 4\sin^2\phi & -2\sin^2\phi & 0 \\ -2\sin^2\phi & 0 & -2\cos^2\phi \\ 0 & -2\cos^2\phi & 4\cos^2\phi \end{pmatrix} . \qquad (3.104)$$

The quasimomenta are p_l, $p_{\bar{l}}$ where $l = 1, 2, 3$, $\bar{l} = \bar{1}, \bar{2}, \bar{3}$. The action of the inversion symmetry on the quasimomenta is given by equation (3.27) with

$$S = \mathbb{1}_3 \otimes \sigma^1 . \qquad (3.105)$$

where the factor of σ_1 exchanges the index l with $\bar{l}$, hence the inversion symmetry is

$$p_l \left(\frac{1}{z} \right) = p_{\bar{l}}(z) . \qquad (3.106)$$

---

[12]This follows from the fact that the only solutions to $\mathbb{E}(k) = \frac{\pi}{2}$ for real k are $k = \pm 1$.

[13]As before, this is by construction, *cf.* equations (3.62) and the discussion in section 3.3.2.

In particular this implies the following for the residues:

$$\kappa_l = -\kappa_{\bar{l}}\,, \qquad m_l = -m_{\bar{l}}\,. \qquad (3.107)$$

The finite-gap equations for $D(2,1;\alpha)^2$ are then given as follows:

$$\frac{\pi n_{1,i}}{\sin^2 \phi} = \frac{(2\kappa_1 - \kappa_2)z + 2\pi(2m_1 - m_2)}{z^2 - 1} + \int dw \frac{2\rho_1(w) - \rho_2(w)}{z - w}$$

$$+ \int \frac{dw}{w^2} \frac{\rho_{\bar{2}}(w) - 2\rho_{\bar{1}}(w)}{z - \frac{1}{w}}\,,$$

$$\pi n_{2,i} = -\frac{(\sin^2 \phi \kappa_1 + \cos^2 \phi \kappa_3)z + 2\pi(\sin^2 \phi m_1 + \cos^2 \phi m_3)}{z^2 - 1}$$

$$-\int dw \frac{sin^2 \phi \rho_1(w) + \cos^2 \phi \rho_3(w)}{z - w}$$

$$+ \int \frac{dw}{w^2} \frac{\sin^2 \phi \rho_{\bar{1}}(w) + \cos^2 \phi \rho_{\bar{3}}(w)}{z - \frac{1}{w}}\,,$$

$$\frac{\pi n_{3,i}}{\cos^2 \phi} = \frac{(2\kappa_3 - \kappa_2)z + 2\pi(2m_1 - m_2)}{z^2 - 1} + \int dw \frac{2\rho_3(w) - \rho_2(w)}{z - w}$$

$$+ \int \frac{dw}{w^2} \frac{\rho_{\bar{2}}(w) - 2\rho_{\bar{3}}(w)}{z - \frac{1}{w}}\,,$$

$$\frac{\pi n_{\bar{1},i}}{\sin^2 \phi} = -\frac{(2\kappa_1 - \kappa_2)z + 2\pi(2m_1 - m_2)}{z^2 - 1} + \int dw \frac{2\rho_{\bar{1}}(w) - \rho_{\bar{2}}(w)}{z - w}$$

$$+ \int \frac{dw}{w^2} \frac{\rho_2(w) - 2\rho_1(w)}{z - \frac{1}{w}}\,,$$

$$\pi n_{\bar{2},i} = \frac{(\sin^2 \phi \kappa_1 + \cos^2 \phi \kappa_3)z + 2\pi(\sin^2 \phi m_1 + \cos^2 \phi m_3)}{z^2 - 1}$$

$$-\int dw \frac{sin^2 \phi \rho_{\bar{1}}(w) + \cos^2 \phi \rho_{\bar{3}}(w)}{z - w}$$

$$+ \int \frac{dw}{w^2} \frac{\sin^2 \phi \rho_1(w) + \cos^2 \phi \rho_3(w)}{z - \frac{1}{w}}\,,$$

$$\frac{\pi n_{\bar{3},i}}{\cos^2 \phi} = -\frac{(2\kappa_3 - \kappa_2)z + 2\pi(2m_1 - m_2)}{z^2 - 1} + \int dw \frac{2\rho_{\bar{3}}(w) - \rho_{\bar{2}}(w)}{z - w}$$

$$+ \int \frac{dw}{w^2} \frac{\rho_2(w) - 2\rho_3(w)}{z - \frac{1}{w}}\,, \qquad (3.108)$$

The quasimomenta $p_l^{\pm}$ describe string states on $AdS_3 \times S^3 \times S^3 \times S^1$ with the exception of the bosonic states on S^1. These can be included by adding a single $U(1)$ quasimomentum p_9. The Cartan matrix and the inversion matrix S for this $U(1)$ factor are both simply a 1×1 identity matrix. The quasimomentum p_9 trivially satisfy their own finite-gap equations with no cuts. In appendix G we give explicit expression for the residues of this quasimomentum in terms of the bosonic $U(1)$ field u_9 in the manner of the previous results for the coset massless boson.

The residues $\kappa_l \pm 2\pi m_l$ are written in terms of functions $f_l^{\pm}(\sigma)$ (cf. equation (3.43)), and these functions $f_l^{\pm}$ satisfy equation the GRC. With the inversion symmetry satisfied (so that we can write the residues of the right-movers in

terms of the left-movers say), the GRC is

$$2A_{lm}f_l^{\pm}f_m^{\pm} + (f_9^{\pm})^2 = 0 \ , \tag{3.109}$$

where A_{lm} here denotes just the 3×3 Cartan matrix for one factor of $D(2, 1\alpha)$. Explicitly this is

$$4\sin^2 \phi \left(f_1^{\pm} - \frac{1}{2}f_2^{\pm} \right)^2 + 4\cos^2 \phi \left(f_3^{\pm} - \frac{1}{2}f_2^{\pm} \right)^2 + \frac{(f_9^{\pm})^2}{2} = (f_2^{\pm})^2 \ . \tag{3.110}$$

We note that it is possible to write the general solution to this condition by introducing functions $\zeta^{\pm}(\sigma)$ and $\chi^{\pm}(\sigma)$ with

$$2\sin \phi \left(f_1^{\pm} - \frac{1}{2}f_2^{\pm} \right) = -\sin \zeta^{\pm} \cos \chi^{\pm} \ f_2^{\pm}$$

$$2\cos \phi \left(f_3^{\pm} - \frac{1}{2}f_2^{\pm} \right) = -\cos \zeta^{\pm} \cos \chi^{\pm} \ f_2^{\pm},$$

$$f_9^{\pm} = \sqrt{2}\sin \chi^{\pm} f_2^{\pm} \ . \tag{3.111}$$

Therefore, the complete proposal for the finite-gap equations with the generalised residue condition is given by equation (3.108), with κ_l and m_l given in terms of $f_l^{\pm}$ via equation (3.43), and $f_1^{\pm}, f_3^{\pm}$ and $f_9^{\pm}$ written in terms of $f_2^{\pm}$ and additional functions $\zeta^{\pm}$ and $\chi^{\pm}$ via equation (3.111).

Now we consider the case of $AdS_3 \times S^3 \times T^4$. Closed strings on $AdS_3 \times S^3$ are described by the coset $\frac{PSU(1,1|2)^2}{SU(1,1) \times SU(2)}$. We take as the Cartan matrix of $PSU(1, 1|2)$:

$$A = \begin{pmatrix} & -1 & \\ -1 & 2 & -1 \\ & -1 & \end{pmatrix} \ . \tag{3.112}$$

The quasimomenta for this space are again p_l, $p_{\bar{l}}$, $l = 1, 2, 3$, $\bar{l} = \bar{1}, \bar{2}, \bar{3}$. The inversion matrix is given by equation (3.105). The residue condition on this coset then reduces to

$$0 = 2A_{lm}f_l^{\pm}f_m^{\pm} = 4f_2^{\pm}(f_2^{\pm} - f_1^{\pm} - f_3^{\pm}) \ . \tag{3.113}$$

For strings on $AdS_3 \times S^3 \times T^4$ we can include the massless modes of T^4 much like we included the massless S^1 mode above. We add 4 additional quasimomenta p_i, $i = 6..9$ each associated to a $U(1)$. These have residues $\kappa_i \pm 2\pi m_i$ given in terms of functions $f_i(\sigma)$ just as for the functions $f_l(\sigma)$ giving the residues of the $PSU(1, 1|2)$ quasimomenta. With the Cartan matrix and inversion matrix for each $U(1)$ taken to be the identity, the GRC is now

$$0 = 2A_{lm}f_l^{\pm}f_m^{\pm} + \delta_{ij}(f_i^{\pm})(f_j^{\pm})$$
$$= 4f_2^{\pm}(f_2^{\pm} - f_1^{\pm} - f_3^{\pm}) + \delta_{ij}(f_i^{\pm})(f_j^{\pm}) \ . \tag{3.114}$$

We can write the general solution to this in terms of functions $\zeta_i^\pm$ with

$$
\begin{aligned}
f_6^\pm &= (f_1^\pm + f_3^\pm) \sin \zeta_6^\pm \ , \\
f_7^\pm &= (f_1^\pm + f_3^\pm) \cos \zeta_6^\pm \sin \zeta_7^\pm \ , \\
f_8^\pm &= (f_1^\pm + f_3^\pm) \cos \zeta_6^\pm \cos \zeta_7^\pm \sin \zeta_8^\pm \ , \\
f_9^\pm &= (f_1^\pm + f_3^\pm) \cos \zeta_6^\pm \cos \zeta_7^\pm \cos \zeta_8^\pm \sin \zeta_9^\pm \ , \\
f_2^\pm &= \frac{1}{2}(f_1^\pm + f_3^\pm)(1 - \cos \zeta_6^\pm \cos \zeta_7^\pm \cos \zeta_8^\pm \cos \zeta_9^\pm) \ .
\end{aligned}
\tag{3.115}
$$

In fact, we can make an additional simplification in this case. The Cartan matrix (3.112) has the null eigenvector $(1, 0, -1)$. Since it is $A_{lm}\kappa_m$ that appears in the finite-gap equations, we can add the any null eigenvector to the residues without changing the finite-gap equations. Therefore we can set $f_1 = f_3$.

The finite-gap equations for the quasimomenta p_l are then given by

$$
\begin{aligned}
2\pi n_{1,i} &= -\frac{\kappa_2 z + 2\pi m_2}{z^2 - 1} - \fint dw \frac{\rho_2(w)}{z - w} + \int \frac{dw}{w^2} \frac{\rho_{\bar{2}}(w)}{z - \frac{1}{w}} \ , \\
2\pi n_{2,i} &= 2\frac{(\kappa_2 - \kappa_1)z + 2\pi(m_2 - m_1)}{z^2 - 1} + \fint dw \frac{2\rho_2(w) - \rho_1(w) - \rho_3(w)}{z - w} \\
&\quad - \int \frac{dw}{w^2} \frac{2\rho_{\bar{2}}(w) - \rho_{\bar{1}}(w) - \rho_{\bar{3}}(w)}{z - \frac{1}{w}} \ , \\
2\pi n_{3,i} &= -\frac{\kappa_2 z + 2\pi m_2}{z^2 - 1} - \fint dw \frac{\rho_2(w)}{z - w} + \int \frac{dw}{w^2} \frac{\rho_{\bar{2}}(w)}{z - \frac{1}{w}} \ , \\
2\pi n_{\bar{1},i} &= \frac{\kappa_2 z + 2\pi m_2}{z^2 - 1} - \fint dw \frac{\rho_{\bar{2}}(w)}{z - w} + \int \frac{dw}{w^2} \frac{\rho_2(w)}{z - \frac{1}{w}} \ , \\
2\pi n_{\bar{2},i} &= 2\frac{(\kappa_1 - \kappa_2)z + 2\pi(m_1 - m_2)}{z^2 - 1} + \fint dw \frac{2\rho_{\bar{2}}(w) - \rho_{\bar{1}}(w) - \rho_{\bar{3}}(w)}{z - w} \\
&\quad - \int \frac{dw}{w^2} \frac{2\rho_2(w) - \rho_1(w) - \rho_3(w)}{z - \frac{1}{w}} \ , \\
2\pi n_{\bar{3},i} &= \frac{\kappa_2 z + 2\pi m_2}{z^2 - 1} - \fint dw \frac{\rho_{\bar{2}}(w)}{z - w} + \int \frac{dw}{w^2} \frac{\rho_2(w)}{z - \frac{1}{w}} \ ,
\end{aligned}
\tag{3.116}
$$

which should be taken together with the fact that the residues are given in terms of the functions f_l via equation (3.43) and these functions satisfy equation (3.114). The quasimomenta p_i associated to the T^4 directions trivially satisfy their own finite-gap equations with no cuts.

3.5 Chapter conclusions and outlook

In this chapter we have seen how to incorporate massless modes into the classical integrability framework of the algebraic curve. The dynamics of massless modes are encoded not in the branch cuts in which all massive mode dynamics are encoded, but purely in the residues of the quasimomenta at $z = \pm 1$. Using the auxiliary linear problem in section 3.2.2, we have seen that the generalised

residue condition is the correct statement of the Virasoro constraints in the algebraic curve setup. The fact that the GRC is not equivalent to the previously used residue condition, and is less restrictive in general on the allowed residues, is the key to why there is the potential for additional dynamics to be contained in the residues. By explicitly constructing quasimomenta for classical solutions in the subspace describing the dynamics for the coset massless boson, we have seen that such quasimomenta do indeed have no branch cuts, and have residues that are consistent solutions to the GRC but would have been ruled out by the previous residue conditions. Thus we can conclude that the GRC is necessarily the correct way to include massless modes.

We have written down finite-gap equations that describe the full classical dynamics, with residues given by the most general solution to the GRC. An important goal is to find the full quantum Bethe equations that reproduce these finite-gap equations with general residues in the classical limit. As well as the classical finite-gap equations, pointers to these come from various directions, including results discussed in the following chapters.

Another direction for future work is to extend the analysis of massless modes in the classical algebraic curve to the case of the mixed-flux backgrounds. The coset action of the classical mixed-flux backgrounds [113] results in quasimomenta which no longer have two simple poles at $z = \pm 1$ but four poles at locations dependent on the NS-NS flux [94]. The Virasoro constraints are still obtained by a condition on these residues, and so it should be possible to repeat the derivation of the GRC and the study of solutions for which the GRC is important for these mixed-flux quasimomenta.

Chapter 4

Quantum corrections to the algebraic curve

We now turn to the subject of semiclassical strings, that is to studying the correction to classical strings at first order in α', or equivalently in λ. In particular, any Noether charge, such as the energy E, can be expanded as

$$E = \sqrt{\lambda} \left(E_0 + \frac{1}{\sqrt{\lambda}} E_1 + \dots \right) , \qquad (4.1)$$

where E_0 is the classical energy and E_1 is the 1-loop correction. The leading order correction to the energy and other charges for given classical solutions can be found directly from the sigma model, by expanding all fields in the form

$$X(\sigma, \tau) = X^{cl}(\sigma, \tau) + \frac{1}{\lambda^{1/4}} \tilde{X}(\sigma, \tau) , \qquad (4.2)$$

and calculating the action to quadratic order in the fluctuations $\tilde{X}$. From this, imposing the equations of motion and Virasoro constraints on these fluctuations leads to generic solutions of the form

$$\tilde{X}(\sigma, \tau) = \sum_n e^{i(\omega_n \tau + n\sigma)} . \qquad (4.3)$$

We refer to ω_n as the fluctuation frequencies.

From the context of integrability, we study quantum fluctuations by considering fluctuations to the classical quasimomenta [121],

$$p_l(z) = p_l^{cl}(z) + \delta p_l(z) . \qquad (4.4)$$

As for the classical quasimomenta, the information contained in the quasimomenta fluctuations is for the most part contained in their analytic structure. We do not consider fluctuations δp_l that have branch cuts. Rather, we allow them to have isolated poles. The interpretation is that the branch cuts in quasimomenta corresponding to classical solutions emerge from the condensation of many poles.

The method for calculating quantum corrections to the energy of classical states using quasimomenta therefore works as follows. Starting from the quasimomenta for the classical solution, a series of poles are added, with their positions determined by the finite-gap equations. The different combinations of quasimomenta sheets onto which poles are added correspond to excitations

of different masses. In addition the residues of the quasimomenta at $z = \pm 1$ are varied, and the Virasoro constraints are imposed on these fluctuations of the residues. Finally asymptotic conditions are used to relate the fluctuations to changes in Noether charges.

In this chapter we will look at the analysis of fluctuations around the class of classical solutions possessing no branch cuts, first in the case of $AdS_3 \times S^3 \times T^4$ and then for $AdS_3 \times S^3 \times S^3 \times S^1$. We will introduce the details of how the set of modes are defined in each case as we go. We obtain original results on the energy fluctuation for a complete class of solutions and then look at specific examples within this class that reduce to other results studied previously. Alongside this we also show how the use of the GRC gives rise to contributions from massless bosons at the semiclassical level, where previously the analysis of fluctuation analysis missed these contributions.

Recently insight into the massless fermions of the BMN limit has been developed [112] by looking at a class of solutions where these fermionic modes have a mass, where a limit can be taken towards the BMN solution such that the energy fluctuation tends smoothly to that of a massless mode. In the results of this chapter we use the prescription of [112] for including these modes. It is still not clear how the method of quasimomenta fluctuations should treat these modes when starting directly from a solution where they give a massless contribution. From the results we obtain in this chapter, we will see that a similar issue arises from some bosonic modes that are massive for the BMN background but become massless around other backgrounds.

4.1 Fluctuations around zero-cut classical solutions in $AdS_3 \times S^3 \times T^4$

We begin by repeating and gathering together some of the key definitions regarding classical quasimomenta from the previous chapter. We denote the quasimomenta for strings on $AdS_3 \times S^3 \times T^4$ as p_l, $p_{\bar{l}}$ and p_i, with $l = 1, \ldots, 3$, $\bar{l} = \bar{1}, \ldots, \bar{3}$, and $i = 6, \ldots, 9$. p_l and $p_{\bar{l}}$ are associated to the Cartan basis for $PSU(1,1|2)^2$. We use the same Cartan basis for each factor, with Cartan matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix} . \tag{4.5}$$

These quasimomenta are related by the inversion symmetry

$$p_l(z) = p_{\bar{l}}\left(\frac{1}{z}\right) . \tag{4.6}$$

The four quasimomenta p_i each arise from a factor of $U(1)$.

As discussed extensively in the previous chapter, the residues of the classi-

cal quasimomenta are related by the generalised residue condition. Denoting the residue of p_l at $z = \pm 1$ by $\frac{1}{2}\kappa_l \pm 2\pi m_l$ (and similarly for the residues of $p_{\bar{l}}$ and p_i), the GRC says that these residues should be given as integrals

$$\frac{1}{2}(\kappa_l \pm 2\pi m_l) = \int_0^{2\pi} \mathrm{d}\sigma f_l^{\pm}(\sigma) \tag{4.7}$$

where these functions are required by the Virasoro constraints to obey the condition

$$A_{lm} f_l^{\pm}(\sigma) f_m^{\pm}(\sigma) + A_{\bar{l}\bar{m}} f_{\bar{l}}^{\pm}(\sigma) f_{\bar{m}}^{\pm}(\sigma) + \delta_{ij} f_i^{\pm}(\sigma) f_j^{\pm}(\sigma) = 0 . \tag{4.8}$$

The inversion symmetry relates the residues of the two $PSU(1,1|2)$ factors via $f_l^{\pm} = -f_{\bar{l}}^{\pm}$ so we can combine both the Virasoro constraints and the action of the inversion symmetry on the residues into the single condition

$$A_{lm} f_l^{\pm}(\sigma) f_m^{\pm}(\sigma) + \frac{1}{2}\delta_{ij} f_i^{\pm}(\sigma) f_j^{\pm}(\sigma) = 0 . \tag{4.9}$$

The large z behaviour of the quasimomenta encodes the Noether charges at $\mathcal{O}\left(\frac{1}{z}\right)$, as well as a constant term if the solution possesses non-zero worldsheet momentum. We write

$$p_l(z) \to P_l + \frac{2\pi}{\sqrt{\lambda}} \frac{Q_l}{z} + \ldots , \quad z \to \infty . \tag{4.10}$$

Fluctuations of classical quasimomenta consist of the addition of extra poles together with fluctuations of the residues at $z = \pm 1$. We will discuss the residue fluctuations presently. To add fluctuation poles we need to know three things: which combinations of quasimomenta to add poles to, what positions in the complex plane the poles can be placed at, and what the residues of these poles should be. The possible combinations of quasimomenta that can have a pole excited simultaneously at the same point give us the set of modes of quantum excitations (with the exception of the massless bosons as we shall see). The possible modes are determined in this way independently of which particular classical solution we are expanding around, as should be expected. By contrast, the position of the poles is determined in a way dependent on the classical background, and gives rise ultimately to the specific energy contribution of each mode to a particular classical background. The residues of the poles can be determined to depend on the pole position in a form universal for each mode and each classical background; this arises from a quantization condition on the action variables as they appear in this setting.

On the left of figure 6 is shown the Dynkin diagram for $PSU(1,1|2)$. The quasimomenta p_l are associated to this Cartan basis. We can think of the three quasimomenta p_l as describing a Riemann surface with a four-sheet cover. By adding poles simultaneously to different combinations of the quasimomenta,

we are connecting different combinations of these sheets with the poles. Equivalently, we can consider the supermatrix realisation of $SU(1,1|2)$ as a subgroup of the supergroup $GL(2|2)$ of $(2|2) \times (2|2)$ supermatrices. There are four diagonal supermatrices, each of which comes with either a bosonic or a fermionic Grassmann grading. An alternative way of defining the $PSU(1,1|2)$ quasimomenta than the group-theoretical way used here is to define four quasimomenta given explicitly through the supermatrix realisation. These four quasimomenta are associated to the four diagonal supermatrices. The three quasimomenta p_l are associated to the Cartan basis of $PSU(1,1|2)$ which requires restricting the diagonal supermatrices to those with zero trace. A convenient basis involves considering three sums of pairs of diagonal supermatrices. This is represented pictorially in figure 6 with the nodes of the $PSU(1,1|2)$ Dynkin diagram drawn in between four lines. Then a branch cut or pole is always shared between precisely two of the "supermatrix basis" quasimomenta, and is related to a cut or pole between the quasimomenta p_l by defining which sheets of the Riemann surface are joined.

In figure 6 we show the possible sets of poles. We denote the position of a possible pole by z_n^r. The number n relates to the position of the pole as we will discuss presently. r defines the mode of the pole, and we will use a notation where $r \subset \{1, 2, 3\}$ denotes precisely which quasimomenta p_l share that pole, as shown in figure 6. The modes can be identified as bosonic or fermionic according to the grading of the diagonal supermatrix elements corresponding to the sheets they join, so that bosonic modes arise from poles joining two sheets of both bosonic or both fermionic grading etc. We refer to modes as massive or massless according to how they appear in the BMN point-particle solution. It was realised that the massless fermions arise from poles as shown in [112], by considering a family of solutions containing the BMN point-particle solution where poles on the relevant sheets give modes with a mass that becomes zero as one takes a limit to the BMN solution. As we will see, it is not yet completely clear how to deal with these modes when starting directly from a solution such as the BMN point-particle where they are massless.

We normalise the pole at $z = z_n^r$ with a canonical residue $\alpha(z_n^r)$ given by

$$\alpha(z) = \frac{2\pi}{\sqrt{\lambda}} \frac{z^2}{z^2 - 1} \ . \tag{4.11}$$

This is done in order to produce an integer filling fraction upon integrating around the pole, as these are the action-variables which are naturally quantized [123, 124].

On branch cuts, the classical quasimomenta are required to satisfy the finite-gap equations which can be written as

$$A_{lm} \not{p}_m(z) = 2\pi n_l \ . \tag{4.12}$$
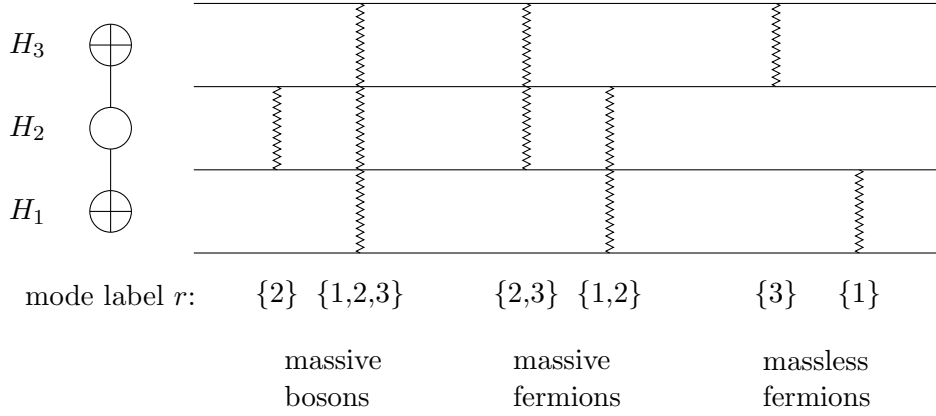
Figure 6: $PSU(1,1|2)$ modes. On the left is shown the Cartan diagram. The quasimomenta describe a four-sheet algebraic curve. A link between two sheets level with the Cartan element H_l corresponds to a mode which excites a pole on the corresponding quasimomentum p_l.

We can determine the permissible positions for the poles of the fluctuated quasimomenta by regarding them as infinitesimal branch cuts, and looking for solutions to the finite-gap equations at a single point. For example, we determine the position of the poles for mode $r = \{2\}$ by solving the equation

$$A_{2m}\, p_m(z_n^{\{2\}}) = 2\pi n \ . \tag{4.13}$$

We consider classical solutions with no branch cuts, so their dynamics are contained solely within the residues at $z = \pm 1$. Furthermore we consider only those solutions with equal residues at both poles, that is we set the winding m_l to zero. Explicitly the classical quasimomenta are

$$p_l(z) = \frac{\kappa_l z}{z^2 - 1} = -p_{\bar{l}}(z) \ , \quad p_i(z) = \frac{\kappa_i z}{z^2 - 1} \ . \tag{4.14}$$

The residue functions $f_l(\sigma)$, $f_i(\sigma)$ satisfy

$$(f_2)^2 + \frac{1}{4}\delta_{ij}f_i f_j = f_2(f_1 + f_3) \ . \tag{4.15}$$

The position $z_n^{\{2\}}$ of the poles added for the mode $r = \{2\}$ to this class of classical solutions is determined by

$$\frac{(2\kappa_2 - \kappa_1 - \kappa_3)z_n^{\{2\}}}{(z_n^{\{2\}})^2 - 1} = 2\pi n \tag{4.16}$$

and the solution to this is[1]

$$z_n^{\{2\}} = \frac{2\kappa_2 - \kappa_1 - \kappa_3}{4\pi n} - \sqrt{1 + \frac{(2\kappa_2 - \kappa_1 - \kappa_3)^2}{16\pi^2 n^2}} \ . \tag{4.17}$$

---

[1]We have chosen the sign of the square root to guarantee a physical pole $|z| > 1$ in the case $\kappa_1 + \kappa_3 > 2\kappa_2$. In the reverse case the opposite sign should be chosen.

Modes that excite poles on multiple quasimomenta should be required to satisfy

$$\sum_{l \in r} A_{lm} p_m(z_n^r) = 2\pi n \ , \tag{4.18}$$

as can be understood from the perspective of these types of poles as "stacks" which are connecting non-adjacent sheets in figure 6. For example, the pole positions $z_n^{\{1,2,3\}}$ of the other massive bosonic mode is determined on our zero-cut classical quasimomenta by

$$-\frac{(\kappa_1 + \kappa_3) z_n^{\{1,2,3\}}}{(z_n^{\{1,2,3\}})^2 - 1} = 2\pi n \ . \tag{4.19}$$

In general we can write the equations determining the pole position for every mode on the zero-cut solutions in the following form

$$\frac{z_n^r}{(z_n^r)^2 - 1} = -\frac{2\pi n s_r}{(\kappa_1 + \kappa_3) m_r} \ , \tag{4.20}$$

where s_r takes different signs on the two coset factors, $s_r = +1$ for $r = \{l\}, \{l, m\}$ etc. and $s_r = -1$ for $r = \{\bar{l}\}, \{\bar{l}, \bar{m}\}$ etc. We will see that the parameters m_r emerge as the masses of the various modes. They are given in terms of the classical residues as follows

$$m_{\{2\}} = m_{\{\bar{2}\}} = 1 - \frac{2\kappa_2}{\kappa_1 + \kappa_3} \ ,$$

$$m_{\{1,2\}} = m_{\{\bar{1},\bar{2}\}} = m_{\{2,3\}} = m_{\{\bar{2},\bar{3}\}} = 1 - \frac{\kappa_2}{\kappa_1 + \kappa_3} \ ,$$

$$m_{\{1\}} = m_{\{\bar{1}\}} = m_{\{3\}} = m_{\{\bar{3}\}} = \frac{\kappa_2}{\kappa_1 + \kappa_3} \ ,$$

$$m_{\{1,2,3\}} = m_{\{\bar{1},\bar{2},\bar{3}\}} = 1 \ , \qquad .$$

The general solution to equation (4.20) giving the position of the poles for each mode in terms of these masses is

$$z_n^r = -s_r \frac{(\kappa_1 + \kappa_3) m_r}{4\pi n} - s_r \sqrt{1 + \frac{(\kappa_1 + \kappa_3)^2 m_r^2}{16\pi^2 n^2}} \ . \tag{4.21}$$

The explicit expression for the quasimomenta fluctuations, fixed already to satisfy the inversion relation, is

$$\delta p_l(z) = \frac{\delta \kappa_l z}{z^2 - 1} + \sum_{r \ni l} \sum_{n=0}^{\infty} \frac{\alpha(z_n^r)}{z - z_n^r} N_n^r - \sum_{r \ni \bar{l}} \sum_{n=0}^{\infty} \frac{\alpha(z_n^r)}{\frac{1}{z} - z_n^r} N_n^r \ ,$$

$$\delta p_{\bar{l}}(z) = -\frac{\delta \kappa_{\bar{l}} z}{z^2 - 1} - \sum_{r \ni \bar{l}} \sum_{n=0}^{\infty} \frac{\alpha(z_n^r)}{z - z_n^r} N_n^r + \sum_{r \ni l} \sum_{n=0}^{\infty} \frac{\alpha(z_n^r)}{\frac{1}{z} - z_n^r} N_n^r \ ,$$

$$\delta p_i(z) = \frac{\delta \kappa_i z}{z^2 - 1} \ . \tag{4.22}$$

As an explicit example to clarify the notation we are using, the fluctuation for

p_1 is given by

$$
\delta p_1(z) = \frac{\delta\kappa_1 z}{z^2-1} + \sum_{n=0}^{\infty} \left(\frac{\alpha(z_n^{\{1\}})}{z-z_n^{\{1\}}} N_n^{\{1\}} + \frac{\alpha(z_n^{\{1,2\}})}{z-z_n^{\{1,2\}}} N_n^{\{1,2\}} + \frac{\alpha(z_n^{\{1,3\}})}{z-z_n^{\{1,3\}}} N_n^{\{1,3\}} \right.
$$
$$
- \frac{\alpha(z_n^{\{\bar{1}\}})}{\frac{1}{z}-z_n^{\{\bar{1}\}}} N_n^{\{\bar{1}\}} - \frac{\alpha(z_n^{\{\bar{1},\bar{2}\}})}{\frac{1}{z}-z_n^{\{\bar{1},\bar{2}\}}} N_n^{\{\bar{1},\bar{2}\}} - \frac{\alpha(z_n^{\{\bar{1},\bar{3}\}})}{\frac{1}{z}-z_n^{\{\bar{1},\bar{3}\}}} N_n^{\{\bar{1},\bar{3}\}}
$$
$$
\left. + \frac{\alpha(z_n^{\{1,2,3\}})}{z-z_n^{\{1,2,3\}}} N_n^{\{1,2,3\}} - \frac{\alpha(z_n^{\{\bar{1},\bar{2},\bar{3}\}})}{\frac{1}{z}-z_n^{\{\bar{1},\bar{2},\bar{3}\}}} N_n^{\{\bar{1},\bar{2},\bar{3}\}} \right) . \quad (4.23)
$$

N_n^r are integers corresponding to the possible multiplicities exciting a single mode r at a single mode number n. We now impose asymptotic conditions on the fluctuations. In accordance with the integer filling fractions resulting from adding poles with the canonical residue (4.11), we have natural quantization conditions on the conserved charges. Hence we require that as $z \to \infty$,[2]

$$
\delta p_l \to c_l + \frac{2\pi}{z\sqrt{\lambda}} \left(\frac{1}{2}\delta\Delta + \sum_{r \ni l} N^r \right) ,
$$
$$
\delta p_{\bar{l}} \to c_{\bar{l}} - \frac{2\pi}{z\sqrt{\lambda}} \left(\frac{1}{2}\delta\Delta + \sum_{r \ni \bar{l}} N^r \right) , \quad (4.24)
$$

where $N^r = \sum_{n=0}^{\infty} N_n^r$. Here $\delta\Delta$ is sometimes called the anomalous energy shift. As discussed in [121], it can be thought of as the part of the energy correction which takes no contribution from any of the zero-modes of the excitations. Imposing the asymptotic conditions (4.24) at $\mathcal{O}\left(\frac{1}{z}\right)$ on the fluctuations (4.22) gives us the relations[3]

$$
\frac{\sqrt{\lambda}}{2\pi}\delta\kappa_l + \sum_{r \ni l,\bar{l}} \sum_{n=0}^{\infty} \frac{1}{(z_n^r)^2-1} N_n^r = \frac{1}{2}\delta\Delta . \quad (4.25)
$$

Now we need to solve the Virasoro constraints, using the GRC derived in the previous chapter, to determine the fluctuations $\delta\kappa_l$ at $z = \pm1$. In particular, we will show that we can choose residue fluctuations that give rise to the massless bosonic contribution to the energy corrections. We solve the GRC condition (4.15) for semiclassical quasimomenta by expanding in λ via

$$
f_l = f_l^{(0)}(\sigma) + \lambda^{-1/4} f_l^{(1)}(\sigma) + \lambda^{-1/2} f_l^{(2)}(\sigma) + \dots \quad (4.26)
$$

(and similarly for f_i). f_l^0 are the classical residue functions. The quasimomenta, and in particular the residues of the quasimomenta, have a semiclassi-

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[2]The relationship between the set of charge corrections δQ_l and the energy correction $\delta\Delta$ also needs to be checked. Equation (4.42) gives the values of the residue functions in terms of worldsheet fields for a zero-cut solution. From the dependence on the worldsheet time t in that equation we can see that the form that $\delta\Delta$ takes in equation (4.24) is correct.

[3]Solving the asymptotic conditions at $\mathcal{O}(1)$ amounts to just fixing the values of the constants c_l, $c_{\bar{l}}$. These values play no role in the analysis of the energy correction.

cal correction at $\mathcal{O}(\lambda^{-1/2})$. However the residue functions f_l arise naturally as derivatives of the bosonic fields, and therefore like the fields should have their first subleading correction at $\mathcal{O}(\lambda^{-1/4})$. This is consistent provided $f_l^{(1)}(\sigma)$ are functions which integrate over the period of σ to zero. We still need to include them however, because the GRC is a non-linear constraint and so products of the form $f_l^{(1)} f_m^{(1)}$ can show up in the residue fluctuations $\delta\kappa_l$. Indeed as we shall see this is exactly how the massless bosonic contributions arise, and we shall have the four massless bosons arising from the four functions $f_i^1(\sigma)$.

With this expansion, the terms of $\mathcal{O}(\lambda^{-1/4})$ in the GRC condition (4.15) give rise to the constraint

$$2f_2^{(0)} f_2^{(1)} + \frac{1}{2}\delta_{ij} f_i^{(0)} f_j^{(1)} = f_2^{(0)}(f_1^{(1)} + f_3^{(1)}) + f_2^{(1)}(f_1^{(0)} + f_3^{(0)}) \ . \qquad (4.27)$$

We can consistently set $f_2^{(1)} = 0$. If $f_2^{(0)} \neq 0$, we can solve this first constraint with

$$f_1^{(1)} + f_3^{(1)} = \frac{1}{2f_2^{(0)}}\delta_{ij} f_i^{(0)} f_j^{(1)} \ . \qquad (4.28)$$

If instead $f_2^{(0)} = 0$ we can consistently set $f_1^{(1)} + f_3^{(1)} = 0$ in addition to $f_2^{(1)} = 0$.[4] Either way, the terms of the GRC condition at $\mathcal{O}(\lambda^{-1/2})$ become

$$f_2^{(2)}(f_1^{(0)} + f_3^{(0)} - 2f_2^{(0)}) + f_2^{(0)}(f_1^{(2)} + f_3^{(2)}) = \frac{1}{4}\delta_{ij}(f_i^{(1)} f_j^{(1)} + 2f_i^{(0)} f_j^{(2)}) \ . \qquad (4.29)$$

We should set $f_i^{(2)} = 0$, as the Noether charges coming from the quasimomenta p_i are simply the four $U(1)$ charges which do not receive quantum corrections.[5] We can also set all the functions $f_l^{(2)}$ to be constant in σ, so that[6]

$$\delta\kappa_l = \frac{2\pi}{\sqrt{\lambda}} f_l^{(2)} \ . \qquad (4.30)$$

Now we can integrate equation (4.29) so as to give the left-hand side in terms of classical residues κ_l and residue fluctuations $\delta\kappa_l$,[7]

$$(\kappa_1 + \kappa_3 - 2\kappa_2)\frac{\sqrt{\lambda}}{2\pi}\delta\kappa_2 + \kappa_2\frac{\sqrt{\lambda}}{2\pi}(\delta\kappa_1 + \delta\kappa_3) = \frac{1}{4}\int_0^{2\pi} \mathrm{d}\sigma \delta_{ij} f_i^{(1)} f_j^{(1)} \ . \qquad (4.31)$$

---

[4]Considering the GRC (4.15) at the leading, classical level, if $f_2^{(0)} = 0$ then $\delta_{ij} f_i^{(0)} f_j^{(0)} = 0$. As the residue functions are real this means that $f_2^{(0)} = 0$ implies that $f_i^{(0)} = 0$ for $i = 6, .., 9$.

