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A family of graded decomposition numbers for diagrammatic Cherednik algebras

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We provide an algorithmic description of a family of graded decomposition numbers for diagrammatic Cherednik
algebras in terms of affine Kazhdan–Lusztig polynomials.

Introduction

Rational Cherednik algebras arise as degenerations of Cherednik's double affine Hecke algebra, a tool which
first rose to prominence in the proof of the MacDonald constant term conjectures. Rational Cherednik algebras
have become hugely popular of late due to their vast array of connections with other mathematical objects – in
particular with the theory of symplectic resolutions, Hilbert schemes, and the representation theory of complex
reflection groups.

Given a *weighting* $\theta \in \mathbb{R}^\ell$ and an *e-multicharge* $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ one can obtain a *charge* $\underline{s} := s(\kappa, \theta) \in \mathbb{Z}^\ell$ via
a process of *Uglovation* as in [Web]. In [Web], Webster defines a finite dimensional, graded, cellular algebra,
$A(n, \theta, \kappa)$, whose module category provides a '2-analogue' of the category $\mathcal{O}^{\underline{s}}$ of the rational Cherednik algebra
of type $G(\ell, 1, n)$ and charge $\underline{s}$ introduced in [GGOR03]. These higher representation theoretic objects often have
forbidding diagrammatic presentations. Their dividends lie in possessing bases and multiplication rules which
are intimately related to their representation theoretic structure.

In this paper, we initiate the combinatorial study of these algebras (this is continued in [BS]). We define the
quiver Temperley–Lieb algebra, $\mathrm{TL}_n(\kappa)$, of type $G(\ell, 1, n)$, to be a certain saturated quotient of $A(n, \theta, \kappa)$ for an
adjacency-free e-multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ and a FLOTW weighting $\theta \in \mathbb{R}^\ell$ (as in [FLOTW99], see Section 3 for
details). We embed affine Kazhdan–Lusztig theory into the combinatorics of $\mathrm{TL}_n(\kappa)$ by interpreting the basis of
this algebra as being indexed by orbits of paths in a Euclidean space under the action of an affine Weyl group
of type $\widehat{A}_{\ell-1}$. Strikingly, one can run Soergel's algorithm internally within the graded basis of the algebra; this
allows us to immediately deduce that the decomposition numbers are given by the associated affine Kazhdan–
Lusztig polynomials! Our proof entirely bypasses the technical machinery of sheaf-theory and D-modules which
are usually essential in proving such results.

Theorem. The graded decomposition numbers for an e-regular block of the quiver Temperley–Lieb algebra of
type $G(\ell, 1, n)$ are given by (non-parabolic) affine Kazhdan–Lusztig polynomials of type $\widehat{A}_{\ell-1}$.   □

The decomposition matrix of $\mathrm{TL}_n(\kappa)$ appears as the submatrix of that of $A(n, \theta, \kappa)$ labelled by one-column
ℓ-multipartitions of n. Therefore, under the equivalence of [Web, Theorem A] we obtain the following corollary.

Corollary. Let $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ be an adjacency-free e-multicharge, $\theta \in \mathbb{R}^\ell$ be a FLOTW weighting, and let
$\underline{s} := s(\kappa, \theta) \in \mathbb{Z}^\ell$ denote the associated charge. The decomposition numbers of the category $\mathcal{O}^{\underline{s}}$ labelled by one-
column e-regular multipartitions are given by the (evaluations at 1 of) non-parobolic affine Kazhdan–Lusztig
polynomials of type $\widehat{A}_{\ell-1}$.   □

In a marked difference from [Gro], we see that it is $\widehat{\mathfrak{sl}}_\ell$ which controls the representation theory of
these rational Cherednik algebras, where ℓ is the the level, rather than the quantum characteristic. Our
alcove-geometric description comes complete with a translation principle; it also allows us to deduce that the
decomposition numbers are stable as the rank n tends to infinity. Thus, we conjecture that the algebras considered

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here are asymptotically related (as the rank tends to infinity) to affine Kac–Moody algebras (see [Kas90]) and in finite rank to the *generalised blob algebras* (see [MW03]). In the level 2 case, the blob algebra first arose in the study of two-dimensional Potts models [MS94], and has subsequently been related to the Virasoro algebra [GJSV13] in the limit as n tends to infinity.

The representation theory of rational Cherednik algebras in characteristic zero has received a great deal of attention of late; in particular Rouquier conjectured in [Rou08, Section 6.5] that the decomposition numbers of these algebras are equal to coefficients in Uglov's 'twisted' Fock spaces. This conjecture now has three independent proofs, [RSVV, Los, Web].