[5]These Noether charges are of the form $Q_i \propto \int_0^{2\pi} u_i \mathrm{d}\sigma$ in the metric (4.41). For a fluctuation δu_i of the form (4.3) we can see that $\delta Q_i = 0$.

[6]The reason we can do this is that we are ultimately interested in the fluctuations of the residues, not of the residue functions $f_l(\sigma)$. In a semiclassical expansion, at any given order in $\lambda^{-1/4}$, the fluctuation of the residue κ_l at that order is clearly proportional to just the constant part of the corresponding residue function $f_l(\sigma)$, since the residue functions are periodic in σ. The GRC is a nonlinear equation and so non-constant parts of the residue functions do play a role in an order-by-order solution of the GRC. However the non-constant parts of $f_l^{(2)}$ would only have an effect if we were considering the residues at higher orders than $\lambda^{-1/2}$

[7]Recall the integration over residue functions to give residues is defined as in equation (4.7).

We can use the asymptotic conditions in the form of equation (4.25) to eliminate all three residue fluctuations $\delta\kappa_l$ from equation (4.31) in terms of $\delta\Delta$ and terms which depend on the pole positions z_n^r. Rearranging for $\delta\Delta$ we find it is given by[8]

$$\delta\Delta = \frac{1}{2(\kappa_1 + \kappa_3)} \int_0^{2\pi} d\sigma \delta_{ij} f_i^1 f_j^1 + \sum_{(\text{All } r)} \sum_{n=0}^{\infty} \frac{2m_r}{(z_n^2)^2 - 1} N_n^r \ . \tag{4.32}$$

in terms of the masses m_r defined in equation (4.21)

Now we pick for the functions $f_i^{(1)}$ the following form

$$f_i^{(1)}(\sigma) = \sum_{n=1}^{\infty} \sqrt{2n} \left((a_i)_n e^{-in\sigma} + (a_i)_n^\dagger e^{in\sigma} \right) \ . \tag{4.33}$$

Through the study of how the residue functions arise in the previous chapter, we can see that this is very natural for a residue function describing the quantum fluctuations of each of the $U(1)$ fields, with creation and annihilation operators $(a_i)_n^\dagger$ and $(a_i)_n$. However, it is also the general form these functions should take purely from the algebraic curve perspective without comparing to the worldsheet, since we know that $f_i^{(1)}$ must be periodic in σ. We then have

$$\int_0^{2\pi} d\sigma (f_i^{(1)})^2 = 8\pi \sum_{n=1}^{\infty} n N_n^i \tag{4.34}$$

with N_n^i being the usual number operator. The energy correction is now given by

$$\delta\Delta = \frac{4\pi}{\kappa_1 + \kappa_3} \sum_{i=6}^{9} \sum_{n=1}^{\infty} n N_n^i + \sum_{(\text{All } r)} \sum_{n=0}^{\infty} \Omega_n^r N_n^r \tag{4.35}$$

with the fluctuation frequencies Ω_n^r for all modes excluding the massless bosons defined to be

$$\Omega_n^r = \Omega(z_n^r) = \frac{2m_r}{(z_n^r)^2 - 1} \ . \tag{4.36}$$

Using the explicit expressions for z_n^r we found previously these are

$$\Omega(z_n^r) = -m_r + \sqrt{m_r^2 + \left(\frac{4\pi}{\kappa_1 + \kappa_3}\right)^2 n^2} \ . \tag{4.37}$$

The physical one-loop correction to the energy is given by

$$E^{1-loop} = \sum_r \sum_{n=0}^{\infty} (-1)^{F_r} \Omega_n^r \ , \tag{4.38}$$

where F_r is equal to $+1$ for bosonic modes and -1 for fermionic modes. In

---

[8]To see fairly quickly that the masses m_r appear in this way note that the left-hand side of equation (4.31) can be written as $\frac{\sqrt{\lambda}}{2\pi}(\kappa_1 + \kappa_2)(m_{\{1\}}\delta\kappa_1 + m_{\{2\}}\delta\kappa_2 + m_{\{3\}}\delta\kappa_3)$ and that $m_{\{1,2\}} = m_{\{1\}} + m_{\{2\}}$ etc.

checking whether this sum is UV divergent, we can approximate the sum by an integral over continuous n and check that there are no divergences in this integral. For the solutions considered here, the masses (4.21) satisfy the relation

$$\sum_r (-1)^{F_r} m_r^2 = 0 \ , \tag{4.39}$$

and it follows from this that

$$\sum_r (-1)^{F_r} \Omega_n^r \sim \mathcal{O}\left(\frac{1}{n^2}\right) \tag{4.40}$$

for large n, and so the one-loop energy correction has no UV divergence.[9]

4.1.1 Examples

We now give some examples of particular classical solutions which fall into this category of having zero-cut algebraic curves and the associated energy spectra that follows from the analysis above, in particular the set of masses m_r. The simplest such solution is the BMN background. In the coordinates for $AdS_3 \times S^3 \times T^4$ given by

$$ds^2 = d\rho^2 - \cosh^2\rho \ dt^2 + \sinh^2\rho \ d\gamma^2$$

$$+ d\theta^2 + \cos^2\theta \ d\psi^2 + \sin^2\theta \ d\varphi^2 + \sum_{i=6}^{9} du_i^2 \ , \tag{4.41}$$

zero cut classical solutions arise from the subspace spanned by t, ψ, u_i, and for such solution the classical residue functions are given in terms of derivatives of the fields by

$$f_1^\pm = f_3^\pm = \frac{1}{4}(\dot{t} \pm \acute{t}) \ , \quad f_2^\pm = \frac{1}{4}(\dot{t} \pm \acute{t}) - \frac{1}{4}(\dot{\psi} \pm \acute{\psi}) \ , \quad f_i^\pm = \frac{1}{2}(\dot{u}_i \pm \acute{u}_i) \ . \tag{4.42}$$

The BMN background is the solution

$$t = \psi = \kappa\tau \ , \tag{4.43}$$

corresponding to classical quasimomenta with $f_1 = f_3 = \frac{1}{2}\kappa$. Hence we find straight away from the general analysis above that there are a set of fluctuation frequencies for massive modes (4 bosons and 4 fermions) with mass $m_r = 1$,

$$\Omega_r = -1 + \sqrt{1 + \frac{n^2}{\kappa^2}} \ , \quad r \ni 2, \bar{2} \ . \tag{4.44}$$

A generalisation of the BMN background is to consider the solution corresponding to an algebraic curve with general constant residue functions, which

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[9]I am grateful to Olof Ohlsson Sax for a discussion of this.

we can write as

$$f_2 = \frac{1}{2}(f_1 + f_3)(1 - \cos \zeta_6 \cos \zeta_7 \cos \zeta_8 \cos \zeta_9) , \qquad f_6 = (f_1 + f_3) \sin \zeta_6 ,$$

$$f_7 = (f_1 + f_3) \cos \zeta_6 \sin \zeta_7 , \qquad f_8 = (f_1 + f_3) \cos \zeta_6 \cos \zeta_7 \sin \zeta_8 ,$$

$$f_9 = (f_1 + f_3) \cos \zeta_6 \cos \zeta_7 \cos \zeta_8 \sin \zeta_9 .$$

$$(4.45)$$

This arises from the worldsheet solution

$$t = \kappa\tau , \qquad u_6 = (\kappa \sin \zeta_6)\tau , \qquad u_7 = (\kappa \cos \zeta_6 \sin \zeta_7)\tau ,$$

$$u_8 = (\kappa \cos \zeta_6 \cos \zeta_7 \sin \zeta_8)\tau , \qquad u_9 = (\kappa \cos \zeta_6 \cos \zeta_7 \cos \zeta_8 \sin \zeta_9)\tau , \quad (4.46)$$

$$\psi = (\kappa \cos \zeta_6 \cos \zeta_7 \cos \zeta_8 \cos \zeta_9)\tau.$$

This gives us a set of energy fluctuations where the masses can be read off from (4.21) with

$$\frac{\kappa_2}{\kappa_1 + \kappa_3} = \frac{1}{2}(1 - \cos \zeta_4 \cos \zeta_5 \cos \zeta_6 \cos \zeta_7) . \qquad (4.47)$$

When $\zeta_i = 0$ for $i = 6\ldots 9$ we have the BMN solution with all massive excitations having the same mass.

The general set of masses (4.21) are given in terms of a single parameter, $\frac{\kappa_2}{\kappa_1 + \kappa_3}$. Therefore we can note that given any zero-cut, no-winding background solution, we can identify a solution of the form (4.46) with the same energy frequencies by finding the value of $\frac{\kappa_2}{\kappa_1 + \kappa_3}$ for the given background and choosing ζ_i that satisfy (4.47). We can also note by examining the set of masses (4.21) that all these solutions have a maximum of four massless fermions, and a minimum of four massless bosons. We get six massless bosons from residues such as

$$f_1 = f_2 = f_3 = \frac{1}{2}\kappa , \quad f_6 = \kappa , \quad f_i = 0 , \quad i = 7\ldots 9 , \qquad (4.48)$$

which corresponds to a BMN-like point-particle solution which instead of rotating around the equator of the S^3 rotates around one of the S^1's,

$$t = u_6 = \kappa\tau . \qquad (4.49)$$

In addition to the six massless bosons, this solution has two bosons of mass 1, and eight fermions with mass $\frac{1}{2}$.

One final example we consider is the following solution

$$t = \kappa\tau , \quad u_6 = a \cos(m\sigma) \cos(m\tau) ,$$

$$\psi = \omega\tau , \quad u_7 = a \cos(m\sigma) \sin(m\tau) . \qquad (4.50)$$

with

$$\kappa^2 = \omega^2 + a^2 m^2 \qquad (4.51)$$

to satisfy the Virasoro constraints. This solution has residues functions given by

$$
f_1^{\pm} = f_3^{\pm} = \frac{1}{2}\kappa \ , \qquad f_2^{\pm} = \frac{1}{2}(\kappa - \omega) \ ,
$$
$$
f_6^{\pm} = \mp am \sin(m\sigma) \ , \qquad f_7^{\pm} = am \cos(m\sigma) \ .
$$

(4.52)

This solution appears to not fall into the category we have considered since it has non-zero winding with $f_6^+ \neq f_6^-$. However it does indeed have fluctuations of the form (4.22) and we do not need to repeat the analysis from scratch to see this. First, the residues for this solution are given by

$$
\kappa_1 = \kappa_3 = 2\pi\kappa \ , \quad \kappa_2 = 2\pi(\kappa - \omega) \ , \quad \kappa_6 = \kappa_7 = 0 \ , \quad m_l = 0 \ ,
$$

(4.53)

and we see that the winding does not enter the residues themselves, so the analysis of adding poles and determining their positions is unaffected. This only leaves the perturbative solution to the GRC, namely equations (4.28) and (4.29). With $f_i^{(0)+} \neq f_i^{(0)-}$, equation (4.28) requires us to also have $f_1^{(1)+} + f_3^{(1)+} \neq f_1^{(1)-} + f_3^{(1)-}$, but this then affects nothing else in the derivation of the fluctuation frequencies. In particular, equation (4.29) does not contain any terms with residue functions that are unequal at ± 1 once we have set $f_i^{(2)} = 0$.

The masses we get for this solution are

$$
m_{\{2\}} = m_{\{\bar{2}\}} = \frac{\omega}{\kappa} \ , \qquad m_{\{1,2,3\}} = m_{\{\bar{1},\bar{2},\bar{3}\}} = 1 \ ,
$$
$$
m_{\{1,2\}} = m_{\{\bar{1},\bar{2}\}} = m_{\{2,3\}} = m_{\{\bar{2},\bar{3}\}} = \frac{1}{2}\left(1 + \frac{\omega}{\kappa}\right) \ ,
$$
$$
m_{\{1\}} = m_{\{\bar{1}\}} = m_{\{3\}} = m_{\{\bar{3}\}} = \frac{1}{2}\left(1 - \frac{\omega}{\kappa}\right) \ .
$$

(4.54)

The examples we have given here represent good candidates within the general set of masses (4.21) derived using the algebraic curve to test against worldsheet calculations. In the next section, where we derive similar results for the $D(2,1;\alpha)$ theory, we give examples of the general results that can be checked against previous worldsheet calculations in the literature.

4.2 Fluctuations around zero-cut classical solutions in $AdS_3 \times S^3 \times S^3 \times S^1$

The quasimomenta for strings on $AdS_3 \times S^3 \times S^3 \times S^1$ are p_l, $p_{\bar{l}}$ associated to $D(2,1;\alpha)^2$ and a single $U(1)$ quasimomentum p_9. The Cartan matrix for $D(2,1;\alpha)^2$ is

$$
A = \begin{pmatrix} 4\sin^2\phi & -2\sin^2\phi & 0 \\ -2\sin^2\phi & 0 & -2\cos^2\phi \\ 0 & -2\cos^2\phi & 4\cos^2\phi \end{pmatrix} \otimes \mathbb{1}_2 \ .
$$

(4.55)

The analysis of semiclassical fluctuations for zero-cut $D(2,1;\alpha)$ solutions is very similar to that done in the section above for $PSU(1,1|2)$ solutions. The main difference is the identification of the excitation modes. The possible mode labels r are as follows:[10]

light bosons:	$\{1\}$,	$\{3\}$,	$\{\bar{1}\}$,	$\{\bar{3}\}$
light fermions:	$\{1,2\}$,	$\{2,3\}$,	$\{\bar{1},\bar{2}\}$,	$\{\bar{2},\bar{3}\}$
heavy bosons:	$\{1,2,2,3\}$,	$\{\bar{1},\bar{2},\bar{2},\bar{3}\}$		
heavy fermions:	$\{1,2,3\}$,	$\{\bar{1},\bar{2},\bar{3}\}$		
massless fermions:	$\{2\}$,	$\{\bar{2}\}$		

Again we label them according to their role in the BMN background. Unlike in $PSU(1,1|2)$, we do not distinguish the masses of the modes as either massive or massless, but distinguish the massive modes into "light" and "heavy". In the associated spin-chain, the light modes are the fundamental excitations that appear, while the heavy modes emerge as composites of the light modes. From the perspective of the quasimomenta, the heavy modes appear as stacks. Thus combining either the light boson $r = \{1\}$ with the light fermion $r = \{2,3\}$, or the boson $r = \{3\}$ with the fermion $r = \{1,2\}$ gives rise to the same mode, the heavy fermion $r = \{1,2,3\}$. The heavy boson arises not by combining two light bosonic modes, but from two light fermions. Hence it is "doubly" excited on p_2 and we will write it as $r = \{1,2,2,3\}$. Explicitly this means that while all other modes consist of poles at positions z_n^r determined by equation (4.18), this heavy bosonic mode has its pole position determined by the condition

$$(A_{1m} + 2A_{2m} + A_{3m})p_m(z_n^{\{1,2,2,3\}}) = 2\pi n , \qquad (4.56)$$

and similarly for the other heavy boson on the other $D(2,1;\alpha)$ factor.

Using the following masses m_r:

$$m_{\{1\}} = m_{\{\bar{1}\}} = \sin^2\phi\left(1 - \frac{2\kappa_1}{\kappa_2}\right) , \quad m_{\{3\}} = m_{\{\bar{3}\}} = \cos^2\phi\left(1 - \frac{2\kappa_3}{\kappa_2}\right) ,$$

$$m_{\{2\}} = m_{\{\bar{2}\}} = \sin^2\phi\frac{\kappa_1}{\kappa_2} + \cos^2\phi\frac{\kappa_3}{\kappa_2} , \qquad m_{\{1,2,2,3\}} = m_{\{\bar{1},\bar{2},\bar{2},\bar{3}\}} = 1 ,$$

$$m_{\{1,2\}} = m_{\{\bar{1},\bar{2}\}} = \sin^2\phi\left(1 - \frac{\kappa_1}{\kappa_2}\right) + \cos^2\phi\frac{\kappa_3}{\kappa_2} ,$$

$$m_{\{2,3\}} = m_{\{\bar{2},\bar{3}\}} = \cos^2\phi\left(1 - \frac{\kappa_3}{\kappa_2}\right) + \sin^2\phi\frac{\kappa_1}{\kappa_2} ,$$

$$m_{\{1,2,3\}} = m_{\{\bar{1},\bar{2},\bar{3}\}} = 1 - \sin^2\phi\frac{\kappa_1}{\kappa_2} - \cos^2\phi\frac{\kappa_3}{\kappa_2} ,$$

$$(4.57)$$

we can write the finite-gap equations determining the pole positions for all modes in a single form, as we did for the modes on $PSU(1,1|2)$. This equation

---

[10]This set of pole fluctuations in the $D(2,1;\alpha)$ algebraic curve was laid out for the massive BMN modes in [109], and for the massless fermions in [112]. We can regard the identification of different pole combinations for light, heavy and massless excitations as a proposition which is validated by applying the final result for the energy correction to the BMN background.

is now

$$\frac{z_n^r}{(z_n^r)^2 - 1} = -\frac{\pi n s_r}{\kappa_2 m_r} \tag{4.58}$$

The explicit pole positions are then

$$z_n^r = -s_r \frac{\kappa_2 m_r}{2\pi n} - s_r \sqrt{1 + \frac{\kappa_2^2 m_r^2}{4\pi^2 n^2}} \; . \tag{4.59}$$

The total form of the quasimomenta fluctuations is again of the form (4.22), with the exception of δp_2, which is given by

$$\delta p_2(z) = \frac{\delta \kappa_2 z}{z^2 - 1} + \sum_{\substack{r=\{2\},\{1,2\}, \\ \{2,3\},\{1,2,3\}}} \sum_{n=0}^{\infty} \frac{\alpha(z_n^r)}{z - z_n^r} N_n^r + \sum_{n=0}^{\infty} \frac{2\alpha(z_n^{\{1,2,2,3\}})}{z - z_n^{\{1,2,2,3\}}} N_n^r$$

$$- \sum_{\substack{r=\{\bar{2}\},\{\bar{1},\bar{2}\}, \\ \{\bar{2},\bar{3}\},\{\bar{1},\bar{2},\bar{3}\}}} \sum_{n=0}^{\infty} \frac{\alpha(z_n^r)}{\frac{1}{z} - z_n^r} N_n^r + \sum_{n=0}^{\infty} \frac{2\alpha(z_n^{\{\bar{1},\bar{2},\bar{2},\bar{3}\}})}{\frac{1}{z} - z_n^{\{\bar{1},\bar{2},\bar{2},\bar{3}\}}} N_n^r \; , \tag{4.60}$$

and $\delta p_{\bar{2}}$ which can be read off from the inversion symmetry. We require the following set of asymptotic conditions as $z \to \infty$:[11]

$$\delta p_1 \to c_1 + \frac{2\pi}{z\sqrt{\lambda}} \left(\frac{1}{2} \delta \Delta + N^{\{1\}} + N^{\{1,2\}} + N^{\{1,2,3\}} + N^{\{1,2,2,3\}} \right) \; ,$$

$$\delta p_2 \to c_2 + \frac{2\pi}{z\sqrt{\lambda}} \left(\delta \Delta + N^{\{2\}} + N^{\{1,2\}} + N^{\{2,3\}} + N^{\{1,2,3\}} + 2N^{\{1,2,2,3\}} \right) \; ,$$

$$\delta p_3 \to c_3 + \frac{2\pi}{z\sqrt{\lambda}} \left(\frac{1}{2} \delta \Delta + N^{\{3\}} + N^{\{2,3\}} + N^{\{1,2,3\}} + N^{\{1,2,2,3\}} \right) \; ,$$

$$\delta p_{\bar{1}} \to c_{\bar{1}} - \frac{2\pi}{z\sqrt{\lambda}} \left(\frac{1}{2} \delta \Delta + N^{\{\bar{1}\}} + N^{\{\bar{1},\bar{2}\}} + N^{\{\bar{1},\bar{2},\bar{3}\}} + N^{\{\bar{1},\bar{2},\bar{2},\bar{3}\}} \right) \; ,$$

$$\delta p_{\bar{2}} \to c_{\bar{2}} - \frac{2\pi}{z\sqrt{\lambda}} \left(\delta \Delta + N^{\{\bar{2}\}} + N^{\{\bar{1},\bar{2}\}} + N^{\{\bar{2},\bar{3}\}} + N^{\{\bar{1},\bar{2},\bar{3}\}} + 2N^{\{\bar{1},\bar{2},\bar{2},\bar{3}\}} \right) \; ,$$

$$\delta p_{\bar{3}} \to c_{\bar{3}} - \frac{2\pi}{z\sqrt{\lambda}} \left(\frac{1}{2} \delta \Delta + N^{\{\bar{3}\}} + N^{\{\bar{2},\bar{3}\}} + N^{\{\bar{1},\bar{2},\bar{3}\}} + N^{\{\bar{1},\bar{2},\bar{2},\bar{3}\}} \right) \; . \tag{4.61}$$

At $\mathcal{O}\left(\frac{1}{z}\right)$ this gives us the conditions

$$\frac{\sqrt{\lambda}}{2\pi} \delta \kappa_l + \sum_{r \ni l, \bar{l}} \sum_{n=0}^{\infty} \frac{1}{(z_n^r)^2 - 1} N_n^r = \frac{1}{2} \delta \Delta \; , \quad l = 1, 3, \bar{1}, \bar{3} \; ,$$

$$\frac{\sqrt{\lambda}}{2\pi} \delta \kappa_2 + \sum_{r \ni 2, \bar{2}} \sum_{n=0}^{\infty} \frac{1}{(z_n^r)^2 - 1} N_n^r = \delta \Delta \; . \tag{4.62}$$

Now we solve the GRC order by order in an expansion in $\lambda^{-1/4}$ as we did for $PSU(1,1|2)$. The exact GRC condition for $D(2,1;\alpha)$ is

$$\sin^2 \phi (f_1)^2 + \cos^2 \phi (f_3)^2 + \frac{1}{8}(f_9)^2 = f_2(\sin^2 \phi f_1 + \cos^2 \phi f_3) \; . \tag{4.63}$$

---

[11]See equation (4.74) to confirm that the normalisations of $\delta \Delta$ here are correct. The integer values for other charges is imposed as before as a quantisation condition.

We expand the residue function f_l in the form (4.26). Then the $\mathcal{O}(\lambda^{-1/4})$ terms of the GRC give the equation

$$2\sin^2\phi f_1^{(0)} f_1^{(1)} + 2\cos^2\phi f_3^{(0)} f_3^{(1)} + \frac{1}{4} f_9^{(0)} f_9^{(1)} = f_2^{(0)}\left(\sin^2\phi f_1^{(1)} + \cos^2\phi f_3^{(1)}\right)$$
$$+ f_2^{(1)}\left(\sin^2\phi f_1^{(0)} + \cos^2\phi f_3^{(0)}\right) .$$
(4.64)

We choose to set $\sin^2\phi f_1^{(1)} + \cos^2\phi f_3^{(1)} = 0$ by defining a new function $g(\sigma)$ via

$$f_1^{(1)} = -\cot\phi\, g\, , \quad f_3^{(1)} = \tan\phi\, g\, . \tag{4.65}$$

In the case $\sin^2\phi f_1^{(0)} + \cos^2\phi f_3^{(0)} = 0$ then the GRC (4.63) at leading, classical level implies that $f_1^{(0)} = f_3^{(3)} = f_9^{(0)} = 0$ since the residue functions are real. We can then consistently set $f_2^{(1)} = 0$. Otherwise we solve equation (4.64) with

$$f_2^{(1)} = \frac{1}{\sin^2\phi f_1^{(0)} + \cos^2\phi f_3^{(0)}}\left(\sin 2\phi (f_3^{(0)} - f_1^{(0)})g + \frac{1}{4} f_9^{(0)} f_9^{(1)}\right) . \tag{4.66}$$

With these choices, and $f_9^{(2)} = 0$ since the $U(1)$ charge does not receive any quantum corrections, the $\mathcal{O}(\lambda^{-1/2})$ terms in the GRC are

$$(\sin^2\phi f_1^{(0)} + \cos^2\phi f_3^{(0)}) f_2^{(2)} + \cos^2\phi (f_2^{(0)} - 2f_3^{(0)}) f_3^{(2)}$$
$$+ \sin^2\phi(f_2^{(0)} - 2f_1^{(0)}) f_1^{(2)} = g^2 + \frac{1}{8}(f_9^{(1)})^2 \tag{4.67}$$

As before[12] we are only interested in σ-independent values for $f_l^{(2)}$, and so we can integrate over σ to write the right-hand side in terms of the classical residues and the residue fluctuations:

$$\int_0^{2\pi} d\sigma\left(g^2 + \frac{1}{8}(f_9^{(1)})^2\right) = (\sin^2\phi\kappa_1 + \cos^2\phi\kappa_3)\frac{\sqrt{\lambda}}{2\pi}\delta\kappa_2$$
$$+ \cos^2\phi(\kappa_2 - 2\kappa_3)\frac{\sqrt{\lambda}}{2\pi}\delta\kappa_3$$
$$+ \sin^2\phi(\kappa_2 - 2\kappa_1)\frac{\sqrt{\lambda}}{2\pi}\delta\kappa_1 \tag{4.68}$$

Now we use the asymptotic conditions (4.62) to eliminate $\delta\kappa_l$ and rearrange to find $\delta\Delta$ as[13]

$$\delta\Delta = \frac{2}{\kappa_2}\int_0^{2\pi} d\sigma\left(g^2 + \frac{1}{4}(f_9^1)^2\right) + \sum_{(\text{All } r)}\sum_{n=0}^{\infty}\frac{2m_r}{(z_n^2)^2 - 1} N_n^r \tag{4.69}$$

We expand the functions g and f_9^1 to give us the two massless bosonic modes,

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[12]See the footnote to equation (4.30).

[13]Similarly to the analysis in the previous section, note that the right-hand side of equation (4.68) can be written as $\frac{\sqrt{\lambda}}{2\pi}\kappa_2(m_{\{1\}}\delta\kappa_1 + m_{\{2\}}\delta\kappa_2 + m_{\{3\}}\delta\kappa_3)$.

just as in the case of $PSU(1,1|2)$ we obtained the four massless bosons from f_i^1. The normalisations we require for these are

$$g(\sigma) = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2} \left(a_n e^{-in\sigma} + a_n^\dagger e^{in\sigma} \right) ,$$

$$f_9^1(\sigma) = \sum_{n=1}^{\infty} \sqrt{2n} \left(b_n e^{-in\sigma} + b_n^\dagger e^{in\sigma} \right) . \tag{4.70}$$

Our final result is

$$\delta\Delta = \frac{2\pi}{\kappa_2} \sum_{n=1}^{\infty} n(N_n^a + N_n^b) + \sum_{(\text{All } r)} \sum_{n=0}^{\infty} \Omega_n^r N_n^r \tag{4.71}$$

with the fluctuation energies Ω_n^r given by

$$\Omega_n^r = \Omega(z_n^r) = -m_r + \sqrt{m_r^2 + \left(\frac{2\pi}{\kappa_2}\right)^2 n^2} . \tag{4.72}$$

Just as in the previous section, we can check that the physical one-loop energy correction from all solutions of the form considered here has no UV divergences, since the masses (4.57) satisfy the condition (4.39).

4.2.1 Examples

The $AdS_3 \times S^3 \times S^3 \times S^1$ metric is

$$ds^2 = d\rho^2 - \cosh^2 \rho \; dt^2 + \sinh^2 \rho \; d\gamma^2$$

$$+ \frac{1}{\sin^2 \phi} \left(d\theta_1^2 + \cos^2 \theta_1 \; d\psi_1^2 + \sin^2 \theta_1 \; d\varphi_1^2 \right)$$

$$+ \frac{1}{\cos^2 \phi} \left(d\theta_2^2 + \cos^2 \theta_2 \; d\psi_2^2 + \sin^2 \theta_2 \; d\varphi_2^2 \right) + du_9^2 . \tag{4.73}$$

Zero-cut algebraic curves arise from solutions in the subspace spanned by t, ψ_1, ψ_2, u_9, and have classical residue functions given by

$$f_1^\pm = \frac{1}{4} \left(\dot{t} \pm \acute{t} - \frac{1}{\sin^2 \phi}(\dot{\psi}_2 \pm \acute{\psi}_2) \right) , \qquad f_2^\pm = \frac{1}{2} \left(\dot{t} \pm \acute{t} \right) ,$$

$$f_3^\pm = \frac{1}{4} \left(\dot{t} \pm \acute{t} - \frac{1}{\cos^2 \phi}(\dot{\psi}_1 \pm \acute{\psi}_1) \right) , \qquad f_9^\pm = \frac{1}{2} \left(\dot{u}_9 \pm \acute{u}_9 \right) . \tag{4.74}$$

The simplest set of solutions giving rise to the energy fluctuations (4.72) come from constant residue functions. The most general possible constant residue functions satisfying the GRC are

$$f_1 = \frac{1}{4} \left(1 - \frac{\sin \zeta \cos \chi}{\sin \phi} \right) , \qquad f_3 = \frac{1}{4} \left(1 - \frac{\cos \zeta \cos \chi}{\cos \phi} \right) , \tag{4.75}$$

$$f_2 = \frac{1}{2}\kappa , \qquad f_9 = \frac{1}{2}\kappa \sin \chi , \tag{4.76}$$

which arise from the point-particle solution given by

$$t = \kappa\tau , \quad \psi_1 = (\kappa \cos\phi \cos\zeta \cos\chi)\tau , \quad \psi_2 = (\kappa \sin\phi \sin\zeta \cos\chi)\tau$$
$$u_9 = (\kappa \sin\chi)\tau . \tag{4.77}$$

This solution gives the fluctuations (4.72) with the masses

$$m_{\{1\}} = m_{\{\bar{1}\}} = \sin\phi \sin\zeta \cos\chi , \quad m_{\{3\}} = m_{\{\bar{3}\}} = \cos\phi \cos\zeta \cos\chi ,$$

$$m_{\{2\}} = m_{\{\bar{2}\}} = \frac{1}{2}(1 - \cos(\phi - \zeta)\cos\chi) , \quad m_{\{1,2,2,3\}} = m_{\{\bar{1},\bar{2},\bar{2},\bar{3}\}} = 1 ,$$

$$m_{\{1,2\}} = m_{\{\bar{1},\bar{2}\}} = \frac{1}{2}(1 - \cos(\phi + \zeta)\cos\chi) ,$$

$$m_{\{2,3\}} = m_{\{\bar{2},\bar{3}\}} = \frac{1}{2}(1 + \cos(\phi + \zeta)\cos\chi) ,$$

$$m_{\{1,2,3\}} = m_{\{\bar{1},\bar{2},\bar{3}\}} = \frac{1}{2}(1 + \cos(\phi - \zeta)\cos\chi) ,$$

$$\tag{4.78}$$

When $\zeta = \phi$, $\chi = 0$ we reproduce the usual BMN spectrum. As we saw for the analogous solution in the $PSU(1,1|2)$ case, within this family of solutions the BMN solution has the minimum possible number of massless bosons and the maximum possible number of massless fermions. When we take $\chi = \frac{\pi}{2}$ (and any value for ζ), we have a solution with six massless bosons, two bosons of mass 1 and eight fermions of mass $\frac{1}{2}$, arising from a BMN-like solution rotating around the S^1. This spectrum is identical to that we obtained for the solution (4.49) in the $PSU(1,1|2)$ theory. When we set $\chi = 0$ but leave ζ unfixed the solution (4.77) becomes the "non-supersymmetric vacuum" solution studied in [112] from both a worldsheet and algebraic curve perspective, and the spectrum (4.78) reproduces that which appears in equation (20) of [112].

Another solution studied in [112] is a folded string solution given by

$$t = \kappa\tau ,$$
$$\psi_1 = \cos^2\phi\,\omega\tau - A\sin\phi\cos\phi\cos(m\tau)\cos(m\sigma) ,$$
$$\psi_2 = \sin^2\phi\,\omega\tau + A\sin\phi\cos\phi\cos(m\tau)\cos(m\sigma) ,$$
$$u_9 = A\sin(m\tau)\cos(m\sigma), \tag{4.79}$$

with

$$\kappa^2 = \omega^2 + A^2 m^2 . \tag{4.80}$$

This has residue functions given by

$$f_1^{\pm}(\sigma) = \frac{1}{4}\left(\kappa + \omega \mp Am\cot\phi\sin(m\sigma)\right) , \quad f_9^{\pm}(\sigma) = \frac{1}{2}Am\cos(m\sigma) ,$$
$$f_3^{\pm}(\sigma) = \frac{1}{4}\left(\kappa + \omega \pm Am\tan\phi\sin(m\sigma)\right) , \quad f_2^{\pm}(\sigma) = \frac{1}{2}\kappa . \tag{4.81}$$

Just as for the solution (4.52) in the T^4 theory, we have winding appearing

at the level of the residue functions with $f_l^+ \neq f_l^-$, but we can still use our zero-winding results. The residues themselves, given by

$$\kappa_1 = \kappa_3 = \pi(\kappa + \omega) \ , \quad \kappa_2 = 2\pi\kappa \ , \quad \kappa_9 = m_l = 0 \tag{4.82}$$

have no winding. This means that the derivation of all modes other than the massless bosons proceeds as before. Meanwhile the use of the GRC to derive the massless bosonic fluctuations also proceeds with only slight adjustment. To solve the GRC at $\mathcal{O}(\lambda^{-1/4})$ as in equation (4.66) we need now to take different values for $f_2^{(1)+}$ and $f_2^{(2)-}$ but we can consistently choose all other higher order terms for the residue functions above classical level to be equal for ± 1. We now have two version of equation (4.67), one with $f_l^{(0)+}$ and another with $f_l^{(0)+}$. If we take the sum of these two equations and integrate over σ we get precisely equation (4.68) as before. Taking the difference of the two equations and integrating we get zero on both sides since $m_l = 0$ and we have set $g^+ = g^- = 0$ etc. consistently.

The masses we obtain from the general result (4.57) using the residues (4.82) are

$$m_{\{1\}} = m_{\{\bar{1}\}} = \frac{\omega}{\kappa} \cos^2 \phi \ , \qquad m_{\{1,2\}} = m_{\{\bar{1},\bar{2}\}} = \frac{1}{2}\left(1 + \frac{\omega}{\kappa}\cos 2\phi\right) \ ,$$

$$m_{\{3\}} = m_{\{\bar{3}\}} = \frac{\omega}{\kappa} \sin^2 \phi \ , \qquad m_{\{2,3\}} = m_{\{\bar{2},\bar{3}\}} = \frac{1}{2}\left(1 - \frac{\omega}{\kappa}\cos 2\phi\right) \ ,$$

$$m_{\{1,2,3\}} = m_{\{\bar{1},\bar{2},\bar{3}\}} = \frac{1}{2}\left(1 - \frac{\omega}{\kappa}\right) \ , \qquad m_{\{2\}} = m_{\{\bar{2}\}} = \frac{1}{2}\left(1 + \frac{\omega}{\kappa}\right) \ ,$$

$$m_{\{1,3\}} = m_{\{\bar{1},\bar{3}\}} = 1 \ . \tag{4.83}$$

This matches with equation (55) of [112] and with the worldsheet calculations in that paper.

4.3   Chapter conclusions and outlook

In this chapter we have examined semiclassical fluctuations around algebraic curve solutions with no classical branch cuts. We have derived expressions for the frequencies of all modes for a broad range of classical backgrounds, namely all zero-cut no-winding solutions in both the $D(2,1;\alpha)^2$ and $PSU(1,1|2)^2$ cosets. As part of this, we have exhibited a perturbative semiclassical solution to the generalised residue conditions which gives rise to massless bosonic fluctuation frequencies in the algebraic curve analysis. This is the first time massless frequencies have been explicitly included in this type of analysis. The use of the GRC to include massless bosons should carry over straightforwardly from the zero-cut solutions studied here to semiclassical analysis of algebraic curves with classical branch cuts as well. This is because, while the presence of branch cuts in the classical algebraic curve affects the positions of fluctuation poles allowed by the no-forcing condition, the dynamics of the massless

excitations are solely contained in the residues at $z = \pm 1$ rather than through
the positions of these additional poles.

In the case of the frequencies around $D(2, 1; \alpha)^2$ backgrounds, we have
shown how particular examples of the results for general zero-cut no-winding
backgrounds given in equations (4.72) and (4.57) match with worldsheet calcu-
lations in [112]. One immediate line of future work that would further confirm
the validity of the results obtained here is to perform similar checks with world-
sheet calculations for energy corrections around solutions in $AdS_3 \times S^3 \times T^4$.
Comparisons between algebraic curve calculations and worldsheet calculations
should be made for physical quantities such as the total one-loop energy shift.
In summing up the contributions of individual mode contributions there can
be issues of regularisation to match these results [125–129]. As we have dis-
cussed here, the one-loop energy correction to the solutions considered in this
chapter has no UV divergences.

We have shown how the massless bosons of the BMN background arise
semiclassically as well as classically solely through the residues of the quasi-
momenta at $z = \pm 1$. We have also made use of the prescription introduced
in [112] to include the massless fermions of the BMN background. However
there remains an open question about these massless fermions. The prescrip-
tion of [112] allows fermionic mode frequencies to be calculated for a set of
solutions that includes the BMN point-particle background in a particular
limit. If one first calculates the energy corrections and then takes the limit
to the BMN background the massless fermionic frequencies are well behaved
and approach the expected contribution for massless excitations. However the
quasimomenta themselves are not well behaved in this limit, with a divergence
over the entire complex plane. We have seen that there are other backgrounds
where different modes become massless. In particular one can take a limit
from a general set of rotating point-particle solutions to one rotating only
on a decoupled S^1. In this case we have shown that two bosons which are
massive for the BMN background become massless. In this limit the quasi-
momenta again diverge everywhere as a result of these massive-to-massless
modes. Understanding these limits will be important for describing how all
massless states appear in the quantum Bethe equations.

The use of the algebraic curve semiclassical corrections to the energy of
classical string solutions can also be used to find important physical informa-
tion about the phases that enter the S-matrix [130] and as such can be used to
compare to expressions for the phases obtained by solving crossing equations.
In the case of the AdS_3 backgrounds, exact S-matrices have recently been
found that include massless interactions [2, 64, 65, 131] (this is the topic of the
next chapter) and so the inclusion of massless excitations into semiclassical
algebraic curve analysis should be used further to test against the phases of
these S-matrices.

Chapter 5

The worldsheet S-matrix of $AdS_3 \times S^3 \times T^4$ with mixed-flux

In this chapter we move from classical and semiclassical integrable strings to some applications of quantum integrability to stings on AdS_3 backgrounds. In particular, instead of looking at the classical limit, we now be looking at a different limit: the "decompactification limit". In this limit the radius of the worldsheet cylinder becomes large so that the worldsheet effectively becomes the 2d plane. Excitations of the strings can then correspond to asymptotic momentum states of a 2d field theory which can scatter amongst themselves, and the fundamental object describing these interactions is the worldsheet S-matrix.

We consider the form of lightcone gauge called uniform lightcone gauge [132], where we set[1]

$$x^+ = t + \psi = \tau \ , \quad p_- = p_\psi - p_t = 2 \ , \tag{5.1}$$

where p_m is the momentum conjugate to x^m. Now, instead of fixing the radius of the worldsheet cylinder, we take the range of σ to be $[-r, r]$, with r unfixed for now. Then we can see that with the uniform lightcone gauge,

$$P_- = \int_{-r}^{r} p_- \mathrm{d}\sigma = E + J = 2r \ . \tag{5.2}$$

In other words the gauge choice (5.1) requires the worldsheet cylinder to have a radius proportional to the lightcone momentum P_-.

We consider all fields to be periodic, $x^m(r, \tau) = x^m(-r, \tau)$.[2] In particular, the level matching condition is that

$$\Delta x^- = \int_{-r}^{r} \acute{x}^- \mathrm{d}\sigma = 0 \ . \tag{5.3}$$

This turns out to be equivalent to the vanishing of the worldsheet momentum p_{ws}, defined by

$$p_{ws} = -\int_{-r}^{r} \mathrm{d}\sigma (p_i \acute{x}^i) \tag{5.4}$$

where $i = 1..8$ are the transverse coordinates.

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[1] This form of lightcone gauge appears to have removed the 1-parameter degree of freedom in the lightcone gauge in the form (2.11) which couldn't be gauged away. However this is illusory. In flat space, we can only reach the gauge (5.1) if we do not completely fix conformal gauge, but only $\gamma_{\tau\sigma} = 0$ and $\det(\gamma) = -1$, leaving a single parameter unfixed in the worldsheet metric.

[2] We ignore the possibility of winding.

To describe consistent and non-trivial scattering of elementary excitations, we need firstly to take the decompactification limit $P_- \to \infty$ so that we have a well-defined notion of asymptotic states, and secondly to consider the theory off-shell, $p_{ws} \neq 0$, so that these states can carry non-zero momentum. Then we have a basis of "in" and "out" asymptotic n-particle momentum states

$$|p_1, \ldots p_n\rangle_{\alpha_1 \ldots \alpha_n}^{(in/out)} \tag{5.5}$$

where α_i are flavour indices. The worldsheet S-matrix is the object which relates in- and out-states.

Scattering in 1+1 dimensions with an infinite number of conserved charges has been studied using the Zamolodchikov-Faddeev (ZF) algebra [133, 134] and features a number of simplifications. One is that the set of momenta of the particles cannot change; rather the only possibility is for different particles to exchange momenta. A corollary to this is that total particle number is conserved. A second major simplification is that multi-particle scattering factorises into a product of two-particle scattering processes. Therefore the fundamental object of interest is the two-particle S-matrix defined by

$$|p_1, p_2\rangle_{\alpha_1, \alpha_2}^{(out)} = S_{\alpha_1 \alpha_2}^{\alpha_3 \alpha_4}(p_1, p_2) |p_1, p_2\rangle_{\alpha_3 \alpha_4}^{(in)} \ . \tag{5.6}$$

One approach to computing the worldsheet S-matrix is to treat it perturbatively and calculate scattering amplitudes at progressively higher loop orders. However, we will instead use an approach which uses the conjectured full quantum integrability of the theory. In this approach we use consistency of the S-matrix with the symmetry algebra $\mathcal{A}$ of the theory together with other constraints to completely determine the S-matrix, up to certain phases. A crucial fact here is that the off-shell symmetry algebra $\mathcal{A}$ is not simply the algebra of the classical superisometries of the string background, but is rather a central extension of that algebra.

The procedure to construct the S-matrix therefore works as follows. First we determine the supercurrents which make up the algebra $\mathcal{A}$ and compute the central charge. In this calculation we use a "hybrid expansion" [135] in which we work exactly in the lightcone bosonic field x^- but to subleading order in transverse fields. This allows us to compute the central charge as an exact function of the worldsheet momentum p_{ws}. Next we determine the two-particle representations of the algebra $\mathcal{A}$, and finally we use consistency with the algebra to compute the two-particle S-matrix.

This approach was originally used to compute the worldsheet S-matrix of $AdS_5 \times S^5$ [135–137], and in these computations the algebra $\mathcal{A}$ was computed by working with the superstring action in its coset formulation, as described in chapter 2. When the same approach was applied to the AdS_3 backgrounds [97,

99][3] it was found that the S-matrix of the massive sector could be computed, but the coset action did not allow for inclusion of the massless modes. This problem is avoided but working not with the coset action but with the Green-Schwarz action. In this way the complete S-matrix, including the massless modes, was computed in [64,65] for the pure R-R $AdS_3 \times S^3 \times T^4$ background. The work of this chapter is an extension of these results to the entire family of mixed-flux $AdS_3 \times S^3 \times T^4$ backgrounds.

We use various different sets of indices in this chapter, see appendix A for index conventions.

5.1 Supergravity background and Killing spinors

The Killing spinors of a supergravity background are the spinorial functions which encode the preserved supersymmetry transformations of that background. In this section we write down the Killing spinors for the mixed-flux $AdS_3 \times S^3 \times T^4$ background. The form of the Killing spinors will be useful when we come to simplify the Green-Schwarz action.

We use a coordinate system that is convenient for expanding around a BMN-like vacuum state. In our choice of coordinates the metric of $AdS_3 \times S^3 \times T^4$ is given by

$$ds^2 = ds^2_{AdS_3} + ds^2_{S^3} + ds^2_{T^4} \ , \tag{5.7}$$

with

$$ds^2_{AdS_3} = -\left(\frac{1 + \frac{z^2}{4}}{1 - \frac{z^2}{4}}\right)^2 dt^2 + \left(\frac{1}{1 - \frac{z^2}{4}}\right)^2 (dz_1^2 + dz_2^2) \tag{5.8}$$

and

$$ds^2_{S^3} = \left(\frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}}\right)^2 d\psi^2 + \left(\frac{1}{1 + \frac{y^2}{4}}\right)^2 (dy_3^2 + dy_4^2) \ , \tag{5.9}$$

where

$$z^2 = z_1^2 + z_2^2 \ , \quad y^2 = y_3^2 + y_4^2 \ . \tag{5.10}$$

The background is supported by a mix of NS-NS and R-R three-form fluxes, and we have a one-parameter family of backgrounds according to the ratio of these fluxes. Explicitly, the non-zero tangent-space components of the NS-NS three-form field H and the R-R three-form field F are

$$F_{012} = F_{345} = 2\tilde{q} \ , \quad H_{012} = H_{345} = 2q \ , \tag{5.11}$$

where the coefficients q, $\tilde{q}$ satisfy

$$q^2 + \tilde{q}^2 = 1 \ , \quad 0 \le q, \tilde{q} \le 1 \ . \tag{5.12}$$

---

[3]The S-matrix computed by [97,99] is the spin-chain S-matrix for the dual gauge theory which is closely related to, but not identical to the string worldsheet S-matrix. See [138] for an explicit comparison of those results with worldsheet computations.

The NS-NS three-form arises from a B-field given by

$$B = \frac{q(z_1 dz_2 - z_2 dz_1) \wedge dt}{\left(1 - \frac{z^2}{4}\right)^2} + \frac{q(y_3 dy_4 - y_4 dy_3) \wedge d\psi}{\left(1 + \frac{y^2}{4}\right)^2} \ . \tag{5.13}$$

The Killing spinor equations for a background with R-R and NS-NS 3-form fluxes can be written as

$$\left(\delta^{IJ}(\partial_m + \frac{1}{4}\slashed{\omega}_m) + \frac{1}{8}\sigma_1^{IJ} \slashed{H}_m + \frac{1}{48}\sigma_3^{IJ} \slashed{F} \slashed{E}_m\right) \tilde{\epsilon}_J = 0 \tag{5.14}$$

with $I, J = 1, 2$. Here we use the contractions

$$\slashed{F} = F_{ABC}\Gamma^{ABC} \ , \slashed{H}_m = H_{mAB}\Gamma^{AB} \ , \slashed{\omega}_m = \omega_m^{AB}\Gamma_{AB} \ , \tag{5.15}$$

and ω_m^{AB} is the spin-connection.

In our cases the non-zero components of the fluxes are given in equation (5.11). $\tilde{\epsilon}_I$ are Majorana-Weyl spinors. In particular they satisfy the chirality condition (2.6). We can also impose a second chirality condition,[4]

$$\tilde{\Gamma}\tilde{\epsilon}_I = -\tilde{\epsilon}_I \tag{5.16}$$

where $\tilde{\Gamma} = \Gamma^{012345}$.

Instead of solving (5.14) directly, we will make a q-dependent change of basis to relate the Killing spinors of the mixed-flux background to those of the pure R-R ($q = 0$) background. This will be useful to us when simplifying the Green-Schwarz action, as well as helping to check the consistency of results for the mixed-flux background with previous results for the pure R-R background when we set q to zero. We make the change of basis

$$\tilde{\epsilon}_1 = \sqrt{\frac{1 + \tilde{q}}{2}}\epsilon_1 - \sqrt{\frac{1 - \tilde{q}}{2}}\epsilon_2 \ ,$$
$$\tilde{\epsilon}_2 = \sqrt{\frac{1 + \tilde{q}}{2}}\epsilon_2 + \sqrt{\frac{1 - \tilde{q}}{2}}\epsilon_1 \ . \tag{5.17}$$

Then the Killing spinor equations become

$$\left[\delta^{IJ}(\partial_m + \frac{1}{4}\slashed{\omega}_m) + \sigma_3^{IJ}\left(\frac{q}{8}\slashed{H}_m + \frac{\tilde{q}}{48}\slashed{F}\slashed{E}_m\right)\right]\epsilon_J$$
$$= \sigma_1^{IJ}\left(\frac{q}{48}\slashed{F}\slashed{E}_m - \frac{\tilde{q}}{8}\slashed{H}_m\right)\epsilon_J \ . \tag{5.18}$$

As shown in appendix H, the right-hand side of this is zero for any spinors

---

[4]This condition arises from the variation of the dilatino in the supergravity background, which is an additional requirement to the Killing spinor equations (5.14) which arise from the variation of the gravitino. For the maximally supersymmetric $AdS_5 \times S^5$ background with constant 5-form R-R flux, the dilatino condition is trivially satisfied and so there are twice as many independent Killing spinors, and hence preserved supersymmetries, as for the maximally supersymmetric AdS_3 backgrounds.

satisfying the chirality condition (5.16) and so this simplifies to

$$\left(\delta^{IJ}(\partial_m + \frac{1}{4}\not\psi_m) + \frac{\sigma_3^{IJ}}{48\tilde{q}} \not{F}\not{E}_m \right)\epsilon_J = 0 \ . \tag{5.19}$$

Since the mixed-flux R-R three-form is related to the pure R-R three-form by a rescaling by $\tilde{q}$, this is exactly the same equation obeyed by the Killing spinors for the pure R-R background.

We can write the solutions to the Killing spinor equation (5.19) as tensor products of two-dimensional spinors as[5]

$$\epsilon_I = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \epsilon_{AdS_3}^I \otimes \eta_{S^3}^I \otimes \psi_0^I \tag{5.20}$$

with ψ_0^I constant two-dimensional spinors. The Killing spinor equations for the full metric then reduce to those of AdS_3 and S^3 separately. We find that all components must be independent of the T^4 coordinates, while $\eta_{S^3}^I$ are S^3 Killing spinors satisfying [139]

$$(\partial_m + \frac{1}{4}\not\psi_m)\eta_{S^3}^I = \sum_{A=3}^{5} \frac{i}{2}E_m^A \gamma_A \sigma_3^{IJ}\eta_{S^3}^J \quad m = \psi, y_3, y_4 \tag{5.21}$$

and $\epsilon_{AdS_3}^I$ are AdS_3 Killing spinors satisfying [140]

$$(\partial_m + \frac{1}{4}\not\psi_m)\epsilon_{AdS_3}^I = \sum_{A=0}^{2} \frac{1}{2}E_m^A \gamma_A \sigma_3^{IJ}\epsilon_{AdS_3}^J \quad m = t, z_1, z_2 \ . \tag{5.22}$$

The solutions to these are given by

$$\eta_{S^3}^1 = \hat{M}_{S^3}\eta_0^1 \ , \quad \epsilon_{AdS_3}^1 = \hat{M}_{AdS_3}\epsilon_0^1 \ ,$$
$$\eta_{S^3}^2 = \check{M}_{S^3}\eta_0^2 \ , \quad \epsilon_{AdS_3}^2 = \check{M}_{AdS_3}\epsilon_0^2 \tag{5.23}$$

where η_0^I and ϵ_0^I are constant two-components spinors and

$$\hat{M}_{S^3} = \frac{1}{\sqrt{1+\frac{y^2}{4}}}\left(1 - \frac{iy_3}{2}\sigma_1 - \frac{iy_4}{2}\sigma_2\right)e^{\frac{i\psi}{2}\sigma_3}$$

$$\check{M}_{S^3} = \frac{1}{\sqrt{1+\frac{y^2}{4}}}\left(1 + \frac{iy_3}{2}\sigma_1 + \frac{iy_4}{2}\sigma_2\right)e^{\frac{-i\psi}{2}\sigma_3}$$

$$\hat{M}_{AdS_3} = \frac{1}{\sqrt{1-\frac{z^2}{4}}}\left(1 - \frac{z_1}{2}\sigma_1 - \frac{z_2}{2}\sigma_2\right)e^{\frac{-t}{2}\sigma_3}$$

$$\check{M}_{AdS_3} = \frac{1}{\sqrt{1-\frac{z^2}{4}}}\left(1 + \frac{z_1}{2}\sigma_1 + \frac{z_2}{2}\sigma_2\right)e^{\frac{t}{2}\sigma_3} \tag{5.24}$$

---

[5]The first and second term in the tensor product arise from the chirality conditions (2.6) and (5.16) respectively in the tensor product basis of gamma matrices given in appendix B.

The Killing spinors for $AdS_3 \times S^3 \times T^4$ with pure R-R flux are therefore given by

$$
\epsilon^1 = \hat{M} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \epsilon_0^1 \otimes \eta_0^1 \otimes \psi_0^1 \right) \, ,
$$

$$
\epsilon^2 = \check{M} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \epsilon_0^2 \otimes \eta_0^2 \otimes \psi_0^2 \right) \tag{5.25}
$$

with

$$
\hat{M} = \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \hat{M}_{AdS_3} \otimes \hat{M}_{S^3} \otimes \mathbb{1}_2 \, ,
$$

$$
\check{M} = \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \check{M}_{AdS_3} \otimes \check{M}_{S^3} \otimes \mathbb{1}_2 \, . \tag{5.26}
$$

The Killing spinors $\tilde{\epsilon}^I$ for the mixed-flux background are given by the q-dependent linear combinations (5.17) of the pure R-R Killing spinors ϵ^I. Finally we note that we can write the matrices $\hat{M}$ and $\check{M}$ in terms of ten-dimensional gamma matrices as

$$
\hat{M} = M_0 M_t \, , \qquad \check{M} = M_0^{-1} M_t^{-1} \tag{5.27}
$$

with

$$
M_0 = \frac{1}{\sqrt{\left(1 - \frac{z^2}{4}\right)\left(1 + \frac{y^2}{4}\right)}} \left(\mathbb{1} - \frac{1}{2} z_i \Gamma^i \Gamma^{012} \right) \left(\mathbb{1} - \frac{1}{2} y_j \Gamma^j \Gamma^{345} \right) \tag{5.28}
$$

and

$$
M_t = e^{-\frac{1}{2}\left(t\Gamma^{12} + \psi\Gamma^{34}\right)} \, . \tag{5.29}
$$

5.2 Bosonic action in lightcone gauge

In this section we examine the bosonic action for the mixed-flux $AdS_3 \times S^3 \times T^4$ background when we fix uniform lightcone gauge. We start with the usual bosonic action (2.1.1), with two slight changes. First we leave the radius of the worldsheet cylinder unfixed, and take the range of σ to be $[-r, r]$. Second we use Weyl invariance to set the determinant of the worldsheet metric γ equal to -1. The bosonic string action is therefore[6]

$$
S_B = -\frac{1}{2} \int \mathrm{d}\tau \int_{-r}^{r} \mathrm{d}\sigma \left(\gamma^{\alpha\beta} G_{mn} + \epsilon^{\alpha\beta} B_{mn} \right) \partial_\alpha X^m \partial_\beta X^n \, . \tag{5.30}
$$

The spacetime metric G and B-field are given in equations (5.7) and (5.13) respectively.

---

[6]We suppress the overall string tension $\frac{\sqrt{\lambda}}{2\pi}$ in the worldsheet action and reinsert it once we compute the central charge.

We introduce the canonical momenta

$$p_m = -\frac{\delta S_B}{\delta \dot{X}^m} = -2\pi \left(\gamma^{\tau\tau} G_{mn} \dot{X}^n + \gamma^{\tau\sigma} G_{mn} \acute{X}^n + B_{mn} \acute{X}^n \right) . \qquad (5.31)$$

Then we can recast the action in the so-called first-order form

$$S_B = 2\pi \int \mathrm{d}\tau \int_{-r}^{r} \mathrm{d}\sigma \left(p_m \dot{X}^m + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2\gamma^{\tau\tau}} C_2 \right) \qquad (5.32)$$

where

$$C_1 = p_m \acute{X}^m \qquad (5.33)$$

and

$$C_2 = G^{mn} p_m p_n + 2 G^{mn} B_{nk} p_m \acute{X}^k$$
$$+ \left(G_{mn} + G^{kl} B_{km} B_{ln} \right) \acute{X}^m \acute{X}^n \qquad (5.34)$$

The Virasoro constraints appear in this first-order form as $C_1 = c_2 = 0$. With uniform lightcone gauge $x^+ = \tau$, $p_- = 1$, the condition $C_1 = 0$ can be solved by

$$\acute{x}^- = -\frac{1}{2}(p_{\underline{i}} \acute{x}^{\underline{i}} + p_i \acute{x}^i) . \qquad (5.35)$$

Hence we see that the vanishing of the worldsheet momentum

$$p_{ws} = -\int_{-r}^{r} \mathrm{d}\sigma (p_{\underline{i}} \acute{x}^{\underline{i}} + p_i \acute{x}^i) \qquad (5.36)$$

is indeed equivalent to the level-matching condition $x^-(r) = x^-(-r)$. $C_2 = 0$ can be solved to find p_+ in terms of the other fields, and the gauge-fixed action can then be written as

$$S_B = \int_{-r}^{r} \mathrm{d}\sigma (p_{\underline{i}} \dot{x}^{\underline{i}} + p_i \dot{x}^i - \mathcal{H}) \qquad (5.37)$$

in terms of the Hamiltonian $\mathcal{H} = -p_+$. To quadratic order in fields this is given by

$$\mathcal{H} = +\frac{1}{2} \left(p_z^2 + p_y^2 + p_x^2 + \acute{z}^2 + \acute{y}^2 + \acute{x}^2 + z^2 + y^2 - 2q\epsilon^{\underline{ij}}(z_{\underline{i}} \acute{z}_{\underline{j}} + y_{\underline{i}} \acute{y}_{\underline{j}}) \right) . \quad (5.38)$$

We can solve the Virasoro constraints explicitly in the uniform lightcone gauge to find x^- in terms of the transverse fields. To quadratic order we get

$$\dot{x}^- = -\frac{1}{4}(\dot{z}^2 + \dot{y}^2 + \dot{x}^2 + \acute{z}^2 + \acute{y}^2 + \acute{x}^2 - z^2 - y^2) ,$$
$$\acute{x}^- = -\frac{1}{2}(\dot{z} \cdot \acute{z} + \dot{y} \cdot \acute{y} + \dot{x} \cdot \acute{x}) . \qquad (5.39)$$

We can then use the $x^\pm$ equations of motion to find the worldsheet metric. To

quadratic order this is

$$\gamma^{\tau\tau} = -1 + \frac{z^2 - y^2}{2} - \frac{q}{2}\epsilon^{\underline{ij}}(z_{\underline{i}}z'_{\underline{j}} - y_{\underline{i}}y'_{\underline{j}}) \ ,$$

$$\gamma^{\tau\sigma} = \frac{q}{2}\epsilon^{\underline{ij}}(z_{\underline{i}}z'_{\underline{j}} - y_{\underline{i}}y'_{\underline{j}}) \ ,$$

$$\gamma^{\sigma\sigma} = +1 + \frac{z^2 - y^2}{2} - \frac{q}{2}\epsilon^{\underline{ij}}(z_{\underline{i}}z'_{\underline{j}} - y_{\underline{i}}y'_{\underline{j}}) \ . \tag{5.40}$$

5.3   The Green-Schwarz Action

In this section we discuss the gauge-fixing of the Green-Schwarz action for the $AdS_3 \times S^3 \times T^4$ mixed-flux background. The key result of this section is the lightcone gauged-fixed action which is given in equations (5.77) and (5.81).

We start with the action up to quadratic order in fermions in the following form [141]. We have the kinetic terms, by which we mean all those proportional to the worldsheet metric, given by

$$\mathcal{L}_{kin} = -i\gamma^{\alpha\beta}\bar{\tilde{\theta}}_I \not{E}_\alpha \left(D_\beta \delta^{IJ} + \frac{1}{8}\sigma_3^{IJ}\not{H}_\beta + \frac{1}{48}\sigma_1^{IJ}\not{F}\not{E}_\beta \right) \tilde{\theta}_J \tag{5.41}$$

and the Wess-Zumino (WZ) terms given by

$$\mathcal{L}_{WZ} = i\epsilon^{\alpha\beta}\bar{\tilde{\theta}}_I \sigma_3^{IJ}\not{E}_\alpha \left(D_\beta \delta^{JK} + \frac{1}{8}\sigma_3^{JK}\not{H}_\beta + \frac{1}{48}\sigma_1^{JK}\not{F}\not{E}_\beta \right) \tilde{\theta}_K \ . \tag{5.42}$$

Here $\not{E}_\alpha$ is the pull-back of the contracted vielbein,

$$\not{E}_\alpha = \partial_\alpha X^m \not{E}_m = \partial_\alpha X^m E_m^A \Gamma_A \ , \tag{5.43}$$

and D_m is the covariant derivative

$$D_m = \partial_m + \frac{1}{4}\omega_m^{AB}\Gamma_{AB} \ . \tag{5.44}$$

We make a q-dependent redefinition of the fermions $\tilde{\theta}_I$ to a new basis θ_I,

$$\tilde{\theta}_1 = \frac{1}{2}\left(\sqrt{1+\tilde{q}} + \sqrt{1-\tilde{q}}\right)\theta_1 + \frac{1}{2}\left(\sqrt{1+\tilde{q}} - \sqrt{1-\tilde{q}}\right)\theta_2 \ ,$$

$$\tilde{\theta}_2 = \frac{1}{2}\left(\sqrt{1+\tilde{q}} - \sqrt{1-\tilde{q}}\right)\theta_1 - \frac{1}{2}\left(\sqrt{1+\tilde{q}} + \sqrt{1-\tilde{q}}\right)\theta_2 \tag{5.45}$$

and then split the fermions θ_I according to their chirality under $\tilde{\Gamma} = \Gamma^{012345}$;

$$\theta_I^\pm = \frac{1}{2}(1 \pm \tilde{\Gamma})\theta_I \ . \tag{5.46}$$

The fermions θ^- obey the same chirality condition as the Killing spinors. This means that, after an appropriate rotation, shifts of these fermions realise the supersymmetry variations, and we will see explicitly that these are the massive fermions of this background. We also note that their chirality under

$\tilde{\Gamma}$ means they obey the condition (H.1) which will be used in simplifying the action presently. The fermions θ^+ will be seen explicitly, after an appropriate rotation, to be the massless fermions of this background.

We define contractions of the vielbeins over the $AdS_3 \times S^3$ and T^4 components separately

$$\breve{E}_\alpha = \sum_{a=0}^{5} E_\alpha^a \Gamma_a \ , \quad \acute{E}_\alpha = \sum_{\dot{a}=6}^{9} E_\alpha^{\dot{a}} \Gamma_{\dot{a}} \ . \tag{5.47}$$

Using the chirality conditions of the fermions we can replace each full vielbein contraction with one of these sub-contractions, giving us the following for the kinetic action

$$\begin{aligned}
\mathcal{L}_{kin} = & - i\gamma^{\alpha\beta}(\bar{\theta}_I^- \breve{E}_\alpha + \bar{\theta}_I^+ \acute{E}_\alpha)\left(\delta^{IJ}D_\beta + \frac{q}{8}\sigma_3^{IJ}\slashed{H}_\beta + \frac{\tilde{q}}{48}\sigma_3^{IJ}\slashed{F}\breve{E}_\beta\right)\theta_J^- \\
& - i\gamma^{\alpha\beta}(\bar{\theta}_I^- \acute{E}_\alpha + \bar{\theta}_I^+ \breve{E}_\alpha)\left(\delta^{IJ}D_\beta + \frac{1}{8}(q\sigma_3^{IJ} + \tilde{q}\sigma_1^{IJ})\slashed{H}_\beta\right)\theta_J^+ \\
& - i\gamma^{\alpha\beta}\frac{1}{48}(\bar{\theta}_I^- \breve{E}_\alpha + \bar{\theta}_I^+ \acute{E}_\alpha)(\tilde{q}\sigma_3^{IJ} - q\sigma_1^{IJ})\slashed{F}\acute{E}_\beta\theta_J^+
\end{aligned} \tag{5.48}$$

and for the WZ action

$$\begin{aligned}
\mathcal{L}_{WZ} = & + iq\epsilon^{\alpha\beta}(\bar{\theta}_I^- \breve{E}_\alpha + \bar{\theta}_I^+ \acute{E}_\alpha)\left(\sigma_3^{IJ}D_\beta + \frac{q}{8}\delta^{IJ}\slashed{H}_\beta + \frac{\tilde{q}}{48}\delta^{IJ}\slashed{F}\breve{E}_\beta\right)\theta_J^- \\
& + i\tilde{q}\epsilon^{\alpha\beta}(\bar{\theta}_I^- \breve{E}_\alpha + \bar{\theta}_I^+ \acute{E}_\alpha)\left(\sigma_1^{IJ}D_\beta - \frac{q}{8}i\sigma_2^{IJ}\slashed{H}_\beta - \frac{\tilde{q}}{48}i\sigma_2^{IJ}\slashed{F}\breve{E}_\beta\right)\theta_J^- \\
& + i\epsilon^{\alpha\beta}(\bar{\theta}_I^- \acute{E}_\alpha + \bar{\theta}_I^+ \breve{E}_\alpha)\left((q\sigma_3^{IJ} + \tilde{q}\sigma_1^{IJ})D_\beta + \frac{1}{8}\delta^{IJ}\slashed{H}_\beta\right)\theta_J^+ \\
& - i\epsilon^{\alpha\beta}\frac{1}{48}(\bar{\theta}_I^- \breve{E}_\alpha + \bar{\theta}_I^+ \acute{E}_\alpha)i\sigma_2^{IJ}\slashed{F}\acute{E}_\beta\theta_J^+
\end{aligned} \tag{5.49}$$

Now we can simplify the "mixed" terms involving both θ^+ and θ^- by making use of Majorana flip identities and equation (H.1) to write all terms with θ^- on the right. In particular we use the following identities

$$\begin{aligned}
\bar{\theta}_I^- \acute{E}_\alpha \partial_\beta \theta_J^+ &= -(\partial_\beta \bar{\theta}_I^-)\acute{E}_\alpha \theta_J^+ = \bar{\theta}_J^+ \acute{E}_\alpha \partial_\beta \theta_I^- \ , \\
\bar{\theta}_I^- \acute{E}_\alpha \slashed{H}_\beta \theta_J^+ &= \bar{\theta}_J^+ \acute{E}_\alpha \slashed{H}_\beta \theta_I^- \ , \\
\bar{\theta}_I^- \breve{E}_\alpha \slashed{F}\acute{E}_\beta \theta_J^+ &= \bar{\theta}_J^+ \acute{E}_\beta \slashed{F}\breve{E}_\alpha \theta_I^- \ ,
\end{aligned} \tag{5.50}$$

where we have dropped a total integral in the first line. Using these we can write the kinetic action as

$$\begin{aligned}
\mathcal{L}_{kin} = & - i\gamma^{\alpha\beta}(\bar{\theta}_I^- \breve{E}_\alpha + 2\bar{\theta}_I^+ \acute{E}_\alpha)\left(\delta^{IJ}D_\beta + \frac{q}{8}\sigma_3^{IJ}\slashed{H}_\beta + \frac{\tilde{q}}{48}\sigma_3^{IJ}\slashed{F}\breve{E}_\beta\right)\theta_J^- \\
& - i\gamma^{\alpha\beta}\bar{\theta}_I^+ \breve{E}_\alpha \left(\delta^{IJ}D_\beta + \frac{1}{8}(q\sigma_3^{IJ} + \tilde{q}\sigma_1^{IJ})\slashed{H}_\beta\right)\theta_J^+ \\
& - i\gamma^{\alpha\beta}\frac{1}{48}\bar{\theta}_I^+ \acute{E}_\alpha(\tilde{q}\sigma_3^{IJ} - q\sigma_1^{IJ})\slashed{F}\acute{E}_\beta\theta_J^+
\end{aligned} \tag{5.51}$$

and the WZ action as

$$
\begin{aligned}
\mathcal{L}_{WZ} = & + iq\epsilon^{\alpha\beta}(\bar{\theta}_I^- \not{E}_\alpha + 2\bar{\theta}_I^+ \not{E}_\alpha)\left(\sigma_3^{IJ} D_\beta + \frac{q}{8}\delta^{IJ}\not{H}_\beta + \frac{\tilde{q}}{48}\delta^{IJ}\not{F}\not{E}_\beta\right)\theta_J^- \\
& + i\tilde{q}\epsilon^{\alpha\beta}(\bar{\theta}_I^- \not{E}_\alpha + 2\bar{\theta}_I^+ \not{E}_\alpha)\left(\sigma_1^{IJ} D_\beta - \frac{q}{8}i\sigma_2^{IJ}\not{H}_\beta - \frac{\tilde{q}}{48}i\sigma_2^{IJ}\not{F}\not{E}_\beta\right)\theta_J^- \\
& + i\epsilon^{\alpha\beta}\bar{\theta}_I^+ \not{E}_\alpha\left((q\sigma_3^{IJ} + \tilde{q}\sigma_1^{IJ})D_\beta + \frac{1}{8}\delta^{IJ}\not{H}_\beta\right)\theta_J^+ \\
& - i\epsilon^{\alpha\beta}\frac{1}{48}\bar{\theta}_I^+ \not{E}_\alpha i\sigma_2^{IJ}\not{F}\not{E}_\beta\theta_J^+
\end{aligned}
\tag{5.52}
$$

We will make use of two different rotations of the fermions. The first is most useful for exhibiting supersymmetry while the second is most useful for fixing lightcone gauge. The first rotation is to define new fermions via

$$
\theta_1^- = \hat{M}\vartheta_1^- , \quad \theta_1^+ = \hat{M}\vartheta_1^+ , \quad \theta_2^- = \check{M}\vartheta_2^- , \quad \theta_2^+ = \check{M}\vartheta_2^+
\tag{5.53}
$$

where $\hat{M}$ and $\check{N}$ are the matrices appearing in the solutions to the Killing spinor equations (5.26). The rotated fermion conjugates are given by

$$
\bar{\theta}_1^- = \bar{\vartheta}_1^- \hat{M}^{-1} , \quad \bar{\theta}_1^+ = \bar{\vartheta}_1^+ \hat{M}^{-1} , \quad \bar{\theta}_2^- = \bar{\vartheta}_2^- \check{M}^{-1} , \quad \bar{\theta}_2^+ = \bar{\vartheta}_2^+ \check{M}^{-1} ,
\tag{5.54}
$$

since $\hat{M}$ and $\check{M}$ satisfy

$$
\hat{M}^\dagger \Gamma^0 = \Gamma^0 \hat{M}^{-1} , \quad \check{M}^\dagger \Gamma^0 = \Gamma^0 \check{M}^{-1} .
\tag{5.55}
$$

Along with rotated fermions we also define rotated vielbeins $\hat{K}$ and $\check{K}$ by

$$
\hat{K}_M^A = \hat{\mathcal{M}}_B^A E_M^B , \quad \check{K}_M^A = \check{\mathcal{M}}_B^A E_M^B
\tag{5.56}
$$

where the orthogonal matrices $\hat{\mathcal{M}}$ and $\check{\mathcal{M}}$ are defined by

$$
\hat{M}^{-1}\Gamma^A \hat{M} = \Gamma^B \hat{\mathcal{M}}_B^A , \quad \check{M}^{-1}\Gamma^A \check{M} = \Gamma^B \check{\mathcal{M}}_B^A .
\tag{5.57}
$$

Acting on the contracted vielbeins, we have $\check{\not{K}} = \not{E}$ and

$$
\hat{\not{K}}_\alpha \equiv \sum_{a=0}^5 \hat{K}_\alpha^a \Gamma_a = \hat{M}^{-1}\not{E}_\alpha \hat{M} , \quad \check{\not{K}}_\alpha \equiv \sum_{a=0}^5 \check{E}_\alpha^a \Gamma_a = \check{M}^{-1}\not{E}_\alpha \check{M}
\tag{5.58}
$$

We can simplify all terms in the action involving θ_I^I immediately via

$$
\begin{aligned}
\left(D_\alpha + \frac{q}{8}\not{H}_\alpha + \frac{\tilde{q}}{48}\not{F}\not{E}_\alpha\right)\theta_1^- &= \hat{M}\partial_\alpha\vartheta_1^- , \\
\left(D_\alpha - \frac{q}{8}\not{H}_\alpha - \frac{\tilde{q}}{48}\not{F}\not{E}_\alpha\right)\theta_2^- &= \check{M}\partial_\alpha\vartheta_2^- .
\end{aligned}
\tag{5.59}
$$

as shown in appendix I. The remaining terms involving the covariant derivative can be rewritten using the spin-connections $\hat{\omega}$ and $\check{\omega}$ for the rotated vielbeins

$\hat{K}$ and $\check{K}$. We have

$$\hat{M}^{-1} D_\alpha \hat{M} \vartheta_I^+ = \partial_\alpha \vartheta_I^+ + \frac{1}{4} \hat{\slashed{\varphi}}_\alpha \vartheta_I^+ = \partial_\alpha \vartheta_I^+ - \frac{1}{2} \hat{K}_\alpha \Gamma^{012} \vartheta_I^+ \ ,$$

$$\check{M}^{-1} D_\alpha \check{M} \vartheta_I^+ = \partial_\alpha \vartheta_I^+ + \frac{1}{4} \check{\slashed{\varphi}} \vartheta_I^+ = \partial_\alpha \vartheta_I^+ + \frac{1}{2} \check{K}_\alpha \Gamma^{012} \vartheta_I^+ \tag{5.60}$$

using the expressions for $\hat{\slashed{\varphi}}$ and $\check{\slashed{\varphi}}$ given in appendix I. Meanwhile the remaining terms proportional to the NS-NS and R-R fluxes are

$$\hat{M}^{-1} \slashed{H}_\alpha \hat{M} \vartheta_I^+ = 4q \hat{\slashed{K}}_\alpha \Gamma^{012} \vartheta_I^+ \ ,$$

$$\check{M}^{-1} \slashed{H}_\alpha \check{M} \vartheta_I^+ = 4q \check{\slashed{K}}_\alpha \Gamma^{012} \vartheta_I^+ \ ,$$

$$\slashed{F} \slashed{\dot{E}}_\alpha \vartheta_I^+ = -\slashed{\dot{E}}_\alpha \slashed{F} \vartheta_I^+ = -24\tilde{q} \slashed{\dot{E}}_\alpha \Gamma^{012} \vartheta_I^+ \ . \tag{5.61}$$

We also note that

$$\gamma^{\alpha\beta} \left(\hat{\slashed{K}}_\alpha \hat{\slashed{K}}_\beta + \slashed{\dot{E}}_\alpha \slashed{\dot{E}}_\beta \right) = \gamma^{\alpha\beta} \left(\check{\slashed{K}}_\alpha \check{\slashed{K}}_\beta + \slashed{\dot{E}}_\alpha \slashed{\dot{E}}_\beta \right) = \gamma^{\alpha\beta} \eta_{AB} E_\alpha^A E_\beta^B \tag{5.62}$$

as $\hat{K}$ and $\check{K}$ come from $SO(1,9)$ transformations of the diagonal vielbein E.