In type $G(1, 1, n)$ there are two beautiful procedures for computing these coefficients, the LLT and Soergel algorithms. The LLT algorithm [LLT96] proceeds via explicit ladder combinatorics and Gaussian elimination; the (computationally more efficient [GW99]) Soergel algorithm is given in terms of paths in an alcove geometry. In type $G(\ell, 1, n)$, there exist several higher-level analogues of the LLT algorithm for computing these coefficients, [Ugl99, Jac05, Yvo07, Fay10], however the beautiful ladder combinatorics developed in [LLT96] is noticeably absent. While our algorithm only computes a single slice of the wider family of these higher-level coefficients, it does benefit from being more explicit and efficient; it also reveals an unforeseen connection between the higher level and classical cases, and brings to light the aforementioned stability and translation principle.

In order to clarify the above, let's consider an example. We will omit technical details and definitions at this stage, and instead concentrate on giving a flavour of the combinatorics that is involved. Let $\ell = 3$, $n = 13$, $e = 8$, $\kappa = (0, 4, 6)$. We shall consider a single block/linkage class of the algebra $\mathrm{TL}_{13}(\kappa)$. We identify a one-column multipartition with a point in $\mathbb{R}^3$ via the map $(1^{\lambda_1}, 1^{\lambda_2}, 1^{\lambda_3}) \mapsto \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \lambda_3 \varepsilon_3$. We then let $\mathbb{R}^2$ denote the quotient space of $\mathbb{R}^3$ by the relation $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. The affine Weyl group of type $\widehat{A}_2$ acts on $\mathbb{R}^2$ fixing the point $-\rho$ for $\rho = e(1, 1, 1) - \kappa = (8, 4, 2)$.
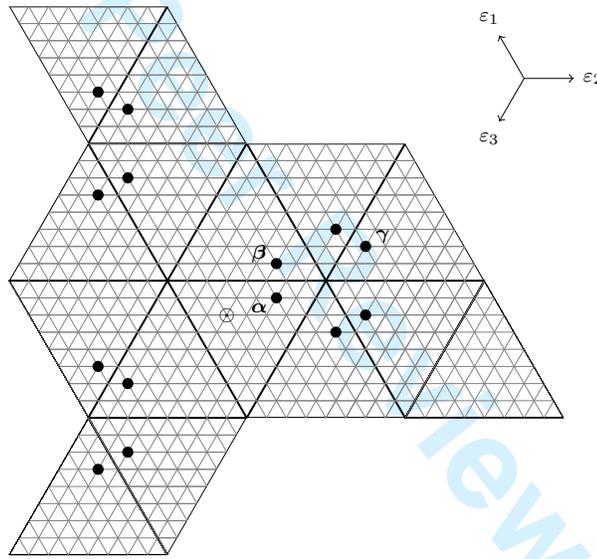
Fig. 1: The black points label the multipartitions of a block of $\mathrm{TL}_{13}(\kappa)$. The origin is labelled as $\odot$, the points $\alpha = (4, 6, 3)$, $\beta = (5, 6, 2)$ and $\gamma = (4, 9, 0)$ are also marked. The thick black lines denote the hyperplanes for the ρ-shifted action of the Weyl group.

For a given μ, we have an associated path ω^μ, from the origin to μ. Given λ a point in this space, we look at paths which may be obtained by folding-up the path ω^μ along hyperplanes so that it terminates at λ (as illustrated shortly); we denote the set of such paths by $\mathrm{Path}(\lambda, \omega^\mu)$.

Each path has an associated degree which can be calculated by running Soergel's (cancellation-free) algorithm along this path. This degree changes by $+1, 0,$ or -1 whenever the path steps onto or off-of an alcove wall. The key to working with the quiver Temperley–Lieb algebras is the following observation,

$$\mathrm{Dim}_t(\Delta_\mu(\lambda)) = \sum_{\omega \in \mathrm{Path}(\lambda, \omega^\mu)} t^{\deg(\omega)}. \tag{†}$$

From this, (and the conditions on our paths) it is immediate that a necessary condition for $[\Delta(\lambda) : L(\mu)] \neq 0$ is that $\ell(\mu) > \ell(\lambda)$ in the length function associated to our geometry. For a fixed λ, we calculate the decomposition

numbers $[\Delta(\lambda) : L(\mu)]$ by running Soergel's algorithm not once, but many times: we run the algorithm to each point μ such that $\ell(\mu) > \ell(\lambda)$. This is a dual set-up to that usually considered.