The final expressions we obtain for the action are

$$
\begin{aligned}
\mathcal{L}_{kin} = -i\gamma^{\alpha\beta} \Big[&\bar{\vartheta}_1^- \check{\slashed{K}}_\alpha \partial_\beta \vartheta_1^- + 2\bar{\vartheta}_1^+ \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_1^- + \bar{\vartheta}_1^+ \check{\slashed{K}}_\alpha \partial_\beta \vartheta_1^+ \\
&+ \bar{\vartheta}_2^- \check{\slashed{K}}_\alpha \partial_\beta \vartheta_2^- + 2\bar{\vartheta}_2^+ \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_2^- + \bar{\vartheta}_2^+ \check{\slashed{K}}_\alpha \partial_\beta \vartheta_2^+ \\
&- \frac{\tilde{q}^2}{2} \sigma_3^{IJ} \bar{\vartheta}_I^+ \Gamma^{012} \vartheta_J^+ E_\alpha^A E_\beta^B \eta_{AB} \\
&+ \frac{q\tilde{q}}{2} \bar{\vartheta}_1^+ \hat{M}^{-1} \check{M} \Gamma^{012} \vartheta_2^+ E_\alpha^A E_\beta^B \eta_{AB} \\
&+ \frac{q\tilde{q}}{2} \bar{\vartheta}_2^+ \check{M}^{-1} \hat{M} \Gamma^{012} \vartheta_1^+ E_\alpha^A E_\beta^B \eta_{AB} \Big] \ ,
\end{aligned}
\tag{5.63}
$$

$$
\begin{aligned}
\mathcal{L}_{WZ} = i\epsilon^{\alpha\beta} \Big[&q \left(\bar{\vartheta}_1^- \check{\slashed{K}}_\alpha \partial_\beta \vartheta_1^- + 2\bar{\vartheta}_1^+ \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_1^- + \bar{\vartheta}_1^+ \check{\slashed{K}}_\alpha \partial_\beta \vartheta_1^+ \right) \\
&- q \left(\bar{\vartheta}_2^- \check{\slashed{K}}_\alpha \partial_\beta \vartheta_2^- + 2\bar{\vartheta}_2^+ \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_2^- + \bar{\vartheta}_2^+ \check{\slashed{K}}_\alpha \partial_\beta \vartheta_2^+ \right) \\
&+ \tilde{q} \left(\bar{\vartheta}_2^- \check{M}^{-1} \hat{M} \check{\slashed{K}}_\alpha \partial_\beta \vartheta_1^- + \bar{\vartheta}_2^+ \check{M}^{-1} \hat{M} \check{\slashed{K}}_\alpha \partial_\beta \vartheta_1^+ \right) \\
&+ \tilde{q} \left(\bar{\vartheta}_1^- \hat{M}^{-1} \check{M} \check{\slashed{K}}_\alpha \partial_\beta \vartheta_2^- + \bar{\vartheta}_1^+ \hat{M}^{-1} \check{M} \check{\slashed{K}}_\alpha \partial_\beta \vartheta_2^+ \right) \\
&+ \tilde{q} \left(\bar{\vartheta}_2^- \check{M}^{-1} \hat{M} \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_1^+ + \bar{\vartheta}_2^+ \check{M}^{-1} \hat{M} \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_1^- \right) \\
&+ \tilde{q} \left(\bar{\vartheta}_1^- \hat{M}^{-1} \check{M} \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_2^+ + \bar{\vartheta}_1^+ \hat{M}^{-1} \check{M} \slashed{\dot{E}}_\alpha \partial_\beta \vartheta_2^- \right) \\
&+ \frac{\tilde{q}}{2} \bar{\vartheta}_2^+ \check{M}^{-1} \hat{M} \left(\check{\slashed{K}}_\alpha \check{\slashed{K}}_\beta + \slashed{\dot{E}}_\alpha \slashed{\dot{E}}_\beta \right) \Gamma^{012} \vartheta_1^+ \\
&+ \frac{\tilde{q}}{2} \bar{\vartheta}_1^+ \hat{M}^{-1} \check{M} \left(\check{\slashed{K}}_\alpha \check{\slashed{K}}_\beta + \slashed{\dot{E}}_\alpha \slashed{\dot{E}}_\beta \right) \Gamma^{012} \vartheta_2^+ \Big] \ .
\end{aligned}
\tag{5.64}
$$

Supersymmetry is realised in the action in this form as a shift of the

fermions ϑ_I^-. The 8 independent real components of each of the Majorana-Weyl spinors ϑ_1^-, ϑ_2^- thus gives us 16 real supersymmetries as is to be expected.

5.3.1 Fixing kappa gauge

To fix kappa gauge we make a different rotation of the fermions, such that the resulting fermions are independent of the lightcone coordinates t and ψ. We make the rotation

$$\theta_1^- = M_0 \eta_1 , \qquad \theta_2^- = M_0^{-1} \eta_2 ,$$
$$\theta_1^+ = M_0 \chi_1 , \qquad \theta_2^+ = M_0^{-1} \chi_2 \tag{5.65}$$

We also use a different set of rotated vielbeins which satisfy

$$\hat{\mathbb{E}}_\alpha = M_0^{-1} \bar{\mathbb{E}}_\alpha M_0 , \quad \check{\mathbb{E}}_\alpha = M_0 \bar{\mathbb{E}}_\alpha M_0^{-1} \tag{5.66}$$

It is simplest to obtain the resulting form of the Green-Schwarz action from the expressions (5.63) and (5.64) rather than from the original action. We have various expressions such as

$$\bar{\vartheta}_1^- \hat{\mathbb{K}}_\alpha \partial_\beta \vartheta_1^- = \bar{\eta}_1 \hat{\mathbb{E}}_\alpha \partial_\beta \eta_1 + \bar{\eta}_1 \hat{\mathbb{E}}_\alpha \left(M_t \partial_\beta M_t^{-1} \right) \eta_1 \tag{5.67}$$

and

$$\bar{\vartheta}_2^- \check{M}^{-1} \hat{M} \hat{\mathbb{E}}_\beta \partial_\beta \vartheta_1^- = \bar{\eta}_2 M_0^2 \hat{\mathbb{E}}_\alpha \partial_\beta \eta_1 + \bar{\eta}_2 M_0^2 \hat{\mathbb{E}}_\alpha \left(M_t \partial_\beta M_t^{-1} \right) \eta_1 \tag{5.68}$$

Note that

$$M_t \partial_\beta M_t^{-1} = -M_t^{-1} \partial_\beta M_t = \frac{1}{2} \left(\Gamma^{12} + \Gamma^{34} \right) \partial_\beta x^+ - \frac{1}{2} \left(\Gamma^{12} - \Gamma^{34} \right) \partial_\beta x^- . \tag{5.69}$$

We introduce the lightcone bosonic coordinates, lightcone vielbeins and lightcone gamma matrices

$$x^\pm = \frac{1}{2} \left(\psi \pm t \right) , \quad E^\pm = \frac{1}{2} \left(E^5 \pm E^0 \right) , \quad \Gamma^\pm = \frac{1}{2} \left(\Gamma^5 \pm \Gamma^0 \right) \tag{5.70}$$

Then we fix lightcone kappa gauge by requiring

$$\Gamma^+ \eta_I = 0 , \quad \Gamma^+ \chi_I = 0 \tag{5.71}$$

One consequence of this is that η_I and χ_I then satisfy

$$\Gamma^{12} \eta_I = \Gamma^{34} \eta_I , \quad \Gamma^{12} \chi_I = -\Gamma^{34} \chi_I \tag{5.72}$$

For any spinors ϵ_1 and ϵ_2 satisfying $\Gamma^+ \epsilon_I = 0$ we have

$$\bar{\epsilon}_1 \Gamma_{A_1 \ldots A_n} \epsilon_2 = 0 \tag{5.73}$$

for any $A_1 \ldots A_n = 1 \ldots 8$, because $\Gamma^+ \Gamma^- + \Gamma^- \Gamma^+ = \mathbb{1}_{10}$ and so

$$\bar{\epsilon}_1 \Gamma_{A_1 \ldots A_n} \epsilon_2 = \bar{\epsilon}_1 \Gamma^+ \Gamma^- \Gamma_{A_1 \ldots A_n} \epsilon_2 + \bar{\epsilon}_1 \Gamma_{A_1 \ldots A_n} \Gamma^+ \Gamma^- \epsilon_2 = 0 \qquad (5.74)$$

as we have Γ^+ acting on η to either the left or right in each term. It then follows that

$$\bar{\epsilon}_1 \hat{\slashed{E}}_\alpha \epsilon_2 = 2\bar{\epsilon}_1 \hat{E}_\alpha^+ \Gamma^- \epsilon_2 \ ,$$
$$\bar{\epsilon}_1 \check{\slashed{E}}_\alpha \epsilon_2 = 2\bar{\epsilon}_1 \check{E}_\alpha^+ \Gamma^- \epsilon_2 \ ,$$
$$\bar{\epsilon}_1 \dot{\slashed{E}}_\alpha \epsilon_2 = 0 \qquad (5.75)$$

Making use of the above and the fact that

$$M_0^2 \dot{\slashed{E}}_\alpha = \dot{\slashed{E}}_\alpha M_0^2 \ , \qquad M_0^{-2} \dot{\slashed{E}}_\alpha = \dot{\slashed{E}}_\alpha M_0^{-2} \qquad (5.76)$$

we obtain the following for the lightcone gauged-fixed kinetic action

$$\mathcal{L}_{kin} = \mathcal{L}_{kin}^{\eta\eta} + \mathcal{L}_{kin}^{\chi\chi} \ , \qquad (5.77)$$

with

$$\mathcal{L}_{kin}^{\eta\eta} = -2i\gamma^{\alpha\beta} \left[\bar{\eta}_1 \hat{E}_\alpha^+ \Gamma^- \left(\partial_\beta \eta_1 + \Gamma^{12} \eta_1 \partial_\beta x^+ \right) \right.$$
$$\left. + \bar{\eta}_2 \check{E}_\alpha^+ \Gamma^- \left(\partial_\beta \eta_2 - \Gamma^{12} \eta_2 \partial_\beta x^+ \right) \right] \ , \qquad (5.78)$$

and

$$\mathcal{L}_{kin}^{\chi\chi} = -2i\gamma^{\alpha\beta} \left[+ \bar{\chi}_1 \hat{E}_\alpha^+ \Gamma^- \left(\partial_\beta \chi_1 - \Gamma^{12} \chi_1 \partial_\beta x^- \right) \right.$$
$$+ \bar{\chi}_2 \check{E}_\alpha^+ \Gamma^- \left(\partial_\beta \chi_2 + \Gamma^{12} \chi_2 \partial_\beta x^- \right)$$
$$- \frac{\tilde{q}^2}{4} \sigma_{IJ}^3 \bar{\chi}_I \Gamma^{012} \chi_J E_\alpha^A E_\beta^B \eta_{AB}$$
$$\left. + \frac{q\tilde{q}}{4} \left(\bar{\chi}_1 M_0^{-2} \Gamma^{012} \chi_2 + \bar{\chi}_2 M_0^2 \Gamma^{012} \chi_1 \right) E_\alpha^A E_\beta^B \eta_{AB} \right] \qquad (5.79)$$

To zeroth order in bosons in lightcone gauge, $\hat{E}_0^+ = \check{E}_0^+ = 1$ with all other vielbein components vanishing, and the worldsheet metric is conformal, $\gamma^{\alpha\beta} = \eta^{\alpha\beta}$. Hence to zeroth order in bosons the kinetic fermion action is

$$\mathcal{L}_{kin} = 2i \left[\bar{\eta}_I \Gamma^- \dot{\eta}_I + \sigma_3^{IJ} \bar{\eta}_I \Gamma^- \Gamma^{12} \eta_J + \bar{\chi}_I \Gamma^- \dot{\chi}_I \right] \ . \qquad (5.80)$$

From this we can see directly that η and χ are indeed massive and massless fermions respectively.

The lightcone gauged-fixed WZ action is

$$\mathcal{L}_{WZ} = \mathcal{L}_{WZ}^{\eta\eta} + \mathcal{L}_{WZ}^{\chi\eta} + \mathcal{L}_{WZ}^{\chi\chi} \, , \qquad (5.81)$$

with

$$\mathcal{L}_{WZ}^{\chi\eta} = 2i\epsilon^{\alpha\beta}\left[\tilde{q}\bar{\chi}_1\acute{E}_\alpha M_0^{-2}\left(\partial_\beta\eta_2 - \Gamma^{12}\eta_2\partial_\beta x^+\right)\right.$$
$$\left. + \tilde{q}\bar{\chi}_2\grave{E}_\alpha M_0^2\left(\partial_\beta\eta_1 + \Gamma^{12}\eta_1\partial_\beta x^+\right)\right] \, , \qquad (5.82)$$

$$\mathcal{L}_{WZ}^{\eta\eta} = 2i\epsilon^{\alpha\beta}\left[\frac{\tilde{q}}{2}\bar{\eta}_1\acute{E}_\alpha M_0^{-2}\left(\partial_\beta\eta_2 - \Gamma^{12}\eta_2\partial_\beta x^+\right)\right.$$
$$+ \frac{\tilde{q}}{2}\bar{\eta}_2\grave{E}_\alpha M_0^2\left(\partial_\beta\eta_1 + \Gamma^{12}\eta_1\partial_\beta x^+\right)$$
$$+ q\bar{\eta}_1\hat{E}_\alpha^+\Gamma^-\left(\partial_\beta\eta_1 + \Gamma^{12}\eta_1\partial_\beta x^+\right)$$
$$\left. - q\bar{\eta}_2\check{E}_\alpha^+\Gamma^-\left(\partial_\beta\eta_2 - \Gamma^{12}\eta_2\partial_\beta x^+\right)\right] \, , \qquad (5.83)$$

and

$$\mathcal{L}_{WZ}^{\chi\chi} = 2i\epsilon^{\alpha\beta}\left[2i\epsilon^{\alpha\beta}\left[\frac{\tilde{q}}{4}\bar{\chi}_1\left(\acute{E}_\alpha\grave{E}_\beta + \grave{E}_\alpha\grave{E}_\beta\right)M_0^{-2}\Gamma^{012}\chi_2\right.\right.$$
$$- \frac{\tilde{q}}{4}\bar{\chi}_2\left(\grave{E}_\alpha\grave{E}_\beta + \grave{E}_\alpha\grave{E}_\beta\right)M_0^2\Gamma^{012}\chi_1$$
$$+ \frac{\tilde{q}}{2}\bar{\chi}_1\acute{E}_\alpha M_0^{-2}\left(\partial_\beta\chi_2 + \Gamma^{12}\chi_2\partial_\beta x^-\right)$$
$$+ \frac{\tilde{q}}{2}\bar{\chi}_2\grave{E}_\alpha M_0^2\left(\partial_\beta\chi_1 - \Gamma^{12}\chi_1\partial_\beta x^-\right)$$
$$+ q\bar{\chi}_1\hat{E}_\alpha^+\Gamma^-\left(\partial_\beta\chi_1 - \Gamma^{12}\chi_1\partial_\beta x^-\right)$$
$$\left.\left. - q\bar{\chi}_2\check{E}_\alpha^+\Gamma^-\left(\partial_\beta\chi_2 + \Gamma^{12}\chi_2\partial_\beta x^-\right)\right]\right. \, . \qquad (5.84)$$

5.4 Supercurrents

Having written down the gauge-fixed action, we can now construct conserved supercurrents for this action j_I^α satisfying

$$\partial_\tau j_I^\tau + \partial_\sigma j_I^\sigma = 0 \, . \qquad (5.85)$$

We construct these supercurrents to leading order in fermions and cubic order in bosons. Our expressions for the action in the previous section are written in terms of ten-dimensional gamma matrices Γ^A and the fermions η_I and χ_I are 32-component spinors. However, they satisfy the Weyl condition (2.6), the kappa-gauge condition (5.71) as well as having definite chirality under Γ^{012345} from their definition in equation (5.65). As such, each has 4 independent physical components. We have therefore 8 independent supercharges coming from

j_1 and j_2. This is also consistent with another role these supercurrents play. They are associated with those supervariations that have to be combined with a kappa transformation in order to preserve lightcone gauge. In practice we do not construct the supercurrents from a Noether procedure which explicitly includes this corresponding kappa transformation. Instead we construct them order by order to satisfy the conservation equation.

The physical spinors transform as bispinors of $\mathfrak{so}(4)_1 \oplus \mathfrak{so}(4)_2$, where $\mathfrak{so}(4)_1$ corresponds to the transverse directions of $AdS_3 \times S^3$ in lightcone-gauge and $\mathfrak{so}(4)_2$ corresponds to T^4. We use indices $\underline{a}, \underline{\dot{a}} = 1, 2$ to represent the $\mathfrak{so}(4)_1$ spinor indices which are respectively positive and negative under Γ^{012345} chirality. Similarly we use indices a and $\dot{a}$ to represent positive and negative chirality $\mathfrak{so}(4)_2$ spinor indices. We also use indices $\underline{i} = 1, 2, 3, 4$, $i = 6, 7, 8, 9$ to represent respectively the transverse $AdS_3 \times S^3$ and T^4 tangent indices of $\mathfrak{so}(4)_1$ and $\mathfrak{so}(4)_2$ gamma matrices. The explicit basis $\gamma^{\underline{i}}$ and τ^i we use is

$$
\begin{aligned}
\gamma^1 = +\sigma_3, \quad & \gamma^2 = -i\mathbb{1}, \quad && \gamma^3 = +\sigma_2, \quad && \gamma^4 = +\sigma_1, \\
\tau^6 = +\sigma_1, \quad & \tau^7 = +\sigma_2, \quad && \tau^8 = +\sigma_3, \quad && \tau^9 = +i\mathbb{1}, \quad &&&.
\end{aligned}
\tag{5.86}
$$

We also define

$$
\tilde{\gamma}^{\underline{i}} = +(\gamma^{\underline{i}})^\dagger , \qquad \tilde{\tau}^i = -(\tau^i)^\dagger ,
\tag{5.87}
$$

and

$$
\begin{aligned}
(\gamma^{\underline{ij}})^{\underline{a}}{}_{\underline{b}} = \frac{1}{2}(\gamma^{\underline{i}}\tilde{\gamma}^{\underline{j}} - \gamma^{\underline{j}}\tilde{\gamma}^{\underline{i}})^{\underline{a}}{}_{\underline{b}}, \qquad & (\tau^{ij})^a{}_b = \frac{1}{2}(\tau^i\tilde{\tau}^j - \tau^j\tilde{\tau}^i)^a{}_b, \\
(\tilde{\gamma}^{\underline{ij}})^{\underline{\dot{a}}}{}_{\underline{\dot{b}}} = \frac{1}{2}(\tilde{\gamma}\gamma^{\underline{j}} - \tilde{\gamma}^{\underline{j}}\gamma^{\underline{i}})^{\underline{\dot{a}}}{}_{\underline{\dot{b}}}, \qquad & (\tilde{\tau}^{ij})^{\dot{a}}{}_b = \frac{1}{2}(\tilde{\tau}^i\tau^j - \tilde{\tau}^j\tau^i)^{\dot{a}}{}_b .
\end{aligned}
\tag{5.88}
$$

The relationship between the ten-dimensional gamma matrices and these $\mathfrak{so}(4)$ gamma matrices is discussed in [65].

The supercurrents are given to quadratic order by

$$
\begin{aligned}
j_1^\tau = ie^{+x^-\gamma^{34}} \big[&(\dot{z}^{\underline{i}} - \dot{y}^{\underline{i}})\gamma_{\underline{i}}\eta_1 + (z^{\underline{i}} + y^{\underline{i}})\gamma^{34}\gamma_{\underline{i}}\eta_1 - (z'^{\underline{i}} - y'^{\underline{i}})\gamma_{\underline{i}}(\tilde{q}\eta_2 + q\eta_1) \\
&+ \dot{x}^i\gamma^{34}\tilde{\tau}_i\chi_1 - x'^i\gamma^{34}\tilde{\tau}_i(\tilde{q}\chi_2 + q\chi_1) \big] ,
\end{aligned}
\tag{5.89}
$$

$$
\begin{aligned}
j_2^\tau = ie^{-x^-\gamma^{34}} \big[&(\dot{z}^{\underline{i}} - \dot{y}^{\underline{i}})\gamma_{\underline{i}}\eta_2 - (z^{\underline{i}} + y^{\underline{i}})\gamma^{34}\gamma_{\underline{i}}\eta_2 - (z'^{\underline{i}} - y'^{\underline{i}})\gamma_{\underline{i}}(\tilde{q}\eta_1 - q\eta_2) \\
&+ \dot{x}^i\gamma^{34}\tilde{\tau}_i\chi_2 - x'^i\gamma^{34}\tilde{\tau}_i(\tilde{q}\chi_1 - q\chi_2) \big] ,
\end{aligned}
\tag{5.90}
$$

$$
\begin{aligned}
j_1^\sigma = ie^{+x^-\gamma^{34}} \big[&(\dot{z}^{\underline{i}} - \dot{y}^{\underline{i}})\gamma_{\underline{i}}(\tilde{q}\eta_2 + q\eta_1) + (z^{\underline{i}} + y^{\underline{i}})\gamma^{34}\gamma_{\underline{i}}(\tilde{q}\eta_2 + q\eta_1) \\
&- (z'^{\underline{i}} - y'^{\underline{i}})\gamma_{\underline{i}}\eta_1 + \dot{x}^i\gamma^{34}\tilde{\tau}_i(\tilde{q}\chi_2 + q\chi_1) - x'^i\gamma^{34}\tilde{\tau}_i)\chi_1 \big] ,
\end{aligned}
\tag{5.91}
$$

$$
\begin{aligned}
j_2^\sigma = ie^{-x^-\gamma^{34}} \big[&(\dot{z}^{\underline{i}} - \dot{y}^i)\gamma_{\underline{i}}(\tilde{q}\eta_1 - q\eta_2) - (z^{\underline{i}} + y^{\underline{i}})\gamma^{34}\gamma_{\underline{i}}(\tilde{q}\eta_1 - q\eta_2) \\
&- (z'^{\underline{i}} - y'^{\underline{i}})\gamma_{\underline{i}}\eta_2 + \dot{x}^i\gamma^{34}\tilde{\tau}_i(\tilde{q}\chi_1 - q\chi_2) - x'^i\gamma^{34}\tilde{\tau}_i\chi_2 \big] .
\end{aligned}
\tag{5.92}
$$

In appendix J we give the supercurrents to quartic order in fields. These expressions all satisfy the conservation equation (5.85) to the appropriate order.

We now give the Poisson brackets of these supercurrents, which will be used in the next section when we discuss in full the algebra $\mathcal{A}$. For this we need the Poisson brackets of the fermions, which are given in appendix K. We then find Poisson brackets of j_1 and j_2 with themselves given by[7]

$$\int d\sigma \, d\sigma' \{j_1^\tau(\sigma), j_1^\tau(\sigma')\}_{\text{PB}} = +\frac{i}{2} \int d\sigma (\mathcal{H} + \mathcal{M}) \, \epsilon \, \epsilon, \qquad (5.93)$$

$$\int d\sigma \, d\sigma' \{j_2^\tau(\sigma), j_2^\tau(\sigma')\}_{\text{PB}} = +\frac{i}{2} \int d\sigma (\mathcal{H} - \mathcal{M}) \, \epsilon \, \epsilon. \qquad (5.94)$$

where the bosonic Hamiltonian density $\mathcal{H}$ is given to quartic order by

$$\begin{aligned}
\mathcal{H} = &+ \frac{1}{2}\left(p_z^2 + p_y^2 + p_x^2 + \acute{z}^2 + \acute{y}^2 + \acute{x}^2 + z^2 + y^2 - 2q\epsilon^{\underline{ij}}(z_{\underline{i}}\acute{z}_{\underline{j}} + y_{\underline{i}}\acute{y}_{\underline{j}})\right) \\
&+ \frac{1}{4}\left(p_z^2 + p_y^2 + p_x^2 + \acute{z}^2 + \acute{y}^2 + \acute{x}^2\right)\left((z^2 - y^2) + q\epsilon^{\underline{ij}}(\acute{z}_{\underline{i}}z_{\underline{j}} - \acute{y}_{\underline{i}}y_{\underline{j}})\right) \\
&+ \frac{1}{4}z^2\left(\acute{z}^2 - p_z^2 + q\epsilon^{\underline{ij}}(\acute{z}_{\underline{i}}z_{\underline{j}} + \acute{y}_{\underline{i}}y_{\underline{j}})\right) - \frac{1}{4}y^2\left(\acute{y}^2 - p_y^2 + q\epsilon^{\underline{ij}}(\acute{z}_{\underline{i}}z_{\underline{j}} + \acute{y}_{\underline{i}}y_{\underline{j}})\right) \\
&- \frac{q}{2}\epsilon^{\underline{ij}}\left(p_{z^i}z_{\underline{j}} + p_{y^i}y_{\underline{j}}\right)\left(p_z \cdot \acute{z} + p_y \cdot \acute{y} + p_x \cdot \acute{x}\right).
\end{aligned} \qquad (5.95)$$

and the "mass" term $\mathcal{M}$ is given by

$$\mathcal{M} = -\epsilon^{\underline{ij}}(p_{\underline{i}}z_{\underline{j}} + p_{\underline{i}}y_{\underline{j}}) - q(p_{\underline{i}}\acute{z}^i + p_{\underline{i}}\acute{y}^i + p_i\acute{x}^i). \qquad (5.96)$$

Note that $\mathcal{M}$ does not have any quartic corrections.

The Poisson bracket of j_1 with j_2 gives

$$\int d\sigma \, d\sigma' \{j_1^\tau(\sigma), j_2^\tau(\sigma')\}_{\text{PB}} = -\frac{i\tilde{q}}{2}e^{+2\gamma^{34}x^-(-\infty)}\left(e^{+\gamma^{34}p_{\text{ws}}} - 1\right)\gamma^{34}\epsilon \, \epsilon. \qquad (5.97)$$

Hence the algebra $\mathcal{A}$ has a central charge

$$C = \frac{i\zeta h}{2}\left(e^{ip_{ws}} - 1\right), \qquad (5.98)$$

where we define $\zeta = \exp(2ix^-(\infty))$ and the coupling h is given, once we put back the string tension which we have been suppressing, by

$$h = \frac{\tilde{q}\sqrt{\lambda}}{2\pi}. \qquad (5.99)$$

This is a rescaling by $\tilde{q}$ of the central charge for the pure R-R theory.

---

[7]We have suppressed spinor indices.

5.5   The algebra $\mathcal{A}$ and its representations

In this section we will examine the off-shell symmetry algebra $\mathcal{A}$ and its representations. We will first consider the near-plane-wave algebra and then deform this to produce the exact algebra. The near-plane-wave generators for $\mathcal{A}$ are precisely the components of the currents j, truncated to quadratic order in fields, as given in the previous section. We will introduce notation for these generators which better exhibits their algebraic structure. In particular we will have two sets of supercharges $\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}$ and $\mathbf{Q}_{\mathrm{R}}{}^{\dot{a}}$, carrying a natural $\mathfrak{su}(2)$ index $\dot{a}$, together with their conjugates $\overline{\mathbf{Q}}_{\mathrm{L}}{}^{\dot{a}}$ and $\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}}$. The anticommutation relations for these supercharges will turn out to be

$$\{\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}, \overline{\mathbf{Q}}_{\mathrm{L}\dot{b}}\} = \frac{1}{2}\delta^{\dot{a}}{}_{\dot{b}}(\mathbf{H} + \mathbf{M}) , \quad \{\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}, \mathbf{Q}_{\mathrm{R}\dot{b}}\} = \delta^{\dot{a}}{}_{\dot{b}}\mathbf{C} ,$$

$$\{\mathbf{Q}_{\mathrm{R}\dot{a}}, \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{b}}\} = \frac{1}{2}\delta^{\dot{a}}{}_{\dot{b}}(\mathbf{H} - \mathbf{M}) , \quad \{\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}}, \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{b}}\} = \delta^{\dot{a}}{}_{\dot{b}}\overline{\mathbf{C}} . \tag{5.100}$$

This algebra is a central extension of $\mathfrak{psu}(1|1)^4$. We will find explicit expressions for the charges $\mathbf{H}$, $\mathbf{M}$, $\mathbf{C}$ and $\overline{\mathbf{C}}$ in the near-plane-wave limit and then in the exact theory.

The key results of this section are as follows. The exact central charges are given in equations (5.140), (5.141) and (5.142). Explicit representations for the algebra which satisfies (5.100) are given in equation (5.143), (5.144) and (5.145) in terms of representation parameters which are defined in equation (5.146) so as to reproduce the correct charges.

5.5.1   Near-plane-wave algebra

We begin by defining the superchargers $\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}$ and $\mathbf{Q}_{\mathrm{R}}{}^{\dot{a}}$ in terms of the currents j_I via

$$\mathbf{Q}_{\mathrm{L}}{}^{1} = -\int \mathrm{d}\sigma \, (j_1^\tau)^{21}, \qquad \mathbf{Q}_{\mathrm{L}}{}^{2} = +\int \mathrm{d}\sigma \, (j_1^\tau)^{22},$$

$$\mathbf{Q}_{\mathrm{R}1} = -\int \mathrm{d}\sigma \, (j_2^\tau)^{12}, \qquad \mathbf{Q}_{\mathrm{R}2} = -\int \mathrm{d}\sigma \, (j_2^\tau)^{11}. \tag{5.101}$$

The Hermitian conjugates of these,

$$\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} = (\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}})^\dagger, \qquad \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} = (\mathbf{Q}_{\mathrm{R}\dot{a}})^\dagger, \tag{5.102}$$

are then given in terms of the remaining components of the currents by:

$$\overline{\mathbf{Q}}_{\mathrm{L}1} = +\int \mathrm{d}\sigma \, (j_1^\tau)^{12}, \qquad \overline{\mathbf{Q}}_{\mathrm{L}2} = +\int \mathrm{d}\sigma \, (j_1^\tau)^{11},$$

$$\overline{\mathbf{Q}}_{\mathrm{R}}{}^{1} = +\int \mathrm{d}\sigma \, (j_2^\tau)^{21}, \qquad \overline{\mathbf{Q}}_{\mathrm{R}}{}^{2} = -\int \mathrm{d}\sigma \, (j_2^\tau)^{22}. \tag{5.103}$$

The give explicit expressions for these supercharges in terms of fields we

need to introduce notation for the components of the fermions η_I and χ_I. We
write the fermion components as follows:

$$(\eta_1)^{\dot{a}b} = \begin{pmatrix} -e^{+i\pi/4}\bar{\eta}_{\mathrm{L}2} & -e^{+i\pi/4}\bar{\eta}_{\mathrm{L}1} \\ e^{-i\pi/4}\eta_{\mathrm{L}}{}^1 & -e^{-i\pi/4}\eta_{\mathrm{L}}{}^2 \end{pmatrix}, \tag{5.104}$$

$$(\eta_2)^{\dot{a}b} = \begin{pmatrix} e^{-i\pi/4}\eta_{\mathrm{R}2} & e^{-i\pi/4}\eta_{\mathrm{R}1} \\ -e^{+i\pi/4}\bar{\eta}_{\mathrm{R}}{}^1 & e^{+i\pi/4}\bar{\eta}_{\mathrm{R}}{}^2 \end{pmatrix}, \tag{5.105}$$

$$(\chi_1)^{\underline{a}b} = \begin{pmatrix} -e^{+i\pi/4}\bar{\chi}_{+2} & e^{+i\pi/4}\bar{\chi}_{+1} \\ -e^{-i\pi/4}\chi_+{}^1 & -e^{-i\pi/4}\chi_+{}^2 \end{pmatrix}, \tag{5.106}$$

$$(\chi_2)^{\underline{a}b} = \begin{pmatrix} e^{-i\pi/4}\chi_-{}^1 & e^{-i\pi/4}\chi_-{}^2 \\ -e^{+i\pi/4}\bar{\chi}_{-2} & e^{+i\pi/4}\bar{\chi}_{-1} \end{pmatrix}. \tag{5.107}$$

To help produce compact expressions we also introduce complex bosonic co-
ordinates

$$Z = -z_2 + iz_1, \quad \bar{Z} = -z_2 - iz_1, \qquad Y = -y_3 - iy_4, \quad \bar{Y} = -y_3 + iy_4,$$
$$X^{11} = -x_6 + ix_7 = (X^{22})^\dagger, \qquad X^{12} = x_8 - ix_9 = -(X^{21})^\dagger, \tag{5.108}$$

with conjugate momenta

$$P_Z = 2\dot{Z}, \quad P_{\bar{Z}} = 2\dot{\bar{Z}}, \qquad P_Z = 2\dot{Z}, \quad P_{\bar{Y}} = 2\dot{\bar{Y}},$$
$$P_{11} = P_{22}^\dagger = 2\dot{X}^{22}, \qquad P_{12} = -P_{21}^\dagger = -2\dot{X}^{21}. \tag{5.109}$$

These fields so defined satisfy the commutation relations

$$[Z(\sigma_1), P_{\bar{Z}}(\sigma_2)] = [\bar{Z}(\sigma_1), P_Z(\sigma_2)] = i\delta(\sigma_1 - \sigma_2) ,$$
$$[Y(\sigma_1), P_{\bar{Y}}(\sigma_2)] = [\bar{Y}(\sigma_1), P_Y(\sigma_2)] = i\delta(\sigma_1 - \sigma_2) ,$$
$$[X^{\dot{a}a}(\sigma_1), P_{\dot{b}b}(\sigma_2)] = i\delta^{\dot{a}}_{\dot{b}}\delta^a_b\delta(\sigma_1 - \sigma_2), \tag{5.110}$$

and the anti-commutation relations

$$\{\bar{\eta}_{\mathrm{L}\dot{a}}(\sigma_1), \eta_{\mathrm{L}}{}^{\dot{b}}(\sigma_2)\} = \{\bar{\eta}_{\mathrm{R}}{}^{\dot{b}}(\sigma_1), \eta_{\mathrm{R}\dot{a}}(\sigma_2)\} = \delta^{\dot{b}}_{\dot{a}}\delta(\sigma_1 - \sigma_2),$$
$$\{\bar{\chi}_{+a}(\sigma_1), \chi_+^b(\sigma_2)\} = \{\bar{\chi}_{-a}(\sigma_1), \chi_-^b(\sigma_2)\} = \delta^b_a\delta(\sigma_1 - \sigma_2). \tag{5.111}$$

The quadratic supercharges are then given in terms of the fields defined

above by

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} = e^{-i\pi/4} \int \mathrm{d}\sigma \Bigg[&\frac{1}{2} P_Z \eta_{\mathrm{L}}{}^{\dot{a}} + Z'(i\tilde{q}\,\bar{\eta}_{\mathrm{R}}{}^{\dot{a}} - q\,\eta_{\mathrm{L}}{}^{\dot{a}}) + iZ\eta_{\mathrm{L}}{}^{\dot{a}} \\
&- \epsilon^{\dot{a}\dot{b}} \left(\frac{i}{2} P_{\bar{Y}} \bar{\eta}_{\mathrm{L}\dot{b}} + \bar{Y}'(\tilde{q}\,\eta_{\mathrm{R}\dot{b}} - iq\,\bar{\eta}_{\mathrm{L}\dot{b}}) + \bar{Y}\bar{\eta}_{\mathrm{L}\dot{b}} \right) \\
&- \frac{1}{2}\epsilon^{\dot{a}\dot{b}} P_{ba}\chi_+{}^{a} - (X^{\dot{a}a})'(i\tilde{q}\,\bar{\chi}_{-a} + q\,\epsilon_{ab}\chi_+{}^{b}) \Bigg], \\
\mathbf{Q}_{\mathrm{R}\dot{a}} = e^{-i\pi/4} \int \mathrm{d}\sigma \Bigg[&\frac{1}{2} P_{\bar{Z}} \eta_{\mathrm{R}\dot{a}} + \bar{Z}'(i\tilde{q}\,\bar{\eta}_{\mathrm{L}\dot{a}} + q\,\eta_{\mathrm{R}\dot{a}}) + i\bar{Z}\eta_{\mathrm{R}\dot{a}} \\
&+ \epsilon_{\dot{a}\dot{b}} \left(\frac{i}{2} P_{Y} \bar{\eta}_{\mathrm{R}}{}^{\dot{b}} + Y'(\tilde{q}\,\eta_{\mathrm{L}}{}^{\dot{b}} + iq\,\bar{\eta}_{\mathrm{R}}{}^{\dot{b}}) + Y\bar{\eta}_{\mathrm{R}}{}^{\dot{b}} \right) \\
&- \frac{1}{2} P_{\dot{a}a}\chi_-{}^{a} + \epsilon_{\dot{a}\dot{b}}(X^{\dot{b}a})'(i\tilde{q}\,\bar{\chi}_{+a} + q\,\epsilon_{ab}\chi_-{}^{b}) \Bigg].
\end{aligned}
\tag{5.112}
$$

It will prove more useful to us to work in momentum space by introducing creation and annihilation operators. We define annihilation operators for the massive bosons as

$$
\begin{aligned}
a_{\mathrm{L}z}(p) &= \frac{1}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\omega_p^{\mathrm{L}}}} \left(\omega_p^{\mathrm{L}} \bar{Z} + \frac{i}{2} P_{\bar{Z}} \right) e^{-ip\sigma}, \\
a_{\mathrm{R}z}(p) &= \frac{1}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\omega_p^{\mathrm{R}}}} \left(\omega_p^{\mathrm{R}} Z + \frac{i}{2} P_{Z} \right) e^{-ip\sigma}, \\
a_{\mathrm{L}y}(p) &= \frac{1}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\omega_p^{\mathrm{L}}}} \left(\omega_p^{\mathrm{L}} \bar{Y} + \frac{i}{2} P_{\bar{Y}} \right) e^{-ip\sigma}, \\
a_{\mathrm{R}y}(p) &= \frac{1}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\omega_p^{\mathrm{R}}}} \left(\omega_p^{\mathrm{R}} Y + \frac{i}{2} P_{Y} \right) e^{-ip\sigma},
\end{aligned}
\tag{5.113}
$$

and for the massless bosons as

$$
a_{\dot{a}a}(p) = \frac{1}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\tilde{\omega}_p}} \left(\tilde{\omega}_p X_{\dot{a}a} + \frac{i}{2} P_{\dot{a}a} \right) e^{-ip\sigma},
\tag{5.114}
$$

where $X_{\dot{a}a} = \left(X^{\dot{a}a} \right)^{\dagger}$. We also define annihilation operators for the massive fermions as

$$
\begin{aligned}
d_{\mathrm{L}\dot{a}}(p) &= +\frac{e^{+i\pi/4}}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\omega_p^{\mathrm{L}}}} \epsilon_{\dot{a}\dot{b}}(f_p^{\mathrm{L}} \eta_{\mathrm{L}}{}^{\dot{b}} + ig_p^{\mathrm{L}} \bar{\eta}_{\mathrm{R}}{}^{\dot{b}}) e^{-ip\sigma}, \\
d_{\mathrm{R}}{}^{\dot{a}}(p) &= -\frac{e^{+i\pi/4}}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\omega_p^{\mathrm{R}}}} \epsilon^{\dot{a}\dot{b}}(f_p^{\mathrm{R}} \eta_{\mathrm{R}\dot{b}} + ig_p^{\mathrm{R}} \bar{\eta}_{\mathrm{L}\dot{b}}) e^{-ip\sigma} ,
\end{aligned}
\tag{5.115}
$$

and for the massless fermions as

$$
\begin{aligned}
\tilde{d}_a(p) &= \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\tilde{\omega}_p}} \left(\tilde{f}_p \bar{\chi}_{+a} - i\tilde{g}_p \epsilon_{ab}\chi_-{}^{b} \right) e^{-ip\sigma}, \\
d_a(p) &= \frac{e^{+i\pi/4}}{\sqrt{2\pi}} \int \frac{\mathrm{d}\sigma}{\sqrt{\tilde{\omega}_p}} \left(\tilde{f}_p \epsilon_{ab}\chi_+{}^{b} - i\tilde{g}_p \bar{\chi}_{-b} \right) e^{-ip\sigma}.
\end{aligned}
\tag{5.116}
$$

Creation operators are then given by complex conjugation.

Our definitions of the annihilation operators above involve the dispersion parameters $\omega_p^{\rm L}$, $\omega_p^{\rm R}$ and $\tilde{\omega}_p$ and the fermion wavefunction parameters $f_p^{\rm L}$, $f_p^{\rm R}$ and $\tilde{f}_p$. In the absence of NS-NS flux, the excitations of the near-plane-wave limit would have the usual relativistic dispersion given as a function of the mass m of the excitation

$$\omega(m, p) = \sqrt{m^2 + p^2} \ . \tag{5.117}$$

However for non-zero q the massive dispersion is altered, and there are two distinct dispersion relations corresponding to the different excitations we have labelled left and right. These are given by

$$\omega_p^{\rm L} = \sqrt{\tilde{q}^2 + (p+q)^2} \ , \quad \omega_p^{\rm R} = \sqrt{\tilde{q}^2 + (p-q)^2} \ . \tag{5.118}$$

In choosing the fermion wavefunction parameters we want to ensure that the creation and annihilation operators all satisfy the canonical commutation and anticommutation relations

$$
\begin{aligned}
[a_{Lz}^\dagger(p_1), a_{Lz}(p_2)] &= [a_{Rz}^\dagger(p_1), a_{Rz}(p_2)] = \delta(p_1 - p_2), \\
[a_{Ly}^\dagger(p_1), a_{Ly}(p_2)] &= [a_{Ry}^\dagger(p_1), a_{Ry}(p_2)] = \delta(p_1 - p_2), \\
[a^{\dot{b}b\dagger}(p_1), a_{\dot{a}a}(p_2)] &= \delta_{\dot{a}}^{\dot{b}} \delta_a^b \delta(p_1 - p_2), \\
\{d_L^{\dot{a}\,\dagger}(p_1), d_{L\dot{b}}(p_2)\} &= \{d_{R\dot{b}}^\dagger(p_1), d_R^{\dot{a}}(p_2)\} = \delta_{\dot{b}}^{\dot{a}} \delta(p_1 - p_2), \\
\{\tilde{d}^{a\dagger}(p_1), \tilde{d}_b(p_2)\} &= \{d^{a\dagger}(p_1), d_b(p_2)\} = \delta_b^a \delta(p_1 - p_2)
\end{aligned}
\tag{5.119}
$$

These (anti)commutators hold provided we choose the wavefunction parameters to satisfy

$$(f_p^{\rm L})^2 + (g_p^{\rm L})^2 = \omega_p^{\rm L} \ , \quad (f_p^{\rm R})^2 + (g_p^{\rm R})^2 = \omega_p^{\rm R} \ , \quad (\tilde{f}_p)^2 + (\tilde{g}_p)^2 = \tilde{\omega}_p \tag{5.120}$$

and

$$f_p^{\rm L} g_{-p}^{\rm R} + f_{-p}^{\rm R} g_p^{\rm L} = \tilde{f}_p \tilde{g}_{-p} + \tilde{f}_{-p} \tilde{g}_p = 0 \ . \tag{5.121}$$

We make the following choice which satisfies these conditions:

$$
\begin{aligned}
f_p^{\rm L} &= \sqrt{\frac{1 + qp + \omega_p^{\rm L}}{2}} \ , & f_p^{\rm R} &= \sqrt{\frac{1 - qp + \omega_p^{\rm R}}{2}} \ , & \tilde{f}_p &= \sqrt{\frac{qp + \tilde{\omega}_p}{2}} \ , \\
g_p^{\rm L} &= -\frac{\tilde{q}p}{2 f_p^{\rm L}} \ , & g_p^{\rm R} &= -\frac{\tilde{q}p}{2 f_p^{\rm R}} \ , & \tilde{g}_p &= -\frac{\tilde{q}p}{2 \tilde{f}_p} \ .
\end{aligned}
\tag{5.122}
$$

In terms of the creation and annihilation operators defined above with

these choices of the wavefunction parameters, the supercharges are given by

$$
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} = \int \mathrm{d}p \bigg[(d_{\mathrm{L}}{}^{\dot{a}\dagger} a_{\mathrm{L}y} + \epsilon^{\dot{a}\dot{b}} a_{\mathrm{L}z}^{\dagger} d_{\mathrm{L}\dot{b}}) f_p^{\mathrm{L}} + (a_{\mathrm{R}y}^{\dagger} d_{\mathrm{R}}{}^{\dot{a}} + \epsilon^{\dot{a}\dot{b}} d_{\mathrm{R}\dot{b}}^{\dagger} a_{\mathrm{R}z}) g_p^{\mathrm{R}}
$$

$$
+ (\epsilon^{\dot{a}\dot{b}} \tilde{d}^{a\dagger} a_{\dot{b}a} + a^{\dot{a}a\dagger} d_a) \tilde{f}_p \bigg],
$$

$$
\mathbf{Q}_{\mathrm{R}\dot{a}} = \int \mathrm{d}p \bigg[(d_{\mathrm{R}\dot{a}}^{\dagger} a_{\mathrm{R}y} - \epsilon_{\dot{a}\dot{b}} a_{\mathrm{R}z}^{\dagger} d_{\mathrm{R}}{}^{\dot{b}}) f_p^{\mathrm{R}} + (a_{\mathrm{L}y}^{\dagger} d_{\mathrm{L}\dot{a}} - \epsilon_{\dot{a}\dot{b}} d_{\mathrm{L}}{}^{\dot{b}\dagger} a_{\mathrm{L}z}) g_p^{\mathrm{L}}
$$

$$
+ (d^{a\dagger} a_{\dot{a}a} - \epsilon_{\dot{a}\dot{b}} a^{\dot{b}a\dagger} \tilde{d}_a) \tilde{g}_p \bigg]. \tag{5.123}
$$

We can now confirm that these supercharges obey the algebra (5.100) with the charges given in terms of oscillators as follows. The Hamiltonian is given by

$$
\mathbf{H} = \int \mathrm{d}p \bigg[\left(a_{\mathrm{L}z}^{\dagger} a_{\mathrm{L}z} + a_{\mathrm{L}y}^{\dagger} a_{\mathrm{L}y} + d_{\mathrm{L}}{}^{\dot{a}\dagger} d_{\mathrm{L}\dot{a}} \right) \left((f_p^{\mathrm{L}})^2 + (g_p^{\mathrm{L}})^2 \right)
$$

$$
+ \left(a_{\mathrm{R}z}^{\dagger} a_{\mathrm{R}z} + a_{\mathrm{R}y}^{\dagger} a_{\mathrm{R}y} + d_{\mathrm{R}}{}^{\dot{a}\dagger} d_{\mathrm{R}\dot{a}} \right) \left((f_p^{\mathrm{R}})^2 + (g_p^{\mathrm{R}})^2 \right)
$$

$$
+ \left(a_{\dot{a}a}^{\dagger} a^{\dot{a}a} + d^{a\dagger} d_a + \tilde{d}^{a\dagger} \tilde{d}_a \right) \left((\tilde{f}_p)^2 + (\tilde{g}_p)^2 \right) \bigg], \tag{5.124}
$$

the mass term is given by

$$
\mathbf{M} = \int \mathrm{d}p \bigg[\left(a_{\mathrm{L}z}^{\dagger} a_{\mathrm{L}z} + a_{\mathrm{L}y}^{\dagger} a_{\mathrm{L}y} + d_{\mathrm{L}}{}^{\dot{a}\dagger} d_{\mathrm{L}\dot{a}} \right) \left((f_p^{\mathrm{L}})^2 - (g_p^{\mathrm{L}})^2 \right)
$$

$$
+ \left(a_{\mathrm{R}z}^{\dagger} a_{\mathrm{R}z} + a_{\mathrm{R}y}^{\dagger} a_{\mathrm{R}y} + d_{\mathrm{R}}{}^{\dot{a}\dagger} d_{\mathrm{R}\dot{a}} \right) \left((f_p^{\mathrm{R}})^2 - (g_p^{\mathrm{R}})^2 \right)
$$

$$
+ \left(a_{\dot{a}a}^{\dagger} a^{\dot{a}a} + d^{a\dagger} d_a + \tilde{d}^{a\dagger} \tilde{d}_a \right) \left((\tilde{f}_p)^2 - (\tilde{g}_p)^2 \right) \bigg], \tag{5.125}
$$

and the central charges are

$$
\mathbf{C} = \overline{\mathbf{C}} = \int \mathrm{d}p \bigg[\left(a_{\mathrm{L}z}^{\dagger} a_{\mathrm{L}z} + a_{\mathrm{L}y}^{\dagger} a_{\mathrm{L}y} + d_{\mathrm{L}}{}^{\dot{a}\dagger} d_{\mathrm{L}\dot{a}} \right) f_p^{\mathrm{L}} g_p^{\mathrm{L}}
$$

$$
+ \left(a_{\mathrm{R}z}^{\dagger} a_{\mathrm{R}z} + a_{\mathrm{R}y}^{\dagger} a_{\mathrm{R}y} + d_{\mathrm{R}}{}^{\dot{a}\dagger} d_{\mathrm{R}\dot{a}} \right) f_p^{\mathrm{R}} g_p^{\mathrm{R}}
$$

$$
+ \left(a_{\dot{a}a}^{\dagger} a^{\dot{a}a} + d^{a\dagger} d_a + \tilde{d}^{a\dagger} \tilde{d}_a \right) \tilde{f}_p \tilde{g}_p \bigg]. \tag{5.126}
$$

We define the 8+8 massive and massless states

$$
|Z^{\mathrm{L},\mathrm{R}}\rangle = a_{\mathrm{L},\mathrm{R}\,z}^{\dagger} |0\rangle, \quad |Y^{\mathrm{L},\mathrm{R}}\rangle = a_{\mathrm{L},\mathrm{R}\,y}^{\dagger} |0\rangle,
$$

$$
|\eta^{\mathrm{L}\dot{a}}\rangle = d_{\mathrm{L}}{}^{\dot{a}\dagger} |0\rangle, \quad |\eta^{\mathrm{R}}_{\dot{a}}\rangle = d_{\mathrm{R}\dot{a}}^{\dagger} |0\rangle, \tag{5.127}
$$

$$
|T^{\dot{a}a}\rangle = a^{\dot{a}a\dagger} |0\rangle, \quad |\chi^a\rangle = d^{a\,\dagger} |0\rangle, \quad |\tilde{\chi}^a\rangle = \tilde{d}^{a\,\dagger} |0\rangle.
$$

The action of the supercharges (5.123) on these states gives rise to three irreducible representations. We have one representation acting on the "left"

massive states, with

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|Y_p^{\mathrm{L}}\right\rangle &= f_p^{\mathrm{L}} \left|\eta_p^{\mathrm{L}\dot{a}}\right\rangle, & \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|\eta_p^{\mathrm{L}\dot{b}}\right\rangle &= \epsilon^{\dot{a}\dot{b}} f_p^{\mathrm{L}} \left|Z_p^{\mathrm{L}}\right\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|Z_p^{\mathrm{L}}\right\rangle &= -\epsilon_{\dot{a}\dot{b}} f_p^{\mathrm{L}} \left|\eta_p^{\mathrm{L}\dot{b}}\right\rangle, & \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|\eta_p^{\mathrm{L}\dot{b}}\right\rangle &= \delta_{\dot{a}}{}^{\dot{b}} f_p^{\mathrm{L}} \left|Y_p^{\mathrm{L}}\right\rangle, \\
\mathbf{Q}_{\mathrm{R}\dot{a}} \left|Z_p^{\mathrm{L}}\right\rangle &= -\epsilon_{\dot{a}\dot{b}} g_p^{\mathrm{L}} \left|\eta_p^{\mathrm{L}\dot{b}}\right\rangle, & \mathbf{Q}_{\mathrm{R}\dot{a}} \left|\eta_p^{\mathrm{L}\dot{b}}\right\rangle &= \delta_{\dot{a}}{}^{\dot{b}} g_p^{\mathrm{L}} \left|Y_p^{\mathrm{L}}\right\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|Y_p^{\mathrm{L}}\right\rangle &= g_p^{\mathrm{L}} \left|\eta_p^{\mathrm{L}\dot{a}}\right\rangle, & \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|\eta_p^{\mathrm{L}\dot{b}}\right\rangle &= \epsilon^{\dot{a}\dot{b}} g_p^{\mathrm{L}} \left|Z_p^{\mathrm{L}}\right\rangle,
\end{aligned}
\tag{5.128}
$$

and

$$
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|Z_p^{\mathrm{L}}\right\rangle = \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|Y_p^{\mathrm{L}}\right\rangle = \mathbf{Q}_{\mathrm{R}\dot{a}} \left|Y_p^{\mathrm{L}}\right\rangle = \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|Z_p^{\mathrm{L}}\right\rangle = 0 .
\tag{5.129}
$$

We have similarly a "right" massive representation given by

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|Z_p^{\mathrm{R}}\right\rangle &= g_p^{\mathrm{R}} \left|\eta_p^{\mathrm{R}\dot{a}}\right\rangle, & \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|\eta_p^{\mathrm{R}\dot{b}}\right\rangle &= -\epsilon^{\dot{a}\dot{b}} g_p^{\mathrm{R}} \left|Y_p^{\mathrm{R}}\right\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|Y_p^{\mathrm{R}}\right\rangle &= \epsilon_{\dot{a}\dot{b}} g_p^{\mathrm{R}} \left|\eta_p^{\mathrm{R}\dot{b}}\right\rangle, & \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|\eta_p^{\mathrm{R}\dot{b}}\right\rangle &= \delta_{\dot{a}}{}^{\dot{b}} g_p^{\mathrm{R}} \left|Z_p^{\mathrm{R}}\right\rangle, \\
\mathbf{Q}_{\mathrm{R}\dot{a}} \left|Y_p^{\mathrm{R}}\right\rangle &= \epsilon_{\dot{a}\dot{b}} f_p^{\mathrm{R}} \left|\eta_p^{\mathrm{R}\dot{b}}\right\rangle, & \mathbf{Q}_{\mathrm{R}\dot{a}} \left|\eta_p^{\mathrm{R}\dot{b}}\right\rangle &= \delta_{\dot{a}}{}^{\dot{b}} f_p^{\mathrm{R}} \left|Z_p^{\mathrm{R}}\right\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|Z_p^{\mathrm{R}}\right\rangle &= f_p^{\mathrm{R}} \left|\eta_p^{\mathrm{R}\dot{a}}\right\rangle, & \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|\eta_p^{\mathrm{R}\dot{b}}\right\rangle &= -\epsilon^{\dot{a}\dot{b}} f_p^{\mathrm{R}} \left|Y_p^{\mathrm{R}}\right\rangle,
\end{aligned}
\tag{5.130}
$$

and

$$
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|Y_p^{\mathrm{R}}\right\rangle = \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|Z_p^{\mathrm{R}}\right\rangle = \mathbf{Q}_{\mathrm{R}\dot{a}} \left|Z_p^{\mathrm{R}}\right\rangle = \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|Y_p^{\mathrm{R}}\right\rangle = 0 ,
\tag{5.131}
$$

and a single massless representation given by

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|T_p^{\dot{b}a}\right\rangle &= \epsilon^{\dot{a}\dot{b}} \tilde{f}_p \left|\widetilde{\chi}_p^{a}\right\rangle, & \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|\chi_p^{a}\right\rangle &= \tilde{f}_p \left|T_p^{\dot{a}a}\right\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|\widetilde{\chi}_p^{a}\right\rangle &= -\epsilon_{\dot{a}\dot{b}} \tilde{f}_p \left|T_p^{\dot{b}a}\right\rangle, & \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|T_p^{\dot{b}a}\right\rangle &= \delta_{\dot{a}}{}^{\dot{b}} \tilde{f}_p \left|\chi_p^{a}\right\rangle, \\
\mathbf{Q}_{\mathrm{R}\dot{a}} \left|T_p^{\dot{b}a}\right\rangle &= \delta_{\dot{a}}{}^{\dot{b}} \tilde{g}_p \left|\chi_p^{a}\right\rangle, & \mathbf{Q}_{\mathrm{R}\dot{a}} \left|\widetilde{\chi}_p^{a}\right\rangle &= -\epsilon_{\dot{a}\dot{b}} \tilde{g}_p \left|T_p^{\dot{b}a}\right\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|\chi_p^{a}\right\rangle &= \tilde{g}_p \left|T_p^{\dot{a}a}\right\rangle, & \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|T_p^{\dot{b}a}\right\rangle &= \epsilon^{\dot{a}\dot{b}} \tilde{g}_p \left|\widetilde{\chi}_p^{a}\right\rangle,
\end{aligned}
\tag{5.132}
$$

and

$$
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \left|\widetilde{\chi}_p^{a}\right\rangle = \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \left|\chi_p^{a}\right\rangle = \mathbf{Q}_{\mathrm{R}\dot{a}} \left|\chi_p^{a}\right\rangle = \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \left|\widetilde{\chi}_p^{a}\right\rangle = 0 .
\tag{5.133}
$$

We can read off the values of the charges $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{C}$ on the states directly from equations (5.124), (5.125) and (5.126). $\mathbf{H}$ takes values

$$
\begin{aligned}
\mathbf{H} \left|Z^{\mathrm{L}}, Y^{\mathrm{L}}, \eta^{\mathrm{L}\dot{a}}\right\rangle &= \sqrt{\tilde{q}^2 + (p+q)^2} \; \left|Z^{\mathrm{L}}, Y^{\mathrm{L}}, \eta^{\mathrm{L}\dot{a}}\right\rangle, \\
\mathbf{H} \left|Z^{\mathrm{R}}, Y^{\mathrm{R}}, \eta^{\mathrm{R}\dot{a}}\right\rangle &= \sqrt{\tilde{q}^2 + (p-q)^2} \; \left|Z^{\mathrm{R}}, Y^{\mathrm{R}}, \eta^{\mathrm{R}\dot{a}}\right\rangle, \\
\mathbf{H} \left|T_p^{\dot{b}a}, \chi_p^{a}, \widetilde{\chi}_p^{a}\right\rangle &= \sqrt{p^2} \; \left|T_p^{\dot{b}a}, \chi_p^{a}, \widetilde{\chi}_p^{a}\right\rangle .
\end{aligned}
\tag{5.134}
$$

$\mathbf{M}$ takes values

$$
\begin{aligned}
\mathbf{M} \left|Z^{\mathrm{L}}, Y^{\mathrm{L}}, \eta^{\mathrm{L}\dot{a}}\right\rangle &= (qp+1) \; \left|Z^{\mathrm{L}}, Y^{\mathrm{L}}, \eta^{\mathrm{L}\dot{a}}\right\rangle, \\
\mathbf{M} \left|Z^{\mathrm{R}}, Y^{\mathrm{R}}, \eta^{\mathrm{R}\dot{a}}\right\rangle &= (qp-1) \; \left|Z^{\mathrm{R}}, Y^{\mathrm{R}}, \eta^{\mathrm{R}\dot{a}}\right\rangle, \\
\mathbf{M} \left|T_p^{\dot{b}a}, \chi_p^{a}, \widetilde{\chi}_p^{a}\right\rangle &= qp \; \left|T_p^{\dot{b}a}, \chi_p^{a}, \widetilde{\chi}_p^{a}\right\rangle
\end{aligned}
\tag{5.135}
$$

which we will write hereafter as

$$\mathbf{M} \left| \mathcal{X}_p \right\rangle = (qp + m) \left| \mathcal{X}_p \right\rangle \qquad (5.136)$$

with m suitably defined according to whether the excitation $\left| \mathcal{X}_p \right\rangle$ is left/right-massive or massless. Finally the value of the central charge is the same for all excitations,

$$\mathbf{C} \left| \mathcal{X}_p \right\rangle = \overline{\mathbf{C}} \left| \mathcal{X}_p \right\rangle = -\frac{\tilde{q}p}{2} \left| \mathcal{X}_p \right\rangle \ . \qquad (5.137)$$

We note that acting on all excitations, the charges satisfy the condition

$$\mathbf{H}^2 = \mathbf{M}^2 + 4\mathbf{C}\overline{\mathbf{C}} \ . \qquad (5.138)$$

This is the shortening condition, meaning that the representations described above are short representations.

5.5.2 Exact representations

Having written down near-plane-wave representations of the algebra $\mathcal{A}$, we now deform these to produce exact representations. We first need the exact values for the charges $\mathbf{H}$, $\mathbf{M}$, $\mathbf{C}$ and $\overline{\mathbf{C}}$. In section 5.4 we calculated the values for $\mathbf{C}, \overline{\mathbf{C}}$ which are exact in worldsheet momentum using the hybrid expansion. We need also to allow for corrections at finite order in λ. The central charges have an overall normalisation proportional to the coupling constant $h(\lambda, \tilde{q})$. From the worldsheet calculations we know this is given to leading order in large λ by

$$h(\lambda, \tilde{q}) = \frac{\sqrt{\lambda}}{2\pi} \tilde{q} + \dots \qquad (5.139)$$

but we allow for subleading terms at finite-λ. The exact central charges are

$$\mathbf{C} \left| \mathcal{X}_p \right\rangle = +\frac{ih}{2} \left(e^{+ip} - 1 \right) \left| \mathcal{X}_p \right\rangle \ , \qquad \overline{\mathbf{C}} \left| \mathcal{X}_p \right\rangle = -\frac{ih}{2} \left(e^{-ip} - 1 \right) \left| \mathcal{X}_p \right\rangle \ . \quad (5.140)$$

The worldsheet calculations of section 5.4 showed that the charge $\mathbf{M}$ receives no corrections at quartic order in fields. From this we expect the value of $\mathbf{M}$ in equation (5.136) to be already exact in momentum. We can strengthen this argument, and argue further that it also receives no corrections in $\frac{1}{\sqrt{\lambda}}$ either by noting that $\mathbf{M}$ plays the role of an angular momentum in the worldsheet theory. Thus we expect it to be quantized with integer values. When we reinsert the factor of worldsheet momentum in equation (5.136) we have

$$\mathbf{M} \left| \mathcal{X}_p \right\rangle = \left(m + \frac{\sqrt{\lambda}}{2\pi} qp \right) \left| \mathcal{X}_p \right\rangle = \left(m + \frac{kp}{2\pi} \right) \left| \mathcal{X}_p \right\rangle \ , \qquad (5.141)$$

where $k \in \mathbb{Z}$ is the WZW level. On-shell, a one-particle state with non-zero winding has momentum $p = 2\pi w$ where $w \in \mathbb{Z}$ is the winding number,

and so $\mathbf{M}$ clearly takes integer value on such a state. We effectively have a second coupling $\mathfrak{k}$ related to the WZW level by $k = 2\pi\mathfrak{k}$. $\mathfrak{k}(\lambda)$ takes the same form (5.139) as $h(\lambda)$ at large λ but unlike h cannot receive any corrections consistent with the integer quantization of $\mathbf{M}$ on a one-particle on-shell state. Also, $\mathfrak{k}$ cannot have any non-linear momentum-dependence consistently with the action of $\mathbf{M}$ on a multiparticle state.

Having exact forms for the charges $\mathbf{M}$, $\mathbf{C}$ and $\overline{\mathbf{C}}$, we find the $\mathbf{H}$ by requiring that the shortening condition (5.138) still holds for exact representations, which must be true since the representations cannot change size in going to the near-plane-wave limit. Thus we have $\mathbf{H}$ given by the all-loop dispersion relation

$$\mathbf{H} |\mathcal{X}_p\rangle = \sqrt{(m + \mathfrak{k}p)^2 + 4h^2 \sin^2\left(\frac{p}{2}\right)} |\mathcal{X}_p\rangle \qquad (5.142)$$

To deform the near-plane-wave representations we look for representation coefficients which reduce to $f_p^{\mathrm{L,R}}$, $g_p^{\mathrm{L,R}}$, $\tilde{f}_p$, $\tilde{g}_p$ in the near-plane-wave limit and produce the exact charges we have constructed. We absorb the mass $|m|$ of the representation into the definition of the representation coefficients instead of defining distinct coefficients for massive and massless representations. Hence we just have left- and right-coefficients. Consistently with our choices in the near-plane-wave limit, we choose the massless representation to be given as a $|m| \to 0$ limit of the left-massive representation. We could find an equivalent massless representation from the limit of the right-massive representation. While the coefficients f_p^{L} etc. were all real, we should no longer assume this since we have seen that the exact central charge is not real. As such we introduce coefficients $a_p^{\mathrm{L,R}}$, $b_p^{\mathrm{L,R}}$ and their complex conjugates. We produce the exact left-massive representation by replacing f_p^{L} everywhere in (5.128) with a_p^{L} or $\bar{a}_p^{\mathrm{L}}$ and g_p^{L} everywhere by b_p^{L} or $\bar{b}_p^{\mathrm{L}}$. Our choices are restricted by the fact that the energy must be positive-definite. In fact this fixes the choices uniquely up to an overall arbitrary choice of whether the product $a_p^{\mathrm{L}} b_p^{\mathrm{L}}$ gives $\mathbf{C}$ or $\overline{\mathbf{C}}$. We then have the exact left-massive representation

$$
\begin{aligned}
&\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} |Y_p^{\mathrm{L}}\rangle = a_p^{\mathrm{L}} |\eta_p^{\mathrm{L}\dot{a}}\rangle, &\qquad &\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} |\eta_p^{\mathrm{L}\dot{b}}\rangle = \epsilon^{\dot{a}\dot{b}} a_p^{\mathrm{L}} |Z_p^{\mathrm{L}}\rangle, \\
&\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} |Z_p^{\mathrm{L}}\rangle = -\epsilon_{\dot{a}\dot{b}} \bar{a}_p^{\mathrm{L}} |\eta_p^{\mathrm{L}\dot{b}}\rangle, &\qquad &\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} |\eta_p^{\mathrm{L}\dot{b}}\rangle = \delta_{\dot{a}}{}^{\dot{b}} \bar{a}_p^{\mathrm{L}} |Y_p^{\mathrm{L}}\rangle, \\
&\mathbf{Q}_{\mathrm{R}\dot{a}} |Z_p^{\mathrm{L}}\rangle = -\epsilon_{\dot{a}\dot{b}} b_p^{\mathrm{L}} |\eta_p^{\mathrm{L}\dot{b}}\rangle, &\qquad &\mathbf{Q}_{\mathrm{R}\dot{a}} |\eta_p^{\mathrm{L}\dot{b}}\rangle = \delta_{\dot{a}}{}^{\dot{b}} b_p^{\mathrm{L}} |Y_p^{\mathrm{L}}\rangle, \\
&\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} |Y_p^{\mathrm{L}}\rangle = \bar{b}_p^{\mathrm{L}} |\eta_p^{\mathrm{L}\dot{a}}\rangle, &\qquad &\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} |\eta_p^{\mathrm{L}\dot{b}}\rangle = \epsilon^{\dot{a}\dot{b}} \bar{b}_p^{\mathrm{L}} |Z_p^{\mathrm{L}}\rangle.
\end{aligned}
\qquad (5.143)
$$

Similarly we have the exact right-massive representation

$$
\begin{aligned}
&\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} |Z_p^{\mathrm{R}}\rangle = b_p^{\mathrm{R}} |\eta_p^{\mathrm{R}\dot{a}}\rangle, &\qquad &\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} |\eta_p^{\mathrm{R}\dot{b}}\rangle = -\epsilon^{\dot{a}\dot{b}} b_p^{\mathrm{R}} |Y_p^{\mathrm{R}}\rangle, \\
&\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} |Y_p^{\mathrm{R}}\rangle = \epsilon_{\dot{a}\dot{b}} \bar{b}_p^{\mathrm{R}} |\eta_p^{\mathrm{R}\dot{b}}\rangle, &\qquad &\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} |\eta_p^{\mathrm{R}\dot{b}}\rangle = \delta_{\dot{a}}{}^{\dot{b}} \bar{b}_p^{\mathrm{R}} |Z_p^{\mathrm{R}}\rangle, \\
&\mathbf{Q}_{\mathrm{R}\dot{a}} |Y_p^{\mathrm{R}}\rangle = \epsilon_{\dot{a}\dot{b}} a_p^{\mathrm{R}} |\eta_p^{\mathrm{R}\dot{b}}\rangle, &\qquad &\mathbf{Q}_{\mathrm{R}\dot{a}} |\eta_p^{\mathrm{R}\dot{b}}\rangle = \delta_{\dot{a}}{}^{\dot{b}} a_p^{\mathrm{R}} |Z_p^{\mathrm{R}}\rangle, \\
&\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} |Z_p^{\mathrm{R}}\rangle = \bar{a}_p^{\mathrm{R}} |\eta_p^{\mathrm{R}\dot{a}}\rangle, &\qquad &\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} |\eta_p^{\mathrm{R}\dot{b}}\rangle = -\epsilon^{\dot{a}\dot{b}} \bar{a}_p^{\mathrm{R}} |Y_p^{\mathrm{R}}\rangle,
\end{aligned}
\qquad (5.144)
$$

and the exact massless representation

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \, |T_p^{\dot{b}a}\rangle &= \epsilon^{\dot{a}\dot{b}} a_p^{\mathrm{L}} \, |\widetilde{\chi}_p^a\rangle, &\qquad \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \, |\chi_p^a\rangle &= a_p^{\mathrm{L}} \, |T_p^{\dot{a}a}\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \, |\widetilde{\chi}_p^a\rangle &= -\epsilon_{\dot{a}\dot{b}} \bar{a}_p^{\mathrm{L}} \, |T_p^{\dot{b}a}\rangle, &\qquad \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}} \, |T_p^{\dot{b}a}\rangle &= \delta_{\dot{a}}^{\ \dot{b}} \bar{a}_p^{\mathrm{L}} \, |\chi_p^a\rangle, \\
\mathbf{Q}_{\mathrm{R}\dot{a}} \, |T_p^{\dot{b}a}\rangle &= \delta_{\dot{a}}^{\ \dot{b}} b_p^{\mathrm{L}} \, |\chi_p^a\rangle, &\qquad \mathbf{Q}_{\mathrm{R}\dot{a}} \, |\widetilde{\chi}_p^a\rangle &= -\epsilon_{\dot{a}\dot{b}} b_p^{\mathrm{L}} \, |T_p^{\dot{b}a}\rangle, \\
\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \, |\chi_p^a\rangle &= \bar{b}_p^{\mathrm{L}} \, |T_p^{\dot{a}a}\rangle, &\qquad \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}} \, |T_p^{\dot{b}a}\rangle &= \epsilon^{\dot{a}\dot{b}} \bar{b}_p^{\mathrm{L}} \, |\widetilde{\chi}_p^a\rangle.
\end{aligned}
\tag{5.145}
$$

We make a choice of representation parameters that produce the correct charges as follows. We take

$$
\begin{aligned}
a_p^{\mathrm{L},\mathrm{R}} &= \eta_p^{\mathrm{L},\mathrm{R}} \, e^{i\xi}, &\qquad \bar{a}_p^{\mathrm{L},\mathrm{R}} &= \eta_p^{\mathrm{L},\mathrm{R}} \, e^{-ip/2} e^{-i\xi}, \\
b_p^{\mathrm{L},\mathrm{R}} &= -\frac{\eta_p^{\mathrm{L},\mathrm{R}}}{x_{\mathrm{L},\mathrm{R}\,p}^-} e^{-ip/2} e^{i\xi}, &\qquad \bar{b}_p^{\mathrm{L},\mathrm{R}} &= -\frac{\eta_p^{\mathrm{L},\mathrm{R}}}{x_{\mathrm{L},\mathrm{R}\,p}^+} e^{-i\xi}
\end{aligned}
\tag{5.146}
$$

where

$$
\eta_p^{\mathrm{L},\mathrm{R}} = e^{ip/4} \sqrt{\frac{ih}{2}(x_{\mathrm{L},\mathrm{R}\,p}^- - x_{\mathrm{L},\mathrm{R}\,p}^+)},
\tag{5.147}
$$

and $x_{\mathrm{L}}^\pm$, $x_{\mathrm{R}}^\pm$ are two sets of so-called Zhukovski variables. We define them here to satisfy

$$
\begin{aligned}
\frac{x_{\mathrm{L}\,p}^+}{x_{\mathrm{L}\,p}^-} &= e^{ip}, &\qquad x_{\mathrm{L}\,p}^+ + \frac{1}{x_{\mathrm{L}\,p}^+} - x_{\mathrm{L}\,p}^- - \frac{1}{x_{\mathrm{L}\,p}^-} &= \frac{2i(|m| + k\,p)}{h}, \\
\frac{x_{\mathrm{R}\,p}^+}{x_{\mathrm{R}\,p}^-} &= e^{ip}, &\qquad x_{\mathrm{R}\,p}^+ + \frac{1}{x_{\mathrm{R}\,p}^+} - x_{\mathrm{R}\,p}^- - \frac{1}{x_{\mathrm{R}\,p}^-} &= \frac{2i(|m| - k\,p)}{h}.
\end{aligned}
\tag{5.148}
$$