As n tends to infinity, we find that there are infinitely many μ such that $\ell(\mu) > \ell(\lambda)$; the dimension of the standard module $\Delta(\lambda)$ and the number of composition factors of $\Delta(\lambda)$ also tend to infinity as n becomes arbitrarily large. Fixing a value of $n \in \mathbb{N}$ truncates the set of weights μ in our Euclidean space to a finite set which labels representations of $\mathrm{TL}_n(\kappa)$. We shall see that the decomposition numbers are stable under this limiting behaviour.

For example, let $\gamma = (4, 9, 0)$; we wish to calculate the dimension of $\Delta_\gamma(\lambda)$ for λ in the above set. Associated to the point γ is the path ω^γ, given by

$$(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2)$$

and pictured in Figure 2. There are a total of 2^3 distinct paths which may be obtained from this path by a series of reflections (as our path passes through three alcove walls). For brevity, we truncate our diagrams so as to only consider alcoves between the origin and γ. The eight paths are listed in Figures 2, 3, and 4.
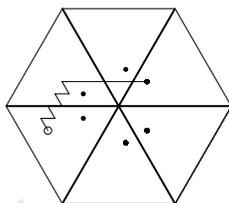
Fig. 2: Our fixed path ω^γ from the origin to $\gamma = (4, 9, 0)$. (The space has been cropped to only include points less than or equal to γ in the dominance ordering.)
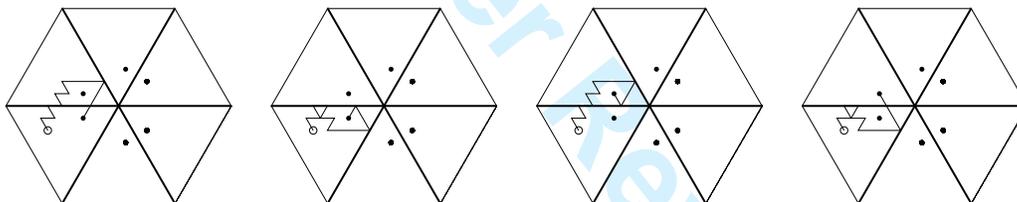
Fig. 3: The leftmost two paths are in $\mathrm{Path}(\alpha, \omega^\gamma)$ and are of degrees 1 and 3 respectively. The final two paths are the elements of $\mathrm{Path}(\beta, \omega^\gamma)$ and are of degrees 0 and 2, respectively.
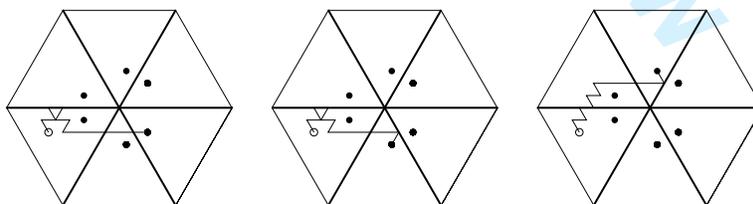
Fig. 4: The unique paths in $\mathrm{Path}((2, 9, 2), \omega^\gamma)$, in $\mathrm{Path}((2, 8, 3), \omega^\gamma)$, and in $\mathrm{Path}((5, 8, 0), \omega^\gamma)$ of degrees 1, 2, and 1 respectively.

Those familiar with Soergel's algorithm will recognise the degrees of the paths listed in the figures, (see Section 1.4 for more details). In general, we will see that at each stage in our algorithm, we remove earlier subpatterns of paths; the surviving paths correspond to decomposition numbers. For example, from Figure 3 and equation (†) we deduce that

$$\mathrm{Dim}_t(\Delta_\gamma(\alpha)) = t^3 + t^1 \quad \mathrm{Dim}_t(\Delta_\gamma(\beta)) = t^2 + t^0.$$

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Grouping together the path of degree zero in $\text{Path}(\beta, \omega^\gamma)$ and the path of degree 1 in $\text{Path}(\alpha, \omega^\gamma)$ we obtain a subpattern that is removed under Soergel's algorithm. The remaining six paths in Figures 2 to 4 survive under Soergel's procedure and so the degrees of these paths provide the second column in the decomposition matrix of (a block of) $\text{TL}_{13}((0, 4, 6))$, in Figure 5. Repeating the procedure, one can obtain the remainder of the decomposition matrix.

$$
\begin{array}{c|ccccccccccccc}
(13,0,0) & 1 \\
(4,9,0) & t & 1 \\
(12,1,0) & t & \cdot & 1 \\
(10,0,3) & t & \cdot & \cdot & 1 \\
(2,0,11) & t & \cdot & \cdot & \cdot & 1 \\
(2,9,2) & t^2 & t & \cdot & \cdot & t & 1 \\
(5,8,0) & t^2 & t & t & \cdot & \cdot & \cdot & 1 \\
(5,0,8) & t^2 & \cdot & \cdot & t & t & \cdot & \cdot & 1 \\
(10,1,2) & t^2 & \cdot & t & t & \cdot & \cdot & \cdot & \cdot & 1 \\
(2,1,10) & t^2 & \cdot & t & \cdot & t & \cdot & \cdot & \cdot & \cdot & 1 \\
(5,6,2) & t+t^3 & t^2 & t^2 & t^2 & t^2 & t & t & t & t & \cdot & 1 \\
(2,8,3) & t^3 & t^2 & t^2 & \cdot & t^2 & t & t & \cdot & \cdot & t & \cdot & 1 \\
(4,1,8) & t^3 & \cdot & t^2 & t^2 & t^2 & \cdot & \cdot & t & t & t & \cdot & \cdot & 1 \\
(4,6,3) & t^2+t^4 & t^3 & t^3 & t^3 & t^3 & t^2 & t^2 & t^2 & t^2 & t^2 & t & t & t & 1 \\
\end{array}
$$

Fig. 5: The decomposition matrix of a block of $\text{TL}_{13}((0, 4, 6))$.

1   Soergel path algebras

In this section (inspired by [MW03, Section 3]), we define an abstract family of algebras whose bases possess desirable properties. The combinatorics of these algebras is controlled by orbits of paths in a Euclidean space.

1.1   Graded cellular algebras with highest weight theories

The algebras in which we shall be interested are all examples of graded cellular algebras, as in [HM10]. We extend their definition by adding two extra axioms, which will afford us extra tools for calculating the graded characters of the modules of these algebras.

Definition 1.1 (cf. Definition 2.1 of [HM10]). Suppose that A is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra which is of finite rank over $\mathbb{C}$. We say that A is a *graded cellular algebra with a highest weight theory* if the following conditions hold.

The algebra is equipped with a *cell datum* $(\Lambda, \mathcal{T}, C, \deg)$, where $(\Lambda, \trianglerighteq)$ is the *weight poset*. For each $\lambda, \mu \in \Lambda$, such that $\lambda \trianglerighteq \mu$, we have a finite set, denoted $\mathcal{T}(\lambda, \mu)$, and we let $\mathcal{T}(\lambda) = \cup_\mu \mathcal{T}(\lambda, \mu)$. There exist maps

$$
C : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \to A; \quad \text{and} \quad \deg : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \to \mathbb{Z}
$$

such that C is injective. We denote $C(\mathsf{S}, \mathsf{T}) = c_{\mathsf{ST}}^\lambda$ for $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, and

1. Each element c_{ST}^λ is homogeneous of degree

$$
\deg(c_{\mathsf{ST}}^\lambda) = \deg(\mathsf{S}) + \deg(\mathsf{T}),
$$

   for $\lambda \in \Lambda$ and $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$.

2. The set $\{c_{\mathsf{ST}}^\lambda \mid \mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda), \lambda \in \Lambda\}$ is a $\mathbb{C}$-basis of A.

3. If $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, for some $\lambda \in \Lambda$, and $a \in A$ then there exist scalars $r_{\mathsf{SU}}(a)$, which do not depend on T, such that

$$
a c_{\mathsf{ST}}^\lambda = \sum_{\mathsf{U} \in \mathcal{T}(\lambda)} r_{\mathsf{SU}}(a) c_{\mathsf{UT}}^\lambda \pmod{A^{\triangleright \lambda}},
$$

   where $A^{\triangleright \lambda}$ is the $\mathbb{C}$-submodule of A spanned by

$$
\{c_{\mathsf{QR}}^\mu \mid \mu \triangleright \lambda \text{ and } \mathsf{Q}, \mathsf{R} \in \mathcal{T}(\mu)\}.
$$

4. The $\mathbb{C}$-linear map $* : A \to A$ determined by $(c_{\mathsf{ST}}^\lambda)^* = c_{\mathsf{TS}}^\lambda$, for all $\lambda \in \Lambda$ and all $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, is an anti-isomorphism of A.

5. The identity 1_A of A has a decomposition $1_A = \sum_{\lambda \in \Lambda} 1_\lambda$ into pairwise orthogonal idempotents 1_λ.