An explicit solution to this is

$$
\begin{aligned}
x_{\mathrm{L}\,p}^\pm &= \frac{(|m| + kp) + \sqrt{(|m| + kp)^2 + 4h^2 \sin^2(\frac{p}{2})}}{2h \sin(\frac{p}{2})} e^{\pm \frac{i}{2}p}, \\
x_{\mathrm{R}\,p}^\pm &= \frac{(|m| - kp) + \sqrt{(|m| - kp)^2 + 4h^2 \sin^2(\frac{p}{2})}}{2h \sin(\frac{p}{2})} e^{\pm \frac{i}{2}p},
\end{aligned}
\tag{5.149}
$$

We end this section with two tensor product representations that we will need in the S-matrix calculations of the next section. First, we note that we can obtain the $\mathfrak{psu}(1|1)^4$ representations (5.143), (5.144) and (5.145) as tensor products of $\mathfrak{su}(1|1)^2$ representations as follows. We consider one $\mathfrak{su}(1|1)^2$ representation which we label ϱ_{L} given by

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}} \, |\phi_p^{\mathrm{L}}\rangle &= a_p^{\mathrm{L}} \, |\psi_p^{\mathrm{L}}\rangle, &\qquad \overline{\mathbf{Q}}_{\mathrm{L}} \, |\psi_p^{\mathrm{L}}\rangle &= \bar{a}_p^{\mathrm{L}} \, |\phi_p^{\mathrm{L}}\rangle, \\
\mathbf{Q}_{\mathrm{R}} \, |\psi_p^{\mathrm{L}}\rangle &= b_p^{\mathrm{L}} \, |\phi_p^{\mathrm{L}}\rangle, &\qquad \overline{\mathbf{Q}}_{\mathrm{R}} \, |\phi_p^{\mathrm{L}}\rangle &= \bar{b}_p^{\mathrm{L}} \, |\psi_p^{\mathrm{L}}\rangle,
\end{aligned}
\tag{5.150}
$$

and another which we label ϱ_{R} given by

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}} \, |\psi_p^{\mathrm{R}}\rangle &= b_p^{\mathrm{R}} \, |\phi_p^{\mathrm{R}}\rangle, &\qquad \overline{\mathbf{Q}}_{\mathrm{L}} \, |\phi_p^{\mathrm{R}}\rangle &= \bar{b}_p^{\mathrm{R}} \, |\psi_p^{\mathrm{R}}\rangle, \\
\mathbf{Q}_{\mathrm{R}} \, |\phi_p^{\mathrm{R}}\rangle &= a_p^{\mathrm{R}} \, |\psi_p^{\mathrm{R}}\rangle, &\qquad \overline{\mathbf{Q}}_{\mathrm{R}} \, |\psi_p^{\mathrm{R}}\rangle &= \bar{a}_p^{\mathrm{R}} \, |\phi_p^{\mathrm{R}}\rangle.
\end{aligned}
\tag{5.151}
$$

We also define a representation $\tilde{\varrho}_L$ which is similar to ϱ_L with fermionic and bosonic states interchanged,

$$
\begin{aligned}
\mathbf{Q}_L \,|\tilde{\psi}_p^L\rangle &= a_p^L \,|\tilde{\phi}_p^L\rangle\,, & \overline{\mathbf{Q}}_L \,|\tilde{\phi}_p^L\rangle &= \bar{a}_p^L \,|\tilde{\psi}_p^L\rangle\,, \\
\mathbf{Q}_R \,|\tilde{\phi}_p^L\rangle &= b_p^L \,|\tilde{\psi}_p^L\rangle\,, & \overline{\mathbf{Q}}_R \,|\tilde{\psi}_p^L\rangle &= \bar{b}_p^L \,|\tilde{\phi}_p^L\rangle\,,
\end{aligned}
\tag{5.152}
$$

We obtain the $\mathfrak{psu}(1|1)^4$ representations by taking appropriate tensor products of these $\mathfrak{su}(1|1)^2$ representations. We define operators of the tensor product $\left(\mathfrak{su}(1|1)^2\right)^2$ by

$$
\begin{aligned}
\mathbf{Q}_L{}^1 &= \mathbf{Q}_L \otimes \mathbf{1}\,, & \mathbf{Q}_L{}^2 &= \mathbf{1} \otimes \mathbf{Q}_L\,, & \mathbf{Q}_{R1} &= \mathbf{Q}_R \otimes \mathbf{1}\,, & \mathbf{Q}_{R2} &= \mathbf{1} \otimes \mathbf{Q}_R\,, \\
\overline{\mathbf{Q}}_{L1} &= \overline{\mathbf{Q}}_L \otimes \mathbf{1}\,, & \overline{\mathbf{Q}}_{L2} &= \mathbf{1} \otimes \overline{\mathbf{Q}}_L\,, & \overline{\mathbf{Q}}_R{}^1 &= \overline{\mathbf{Q}}_R \otimes \mathbf{1}\,, & \overline{\mathbf{Q}}_R{}^2 &= \mathbf{1} \otimes \overline{\mathbf{Q}}_R\,,
\end{aligned}
\tag{5.153}
$$

Then we obtain the representation (5.143) from $\varrho_L \otimes \varrho_L$ with states given by

$$
\begin{aligned}
Y^L &= \phi^L \otimes \phi^L\,, & \eta^{L1} &= \psi^L \otimes \phi^L\,, \\
Z^L &= \psi^L \otimes \psi^L\,, & \eta^{L2} &= \phi^L \otimes \psi^L\,.
\end{aligned}
\tag{5.154}
$$

The representation (5.144) is obtained from $\varrho_R \otimes \varrho_R$ with states

$$
\begin{aligned}
Y^R &= \phi^R \otimes \phi^R\,, & \eta^R{}_1 &= \psi^R \otimes \phi^R\,, \\
Z^R &= \psi^R \otimes \psi^R & \eta^R{}_2 &= \phi^R \otimes \psi^R\,.
\end{aligned}
\tag{5.155}
$$

The massless representations (5.145) are obtained from $\left(\varrho_L \otimes \tilde{\varrho}_L\right)^{\oplus 2}$ with states

$$
\begin{aligned}
T^{1a} &= \left(\psi^L \otimes \tilde{\psi}^L\right)^a\,, & \widetilde{\chi}^a &= \left(\psi^L \otimes \tilde{\phi}^L\right)^a\,, \\
T^{2a} &= \left(\phi^L \otimes \tilde{\phi}^L\right)^a\,, & \chi^a &= \left(\phi^L \otimes \tilde{\psi}^L\right)^a\,.
\end{aligned}
\tag{5.156}
$$

In the next section we will first construct S-matrices which are invariant under these $\mathfrak{su}(1|1)^2$ representations and then the full $\mathfrak{psu}(1|1)^4$-invariant S-matrices will follow from taking appropriate tensor products.

We also need to describe the two-particle representations of the algebra. We construct these as tensor products of one-particle representations. A complicating factor is that we need to pick the phase ξ which appears in the representations parameters in equation (5.146) appropriately. For the one-particle representations we can choose $\xi = 0$. However, for a two-particle representation with two phases ξ_1 and ξ_2, the choice $\xi_1 = \xi_2 = 0$ is not consistent with the correct values for the central charges. Given a general two-particle state $|\mathcal{X}_p \mathcal{Y}_q\rangle$, we want the central charge to take the same form as before but with the one-particle momentum replaced with the total momentum $p + q$. That is, we require

$$
\mathbf{C} \,|\mathcal{X}_p \mathcal{Y}_q\rangle = \frac{ih}{2}\left(e^{i(p+q)} - 1\right)|\mathcal{X}_p \mathcal{Y}_q\rangle\,.
\tag{5.157}
$$

This requires us to choose the two phases ξ_1 and ξ_2 as either

$$\xi_1 = 0 \ , \qquad \xi_2 = p/2 \ , \tag{5.158}$$

or

$$\xi_1 = q/2 \ , \qquad \xi_2 = 0 \ . \tag{5.159}$$

We use the first choice. The two-particle supercharges $\mathbf{Q}_{(12)}$ are then defined by

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}{}_{(12)}(p,q) &= \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}(p) \otimes \mathbf{1} + e^{+\frac{i}{2}p}\Sigma \otimes \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}(q) \, , \\
\mathbf{Q}_{\mathrm{R}\dot{a}(12)}(p,q) &= \mathbf{Q}_{\mathrm{R}\dot{a}}(p) \otimes \mathbf{1} + e^{+\frac{i}{2}p}\Sigma \otimes \mathbf{Q}_{\mathrm{R}\dot{a}}(q) \, , \\
\overline{\mathbf{Q}}_{\mathrm{L}\dot{a}(12)}(p,q) &= \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}}(p) \otimes \mathbf{1} + e^{-\frac{i}{2}p}\Sigma \otimes \overline{\mathbf{Q}}_{\mathrm{L}\dot{a}}(q) \, , \\
\overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}}{}_{(12)}(p,q) &= \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}}(p) \otimes \mathbf{1} + e^{-\frac{i}{2}p}\Sigma \otimes \overline{\mathbf{Q}}_{\mathrm{R}}{}^{\dot{a}}(q) \, ,
\end{aligned}
\tag{5.160}
$$

where Σ is the fermion-sign matrices taking values of $+1$ and -1 on bosons and fermions respectively. Note that in equation (5.160) we have explicitly put the phase factor in such a way that the one-particle supercharges in both parts of the tensor product still have zero phase.

5.6   The all-loop S-matrix

In this section we describe the derivation of the all-loop S-matrix from the symmetry algebra described in the previous section. We begin by constructing a set of S-matrices which are invariant under the various two-particle representations of $\mathfrak{su}(1|1)^2$: one for the scattering of two particles both transforming under ϱ_{L}; one for the scattering of a particle transforming under ϱ_{L} with one transforming under ϱ_{R} etc. We will then discuss the appropriate tensor products of these $\mathfrak{su}(1|1)^2$ S-matrices to produce the full $\mathfrak{psu}(1|1)^4$ S-matrix.

As defined in equation (5.6), the S-matrix relates a basis of in-states to one of out-states. These two bases are naturally related however, and we can choose to use a single basis of states with in-states and out-states distinguished by the ordering of momentum. A natural basis to use is one where the two-particle state $|\mathcal{X}_p \mathcal{Y}_q\rangle$ describes an in-state if $p > q$ and an out-state if $q > p$, see figure 7. We will choose to always order momenta such that $p > q$, hence $|\mathcal{X}_p \mathcal{Y}_q\rangle$ is unambiguously an in-state while $|\mathcal{Y}_q \mathcal{X}_p\rangle$ is an out-state.

The key condition we use to determine the S-matrix $\mathcal{S}_{(12)}$ is consistency with the symmetries described by the supercharges $\mathbf{Q}_{(12)}$. This consistency demands that $\mathcal{S}_{(12)}$ commutes with all the supercharges, i.e.

$$\mathcal{S}_{(12)}(p,q) \, \mathbf{Q}_{(12)}(p,q) = \mathbf{Q}_{(12)}(q,p) \, \mathcal{S}_{(12)}(p,q). \tag{5.161}$$

The different ordering of momenta in $\mathbf{Q}_{(12)}$ on each side is due to the fact that on the left $\mathbf{Q}_{(12)}$ acts on an in-state while on the right it acts on an out-state.

Scattering processes in 1+1 dimensions can be divided between transmis-
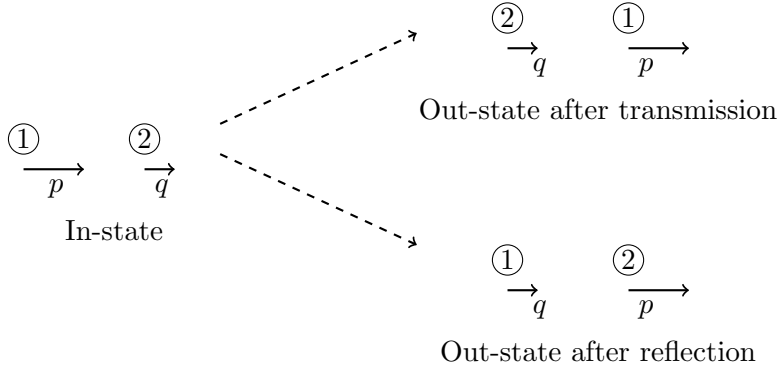
Figure 7: In-states and out-states after scattering processes of reflection and transmission.

sion processes and reflection processes, see figure 7. Since the S-matrix acts non-trivially on the flavour indices, this distinction only matters for scattering of two particles transforming under different representations. For example, if $\mathcal{X}_p^{\text{L}}$ transforms in ϱ_{L} and $\mathcal{X}_q^{\text{R}}$ transforms under ϱ_{R} then we can write[8]

$$\mathcal{S} \, |\mathcal{X}_p^{\text{L}} \mathcal{X}_q^{\text{R}}\rangle = T_{pq} \sum_{\mathcal{Y}} |\mathcal{Y}_q^{\text{R}} \mathcal{Y}_p^{\text{L}}\rangle + R_{pq} \sum_{\mathcal{Y}} |\mathcal{Y}_q^{\text{L}} \mathcal{Y}_p^{\text{R}}\rangle \,, \tag{5.162}$$

where T_{pq} and R_{pq} are transmission and reflection amplitudes respectively. However, we can set $R_{pq} = 0$ immediately in this case by the following argument. Since the symmetry condition (5.161) has to hold for all supercharges it also must hold for all anticommutators of the supercharges and in particular for $\mathbf{H}$. This is just the statement that scattering preserves the total energy. But, with a value for the angular momentum m of $+1$ in ϱ_{L} and -1 in ϱ_{R}, a non-zero reflection amplitude in the symmetry condition would require

$$E(p, 1) + E(q, -1) = E(q, 1) + E(p, -1) \tag{5.163}$$

which does not hold for the energy $E(p, m)$ in equation (5.142). We will determine the scattering amplitudes as functions of the Zhukovski variables and so only implicitly as functions of m, but we will assume that whenever we scatter particles in different representations a different value for the angular momentum is being used for each and so we have only transmission. When we come to describe the tensor products of $\mathfrak{su}(1|1)^2$ S-matrices we will confirm that this assumption was correct.

5.6.1 $\mathfrak{su}(1|1)^2$-invariant S-matrices

To show explicitly how the S-matrix is determined by symmetry conditions, consider the scattering of two particles in the ϱ_{L} representation. In the basis

---

[8]By $\sum_{\mathcal{Y}}$ we mean a sum over possible flavours of out-states $|\mathcal{Y}\rangle$.

$(|\phi_p^{\mathrm{L}}\phi_q^{\mathrm{L}}\rangle, |\phi_p^{\mathrm{L}}\psi_q^{\mathrm{L}}\rangle, |\psi_p^{\mathrm{L}}\phi_q^{\mathrm{L}}\rangle, |\psi_p^{\mathrm{L}}\psi_q^{\mathrm{L}}\rangle)$, the two-particle supercharges are given by

$$
\mathbf{Q}_{\mathrm{L}}(p,q) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
e^{\frac{ip}{2}}\eta_q^{\mathrm{L}} & 0 & 0 & 0 \\
\eta_p^{\mathrm{L}} & 0 & 0 & 0 \\
0 & \eta_p^{\mathrm{L}} & -e^{\frac{ip}{2}}\eta_q^{\mathrm{L}} & 0
\end{pmatrix},
$$

$$
\mathbf{Q}_{\mathrm{R}}(p,q) = \begin{pmatrix}
0 & -\frac{e^{\frac{i}{2}(p-q)}}{x_{\mathrm{L}q}^-}\eta_q^{\mathrm{L}} & -\frac{e^{\frac{-ip}{2}}}{x_{\mathrm{L}p}^-}\eta_p^{\mathrm{L}} & 0 \\
0 & 0 & 0 & -\frac{e^{\frac{-ip}{2}}}{x_{\mathrm{L}p}^-}\eta_p^{\mathrm{L}} \\
0 & 0 & 0 & \frac{e^{\frac{i}{2}(p-q)}}{x_{\mathrm{L}q}^-}\eta_q^{\mathrm{L}} \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{5.164}
$$

Now requiring that equation (5.161) holds for both these supercharges and their conjugates, the S-matrix is determined to be

$$
S^{\mathrm{LL}}(p,q) = \begin{pmatrix}
A_{pq}^{\mathrm{LL}} & 0 & 0 & 0 \\
0 & C_{pq}^{\mathrm{LL}} & D_{pq}^{\mathrm{LL}} & 0 \\
0 & B_{pq}^{\mathrm{LL}} & E_{pq}^{\mathrm{LL}} & 0 \\
0 & 0 & 0 & F_{pq}^{\mathrm{LL}}
\end{pmatrix} \tag{5.165}
$$

where expressions for the elements $A_{pq}^{\mathrm{LL}}, \ldots$ are given in appendix L. As equation (5.161) is linear, there is an overall undetermined normalisation. We choose to set $A_{pq}^{\mathrm{LL}} = 1$, and then the overall normalisation will reappear as a dressing factor later.

Similarly we compute the S-matrix for scattering in other representations. We noted earlier that the representation $\tilde{\varrho}_{\mathrm{L}}$ is related to ϱ_{L} by a choice of different highest weight state. This ensures that the S-matrices for scattering of a particle transforming under ϱ_{L} with one transforming under $\tilde{\varrho}_{\mathrm{L}}$, and for scattering of two particles transforming under $\tilde{\varrho}_{\mathrm{L}}$, are related to the S-matrix above by a change of basis. Altogether we write the action of the S-matrices we compute as follows. First for all "left-left" scattering:

$$
\begin{aligned}
\mathcal{S}^{\mathrm{LL}}|\phi_p^{\mathrm{L}}\phi_q^{\mathrm{L}}\rangle &= A_{pq}^{\mathrm{LL}}|\phi_q^{\mathrm{L}}\phi_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\mathrm{LL}}|\phi_p^{\mathrm{L}}\psi_q^{\mathrm{L}}\rangle &= B_{pq}^{\mathrm{LL}}|\psi_q^{\mathrm{L}}\phi_p^{\mathrm{L}}\rangle + C_{pq}^{\mathrm{LL}}|\phi_q^{\mathrm{L}}\psi_p^{\mathrm{L}}\rangle, \\
\mathcal{S}^{\mathrm{LL}}|\psi_p^{\mathrm{L}}\psi_q^{\mathrm{L}}\rangle &= F_{pq}^{\mathrm{LL}}|\psi_q^{\mathrm{L}}\psi_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\mathrm{LL}}|\psi_p^{\mathrm{L}}\phi_q^{\mathrm{L}}\rangle &= D_{pq}^{\mathrm{LL}}|\phi_q^{\mathrm{L}}\psi_p^{\mathrm{L}}\rangle + E_{pq}^{\mathrm{LL}}|\psi_q^{\mathrm{L}}\phi_p^{\mathrm{L}}\rangle,
\end{aligned} \tag{5.166}
$$

$$
\begin{aligned}
\mathcal{S}^{\tilde{\mathrm{L}}\tilde{\mathrm{L}}}|\tilde{\phi}_p^{\mathrm{L}}\tilde{\phi}_q^{\mathrm{L}}\rangle &= -F_{pq}^{\mathrm{LL}}|\tilde{\phi}_q^{\mathrm{L}}\tilde{\phi}_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\tilde{\mathrm{L}}\tilde{\mathrm{L}}}|\tilde{\phi}_p^{\mathrm{L}}\tilde{\psi}_q^{\mathrm{L}}\rangle &= D_{pq}^{\mathrm{LL}}|\tilde{\psi}_q^{\mathrm{L}}\tilde{\phi}_p^{\mathrm{L}}\rangle - E_{pq}^{\mathrm{LL}}|\tilde{\phi}_q^{\mathrm{L}}\tilde{\psi}_p^{\mathrm{L}}\rangle, \\
\mathcal{S}^{\tilde{\mathrm{L}}\tilde{\mathrm{L}}}|\tilde{\psi}_p^{\mathrm{L}}\tilde{\psi}_q^{\mathrm{L}}\rangle &= -A_{pq}^{\mathrm{LL}}|\tilde{\psi}_q^{\mathrm{L}}\tilde{\psi}_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\tilde{\mathrm{L}}\tilde{\mathrm{L}}}|\tilde{\psi}_p^{\mathrm{L}}\tilde{\phi}_q^{\mathrm{L}}\rangle &= B_{pq}^{\mathrm{LL}}|\tilde{\phi}_q^{\mathrm{L}}\tilde{\psi}_p^{\mathrm{L}}\rangle - C_{pq}^{\mathrm{LL}}|\tilde{\psi}_q^{\mathrm{L}}\tilde{\phi}_p^{\mathrm{L}}\rangle,
\end{aligned} \tag{5.167}
$$

$$\begin{aligned}
\mathcal{S}^{\mathrm{L}\tilde{\mathrm{L}}} \, |\phi_p^{\mathrm{L}}\tilde{\phi}_q^{\mathrm{L}}\rangle &= B_{pq}^{\mathrm{LL}} \, |\tilde{\phi}_q^{\mathrm{L}}\phi_p^{\mathrm{L}}\rangle - C_{pq}^{\mathrm{LL}} \, |\tilde{\psi}_q^{\mathrm{L}}\psi_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\mathrm{L}\tilde{\mathrm{L}}} \, |\phi_p^{\mathrm{L}}\tilde{\psi}_q^{\mathrm{L}}\rangle &= A_{pq}^{\mathrm{LL}} \, |\tilde{\psi}_q^{\mathrm{L}}\phi_p^{\mathrm{L}}\rangle, \\
\mathcal{S}^{\mathrm{L}\tilde{\mathrm{L}}} \, |\psi_p^{\mathrm{L}}\tilde{\psi}_q^{\mathrm{L}}\rangle &= -D_{pq}^{\mathrm{LL}} \, |\tilde{\psi}_q^{\mathrm{L}}\psi_p^{\mathrm{L}}\rangle + E_{pq}^{\mathrm{LL}} \, |\tilde{\phi}_q^{\mathrm{L}}\phi_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\mathrm{L}\tilde{\mathrm{L}}} \, |\psi_p^{\mathrm{L}}\tilde{\phi}_q^{\mathrm{L}}\rangle &= -F_{pq}^{\mathrm{LL}} \, |\tilde{\phi}_q^{\mathrm{L}}\psi_p^{\mathrm{L}}\rangle,
\end{aligned}$$
$$(5.168)$$

$$\begin{aligned}
\mathcal{S}^{\tilde{\mathrm{L}}\mathrm{L}} \, |\tilde{\phi}_p^{\mathrm{L}}\phi_q^{\mathrm{L}}\rangle &= D_{pq}^{\mathrm{LL}} \, |\phi_q^{\mathrm{L}}\tilde{\phi}_p^{\mathrm{L}}\rangle + E_{pq}^{\mathrm{LL}} \, |\psi_q^{\mathrm{L}}\tilde{\psi}_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\tilde{\mathrm{L}}\mathrm{L}} \, |\tilde{\phi}_p^{\mathrm{L}}\psi_q^{\mathrm{L}}\rangle &= -F_{pq}^{\mathrm{LL}} \, |\psi_q^{\mathrm{L}}\tilde{\phi}_p^{\mathrm{L}}\rangle, \\
\mathcal{S}^{\tilde{\mathrm{L}}\mathrm{L}} \, |\tilde{\psi}_p^{\mathrm{L}}\psi_q^{\mathrm{L}}\rangle &= -B_{pq}^{\mathrm{LL}} \, |\psi_q^{\mathrm{L}}\tilde{\psi}_p^{\mathrm{L}}\rangle - C_{pq}^{\mathrm{LL}} \, |\phi_q^{\mathrm{L}}\tilde{\phi}_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\tilde{\mathrm{L}}\mathrm{L}} \, |\tilde{\psi}_p^{\mathrm{L}}\phi_q^{\mathrm{L}}\rangle &= A_{pq}^{\mathrm{LL}} \, |\phi_q^{\mathrm{L}}\tilde{\psi}_p^{\mathrm{L}}\rangle.
\end{aligned}$$
$$(5.169)$$

Next for "left-right" scattering:

$$\begin{aligned}
\mathcal{S}^{\mathrm{LR}} \, |\phi_p^{\mathrm{L}}\phi_q^{\mathrm{R}}\rangle &= A_{pq}^{\mathrm{LR}} \, |\phi_q^{\mathrm{R}}\phi_p^{\mathrm{L}}\rangle + B_{pq}^{\mathrm{LR}} \, |\psi_q^{\mathrm{R}}\psi_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\mathrm{LR}} \, |\phi_p^{\mathrm{L}}\psi_q^{\mathrm{R}}\rangle &= C_{pq}^{\mathrm{LR}} \, |\psi_q^{\mathrm{R}}\phi_p^{\mathrm{L}}\rangle, \\
\mathcal{S}^{\mathrm{LR}} \, |\psi_p^{\mathrm{L}}\psi_q^{\mathrm{R}}\rangle &= E_{pq}^{\mathrm{LR}} \, |\psi_q^{\mathrm{R}}\psi_p^{\mathrm{L}}\rangle + F_{pq}^{\mathrm{LR}} \, |\phi_q^{\mathrm{R}}\phi_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\mathrm{LR}} \, |\psi_p^{\mathrm{L}}\phi_q^{\mathrm{R}}\rangle &= D_{pq}^{\mathrm{LR}} \, |\phi_q^{\mathrm{R}}\psi_p^{\mathrm{L}}\rangle,
\end{aligned}$$
$$(5.170)$$

$$\begin{aligned}
\mathcal{S}^{\mathrm{RL}} \, |\phi_p^{\mathrm{R}}\phi_q^{\mathrm{L}}\rangle &= A_{pq}^{\mathrm{RL}} \, |\phi_q^{\mathrm{L}}\phi_p^{\mathrm{R}}\rangle + B_{pq}^{\mathrm{LR}} \, |\psi_q^{\mathrm{L}}\psi_p^{\mathrm{R}}\rangle, & \mathcal{S}^{\mathrm{RL}} \, |\phi_p^{\mathrm{R}}\psi_q^{\mathrm{L}}\rangle &= C_{pq}^{\mathrm{RL}} \, |\psi_q^{\mathrm{L}}\phi_p^{\mathrm{R}}\rangle, \\
\mathcal{S}^{\mathrm{RL}} \, |\psi_p^{\mathrm{R}}\psi_q^{\mathrm{L}}\rangle &= E_{pq}^{\mathrm{RL}} \, |\psi_q^{\mathrm{L}}\psi_p^{\mathrm{R}}\rangle + F_{pq}^{\mathrm{RL}} \, |\phi_q^{\mathrm{L}}\phi_p^{\mathrm{R}}\rangle, & \mathcal{S}^{\mathrm{RL}} \, |\psi_p^{\mathrm{R}}\phi_q^{\mathrm{L}}\rangle &= D_{pq}^{\mathrm{RL}} \, |\phi_q^{\mathrm{L}}\psi_p^{\mathrm{R}}\rangle,
\end{aligned}$$
$$(5.171)$$

$$\begin{aligned}
\mathcal{S}^{\tilde{\mathrm{L}}\mathrm{R}} \, |\tilde{\phi}_p^{\mathrm{L}}\phi_q^{\mathrm{R}}\rangle &= +D_{pq}^{\mathrm{LR}} \, |\phi_q^{\mathrm{R}}\tilde{\phi}_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\tilde{\mathrm{L}}\mathrm{R}} \, |\tilde{\phi}_p^{\mathrm{L}}\psi_q^{\mathrm{R}}\rangle &= -E_{pq}^{\mathrm{LR}} \, |\psi_q^{\mathrm{R}}\tilde{\phi}_p^{\mathrm{L}}\rangle - F_{pq}^{\mathrm{LR}} \, |\phi_q^{\mathrm{R}}\tilde{\psi}_p^{\mathrm{L}}\rangle, \\
\mathcal{S}^{\tilde{\mathrm{L}}\mathrm{R}} \, |\tilde{\psi}_p^{\mathrm{L}}\psi_q^{\mathrm{R}}\rangle &= -C_{pq}^{\mathrm{LR}} \, |\psi_q^{\mathrm{R}}\tilde{\psi}_p^{\mathrm{L}}\rangle, & \mathcal{S}^{\tilde{\mathrm{L}}\mathrm{R}} \, |\tilde{\psi}_p^{\mathrm{L}}\phi_q^{\mathrm{R}}\rangle &= +A_{pq}^{\mathrm{LR}} \, |\phi_q^{\mathrm{R}}\tilde{\psi}_p^{\mathrm{L}}\rangle - B_{pq}^{\mathrm{LR}} \, |\psi_q^{\mathrm{R}}\tilde{\phi}_p^{\mathrm{L}}\rangle,
\end{aligned}$$
$$(5.172)$$

$$\begin{aligned}
\mathcal{S}^{\mathrm{R}\tilde{\mathrm{L}}} \, |\phi_p^{\mathrm{R}}\tilde{\phi}_q^{\mathrm{L}}\rangle &= +C_{pq}^{\mathrm{RL}} \, |\tilde{\phi}_q^{\mathrm{L}}\phi_p^{\mathrm{R}}\rangle, & \mathcal{S}^{\mathrm{R}\tilde{\mathrm{L}}} \, |\phi_p^{\mathrm{R}}\tilde{\psi}_q^{\mathrm{L}}\rangle &= +A_{pq}^{\mathrm{RL}} \, |\tilde{\psi}_q^{\mathrm{L}}\phi_p^{\mathrm{R}}\rangle - B_{pq}^{\mathrm{RL}} \, |\tilde{\phi}_q^{\mathrm{L}}\psi_p^{\mathrm{R}}\rangle, \\
\mathcal{S}^{\mathrm{R}\tilde{\mathrm{L}}} \, |\psi_p^{\mathrm{R}}\tilde{\psi}_q^{\mathrm{L}}\rangle &= -D_{pq}^{\mathrm{RL}} \, |\tilde{\psi}_q^{\mathrm{L}}\psi_p^{\mathrm{R}}\rangle, & \mathcal{S}^{\mathrm{R}\tilde{\mathrm{L}}} \, |\psi_p^{\mathrm{R}}\tilde{\phi}_q^{\mathrm{L}}\rangle &= -E_{pq}^{\mathrm{RL}} \, |\tilde{\phi}_q^{\mathrm{L}}\psi_p^{\mathrm{R}}\rangle + F_{pq}^{\mathrm{RL}} \, |\tilde{\psi}_q^{\mathrm{L}}\phi_p^{\mathrm{R}}\rangle.
\end{aligned}$$
$$(5.173)$$

The S-matrices for "right-left" and "right-right" scattering are obtained by the use of left-right symmetry on the above.

5.6.2 $\mathfrak{psu}(1|1)^4$-invariant S-matrices

We have found the $\mathfrak{su}(1|1)^2$-invariant S-matrices. To find the full $\mathfrak{psu}(1|1)^4$-invariant S-matrices, we need in principle to look for 16 x 16 matrices which satisfy (5.161) for the two-particle supercharges of the full algebra. However, since the $\mathfrak{psu}(1|1)^4$ supercharges are obtained by tensor products of the $\mathfrak{su}(1|1)^2$ supercharges, we can obtain the full S-matrix by taking tensor products of the S-matrices we have already computed. We have to take a graded tensor product however, to account for the correct ordering of the two tensor products taking us from $\mathfrak{su}(1|1)^2$ to $\mathfrak{psu}(1|1)^4$, and from one-particle to two-particle representations, the latter of which is graded. We define the graded tensor product we need as follows. Consider a basis for two-particle $\mathfrak{psu}(1|1)^4$ states

given naturally in terms of $\mathfrak{su}(1|1)^2$ states as follows:

$$(|\phi_p \phi_p \phi_q \phi_q\rangle , |\phi_p \phi_p \phi_q \psi_q\rangle , |\phi_p \phi_p \psi_q \phi_q\rangle , \ldots |\psi_p \psi_p \psi_q \psi_q\rangle) \tag{5.174}$$

where each particle is in an appropriate representation. For example, in defining the graded tensor product $\mathcal{S}^{\mathrm{LL}} \hat{\otimes} \, \mathcal{S}^{\mathrm{L\tilde{L}}}$, this basis is

$$(|\phi_p^{\mathrm{L}} \phi_p^{\mathrm{L}} \phi_q^{\mathrm{L}} \phi_q^{\tilde{\mathrm{L}}}\rangle , |\phi_p^{\mathrm{L}} \phi_p^{\mathrm{L}} \phi_q^{\mathrm{L}} \psi_q^{\tilde{\mathrm{L}}}\rangle , |\phi_p^{\mathrm{L}} \phi_p^{\mathrm{L}} \psi_q^{\mathrm{L}} \phi_q^{\tilde{\mathrm{L}}}\rangle , \ldots |\psi_p^{\mathrm{L}} \psi_p^{\mathrm{L}} \psi_q^{\mathrm{L}} \psi_q^{\tilde{\mathrm{L}}}\rangle)$$
$$=(|Y_p^{\mathrm{L}} T_q^{2a}\rangle , |Y_p^{\mathrm{L}} \chi_q^a\rangle , |Y_p^{\mathrm{L}} \tilde{\chi}_q^a\rangle , \ldots |Z_p^{\mathrm{L}} T_q^{1a}\rangle) \tag{5.175}$$

and so this clearly describes scattering between a left-massive and massless particle. We define the graded tensor product $\mathcal{S}^{(1)}_{\mathfrak{su}(1|1)^2} \hat{\otimes} \, \mathcal{S}^{(2)}_{\mathfrak{su}(1|1)^2}$ in such a basis in terms of the normal tensor product by

$$\mathcal{S}^{(1)}_{\mathfrak{su}(1|1)^2} \hat{\otimes} \, \mathcal{S}^{(2)}_{\mathfrak{su}(1|1)^2} = P \left(\mathcal{S}^{(1)}_{\mathfrak{su}(1|1)^2} \otimes \, \mathcal{S}^{(2)}_{\mathfrak{su}(1|1)^2} \right) P \tag{5.176}$$

where

$$P = \mathbb{1}_2 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes \mathbb{1}_2 \tag{5.177}$$

The full S-matrix $\mathcal{S}$ factorises into sectors for massive-only, massless-only and massive-massive scattering. We denote the massive-massive S-matrix by $\mathcal{S}^{\bullet\bullet}$, the massless-massless S-matrix by $\mathcal{S}^{\circ\circ}$ and the mixed-mass S-matrices by $\mathcal{S}^{\circ\bullet}$, $\mathcal{S}^{\bullet\circ}$. The massive $\mathfrak{psu}(1|1)^4$ representations are $\varrho_{\mathrm{L}} \otimes \varrho_{\mathrm{L}}$ and $\varrho_{\mathrm{R}} \otimes \varrho_{\mathrm{R}}$. Therefore the graded tensor products of $\mathfrak{su}(1|1)^2$ S-matrices giving rise to the massive $\mathfrak{psu}(1|1)^4$ S-matrix $\mathcal{S}^{\bullet\bullet}$ are $\mathcal{S}^{\mathrm{LL}} \hat{\otimes} \mathcal{S}^{\mathrm{LL}}$, $\mathcal{S}^{\mathrm{RL}} \hat{\otimes} \mathcal{S}^{\mathrm{RL}}$, $\mathcal{S}^{\mathrm{LR}} \hat{\otimes} \mathcal{S}^{\mathrm{LR}}$ and $\mathcal{S}^{\mathrm{RR}} \hat{\otimes} \mathcal{S}^{\mathrm{RR}}$. Each of these comes with its own undetermined dressing factor since we have not determined the normalisations of the $\mathfrak{su}(1|1)^2$ S-matrices. Hence the S-matrix for the massive-massive sector is given by

$$\mathcal{S}^{\bullet\bullet} = \begin{pmatrix} \sigma_{\mathrm{LL}}^{\bullet\bullet} \, \mathcal{S}^{\mathrm{LL}} \hat{\otimes} \mathcal{S}^{\mathrm{LL}} & \tilde{\sigma}_{\mathrm{RL}}^{\bullet\bullet} \, \mathcal{S}^{\mathrm{RL}} \hat{\otimes} \mathcal{S}^{\mathrm{RL}} \\ \tilde{\sigma}_{\mathrm{LR}}^{\bullet\bullet} \, \mathcal{S}^{\mathrm{LR}} \hat{\otimes} \mathcal{S}^{\mathrm{LR}} & \sigma_{\mathrm{RR}}^{\bullet\bullet} \, \mathcal{S}^{\mathrm{RR}} \hat{\otimes} \mathcal{S}^{\mathrm{RR}} \end{pmatrix}. \tag{5.178}$$

The dressing factors are related by left-right symmetry, so we can define functions $\sigma^{\bullet\bullet}$ and $\tilde{\sigma}^{\bullet\bullet}$ of the Zhukovski variables such that

$$\sigma_{\mathrm{LL}}^{\bullet\bullet}(p,q) = \sigma^{\bullet\bullet}(x_{p\,\mathrm{L}}^{\pm}, x_{q\,\mathrm{L}}^{\pm}), \qquad \sigma_{\mathrm{RR}}^{\bullet\bullet}(p,q) = \sigma^{\bullet\bullet}(x_{p\,\mathrm{R}}^{\pm}, x_{q\,\mathrm{R}}^{\pm}),$$
$$\tilde{\sigma}_{\mathrm{LR}}^{\bullet\bullet}(p,q) = \tilde{\sigma}^{\bullet\bullet}(x_{p\,\mathrm{L}}^{\pm}, x_{q\,\mathrm{R}}^{\pm}), \qquad \tilde{\sigma}_{\mathrm{RL}}^{\bullet\bullet}(p,q) = \tilde{\sigma}^{\bullet\bullet}(x_{p\,\mathrm{R}}^{\pm}, x_{q\,\mathrm{L}}^{\pm}), \tag{5.179}$$

The massless $\mathfrak{psu}(1|1)^4$ representations are $(\varrho_{\mathrm{L}} \otimes \varrho_{\tilde{\mathrm{L}}})^{\oplus 2}$. The tensor sum corresponds to transformations under $\mathfrak{su}(2)_{\circ}$ which acts on the indices $a, b \ldots$. Hence the graded tensor products of $\mathfrak{su}(1|1)^2$ S-matrices giving rise to the massive-massless $\mathfrak{psu}(1|1)^4$ S-matrix $\mathcal{S}^{\bullet\circ}$ are $\mathcal{S}^{\mathrm{LL}} \hat{\otimes} \mathcal{S}^{\mathrm{L\tilde{L}}}$ and $\mathcal{S}^{\mathrm{RL}} \hat{\otimes} \mathcal{S}^{\mathrm{R\tilde{L}}}$, and

each comes as a doublet under $\mathfrak{su}(2)_\circ$. Hence $\mathcal{S}^{\bullet\circ}$ is given by

$$\mathcal{S}^{\bullet\circ} = \left[\sigma_{\mathrm{L}}^{\bullet\circ}(\mathcal{S}^{\mathrm{LL}} \otimes \mathcal{S}^{\mathrm{L}\tilde{\mathrm{L}}})^{\oplus 2}\right] \oplus \left[\sigma_{\mathrm{R}}^{\bullet\circ}(\mathcal{S}^{\mathrm{RL}} \otimes \mathcal{S}^{\mathrm{R}\tilde{\mathrm{L}}})^{\oplus 2}\right] . \tag{5.180}$$

Left-right symmetry again means we can write the two dressing factors in terms of a single function $\sigma^{\bullet\circ}$ of the Zhukovski variables, with

$$\sigma_{\mathrm{L}}^{\bullet\circ}(p,q) = \sigma^{\bullet\circ}(x_{\mathrm{L}\,p}^{\pm}, x_{\mathrm{L}\,q}^{\pm}), \qquad \sigma_{\mathrm{R}}^{\bullet\circ}(p,q) = \sigma^{\bullet\circ}(x_{\mathrm{R}\,p}^{\pm}, x_{\mathrm{R}\,q}^{\pm}) . \tag{5.181}$$

Similarly, $\mathcal{S}^{\circ\bullet}$ is given by

$$\mathcal{S}^{\circ\bullet} = \left[\sigma_{\mathrm{L}}^{\circ\bullet}(\mathcal{S}^{\mathrm{LL}} \otimes \mathcal{S}^{\tilde{\mathrm{L}}\mathrm{L}})^{\oplus 2}\right] \oplus \left[\sigma_{\mathrm{R}}^{\circ\bullet}(\mathcal{S}^{\mathrm{LR}} \otimes \mathcal{S}^{\tilde{\mathrm{L}}\mathrm{R}})^{\oplus 2}\right], \tag{5.182}$$

with

$$\sigma_{\mathrm{L}}^{\circ\bullet}(p,q) = \sigma^{\circ\bullet}(x_{\mathrm{L}\,p}^{\pm}, x_{\mathrm{L}\,q}^{\pm}), \qquad \sigma_{\mathrm{R}}^{\circ\bullet}(p,q) = \sigma^{\circ\bullet}(x_{\mathrm{R}\,p}^{\pm}, x_{\mathrm{R}\,q}^{\pm}) . \tag{5.183}$$

Taking two copies of the massless representation $(\varrho_{\mathrm{L}} \otimes \varrho_{\tilde{\mathrm{L}}})^{\oplus 2}$, we see that the massless $\mathfrak{psu}(1|1)^4$ S-matrix $\mathcal{S}^{\circ\circ}$ arises from four copies of $\mathcal{S}^{\mathrm{LL}} \hat{\otimes} \mathcal{S}^{\tilde{\mathrm{L}}\tilde{\mathrm{L}}}$, that transform under $\mathfrak{su}(2)_\circ$, so $\mathcal{S}^{\circ\circ}$ is given as a tensor product of $\mathcal{S}^{\mathrm{LL}} \hat{\otimes} \mathcal{S}^{\tilde{\mathrm{L}}\tilde{\mathrm{L}}}$ with an $\mathfrak{su}(2)$-invariant S-matrix, that is

$$\mathcal{S}^{\circ\circ} = \sigma^{\circ\circ} \, \mathcal{S}_{\mathfrak{su}(2)} \otimes \left(\mathcal{S}^{\mathrm{LL}} \hat{\otimes} \mathcal{S}^{\tilde{\mathrm{L}}\tilde{\mathrm{L}}}\right) . \tag{5.184}$$

The full S-matrix has been determined up to the dressing factors. These are subject to several constraints namely, unitarity of the full S-matrix and crossing symmetry [142] which gives rise to the crossing equations. These are discussed for the dressing factors here in [2].