6. For $\mathsf{S} \in \mathcal{T}(\lambda, \mu)$, $\mathsf{T} \in \mathcal{T}(\lambda, \nu)$, we have that $1_\mu c_{\mathsf{ST}}^\lambda 1_\nu = c_{\mathsf{ST}}^\lambda$. There exists a unique element $\mathsf{T}^\lambda \in \mathcal{T}(\lambda, \lambda)$, and $c_{\mathsf{T}^\lambda \mathsf{T}^\lambda}^\lambda = 1_\lambda$.

$\square$

Unless otherwise stated, all results in this section follow from [HM10]. Suppose that A is a graded cellular algebra with a highest weight theory. Given any $\lambda \in \Lambda$, the *graded standard module* $\Delta(\lambda)$ is the graded left A-module

$$\Delta(\lambda) = \bigoplus_{\substack{\mu \in \Lambda \\ z \in \mathbb{Z}}} \Delta_\mu(\lambda)_z,$$

where $\Delta_\mu(\lambda)_z$ is the $\mathbb{C}$ vector-space with basis $\{c_{\mathsf{S}}^\lambda \mid \mathsf{S} \in \mathcal{T}(\lambda, \mu)$ and $\deg(\mathsf{S}) = z\}$. The action of A on $\Delta(\lambda)$ is given by

$$a c_{\mathsf{S}}^\lambda = \sum_{\mathsf{U} \in \mathcal{T}(\lambda)} r_{\mathsf{SU}}(a) c_{\mathsf{U}}^\lambda,$$

where the scalars $r_{\mathsf{SU}}(a)$ are the scalars appearing in condition (3) of Definition 1.1. Suppose that $\lambda \in \Lambda$. There is a bilinear form $\langle \ , \ \rangle_\lambda$ on $\Delta(\lambda)$ which is determined by

$$c_{\mathsf{US}}^\lambda c_{\mathsf{TV}}^\lambda \equiv \langle c_{\mathsf{S}}^\lambda, c_{\mathsf{T}}^\lambda \rangle_\lambda c_{\mathsf{UV}}^\lambda \pmod{A^{\rhd \lambda}},$$

for any $\mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V} \in \mathcal{T}(\lambda)$.

Proposition 1.2. If A is a graded cellular algebra with a highest weight theory, then A is a quasi-hereditary algebra in the sense of [CPS88]. $\square$

Proof. Condition (6) of the definition implies that $\Delta_\lambda(\lambda)$ is 1-dimensional and spanned by $c_{\mathsf{T}^\lambda}^\lambda$. Therefore

$$\langle c_{\mathsf{S}}, c_{\mathsf{T}^\lambda} \rangle_\lambda = \begin{cases} 1 & \text{if } \mathsf{S} = \mathsf{T}^\lambda \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the rank of the Gram matrix is non-zero and so the algebra is quasi-hereditary, by [GL96, Remark 3.10]. $\blacksquare$

Let t be an indeterminate over $\mathbb{N}_0$. If $M = \oplus_{z \in \mathbb{Z}} M_z$ is a free graded $\mathbb{C}$-module, then its *graded dimension* is the Laurent polynomial

$$\mathrm{Dim}_t(M) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{C}} M_k) t^k.$$

If M is a graded A-module and $k \in \mathbb{Z}$, define $M\langle k \rangle$ to be the same module with $(M\langle k \rangle)_i = M_{i-k}$ for all $i \in \mathbb{Z}$. We call this a *degree shift* by k. If M is a graded A-module and L is a graded simple module let $[M : L\langle k \rangle]$ be the multiplicity of $L\langle k \rangle$ as a graded composition factor of M, for $k \in \mathbb{Z}$.

Suppose that A is a graded cellular algebra with a highest weight theory. For every $\lambda \in \Lambda$, define $L(\lambda)$ to be the quotient of the corresponding standard module $\Delta(\lambda)$ by the radical of the bilinear form $\langle \ , \ \rangle_\lambda$. This module is graded (by [HM10, Lemma 2.7]) and simple, and in fact every simple module is of the form $L(\lambda)\langle k \rangle$ for some $k \in \mathbb{Z}$, $\lambda \in \Lambda$. We let $L_\mu(\lambda)$ denote the μ-weight space $1_\mu L(\lambda)$. The *graded decomposition matrix* of A is the matrix $\mathbf{D}_A(t) = (d_{\lambda\mu}(t))$, where

$$d_{\lambda\mu}(t) = \sum_{k \in \mathbb{Z}} [\Delta(\lambda) : L(\mu)\langle k \rangle] \, t^k,$$

for $\lambda, \mu \in \Lambda$. The following proposition is a key ingredient in our proof of the main result of this paper, and follows immediately from [HM10, Proposition 2.18].

Proposition 1.3. If $\mu \in \Lambda$ then $\mathrm{Dim}_t(L(\mu)) \in \mathbb{N}_0[t + t^{-1}]$. $\square$

Given $\lambda, \mu \in \Lambda$ such that $\lambda \rhd \mu$, we say that λ and μ are *tableau-linked* if the set $\mathcal{T}(\lambda, \mu)$ is non-empty. The equivalence classes of the equivalence relation on Λ generated by this tableau-linkage are called the *tableau-blocks* of A.

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Proposition 1.4. [The Linkage Principle] If $\lambda, \mu \in \Lambda$ label simple modules in the same block of A, then λ and μ are tableau-linked. $\qquad\qquad\square$

Proof. It is clear that a necessary condition for $\text{Dim}_t(\text{Hom}(P(\lambda), \Delta(\mu))) = d_{\lambda\mu}(t) \neq 0$, is that $\mathcal{T}(\lambda, \mu) \neq \emptyset$. The result then follows from [GL96, (3.9.8)]. $\qquad\qquad\blacksquare$

This result inspires the next section, in which we connect tableaux to paths in an alcove geometry.

1.2   The alcove geometry

We shall assume standard facts concerning root systems, see [Bou02]. Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ be a set of formal symbols and set

$$E_r = \bigoplus_{i=1}^r \mathbb{R}\varepsilon_i$$

to be the r-dimensional real vector space with basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$. We have an inner product $\langle\,,\,\rangle$ given by extending linearly the relations

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}$$

for all $1 \leqslant i, j \leqslant r$, where $\delta_{i,j}$ is the Kronecker delta.

Let Φ denote a root system embedded in E_r as in [Bou02, Plates I to IX]. We take R^+ to be the set of positive roots. For each $\alpha \in \Phi$ there is a unique coroot $\alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. For $e \in \mathbb{N}$ we let W^e denote the affine reflection group generated by the reflections $s_{\alpha,me}$ (for $\alpha \in \Phi$, $m \in \mathbb{Z}$) given by

$$s_{\alpha,me}(x) = x - (\langle x, \alpha^\vee \rangle - me)\alpha$$

for all $x \in E_r$. Define $W = W^\infty$ to be the subgroup of W^e generated by the reflections $s_{\alpha,0}$ for $\alpha \in \Phi$.

Now, given $\rho \in E_r$, we shall always consider the shifted action of W^e by ρ given by

$$w \cdot x = w(x + \rho) - \rho$$

for all $w \in W^e$ and $x \in E_r$. We regard $s_{\alpha,me}$ as a reflection with respect to the hyperplane

$$h_{\alpha,me} = \{\lambda \in E_r \mid \langle \lambda + \rho, \alpha^\vee \rangle = me\}.$$

The reflection group W^e acting on E_r defines a system of facets. A *facet* is a non-empty subset of E_r of the form

$$\mathfrak{f} = \{\lambda \in E_r \mid \langle \lambda + \rho, \alpha^\vee \rangle = m_\alpha e \text{ for all } \alpha \in R_+^0(\mathfrak{f}),$$
$$(m_\alpha - 1)e < \langle \lambda + \rho, \alpha^\vee \rangle < m_\alpha e \text{ for all } \alpha \in R_+^1(\mathfrak{f})\},$$

for suitable integers $m_\alpha \in \mathbb{Z}$ and a disjoint decomposition $R^+ = R_+^0(\mathfrak{f}) \cup R_+^1(\mathfrak{f})$. A facet, $\mathfrak{f}$, is called an *alcove* if $|R_+^0(\mathfrak{f})| = 0$ and a *wall* if $|R_+^0(\mathfrak{f})| = 1$. A point $x \in \mathfrak{f}$ is called *e-regular* (respectively *e-singular*) if $|R_+^0(\mathfrak{f})| = 0$ (respectively $|R_+^0(\mathfrak{f})| \geqslant 1$). We extend this terminology to the orbit, $W^e \cdot x$, in the obvious fashion.

Let h denote the Coxeter number of the chosen root system Φ in E_r. We assume that $e > h$ and that ρ is chosen so that the origin is always contained in an alcove, which we refer to as the *fundamental alcove*. The *closure*, $\bar{\mathfrak{f}}$, of a facet, $\mathfrak{f}$, is defined as follows

$$\bar{\mathfrak{f}} = \{\lambda \in E_r \mid \langle \lambda + \rho, \alpha^\vee \rangle = m_\alpha e \text{ for all } \alpha \in R_+^0(\mathfrak{f}),$$
$$(m_\alpha - 1)e \leqslant \langle \lambda + \rho, \alpha^\vee \rangle \leqslant m_\alpha e \text{ for all } \alpha \in R_+^1(\mathfrak{f})\}.$$