5.7 Chapter conclusions and outlook

In this chapter we have seen how the complete all-loop S-matrix for scattering of fundamental excitations in $AdS_3 \times S^3 \times T^4$ with mixed-flux can be derived from the symmetry algebra of the theory. By deriving the algebra of symmetry currents from the Green-Schwarz action, rather than the coset action, we avoid the problems associated with massless excitations in the coset action and thus find an S-matrix which includes massless excitations. The massive sector of the S-matrix derived here reproduces the results found in [115, 116]. In particular the results here support the dispersion relation proposed in [116][9]. The dispersion relation for mixed-flux excitations represents an important difference from the results of the pure R-R background. In particular, as we have seen, the left- and right-massive representations are different for $q \neq 0$.

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[9]However recently perturbative worldsheet calculations to two loops [108] have suggested a disagreement with the proposed exact massless dispersion relation.

When we take the limit $q \to 0$ of the S-matrix computed in this chapter we reproduce the exact complete pure R-R S-matrix found in [64, 65].

One natural development from the work described in this chapter and related papers [64, 65, 131] is to study the inclusion of massless excitations in the S-matrices of other AdS backgrounds where these massless modes are present, such as in the integrable AdS_2 backgrounds. Some progress in this direction was made in [143]. There expressions for massless scattering constructed by taking massless limits in representations obtained from the coset symmetry algebra were tested against arguments from Yangian symmetry [144].

An interesting prospect raised by the developments in integrability for the mixed-flux AdS_3 dualities is to consider the limit $q \to 1$ and thus describe the pure NS-NS backgrounds using integrability. AdS_3 backgrounds with pure NS-NS flux were studied before the discovery of integrability in holography using worldsheet CFT techniques [29–36] and so it might be possible to make a connection between these approaches and integrability. The pure NS-NS backgrounds have been studied from an integrability perspective recently in [145].

The exact S-matrix derived in this chapter has two parameters: the flux parameter q and the coupling constant λ. In [131] where the exact S-matrix for $AdS_3 \times S^3 \times S^3 \times S^1$ was found using the same approach, this was extended to a three-parameter family with the addition of the parameter α describing the relative S^3 radii. The complex set of quantum integrable models being explored by these techniques provides good opportunities for exploring integrability in holography in quite general settings.

In the context of *relativistic* two-dimensional quantum field theories, massless integrable models had been studied in the early 1990s in an effort to understand renormalization group flows [146–151]. These results relate only to relativistic field theories, while the massless modes that enter the AdS_3/CFT_2 correspondence are non-relativistic. It would be interesting to look at the near-relativistic limit of the S-matrix constructed in this chapter in order to see whether some of the relativistic S-matrices found previously can be obtained as limits of the AdS_3/CFT_2 S-matrix.

Chapter 6

Conclusion

The gauge/string correspondence is a completely new way of understanding quantum gauge and gravity theories. Since it is a weak/strong duality it opens the door for understanding strongly-coupled gauge theories and highly curved gravity in terms of the weakly-coupled duals. Perhaps more significantly, the duality states that these very different theories in fact describe the same physics. As a result understanding exactly *how* holography works remains of considerable interest.

In highly supersymmetric settings such as $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$, integrability has proven to be a powerful tool in unravelling the inner workings of the correspondence. In these theories integrability provides an explicit tool for calculating anomalous dimensions of operators, or equivalently the energies of closed string states, at all values of the coupling. As such it helps explain exactly how these dualities work.

While integrability is unlikely to explain generic gauge/string duals, it can be applied to other classes of duals. It was observed in [61] that AdS_3/CFT_2 backgrounds with 8+8 supersymmetries are likely to be integrable. However, initial progress was not as rapid as in the case of AdS_4/CFT_3 as it was found that these backgrounds possessed so called massless modes which were at first more difficult to incorporate within the methods of integrability.

In this thesis we have investigated the AdS_3/CFT_2 correspondence by studying two families of backgrounds: $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$, supported by mixed R-R and NS-NS fluxes. Our work has focused on the role that the massless modes play in these settings. In chapter 3 we incorporated massless modes into the classical integrability machinery as encoded in the algebraic curve and finite-gap equations. Our work focused on the zero-cut sector which contained precisely the massless bosonic modes of the theory. Previous implementations of the Virasoro constraints on the finite-gap equations were appropriate to $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ backgrounds, but were not sufficiently general to take into account the multiple factors in the geometry of the AdS_3 backgrounds. We demonstrated how the Virasoro constraints had to be implemented in this setting in a way that included these massless modes. This led us to the so-called Generalised Residue Condition (GRC) (3.44).

In chapter 4 we saw how fluctuations around this zero-cut sector can be studied quite generally, and that quantum massless excitations remain decoupled as $z = \pm 1$ residue fluctuations. There is still a puzzle regarding what could be thought of as the point where this distinction between cut-dynamics

and residue-dynamics breaks down: for particular backgrounds where massive excitations become massless. Nonetheless our analysis provides the tools necessary to investigate strings in a semi-classical setting. Such calculations provide important information about the expressions for the phases of the worldsheet S-matrix that cannot be fixed from symmetries alone, and allow for comparisons with expressions for the phases obtained by solving crossing equations.

In chapter 5 we saw how it is possible to incorporate massless excitations into the all-loop S-matrix, with a resultant complete S-matrix factorising into massive-massive, massless-massless and massive-massless sectors. As well as being important for understanding 3d/2d dualities, progress in incorporating massless sectors will have applications elsewhere, e.g. 2d/1d dualities, and other cases with less supersymmetry.

As well as progress in applying integrability techniques to the massless sectors of the theories, in this thesis we have also seen progress in extending integrability to mixed-flux backgrounds. In chapter 5 we obtained mixed-flux, q-dependent expressions for the quadratic action, supercharges and S-matrix. As well as developing insight into a greater range of dualities, we have seen how studying the mixed-flux backgrounds can provide greater insight into the pure R-R backgrounds as well. For example, we saw that whereas in the derivation of the exact S-matrix for the pure R-R theory from symmetries one has to appeal to perturbative calculations to choose the pure-transmission S-matrix over the pure-reflection S-matrix, in the mixed-flux case this choice can be deduced solely from the symmetries.

The results obtained in this thesis using the semiclassical algebraic curve and exact S-matrix techniques can be compared with results from perturbative worldsheet calculations. The results in chapter 4 reproduce worldsheet results in [112] and suggest further worldsheet calculations that could be carried out in comparison. The exact results obtained in chapter 5 can be compared with worldsheet results for dispersion relations and scattering in [107,108,114,152].

In this thesis we have focused on integrability on the string side of the dualities. Exact results from the string side such as the all-loop S-matrix lead to integrable spin-chains [95,96,98,99] that must describe certain weakly-coupled sectors of the dual CFTs. However, for several years there was no real evidence of integrability directly on the CFT side. One natural place where one might have looked for signs of integrability on the CFT$_2$ side was near the $Sym^N(T^4)$ orbifold point. An extensive study by Pakman, Rastelli and Razamat [153] searched for signs of an integrable spin-chain by computing anomalous dimensions of operators in the $Sym^N(T^4)$ theory deformed by a marginal operator. Despite a large effort, the authors were unable to identify a spin-chain picture of the type discovered by Minahan and Zarembo [24]. More recently [39], evidence of integrability on the CFT$_2$ side was found by

studying the IR dynamics of two-dimensional gauge theories [154].

Integrability has shown itself to be an invaluable tool for understanding low supersymmetry holography. In this thesis we have seen various contexts where the scope of integrability to study these dualities has been widened beyond its previous boundaries. By understanding more examples of exact dualities, it can be hoped that we will gain general insights into the physics of holography.

Appendix A

Index conventions

In this appendix we give our conventions for indices. We denote worldsheet coordinates by $\alpha, \beta, \ldots = \tau, \sigma$; spacetime coordinates by indices $m, n, \ldots = 0, \ldots 9$; and $\mathfrak{so}(1,9)$ tangent coordinates by $A, B, \ldots = 0, \ldots 9$. We label the two sets of spacetime spinors with indices $I, J, \ldots = 1, 2$. Indices $l, m \ldots$ are associated to the Cartan bases of supergroups and are carried by the associated quasimomenta.

We also use indices referring to representations of the algebra $\mathfrak{so}(4)_1 \times \mathfrak{so}(4)_2$, where $\mathfrak{so}(4)_1$ corresponds to rotations along the $AdS_3 \times S^3$ directions transverse to the light-cone directions t and ψ and $\mathfrak{so}(4)_2$ corresponds to rotations along T^4. We use indices $\underline{a}, \underline{b}, \ldots = 1, 2$ and $\underline{\dot{a}}, \underline{\dot{b}}, \ldots = 1, 2$ for the two Weyl spinors of $\mathfrak{so}(4)_1$; and indices $a, b, \ldots = 1, 2$ and $\dot{a}, \dot{b}, \ldots = 1, 2$ for the two Weyl spinors of $\mathfrak{so}(4)_2$. We use indices $\underline{i}, \underline{j}, \ldots = 1, \ldots, 4$ for the vector of $\mathfrak{so}(4)_1$. We use the same indices for the transverse coordinates of AdS_3 and S^3 themselves ($z_{\underline{i}}$ and $y_{\underline{i}}$ respectively) with the understanding that $z_3 = z_4 = y_1 = y_2 = 0$. We use summation conventions on all indices listed above.

We raise and lower spinor indices with epsilon symbols normalised as

$$\epsilon^{12} = -\epsilon_{12} = +1 \ . \tag{A.1}$$

We also occasionally write $\epsilon^{\underline{ij}}$, by this we will always mean an expression of the following form

$$\epsilon^{\underline{ij}} z_{\underline{i}} \partial_\alpha z_{\underline{j}} = z_1 \partial_\alpha z_2 - z_2 \partial_\alpha z_1 \ , \quad \epsilon^{\underline{ij}} y_{\underline{i}} \partial_\alpha y_{\underline{j}} = y_3 \partial_\alpha y_4 - y_4 \partial_\alpha y_3 \ . \tag{A.2}$$

Similarly in our conventions

$$\dot{z} \cdot \dot{z}' = \dot{z}_{\underline{i}} \dot{z}'^{\underline{i}} = \dot{z}_1 \dot{z}'_1 + \dot{z}_2 \dot{z}'_2 \ , \quad \dot{y} \cdot \dot{y}' = \dot{y}_{\underline{i}} \dot{y}'^{\underline{i}} = \dot{y}_3 \dot{y}'_3 + \dot{y}_4 \dot{y}'_4 \ . \tag{A.3}$$

Appendix B

Gamma matrices

We define AdS_3 and S^3 gamma matrices γ^A as

$$\gamma^0 = -i\sigma_3, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_2, \quad \gamma^3 = \sigma_1, \quad \gamma^4 = \sigma_2, \quad \gamma^5 = \sigma_3. \quad \text{(B.1)}$$

In addition we define

$$\gamma^6 = \sigma_1, \quad \gamma^7 = \sigma_2, \quad \gamma^8 = \sigma_3. \quad \text{(B.2)}$$

In terms of these we define ten-dimensional gamma matrices Γ^A by

$$
\begin{aligned}
\Gamma^A &= +\sigma_1 \otimes \sigma_2 \otimes \gamma^A \otimes \mathbb{1} \otimes \mathbb{1}, \quad A = 0,1,2, \\
\Gamma^A &= +\sigma_1 \otimes \sigma_1 \otimes \mathbb{1} \otimes \gamma^A \otimes \mathbb{1}, \quad A = 3,4,5, \\
\Gamma^A &= +\sigma_1 \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma^A, \quad A = 6,7,8, \\
\Gamma^9 &= -\sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}.
\end{aligned}
\quad \text{(B.3)}
$$

Further defining the matrices

$$T = -i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes , \quad C = \Gamma^0 , \quad B = -\Gamma^0 T , \quad \text{(B.4)}$$

Majorana spinors θ are those whose Dirac and Majorana conjugates are equal,

$$\bar{\theta} = \theta^\dagger C = \theta^t T , \quad \text{(B.5)}$$

which is equivalent to the reality condition

$$\theta^* = B\theta . \quad \text{(B.6)}$$

Note that the matrices T, C and B satisfy the relations

$$
\begin{aligned}
T^\dagger T = C^\dagger C = B^\dagger B &= 1 , \\
T^\dagger = T^t &= -T , \\
C^\dagger = C^t &= -C , \\
B^\dagger = B^t &= B .
\end{aligned}
\quad \text{(B.7)}
$$

Appendix C

Quasimomenta residues for general solutions on $\mathbb{R} \times S^3 \times S^1$

The metric is

$$ds^2 = R^2 \left[-dt^2 + \frac{1}{\cos^2 \phi} (d\theta^2 + \cos^2 \theta d\psi_1^2 + \sin^2 \theta d\varphi^2) + \frac{1}{\sin^2 \phi} d\psi_2^2 \right] . \quad \text{(C.1)}$$

The group representative g is a direct sum $g = g_0 \oplus g_1 \oplus g_2$ as before. g_0 and g_2 are chosen exactly as in (3.49) and (3.48), but for g_1 corresponding to the full S^3 we take

$$g_1 = \sqrt{\frac{1}{2\cos\phi}} \begin{pmatrix} \cos\theta e^{i\psi_1} & -\sin\theta e^{-i\varphi} & 0 & 0 \\ \sin\theta e^{i\varphi} & \cos\theta e^{-i\psi_1} & 0 & 0 \\ 0 & 0 & i\sin\theta e^{-i\varphi} & -i\cos\theta e^{i\psi_1} \\ 0 & 0 & i\cos\theta e^{-i\psi_1} & -i\sin\theta e^{i\varphi} \end{pmatrix} . \quad \text{(C.2)}$$

The current j is (with the first and third terms in the direct sum unchanged from equation (3.50))

$$j = \frac{dt}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \oplus \frac{1}{2\cos\phi} \begin{pmatrix} iu & -v+iw & 0 & 0 \\ v+iw & -iu & 0 & 0 \\ 0 & 0 & iu & -v-iw \\ 0 & 0 & v-iw & -iu \end{pmatrix}$$

$$\oplus \frac{i}{\sin\phi} \frac{d\psi_2}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \quad \text{(C.3)}$$

where u, v and w are all real one-forms given by

$$u = \cos^2 \theta d\psi_1 + \sin^2 \theta d\varphi$$
$$v + iw = e^{i(\psi_1 + \varphi)} (d\theta + i\sin\theta\cos\theta(d\varphi - d\psi_1)) . \quad \text{(C.4)}$$

As in section 3.3, we have again chosen a group representative satisfying $\Omega(j) = -j$ and so $j^{(2)} = \frac{1}{2}(j - \Omega(j)) = j$. We can confirm that

$$
\begin{aligned}
\mathrm{tr}\left[(j^{(2)})^2\right] = \mathrm{tr}(j^2) &= 2\left[dt^2 - \frac{1}{\cos^2\phi}\left(u^2 + v^2 + w^2\right) - \frac{1}{\sin^2\phi}d\psi_2^2\right] \\
&= 2\left[dt^2 - \frac{1}{\cos^2\phi}\left(d\theta^2 + \cos^2\theta d\psi_1^2 + \sin^2\theta d\varphi^2\right) - \frac{1}{\sin^2\phi}d\psi_2^2\right] .
\end{aligned}
\tag{C.5}
$$

The relevant (S^3) part of the Lax operator L_σ obtained from the current in (C.3) is given by

$$
L_\sigma = \begin{pmatrix}
ia & -b+ic & 0 & 0 \\
b+ic & -ia & 0 & 0 \\
0 & 0 & ia & -b-ic \\
0 & 0 & b-ic & -ia
\end{pmatrix} ,
\tag{C.6}
$$

with a, b and c given by

$$
\begin{aligned}
a &= \frac{1}{2\cos\phi}\frac{1}{z^2-1}\left[(z^2+1)u_\sigma + 2zu_\tau\right] , \\
b &= \frac{1}{2\cos\phi}\frac{1}{z^2-1}\left[(z^2+1)v_\sigma + 2zv_\tau\right] , \\
c &= \frac{1}{2\cos\phi}\frac{1}{z^2-1}\left[(z^2+1)w_\sigma + 2zw_\tau\right] .
\end{aligned}
\tag{C.7}
$$

We can find the residues of the quasimomenta on this space using the WKB analysis (see section 3.2.2). We need the eigenvalues of $V = -ihL_\sigma$ in the limit $h = z \mp 1 \to 0$. With L_σ as in equation (C.6), there is the following eigenvalue of multiplicity 2:

$$
\frac{1}{2\cos\phi}\sqrt{(u_\tau \pm u_\sigma)^2 + (v_\tau \pm v_\sigma)^2 + (w_\tau \pm w_\sigma)^2}
\tag{C.8}
$$

and of course the negative of this. Note that $\pm$ in this expression refers to the limit $z \to \pm 1$.

We therefore have expressions for the residues of the quasimomenta on this space as follows. There are residues $\kappa_0 \pm 2\pi m_0$ and $\kappa_3 \pm 2\pi m_2$ given as in equation (3.58) for the quasimomenta associated to $\mathbb{R}$ and S^1. There are generically two distinct quasimomenta p_1^+ and p_1^- associated to S^3, but they both have the same residues (with opposite signs as required by the inversion symmetry); this equality of residues is seen in the fact that the residues of V have multiplicity two. These residues are

$$
\kappa_1 \pm 2\pi m_1 = \frac{1}{\cos\phi}\int_0^{2\pi} d\sigma\sqrt{(u_\tau \pm u_\sigma)^2 + (v_\tau \pm v_\sigma)^2 + (w_\tau \pm w_\sigma)^2} .
\tag{C.9}
$$

We can therefore see that the residues for all quasimomenta, including those on

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S^3, are given naturally in terms of integrals of functions $f_l^\pm(\sigma)$. Furthermore, using equation (C.5), we can see that the condition (3.44) on these functions is exactly the more familiar form of the Virasoro constraints on classical bosonic strings on a curved background, here $\mathbb{R} \times S^3 \times S^1$, namely

$$G_{mn}(\dot{X}^m \pm X'^m)(\dot{X}^n \pm X'^m) = 0 \tag{C.10}$$

where X^m are the spacetime fields and G_{mn} is the spacetime metric.

Similarly for the quasimomenta for the full coset space of $AdS_3 \times S^3 \times S^3$, the Virasoro constraints in the form (C.10) can be seen to be equivalent to the generalised residue conditions (3.43) and (3.44), not the null residue condition (3.46).

Appendix D

Generalised residue condition for $AdS_5 \times S^5$

The coset for strings on $AdS_5 \times S^5$ is $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$. We follow the conventions of the review [87]. The Cartan matrix for $PSU(2,2|4)$ is

$$
A = \begin{pmatrix}
 & 1 & & & & & \\
1 & -2 & 1 & & & & \\
 & 1 & & -1 & & & \\
 & & -1 & 2 & -1 & & \\
 & & & -1 & & 1 & \\
 & & & & 1 & -2 & 1 \\
 & & & & & 1 &
\end{pmatrix}
\tag{D.1}
$$

and the matrix S giving the inversion symmetry through equation (3.27) is

$$
S = \begin{pmatrix}
 & & 1 & -1 & & & \\
 & 1 & & -1 & & & \\
1 & & & -1 & & & \\
 & & & -1 & & & \\
 & & & -1 & & 1 & \\
 & & & -1 & 1 & & \\
 & & & -1 & 1 & &
\end{pmatrix} .
\tag{D.2}
$$

The quasimomenta are p_l with the index l running from 1 to 7. The residues are given in terms of functions $f_l(\sigma)$ as in equation (3.43). The action of the inversion symmetry on the residues (see equation (3.30)) means f_l must satisfy

$$
\sum_{m=1}^{7} S_{lm} f_m = -f_l .
\tag{D.3}
$$

Solving this inversion symmetry, we find that we can choose f_1, f_4 and f_7 to be independent, while the remaining functions are given in terms of these three:

$$
f_2 = f_6 = \frac{1}{2} f_4, \quad f_3 = f_4 - f_1, \quad f_5 = f_4 - f_7 .
\tag{D.4}
$$

With these substitutions made, the version of the condition (3.109) on this space is

$$
0 = \sum_{l,m=1}^{7} A_{lm} f_l f_m = f_4 \left(f_1 + f_7 - \frac{1}{2} f_4 \right) .
\tag{D.5}
$$

The values of f_l for the BMN vacuum are

$$f_1 + f_7 = \kappa, \quad f_4 = 0 \ . \tag{D.6}$$

For the residues of $D(2, 1; \alpha)^2$ we were able to solve the constraint on the functions f_l in such a way that we could have a range of solutions including the BMN vacuum with residues for solutions away from the BMN vacuum having qualitively different values. In particular in equation (3.111) the BMN vacuum solution has $\zeta^{\pm} - \phi = \chi^{\pm} = 0$, but other solutions could vary smoothly away from this. In contrast, any solution to (D.5) must either have residues identical to the BMN vacuum for some value of κ or be disconnected from it: the solution $f_1 + f_7 = \frac{1}{2} f_4$ can only be smoothly connected to the BMN vacuum when $\kappa = 0$.

Appendix E

Generalised residue condition for $AdS_4 \times CP^3$

The coset for strings on $AdS_4 \times CP^3$ is $\frac{OSp(6|4)}{U(3)\times SO(3,1)}$. The Cartan matrix of $OSp(6|4)$ is

$$
A = \begin{pmatrix}
 & 1 & & & \\
1 & -2 & 1 & & \\
 & 1 & & -1 & -1 \\
 & & -1 & 2 & \\
 & & -1 & & 2
\end{pmatrix}
\tag{E.1}
$$

and the inversion symmetry matrix S is

$$
S = \begin{pmatrix}
 & & 1 & -1 & -1 \\
 & 1 & & -1 & -1 \\
1 & & & -1 & -1 \\
 & & & -1 & \\
 & & -1 & &
\end{pmatrix} .
\tag{E.2}
$$

Now the quasimomenta are p_l with l running from 1 to 5. The action of the inversion symmetry on the residues means that there are 2 independent functions f_1 and f_4, with the others given by

$$
f_2 = f_5 = f_4, \quad f_3 = 2f_4 - f_1 .
\tag{E.3}
$$

Then in terms of f_1 and f_4, the condition the functions need to satisfy is

$$
0 = \sum_{l,m=1}^{5} A_{lm} f_l f_m = 2f_4(2f_1 - f_4) .
\tag{E.4}
$$

We see that this is very similar in form to the condition (D.5), and the argument from this point is identical to that in the last section. The BMN vacuum has $f_4 = 0$ and $f_1 = \kappa$, and there is no other solution smoothly connected to this.

Appendix F

Generalised residue condition for $D(2,1;\alpha)^2$ in mixed grading

In section 3.4, we used a grading for $D(2,1;\alpha)^2$ which involves bosonic Cartan generators only. In [98] an alternative grading was used, involving bosonic Cartan generators on one factor of $D(2,1;\alpha)$ and fermionic generators on the other. The Cartan matrix is given in this mixed grading by

$$A = \begin{pmatrix} 4\sin^2\phi & -2\sin^2\phi & & & & \\ -2\sin^2\phi & & -2\cos^2\phi & & & \\ & -2\cos^2\phi & 4\cos^2\phi & & & \\ & & & 2\sin^2\phi & -2 & \\ & & & 2\sin^2\phi & & 2\cos^2\phi \\ & & & -2 & 2\cos^2\phi & \end{pmatrix} \tag{F.1}$$

and the matrix S defining the action of the inversion symmetry on the quasi-momenta through equation (3.27) is given by

$$S = \begin{pmatrix} & & -1 \\ -1 & 1 & -1 \\ & & -1 \end{pmatrix} \otimes \sigma_1 . \tag{F.2}$$

Following the notation in [98], we take the index structure on the quasimomenta as follows: we have quasimomenta p_l and $p_{\bar{l}}$ with $l, \bar{l} = 1, 2, 3$. The upper left quadrant of A corresponds to indies l, the lower right to indices $\bar{l}$, and the factor of σ_1 in S interchanges l and $\bar{l}$.

The action of the inversion symmetry on the residues via equation (3.30) means we can determine the functions $f_{\bar{l}}$ in terms of f_l. We have:

$$f_{\bar{1}} = f_1, \quad f_{\bar{3}} = f_3, \quad f_{\bar{2}} = f_1 - f_2 + f_3 . \tag{F.3}$$

We can insert this into the relevant equivalent of the condition (3.109) and we find that:[1]

$$\sum_{l,m}^{3} A_{lm} f_l f_m = \sum_{\bar{l},\bar{m}=1}^{3} A_{\bar{l}\bar{m}} f_{\bar{l}} f_{\bar{m}}$$

$$= 4\sin^2\phi f_1(f_1 - f_2) + 4\cos^2\phi f_3(f_3 - f_2) . \tag{F.4}$$

---

[1] A_{lm} referring only to the upper-left components of A and $A_{\bar{l}\bar{m}}$ to the lower-right components.

In other words, in the mixed grading just as in the bosonic grading, the residue condition is identical when considered either solely on left-movers or right-movers. The full condition in this case is

$$\sum_{l,m} A_{lm} f_l f_m + \sum_{\bar{l},\bar{m}=1}^{3} A_{\bar{l}\bar{m}} f_{\bar{l}} f_{\bar{m}} = 0 \tag{F.5}$$

and so we have exactly the same condition with exactly the same analysis for quasimomenta in the mixed grading as in bosonic grading.

Appendix G

Decoupled S^1 mode

In section 3.3 we showed that classical solutions corresponding to the "coset" massless boson have residues that satisfy the GRC but not the old residue conditions. Here we present the quasimomenta for classical solutions in $AdS_3 \times S^3 \times S^3 \times S^1$ that correspond to the other massless boson associated to the S^1. We give these results only in lightcone gauge.

We consider the most general solution for the decoupled mode in lightcone gauge, so explicitly the solution is[1]

$$x^+ = \kappa\tau$$

$$x_8 = x_0 + \alpha' p_0 \tau + w\sigma$$

$$+ \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(a_n e^{-in(\tau+\sigma)} + a_n^\dagger e^{in(\tau+\sigma)} + \tilde{a}_n e^{-in(\tau-\sigma)} + \tilde{a}_n^\dagger e^{in(\tau-\sigma)} \right) .$$

$$(G.1)$$

The results are then very similar to those of the analogous solution for the "coset mode" in section 3.3.2. The Virasoro constraints determine x^-. Just as for the coset solution, we do not write down the full expression for x_1 as we only need integrals over σ of its derivatives. We note however, that the expression for x^- for this decoupled solution is very similar to that for the coset solution; the only difference is to the zero mode contribution which arose previously from x_1, see equation (3.2). We must now also account for the additional quasimomentum p_9 coming from the coordinate u_9 of the S^1. It is again purely analytic, so is given by equation (3.57) where the residues $\kappa_9 \pm 2\pi m_9$ are now given by

$$\kappa_9 = \frac{1}{R} \int_0^{2\pi} \mathrm{d}\sigma \partial_\tau x_8 , \quad 2\pi m_9 = \frac{1}{R} \int_0^{2\pi} \mathrm{d}\sigma \partial_\sigma x_8 . \qquad (G.2)$$

We then have the final solution for the quasimomenta of this decoupled mode solution as follows: p_l are given by equation (3.57) for $l = 0, 1, 2, 9$, with the residues given as follows:

$$\kappa_0 = 2\pi i\kappa + \frac{i\pi\alpha'}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger + \tilde{a}_n \tilde{a}_n^\dagger) + \frac{i\pi(\alpha'^2 p_0^2 + w^2)}{2\kappa R^2} , \qquad (G.3)$$

$$\kappa_1 = -2\pi\kappa\cos\phi + \frac{\pi\alpha'\cos\phi}{\kappa R^2} \sum_{n=1}^{\infty} n(a_n a_n^\dagger + \tilde{a}_n \tilde{a}_n^\dagger) + \frac{\pi(\alpha'^2 p_0^2 + w^2)\cos\phi}{2\kappa R^2} , \quad (G.4)$$

---

[1] x_8 is defined in equation (3.2).

$$\kappa_2 = -2\pi\kappa\sin\phi + \frac{\pi\alpha'\sin\phi}{\kappa R^2}\sum_{n=1}^{\infty} n(a_n a_n^\dagger + \tilde{a}_n \tilde{a}_n^\dagger) + \frac{\pi(\alpha'^2 p_0^2 + w^2)\sin\phi}{2\kappa R^2} \ , \quad \text{(G.5)}$$

$$2\pi m_0 = \frac{i\pi\alpha'}{\kappa R^2}\sum_{n=1}^{\infty} n(a_n a_n^\dagger - \tilde{a}_n \tilde{a}_n^\dagger) + \frac{i\pi\alpha' p_0 w}{\kappa R^2} \ , \quad \text{(G.6)}$$

$$2\pi m_1 = \frac{\pi\alpha'\cos\phi}{\kappa R^2}\sum_{n=1}^{\infty} n(a_n a_n^\dagger - \tilde{a}_n \tilde{a}_n^\dagger) + \frac{\pi\alpha' p_0 w\cos\phi}{\kappa R^2} \ , \quad \text{(G.7)}$$

$$2\pi m_2 = \frac{\pi\alpha'\sin\phi}{\kappa R^2}\sum_{n=1}^{\infty} n(a_n a_n^\dagger - \tilde{a}_n \tilde{a}_n^\dagger) + \frac{\pi\alpha' p_0 w\sin\phi}{\kappa R^2} \ , \quad \text{(G.8)}$$

$$\kappa_9 = \frac{\alpha' p_0}{R} \ , \qquad 2\pi m_9 = \frac{w}{R} \ . \quad \text{(G.9)}$$

This should be compared with the very similar expression (3.76) for the coset mode solution. As in that case, we impose the level matching condition (3.77) from σ-periodicity of t, and in this case this fixes

$$m_1 = m_2 = 0 \quad \text{(G.10)}$$

so the only non-zero winding mode is m_9.

Appendix H

Spinor identity

In this appendix we show that for all spinors ϵ which are anti-chiral under $\tilde{\Gamma}$, the following relation holds:

$$\frac{\tilde{q}}{8} \not{H}_m \epsilon = \frac{q}{48} \not{F} \not{E}_m \epsilon \ . \tag{H.1}$$

This identity is used several times in chapter 5.

We note first that $\not{F}$ can be expressed in any of the following ways:

$$\not{F} = 12\tilde{q}(\Gamma^{012} + \Gamma^{345}) = 12\tilde{q}\Gamma^{012}(1 + \tilde{\Gamma}) = 12\tilde{q}\Gamma^{345}(1 + \tilde{\Gamma}) \ . \tag{H.2}$$

and that $\tilde{\Gamma}$ commutes with Γ_a for $a = 6 \ldots 9$ while for $a = 0 \ldots 5$:

$$(1 \pm \tilde{\Gamma})\Gamma_a = \Gamma_a(1 \mp \tilde{\Gamma}) \ . \tag{H.3}$$

Now for $m = X^i$ (the T^4 coordinates), (H.1) is trivially true with both sides equal to zero. For other m, we expand the right-hand side over the AdS_3 and S^3 coordinates separately:

$$\begin{aligned}
\frac{q}{48} \not{F} \not{E}_m \epsilon &= \frac{q\tilde{q}}{4} \left(\Gamma^{012} \sum_{a=0}^{2} E_m^a \Gamma_a + \Gamma^{345} \sum_{a=3}^{5} E_m^a \Gamma_a \right) (1 - \tilde{\Gamma})\epsilon \\
&= \frac{q\tilde{q}}{2} \left(\Gamma^{012} \sum_{a=0}^{2} E_m^a \Gamma_a + \Gamma^{345} \sum_{a=3}^{5} E_m^a \Gamma_a \right) \epsilon \\
&= \frac{q\tilde{q}}{4} \left(\sum_{a,b,c=0}^{2} \epsilon_{abc}\Gamma^{bc} E_m^a + \sum_{a,b,c=3}^{5} \epsilon_{abc}\Gamma^{bc} E_m^a \right) \epsilon \tag{H.4}
\end{aligned}$$

where $\epsilon_{012} = \epsilon_{345} = 1$. This last line is precisely the left-hand side of (H.1) once we expand it in coordinates.