We define a length function on the set of alcoves as follows. We say that two alcoves, $\mathfrak{a}_i, \mathfrak{a}_j$ are adjacent if $\bar{\mathfrak{a}}_i \cap \bar{\mathfrak{a}}_j$ contains a wall. Given any pair of alcoves $\mathfrak{a}$ and $\mathfrak{b}$, there exists a chain of adjacent alcoves,

$$\mathfrak{a} = \mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell = \mathfrak{b},$$

and we define the *length* $\ell(\mathfrak{a}, \mathfrak{b})$ to be the minimal value of ℓ in such a chain. We extend this notation to points in alcoves in the obvious manner.

1.3 **Paths in an alcove geometry**

In this section we fix $e > h$ (so that our alcoves contain lattice points) and will define certain paths in our alcove geometry. It will be convenient to associate paths with non-negative linear combinations of elements, and so we will first consider a larger space of paths from which our alcove geometry will arise as a quotient. Fix some $s > r$ and consider the space E_s and set $\Pi = \bigoplus_{i=1}^{s} \mathbb{N}\varepsilon_i \subset E_s$. The set $\Pi_n = \left\{ \sum_{i=1}^{s} a_i \varepsilon_i \mid \sum_{i=1}^{s} a_i = n \right\} \subset \Pi$ will correspond to the set of terminating points of our paths of length n.

We will assume that there is a projection map $\phi : E_s \to E_r$ such that $\phi(\varepsilon_i) = \varepsilon_i$ for $1 \leqslant i \leqslant r$ and

$$\phi(\Pi) = \bigoplus_{i=1}^{r} \mathbb{Z}\varepsilon_i \subset E_r.$$

We will abuse notation and write ε_i for the image of ε_i in E_r for all $1 \leqslant i \leqslant s$.

Example 1.5. Let Φ be a root system of type A_r, in which case we can take $s = r + 1$ and

$$\phi(\varepsilon_i) = \begin{cases} \varepsilon_i & \text{if } 1 \leqslant i \leqslant r \\ -(\varepsilon_1 + \cdots + \varepsilon_r) & \text{if } i = r + 1. \end{cases}$$

Here we have identified E_r with the hyperplane in E_{r+1} given by $\varepsilon_1 + \cdots + \varepsilon_r = 0$. This will be the example we focus on for the rest of the paper.

Now let Φ be any root system embedded into E_r as in [Bou02, Plates I to IX]. We can take $s = 2r$ and

$$\phi(\varepsilon_i) = \begin{cases} \varepsilon_i & \text{if } 1 \leqslant i \leqslant r \\ -\varepsilon_{2r+1-i} & \text{if } r + 1 \leqslant i \leqslant 2r. \end{cases}$$

$\square$

Given $k \in \mathbb{N}$, we let **k** denote the set $\{1, 2, \ldots, k\}$. Given a map $w : \mathbf{n} \to \mathbf{s}$ we define points $\omega(k) \in E_r$ by

$$\omega(k) = \sum_{1 \leqslant i \leqslant k} \varepsilon_{w(i)},$$

for $1 \leqslant k \leqslant n$. We define the associated path of length n in our alcove geometry E_r by $\omega = (\omega(0), \omega(1), \omega(2), \ldots, \omega(n))$, where we fix all paths to begin at the origin, so that $\omega(0) = \odot \in E_r$. We let $\omega_{\leqslant k}$ denote the subpath of ω of length k corresponding to $w_{\leqslant k} = w|_{\mathbf{k}} : \mathbf{k} \to \mathbf{s}$.

Definition 1.6. Fix a path $\omega = (\omega(0), \omega(1), \omega(2), \ldots, \omega(n))$ such that $\omega(0) = \odot \in E_r$. We define a degree function on ω by induction. We set $\deg(\omega(0)) = 0$ and set

$$\deg(\omega_{\leqslant k}) = \deg(\omega_{\leqslant k-1}) + \sum_{\alpha \in \Phi} d_\alpha(\omega, k)$$

where $d_\alpha(\omega, k)$ is defined as follows. Fix $\alpha \in \Phi$, and consider the hyperplanes $h_{\alpha, me}$ for $m \in \mathbb{Z}$. If $\omega(k)$ and $\omega(k+1)$ both lie on some $h_{\alpha, me}$ or if neither lie on some $h_{\alpha, me}$ for $m \in \mathbb{Z}$, then $d_\alpha(\omega, k) = 0$. Otherwise, exactly one of $\omega(k)$ and $\omega(k-1)$ lies on some hyperplane $h_{\alpha, me}$. Removing the hyperplane $h_{\alpha, me}$ leaves two distinct subsets $E_r^+(\alpha, me)$ and $E_r^-(\alpha, me)$ where $\odot \in E_r^-(\alpha, me)$. If $\omega(k-1) \in E_r^-(\alpha, me)$, or $\omega(k) \in E_r^+(\alpha, me)$, then set $d_\alpha(\omega, k) = 0$. If $\omega(k-1) \in E_r^+(\alpha, me)$, then $d_\alpha(\omega, k) = -1$. If $\omega(k) \in E_r^-(\alpha, me)$, then $d_\alpha(\omega, k) = +1$. $\square$