Appendix I

Spin-connections

In this appendix we give explicit expressions for the spin-connection components ω_m^{AB} and the rotated spin-connection components $\hat{\omega}_m^{AB}$ and $\check{\omega}_m^{AB}$. We also prove some identities with these rotated spin-connections that are used in the main text.

The natural diagonal dreibein for the AdS_3 metric (5.8) is

$$E_m{}^A = \frac{1}{1 + \frac{y^2}{4}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \frac{y^2}{4} \end{pmatrix} . \tag{I.1}$$

The tangent components of the spin connection for this dreibein are given by

$$\omega_{AB} = \frac{1}{2} \frac{1}{1 + \frac{y^2}{4}} \begin{pmatrix} 0 & y_3 dy_4 - y_4 dy_3 & 2y_3 d\psi \\ y_4 dy_3 - y_3 dy_4 & 0 & 2y_4 d\psi \\ -2y_3 d\psi & -2y_4 d\psi & 0 \end{pmatrix} . \tag{I.2}$$

Similarly we have the diagonal dreibein for the AdS_3 metric

$$E_m{}^A = \frac{1}{1 - \frac{z^2}{4}} \begin{pmatrix} 1 + \frac{z^2}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{I.3}$$

with spin connection components given by

$$\omega_{AB} = \frac{1}{2} \frac{1}{1 - \frac{z^2}{4}} \begin{pmatrix} 0 & 2z_1 dt & 2z_2 dt \\ -2z_1 dt & 0 & z_2 dz_1 - z_1 dz_2 \\ -2z_2 dt & z_1 dz_2 - z_2 dz_1 & 0 \end{pmatrix} . \tag{I.4}$$

We change basis to the "rotated" dreibeins as follows. We note that

$$\hat{M}_{S^3}^{-1} \gamma_A \hat{M}_{S^3} E_m^A = \gamma_A \hat{\mathcal{M}}^A{}_B E_m^B , \quad \check{M}_{S^3}^{-1} \gamma_A \check{M}_{S^3} E_m^A = \gamma_A \check{\mathcal{M}}^A{}_B E_m^B \tag{I.5}$$

where $\hat{\mathcal{M}}^A{}_B$ and $\check{\mathcal{M}}^A{}_B$ the orthogonal matrices. Then we define the rotated S^3 dreibeins $\hat{K}$ and $\check{K}$ by

$$\hat{K}_m^A = \hat{\mathcal{M}}^A{}_B E_m^B , \quad \check{K}_m^a = \check{\mathcal{M}}^A{}_B E_m^B . \tag{I.6}$$

Explicitly the components of the rotated dreibeins are

$$
\hat{K}_m{}^A = \begin{pmatrix} 1 + \frac{y_3^2 - y_4^2}{4} & +\frac{y_3 y_4}{2} & +y_4 \\ +\frac{y_3 y_4}{2} & 1 - \frac{y_3^2 - y_4^2}{4} & -y_3 \\ -y_4 \left(1 - \frac{y^2}{4}\right) & +y_3 \left(1 - \frac{y^2}{4}\right) & 1 \end{pmatrix} \begin{pmatrix} +\cos\psi & -\sin\psi & 0 \\ +\sin\psi & +\cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\check{K}_m{}^A = \begin{pmatrix} 1 + \frac{y_3^2 - y_4^2}{4} & +\frac{y_3 y_4}{2} & -y_4 \\ +\frac{y_3 y_4}{2} & 1 - \frac{y_3^2 - y_4^2}{4} & +y_3 \\ +y_4 \left(1 - \frac{y^2}{4}\right) & -y_3 \left(1 - \frac{y^2}{4}\right) & 1 \end{pmatrix} \begin{pmatrix} +\cos\psi & +\sin\psi & 0 \\ -\sin\psi & +\cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$\tag{I.7}$$

Defining rotated drebeins for AdS_3 similarly we find components for these given by

$$
\hat{K}_m{}^A = \begin{pmatrix} 1 & +z_2 \left(1 + \frac{z^2}{4}\right) & -z_1 \left(1 + \frac{z^2}{4}\right) \\ +z_2 & 1 - \frac{z_1^2 - z_2^2}{4} & -\frac{z_1 z_2}{2} \\ -z_1 & -\frac{z_1 z_2}{2} & 1 + \frac{z_1^2 - z_2^2}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos t & -\sin t \\ 0 & +\sin t & +\cos t \end{pmatrix}
$$

$$
\check{K}_m{}^A = \begin{pmatrix} 1 & -z_2 \left(1 + \frac{z^2}{4}\right) & +z_1 \left(1 + \frac{z^2}{4}\right) \\ -z_2 & 1 - \frac{z_1^2 - z_2^2}{4} & -\frac{z_1 z_2}{2} \\ +z_1 & -\frac{z_1 z_2}{2} & 1 + \frac{z_1^2 - z_2^2}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos t & +\sin t \\ 0 & -\sin t & +\cos t \end{pmatrix}
$$

$$\tag{I.8}$$

When we calculate the components of the spin-connections for these rotated dreibeins we find that they can be given simply in terms of the dreibein components. We find the relations

$$
\hat{\omega}_{mAB} = -\epsilon_{ABC} \hat{K}_m^C \,, \quad \check{\omega}_{mAB} = +\epsilon_{ABC} \check{K}_m^C \tag{I.9}
$$

holding for both the AdS_3 and S^3 spin-connections, where ϵ_{ABC} denotes epsilon symbols with $\epsilon_{012} = \epsilon_{345} = 1$. We can therefore write the contracted spin-connection $\hat{\slashed{\phi}}_m = \omega_{mAB} \Gamma^{AB}$ as

$$
\frac{1}{4}\hat{\slashed{\phi}}_m = -\frac{1}{2} \sum_{A=0}^{2} \hat{K}_m^A \Gamma_A \Gamma^{012} - \frac{1}{2} \sum_{A=3}^{5} \hat{K}_m^A \Gamma_A \Gamma^{345}
$$

$$
= -\frac{1}{4}\hat{\slashed{K}}_\alpha \left(\Gamma^{012} + \Gamma^{345}\right) - \frac{1}{4}\left(\Gamma^{012} + \Gamma^{345}\right)\hat{\slashed{K}}_\alpha \tag{I.10}
$$

and similarly

$$
\frac{1}{4}\check{\slashed{\phi}}_m = +\frac{1}{4}\check{\slashed{K}}_\alpha \left(\Gamma^{012} + \Gamma^{345}\right) + \frac{1}{4}\left(\Gamma^{012} + \Gamma^{345}\right)\check{\slashed{K}}_\alpha \tag{I.11}
$$

We can see from these expressions that $\hat{\slashed{\phi}}_\alpha$ is proportional to $\hat{\slashed{H}}_\alpha$.

Using these expressions for the rotated spin-connections and the fact that

the rotated covariant derivative satisfies the relations

$$\hat{M}^{-1}\left(\partial_m + \frac{1}{4}\phi_m\right)\hat{M} = \partial_m + \frac{1}{4}\hat{\phi}_m \,,$$

$$\check{M}^{-1}\left(\partial_m + \frac{1}{4}\phi_m\right)\check{M} = \partial_m + \frac{1}{4}\check{\phi}_m \,, \tag{I.12}$$

we can derive the following expressions:

$$\hat{M}^{-1}\left(D_\alpha + \frac{q}{8}\slashed{H}_\alpha + \frac{\tilde{q}}{48}\slashed{F}\slashed{E}_\alpha\right)\hat{M} = \partial_\alpha - \frac{\tilde{q}^2}{2}\hat{K}_\alpha\left(\Gamma^{012} + \Gamma^{345}\right) \,,$$

$$\check{M}^{-1}\left(D_\alpha - \frac{q}{8}\slashed{H}_\alpha - \frac{\tilde{q}}{48}\slashed{F}\slashed{E}_\alpha\right)\check{M} = \partial_\alpha + \frac{\tilde{q}^2}{2}\hat{K}_\alpha\left(\Gamma^{012} + \Gamma^{345}\right) \,. \tag{I.13}$$

Using these expressions we can see directly that the spinors (5.20) do indeed satisfy the Killing spinor equation (5.18), and that equation (5.59) holds for the massive fermions in the Green-Schwarz action.

Appendix J
Quartic Supercurrents

In this appendix we give expressions for the supercurrents of section 5.4 to cubic order in bosonic fields and leading order in bosons. We split the expressions up in the form

$$j_I = j_{I,\text{massless}} + j_{I,\text{massive}} + j_{I,\text{mixed}} \ . \tag{J.1}$$

The massless currents receive no corrections at quartic order, and are given by

$$
\begin{aligned}
j_{1,\text{massless}}^{\tau} &= i\gamma^{34}e^{+x^-\gamma^{34}}\Big(+\dot{x}^i\tilde{\tau}_i\chi_1 - \tilde{q}\acute{x}^i\tilde{\tau}_i\chi_2 - q\dot{x}^i\tilde{\tau}_i\chi_1\Big),\\
j_{2,\text{massless}}^{\tau} &= i\gamma^{34}e^{-x^-\gamma^{34}}\Big(+\dot{x}^i\tilde{\tau}_i\chi_2 - \tilde{q}\acute{x}^i\tilde{\tau}_i\chi_1 + q\dot{x}^i\tilde{\tau}_i\chi_2\Big),\\
j_{1,\text{massless}}^{\sigma} &= i\gamma^{34}e^{+x^-\gamma^{34}}\Big(-\acute{x}^i\tilde{\tau}_i\chi_1 + \tilde{q}\dot{x}^i\tilde{\tau}_i\chi_2 + q\dot{x}^i\tilde{\tau}_i\chi_1\Big),\\
j_{2,\text{massless}}^{\sigma} &= i\gamma^{34}e^{-x^-\gamma^{34}}\Big(-\acute{x}^i\tilde{\tau}_i\chi_2 + \tilde{q}\dot{x}^i\tilde{\tau}_i\chi_1 - q\dot{x}^i\tilde{\tau}_i\chi_2\Big).
\end{aligned}
\tag{J.2}
$$

The massive currents are given by

$$
\begin{aligned}
j_{1,\text{massive}}^{\tau} = ie^{+x^-\gamma^{34}}\Big[&+(\dot{z}^{\underline{i}} - \dot{y}^{\underline{i}})\gamma_{\underline{i}}\eta_1 + (z^{\underline{i}} - y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 - (\acute{z}^{\underline{i}} - \acute{y}^{\underline{i}})\gamma_{\underline{i}}(\tilde{q}\eta_2 + q\eta_1)\\
&-\tfrac{1}{2}\Big((z^2 - y^2)(\dot{z}^{\underline{i}} - \dot{y}^{\underline{i}}) - \tfrac{3}{2}(z^2\dot{z}^{\underline{i}} + y^2\dot{y}^{\underline{i}})\\
&\qquad + (z\cdot\dot{z}z^{\underline{i}} + y\cdot\dot{y}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_1\\
&-\tfrac{1}{4}\Big((\dot{z}^2 + \acute{z}^2 + \dot{y}^2 + \acute{y}^2)(z^{\underline{i}} + y^{\underline{i}}) - (y^2 z^{\underline{i}} + z^2 y^{\underline{i}})\Big)\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1\\
&-\tfrac{\tilde{q}}{2}\Big((z^2 - y^2)(\acute{z}^{\underline{i}} - \acute{y}^{\underline{i}}) - \tfrac{1}{2}(z^2\acute{z}^{\underline{i}} + y^2\acute{y}^{\underline{i}})\\
&\qquad + 2(z\cdot\acute{z} - y\cdot\acute{y})(z^{\underline{i}} - y^{\underline{i}}) - (z\cdot\acute{z}z^{\underline{i}} + y\cdot\acute{y}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_2\\
&-\tfrac{\tilde{q}}{2}(\dot{z}\cdot\acute{z} + \dot{y}\cdot\acute{y})(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2\\
&-\tfrac{q}{4}\Big((z^2\acute{z}^{\underline{k}} + y^2\acute{y}^{\underline{k}}) + 2\epsilon_{\underline{ij}}(z^i\dot{z}^{\underline{j}} - y^i\dot{y}^{\underline{j}})(\acute{z}^{\underline{k}} - \acute{y}^{\underline{k}})\\
&\qquad + 2\epsilon_{\underline{ij}}(y^i\acute{y}^{\underline{j}}\dot{z}^{\underline{k}} + z^i\acute{z}^{\underline{j}}\dot{y}^{\underline{k}})\Big)\gamma_{\underline{k}}\eta_1\\
&-\tfrac{q}{2}\Big((z\cdot\dot{z}\acute{z}^{\underline{k}} + y\cdot\dot{y}\acute{y}^{\underline{k}}) - (\dot{y}\cdot\acute{y}z^{\underline{k}} + \dot{z}\cdot\acute{z}y^{\underline{k}})\\
&\qquad - \epsilon_{\underline{ij}}(y^i\acute{y}^{\underline{j}}z^{\underline{k}} + z^i\acute{z}^{\underline{j}}y^{\underline{k}})\Big)\gamma_{\underline{k}}\tilde{\gamma}^{34}\eta_1\Big],
\end{aligned}
\tag{J.3}
$$

$$j_{2,\text{massive}}^{\tau} = ie^{-x^{-}\gamma^{34}}\Big[+(\dot{z}^{\underline{i}}-\dot{y}^{\underline{i}})\gamma_{\underline{i}}\eta_2 - (z^{\underline{i}}-y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 - (\acute{z}^{\underline{i}}-\acute{y}^{\underline{i}})\gamma_{\underline{i}}(\tilde{q}\eta_1 - q\eta_2)$$

$$-\frac{1}{2}\Big((z^2-y^2)(\dot{z}^{\underline{i}}-\dot{y}^{\underline{i}}) - \frac{3}{2}(z^2\dot{z}^{\underline{i}}+y^2\dot{y}^{\underline{i}})$$

$$+ (z\cdot\dot{z}z^{\underline{i}}+y\cdot\dot{y}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_2$$

$$-\frac{1}{4}\Big((\dot{z}^2+\acute{z}^2+\dot{y}^2+\acute{y}^2)(z^{\underline{i}}+y^{\underline{i}}) + (y^2z^{\underline{i}}+z^2y^{\underline{i}})\Big)\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2$$

$$-\frac{\tilde{q}}{2}\Big((z^2-y^2)(\acute{z}^{\underline{i}}-\acute{y}^{\underline{i}}) - \frac{1}{2}(z^2\acute{z}^{\underline{i}}+y^2\acute{y}^{\underline{i}})$$

$$+ 2(z\cdot\acute{z}-y\cdot\acute{y})(z^{\underline{i}}-y^{\underline{i}}) - (z\cdot\acute{z}z^{\underline{i}}+y\cdot\acute{y}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_1$$

$$-\frac{\tilde{q}}{2}(\dot{z}\cdot\acute{z}+\dot{y}\cdot\acute{y})(z^{\underline{i}}+y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1$$

$$+\frac{q}{4}\Big((z^2\acute{z}^{\underline{k}}+y^2\acute{y}^{\underline{k}}) - 2\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}}-y^{\underline{i}}\dot{y}^{\underline{j}})(\acute{z}^{\underline{k}}-\acute{y}^{\underline{k}})$$

$$- 2\epsilon_{\underline{ij}}(y^{\underline{i}}\acute{y}^{\underline{j}}z^{\underline{k}}+z^{\underline{i}}\acute{z}^{\underline{j}}y^{\underline{k}})\Big)\gamma_{\underline{k}}\eta_2$$

$$-\frac{q}{2}\Big((z\cdot\dot{z}\acute{z}^{\underline{k}}+y\cdot\dot{y}\acute{y}^{\underline{k}}) - (\dot{y}\cdot\acute{y}z^{\underline{k}}+\dot{z}\cdot\acute{z}y^{\underline{k}})$$

$$- \epsilon_{\underline{ij}}(y^{\underline{i}}\acute{y}^{\underline{j}}z^{\underline{k}}+z^{\underline{i}}\acute{z}^{\underline{j}}y^{\underline{k}})\Big)\gamma_{\underline{k}}\tilde{\gamma}^{34}\eta_2\Big], \qquad (J.4)$$

$$j_{1,\text{massive}}^{\sigma} = ie^{+x^{-}\gamma^{34}}\Big[-(\acute{z}^{\underline{i}}-\acute{y}^{\underline{i}})\gamma_{\underline{i}}\eta_1 - (z^{\underline{i}}-y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 + (\dot{z}^{\underline{i}}-\dot{y}^{\underline{i}})\gamma_{\underline{i}}(\tilde{q}\eta_2 + q\eta_1)$$

$$-\frac{1}{2}\Big((z^2-y^2)(\acute{z}^{\underline{i}}-\acute{y}^{\underline{i}}) + \frac{3}{2}(z^2\acute{z}^{\underline{i}}+y^2\acute{y}^{\underline{i}})$$

$$- (z\cdot\acute{z}z^{\underline{i}}+y\cdot\acute{y}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_1$$

$$-\frac{1}{2}(\dot{z}\cdot\acute{z}+\dot{y}\cdot\acute{y})(z^{\underline{i}}+y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1$$

$$+\frac{\tilde{q}}{4}\Big((z^2-y^2)(\dot{z}^{\underline{i}}-\dot{y}^{\underline{i}}) - (z^2\dot{z}^{\underline{i}}+y^2\dot{y}^{\underline{i}})$$

$$+ 2(z\cdot\dot{z}-y\cdot\dot{y})(z^{\underline{i}}-y^{\underline{i}}) - 2(y\cdot\dot{y}z^{\underline{i}}+z\cdot\dot{z}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_2$$

$$-\frac{\tilde{q}}{4}\Big((\dot{z}^2+\acute{z}^2+z^2+\dot{y}^2+\acute{y}^2+y^2)(z^{\underline{i}}+y^{\underline{i}})$$

$$- 3(z^2z^{\underline{i}}+y^2y^{\underline{i}})\Big)\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2$$

$$+\frac{q}{4}\Big((z^2\dot{z}^{\underline{k}}+y^2\dot{y}^{\underline{k}}) - 2\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}}-y^{\underline{i}}\dot{y}^{\underline{j}})(\dot{z}^{\underline{k}}-\dot{y}^{\underline{k}})$$

$$+ 2\epsilon_{\underline{ij}}(z^{\underline{i}}\acute{z}^{\underline{j}}-y^{\underline{i}}\acute{y}^{\underline{j}})(\acute{z}^{\underline{k}}-\acute{y}^{\underline{k}})\Big)\gamma_{\underline{k}}\eta_1$$

$$+\frac{q}{4}\Big((\dot{z}^2+\acute{z}^2-z^2+\dot{y}^2+\acute{y}^2-y^2)(z^{\underline{k}}+y^{\underline{k}}) + 3(z^2z^{\underline{k}}+y^2y^{\underline{k}})$$

$$+ 2\epsilon_{\underline{ij}}(y^{\underline{i}}\dot{y}^{\underline{j}}z^{\underline{k}}+z^{\underline{i}}\dot{z}^{\underline{j}}y^{\underline{k}})\Big)\gamma_{\underline{k}}\tilde{\gamma}^{34}\eta_1\Big], \qquad (J.5)$$

$$
\begin{aligned}
j_{2,\text{massive}}^{\sigma} = ie^{-x^{-}\gamma^{34}} \Big[&-(\dot{z}^{\underline{i}} - \dot{y}'^{\underline{i}})\gamma_{\underline{i}}\eta_2 + (z^{\underline{i}} - y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 + (\dot{z}^{\underline{i}} - \dot{y}^{\underline{i}})\gamma_{\underline{i}}(\tilde{q}\eta_1 - q\eta_2) \\
&-\tfrac{1}{2}\Big((z^2 - y^2)(\dot{z}'^{\underline{i}} - \dot{y}'^{\underline{i}}) + \tfrac{3}{2}(z^2\dot{z}^{\underline{i}} + y^2\dot{y}^{\underline{i}}) \\
&\qquad - (z\cdot\dot{z}z^{\underline{i}} + y\cdot\dot{y}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_2 \\
&+\tfrac{1}{2}(\dot{z}\cdot\dot{z} + \dot{y}\cdot\dot{y})(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 \\
&+\tfrac{\tilde{q}}{4}\Big(z^2 - y^2)(\dot{z}^{\underline{i}} - \dot{y}^{\underline{i}}) - (z^2\dot{z}^{\underline{i}} + y^2\dot{y}^{\underline{i}}) \\
&\qquad + 2(z\cdot\dot{z} - y\cdot\dot{y})(z^{\underline{i}} - y^{\underline{i}}) - 2(y\cdot\dot{y}z^{\underline{i}} + z\cdot\dot{z}y^{\underline{i}})\Big)\gamma_{\underline{i}}\eta_1 \\
&-\tfrac{\tilde{q}}{4}\Big((\dot{z}^2 + \dot{z}'^2 + z^2 + \dot{y}^2 + \dot{y}'^2 + y^2)(z^{\underline{i}} + y^{\underline{i}}) \\
&\qquad + 3(z^2 z^{\underline{i}} + y^2 y^{\underline{i}})\Big)\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 \\
&-\tfrac{q}{4}\Big((z^2\dot{z}^{\underline{k}} + y^2\dot{y}^{\underline{k}}) + 2\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}} - y^{\underline{i}}\dot{y}^{\underline{j}})(\dot{z}^{\underline{k}} - \dot{y}^{\underline{k}}) \\
&\qquad - 2\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}} - y^{\underline{i}}\dot{y}'^{\underline{j}})(\dot{z}^{\underline{k}} - \dot{y}'^{\underline{k}}))\Big)\gamma_{\underline{k}}\eta_2 \\
&+\tfrac{q}{4}\Big((\dot{z}^2 + \dot{z}'^2 - z^2 + \dot{y}^2 + \dot{y}'^2 - y^2)(z^{\underline{k}} + y^{\underline{k}}) + 3(z^2 z^{\underline{k}} + y^2 y^{\underline{k}}) \\
&\qquad - 2\epsilon_{\underline{ij}}(y^{\underline{i}}\dot{y}^{\underline{j}}z^{\underline{k}} + z^{\underline{i}}\dot{z}^{\underline{j}}y^{\underline{k}})\Big)\gamma_{\underline{k}}\tilde{\gamma}^{34}\eta_2 \Big].
\end{aligned} \tag{J.6}
$$

The mixed currents are given by

$$
\begin{aligned}
j_{1,\text{mixed}}^{\tau} = ie^{+x^{-}\gamma^{34}} \Big(&-\tfrac{1}{2}(z^2 - y^2)(\dot{x}^i\gamma^{34}\tilde{\tau}_i\chi_1 + \tilde{q}\,\dot{x}'^i\gamma^{34}\tilde{\tau}_i\chi_2) + \tilde{q}\,z^{\underline{i}}y^{\underline{j}}\dot{x}'^k\gamma^{34}\gamma_{\underline{ij}}\tilde{\tau}_k\chi_2 \\
&+\tfrac{1}{4}(\dot{x}^2 + \dot{x}'^2)(z^{\underline{i}} + y^{\underline{i}})\gamma_i\tilde{\gamma}^{34}\eta_1 + \tfrac{\tilde{q}}{2}(\dot{x}\cdot\dot{x}')(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 \\
&+\tfrac{q}{2}\big(\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}} - y^{\underline{i}}\dot{y}'^{\underline{j}})\dot{x}^k - \epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}} - y^{\underline{i}}\dot{y}^{\underline{j}})\dot{x}'^k\big)\gamma^{34}\tilde{\tau}_k\chi_1 \\
&-\tfrac{q}{2}(\dot{x}\cdot\dot{x}')(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 \Big) \,,
\end{aligned} \tag{J.7}
$$

$$
\begin{aligned}
j_{2,\text{mixed}}^{\tau} = ie^{-x^{-}\gamma^{34}} \Big(&-\tfrac{1}{2}(z^2 - y^2)(\dot{x}^i\gamma^{34}\tilde{\tau}_i\chi_2 + \tilde{q}\,\dot{x}'^i\gamma^{34}\tilde{\tau}_i\chi_1) + \tilde{q}\,z^{\underline{i}}y^{\underline{j}}\dot{x}'^k\gamma^{34}\gamma_{\underline{ij}}\tilde{\tau}_k\chi_1 \\
&+\tfrac{1}{4}(\dot{x}^2 + \dot{x}'^2)(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 - \tfrac{\tilde{q}}{2}(\dot{x}\cdot\dot{x}')(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 \\
&+\tfrac{q}{2}\big(\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}'^{\underline{j}} - y^{\underline{i}}\dot{y}'^{\underline{j}})\dot{x}^k - \epsilon_{\underline{i}}(z^{\underline{i}}\dot{z}^{\underline{j}} - y^{\underline{i}}\dot{y}^{\underline{j}})\dot{x}'^k\big)\gamma^{34}\tilde{\tau}_k\chi_2 \\
&-\tfrac{q}{2}(\dot{x}\cdot\dot{x}')(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 \Big) \,,
\end{aligned} \tag{J.8}
$$

$$
\begin{aligned}
j_{1,\text{mixed}}^{\sigma} = ie^{+x^{-}\gamma^{34}} \Big(&-\tfrac{1}{2}(z^2 - y^2)(\dot{x}'^i\gamma^{34}\tilde{\tau}_i\chi_1 - \tilde{q}\,\dot{x}^i\gamma^{34}\tilde{\tau}_i\chi_2) - \tilde{q}\,z^{\underline{i}}y^{\underline{j}}\dot{x}^k\gamma^{34}\gamma_{\underline{ij}}\tilde{\tau}_k\chi_2 \\
&-\tfrac{\tilde{q}}{4}(\dot{x}^2 + \dot{x}'^2)(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 - \tfrac{1}{2}(\dot{x}\cdot\dot{x}')(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 \\
&-\tfrac{q}{2}\big(\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}} - y^{\underline{i}}\dot{y}^{\underline{j}})\dot{x}^k - \epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}'^{\underline{j}} - y^{\underline{i}}\dot{y}'^{\underline{j}})\dot{x}'^k\big)\gamma^{34}\tilde{\tau}_k\chi_1 \\
&+\tfrac{q}{4}(\dot{x}^2 + \dot{x}'^2)(z^{\underline{i}} + y^{\underline{i}})\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 \Big) \,,
\end{aligned} \tag{J.9}
$$

$$
\begin{aligned}
j^{\sigma}_{2,\mathrm{mixed}} = ie^{-x^{-}\gamma^{34}} \Big(&-\tfrac{1}{2}(z^2 - y^2)(\acute{x}^i\gamma^{34}\tilde{\tau}_i\chi_2 - \tilde{q}\,\dot{x}^i\gamma^{34}\tilde{\tau}_i\chi_1) - \tilde{q}\,z^i\underline{y^j}\dot{x}^k\gamma^{34}\gamma_{\underline{ij}}\tilde{\tau}_k\chi_1 \\
&+\tfrac{\tilde{q}}{4}(\dot{x}^2 + \acute{x}^2)(z^{\underline{i}} + y^i)\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_1 + \tfrac{1}{2}(\dot{x}\cdot\acute{x})(z^{\underline{i}} + y^i)\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 \\
&-\tfrac{q}{2}\big(\epsilon_{\underline{ij}}(z^{\underline{i}}\dot{z}^{\underline{j}} - y^i\dot{y}^{\underline{j}})\dot{x}^k - \epsilon_{\underline{ij}}(z^{\underline{i}}\acute{z}^{\underline{j}} - y^{\underline{i}}\acute{y}^{\underline{j}})\acute{x}^k\big)\gamma^{34}\tilde{\tau}_k\chi_2 \\
&+\tfrac{q}{4}(\dot{x}^2 + \acute{x}^2)(z^{\underline{i}} + y^i)\gamma_{\underline{i}}\tilde{\gamma}^{34}\eta_2 \Big) \, .
\end{aligned}
\tag{J.10}
$$

Appendix K

Poisson brackets

In this appendix we give the Poisson brackets of the fermion fields η_I, χ_I which are used to compute the algebra of supercurrents in section (5.4). They are

$$\{(\eta_1)^{\underline{\dot{a}}\dot{a}}, (\eta_1)^{\underline{\dot{b}}\dot{b}}\}_{\text{PB}} = -\frac{i}{4}(1 + A_1)\,\epsilon^{\underline{\dot{a}\dot{b}}}\,\epsilon^{\dot{a}\dot{b}},$$

$$\{(\eta_1)^{\underline{\dot{a}}\dot{a}}, (\eta_2)^{\underline{\dot{b}}\dot{b}}\}_{\text{PB}} = -\frac{i}{4}(A_2)^{\underline{\dot{a}}}{}_{\underline{\dot{c}}}\,\epsilon^{\underline{\dot{c}\dot{b}}}\,\epsilon^{\dot{a}\dot{b}},$$

$$\{(\eta_2)^{\underline{\dot{a}}\dot{a}}, (\eta_2)^{\underline{\dot{b}}\dot{b}}\}_{\text{PB}} = -\frac{i}{4}(1 - A_1)\,\epsilon^{\underline{\dot{a}\dot{b}}}\,\epsilon^{\dot{a}\dot{b}},$$

$$\{(\chi_1)^{\underline{a}a}, (\chi_2)^{\underline{b}b}\}_{\text{PB}} = -\frac{i}{4}(A_3)^{\underline{a}}{}_{\underline{c}}\,\epsilon^{\underline{cb}}\,\epsilon^{ab},$$

$$\{(\chi_1)^{\underline{a}a}, (\chi_1)^{\underline{b}b}\}_{\text{PB}} = -\frac{i}{4}(1 + A_1)\,\epsilon^{\underline{ab}}\,\epsilon^{ab}, \qquad\qquad \text{(K.1)}$$

$$\{(\eta_1)^{\underline{\dot{a}}\dot{a}}, (\chi_2)^{\underline{b}b}\}_{\text{PB}} = -\frac{i}{4}(A_4)^{\underline{\dot{a}}\dot{a}}{}_{\underline{c}c}\,\epsilon^{\underline{cb}}\,\epsilon^{cb},$$

$$\{(\chi_2)^{\underline{a}a}, (\chi_2)^{\underline{b}b}\}_{\text{PB}} = -\frac{i}{4}(1 - A_1)\,\epsilon^{\underline{ab}}\,\epsilon^{ab},$$

$$\{(\eta_2)^{\underline{\dot{a}}\dot{a}}, (\chi_1)^{\underline{b}b}\}_{\text{PB}} = +\frac{i}{4}(A_4)^{\underline{\dot{a}}\dot{a}}{}_{\underline{c}c}\,\epsilon^{\underline{cb}}\,\epsilon^{cb},$$

with the coefficients A_i given to quadratic order by

$$
\begin{aligned}
A_1 &= -\tfrac{1}{2}\epsilon^{\underline{ij}}(z_{\underline{i}}\dot{z}_{\underline{j}} - y_{\underline{i}}\dot{y}_{\underline{j}}), \\
A_2 &= -\tfrac{\tilde{q}}{2}\tilde{\gamma}^{34}(z\cdot\dot{z} - y\cdot\dot{y}) + \tfrac{\tilde{q}}{2}\tilde{\gamma}^{34}\tilde{\gamma}^{\underline{ij}}(z_{\underline{i}}\dot{y}_{\underline{j}} + \dot{z}_{\underline{i}}y_{\underline{j}}), \\
A_3 &= +\tfrac{\tilde{q}}{2}\gamma^{34}(z\cdot\dot{z} + y\cdot\dot{y}) + \tfrac{\tilde{q}}{2}\gamma^{34}\gamma^{\underline{ij}}(z_{\underline{i}}\dot{y}_{\underline{j}} - \dot{z}_{\underline{i}}y_{\underline{j}}), \\
A_4 &= +\tfrac{\tilde{q}}{2}\tilde{\gamma}^{\underline{i}}\tilde{\tau}^{k}(z_{\underline{i}} - y_{\underline{i}})\dot{x}_k.
\end{aligned}
\qquad\qquad \text{(K.2)}
$$

Appendix L

S-matrix elements

Here we give the components of the $\mathfrak{su}(1|1)$-invariant S-matrices of section 5.6.1, in terms of the Zhukovski variables (5.149). The components of $\mathcal{S}^{\mathrm{LL}}$ are

$$A_{pq}^{\mathrm{LL}} = 1,$$

$$B_{pq}^{\mathrm{LL}} = \left(\frac{x_{\mathrm{L}\,p}^-}{x_{\mathrm{L}\,p}^+}\right)^{1/2} \frac{x_{\mathrm{L}\,p}^+ - x_{\mathrm{L}\,q}^+}{x_{\mathrm{L}\,p}^- - x_{\mathrm{L}\,q}^+},$$

$$C_{pq}^{\mathrm{LL}} = \left(\frac{x_{\mathrm{L}\,p}^- \, x_{\mathrm{L}\,q}^+}{x_{\mathrm{L}\,p}^+ \, x_{\mathrm{L}\,q}^-}\right)^{1/2} \frac{x_{\mathrm{L}\,q}^- - x_{\mathrm{L}\,q}^+}{x_{\mathrm{L}\,p}^- - x_{\mathrm{L}\,q}^+} \frac{\eta_p^{\mathrm{L}}}{\eta_q^{\mathrm{L}}},$$

$$D_{pq}^{\mathrm{LL}} = \left(\frac{x_{\mathrm{L}\,q}^+}{x_{\mathrm{L}\,q}^-}\right)^{1/2} \frac{x_{\mathrm{L}\,p}^- - x_{\mathrm{L}\,q}^-}{x_{\mathrm{L}\,p}^- - x_{\mathrm{L}\,q}^+}, \tag{L.1}$$

$$E_{pq}^{\mathrm{LL}} = \frac{x_{\mathrm{L}\,p}^- - x_{\mathrm{L}\,p}^+}{x_{\mathrm{L}\,p}^- - x_{\mathrm{L}\,q}^+} \frac{\eta_q^{\mathrm{L}}}{\eta_p^{\mathrm{L}}},$$

$$F_{pq}^{\mathrm{LL}} = -\left(\frac{x_{\mathrm{L}\,p}^- \, x_{\mathrm{L}\,q}^+}{x_{\mathrm{L}\,p}^+ \, x_{\mathrm{L}\,q}^-}\right)^{1/2} \frac{x_{\mathrm{L}\,p}^+ - x_{\mathrm{L}\,q}^-}{x_{\mathrm{L}\,p}^- - x_{\mathrm{L}\,q}^+},$$

and the components of $\mathcal{S}^{\mathrm{LR}}$ are

$$A_{pq}^{\mathrm{LR}} = \sqrt{\frac{x_{\mathrm{L}\,p}^+}{x_{\mathrm{L}\,p}^-}} \frac{1 - \frac{1}{x_{\mathrm{L}\,p}^+ x_{\mathrm{R}\,q}^-}}{1 - \frac{1}{x_{\mathrm{L}\,p}^- x_{\mathrm{R}\,q}^-}},$$

$$C_{pq}^{\mathrm{LR}} = 1,$$

$$B_{pq}^{\mathrm{LR}} = -\frac{2i}{h} \sqrt{\frac{x_{\mathrm{L}\,p}^- \, x_{\mathrm{R}\,q}^+}{x_{\mathrm{L}\,p}^+ \, x_{\mathrm{R}\,q}^-}} \frac{\eta_p^{\mathrm{L}} \eta_q^{\mathrm{R}}}{x_{\mathrm{L}\,p}^- x_{\mathrm{R}\,q}^+} \frac{1}{1 - \frac{1}{x_{\mathrm{L}\,p}^- x_{\mathrm{R}\,q}^-}},$$

$$D_{pq}^{\mathrm{LR}} = \sqrt{\frac{x_{\mathrm{L}\,p}^+ \, x_{\mathrm{R}\,q}^+}{x_{\mathrm{L}\,p}^- \, x_{\mathrm{R}\,q}^-}} \frac{1 - \frac{1}{x_{\mathrm{L}\,p}^+ x_{\mathrm{R}\,q}^+}}{1 - \frac{1}{x_{\mathrm{L}\,p}^- x_{\mathrm{R}\,q}^-}}, \tag{L.2}$$

$$F_{pq}^{\mathrm{LR}} = \frac{2i}{h} \sqrt{\frac{x_{\mathrm{L}\,p}^+ \, x_{\mathrm{R}\,q}^+}{x_{\mathrm{L}\,p}^- \, x_{\mathrm{R}\,q}^-}} \frac{\eta_p^{\mathrm{L}} \eta_q^{\mathrm{R}}}{x_{\mathrm{L}\,p}^+ x_{\mathrm{R}\,q}^+} \frac{1}{1 - \frac{1}{x_{\mathrm{L}\,p}^- x_{\mathrm{R}\,q}^-}},$$

$$E_{pq}^{\mathrm{LR}} = -\sqrt{\frac{x_{\mathrm{R}\,q}^+}{x_{\mathrm{R}\,q}^-}} \frac{1 - \frac{1}{x_{\mathrm{L}\,p}^- x_{\mathrm{R}\,q}^+}}{1 - \frac{1}{x_{\mathrm{L}\,p}^- x_{\mathrm{R}\,q}^-}}.$$

Components of the other S-matrices can be found using left-right symmetry.

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