Figure 6 illustrates the four subcases outlined above. In each case the diagram depicts a hyperplane, labelled by $h_{\alpha, me}$, with the corresponding subsets $E_r^+(\alpha, me)$ and $E_r^-(\alpha, me)$ labelled. The incoming/outgoing arrows labels steps onto and off of the hyperplane and the corresponding $d_\alpha(\omega, k)$.

Let ω be a path which passes through a hyperplane $h_{\alpha, me}$ at point $\omega(k)$ (note that k is not necessarily unique). Then, let ω' be the path obtained from ω by applying the reflection $s_{\alpha, me}$ to all the steps in ω after the point $\omega(k)$. In other words, $\omega'(i) = \omega(i)$ for all $1 \leqslant i \leqslant k$ and $\omega(i') = s_{\alpha, me} \cdot \omega(i)$ for $k \leqslant i \leqslant n$. We refer to the path ω' as the reflection of ω in $h_{\alpha, me}$ at point $\omega(k)$ and denote this by $s_{\alpha, me}^k \cdot \omega$. We write $\omega \sim \omega'$ if the path ω can be obtained from ω' by a series of reflections in W^e.

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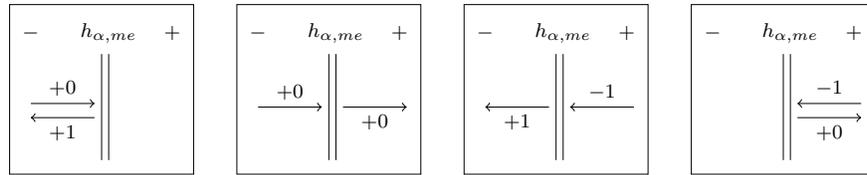
Fig. 6: The four subcases for the values of $d_\alpha(\omega, k)$ as ω crosses a wall. The $\pm$ indicate the distinct subsets E_r^+ and E_r^- of E_r. In each case the first (respectively second) step has its degree recorded as a superscript (respectively subscript).

1.4 Soergel's algorithm for paths

Fix $e > h$. We now recall the classical construction of Soergel's algorithm with respect to a path in the geometry. The procedure outlined below is somewhat simpler, as all points in our geometry belong to the *dominant chamber* [Soe97, Section 4].

Definition 1.7. Let $e > h$, and assume $\mu \in \phi(\Pi_n)$ is an e-regular point. We say that a path ω from $\odot$ to μ of length n is *admissible* if (i) $\deg(\omega_{\leqslant k}) = 0$ for all $1 \leqslant k \leqslant n$, and (ii) whenever $\omega(k)$ lies on two distinct hyperplanes $h_{\alpha, m_1 e}$ and $h_{\beta, m_2 e}$ for some $1 \leqslant k \leqslant n$ this implies that $\langle \alpha, \beta^\vee \rangle = 0$ (we say that the hyperplanes are orthogonal). □

Given any e-regular point $\mu \in \phi(\Pi_n)$, it is easy to see that there exists an admissible path from $\odot$ to μ.

Definition 1.8. For each $\mu \in \phi(\Pi_n)$, we fix one such path, ω^μ. We let $\mathrm{Path}(\lambda, \omega^\mu)$ denote the set of all paths from $\odot$ to λ which may be obtained from ω^μ by a series of reflections. □

Example 1.9. Recall the example from the introduction. Here the geometry is of type $\widehat{A}_2$, $n = 13$, $e = 8$ and $\rho = (8, 4, 2)$. The path ω^γ is recorded in Figure 2. We clearly have that $d_\alpha(\omega, k) = 0$ at all points $1 \leqslant k \leqslant n$ and all $\alpha \in \Phi$. The two leftmost diagrams in Figure 3 are the elements of $\mathrm{Path}((4, 6, 3), \omega^\gamma)$. Let ω (respectively ω') denote the path in the leftmost (respectively rightmost) case. We have that

$$d_{\varepsilon_2 - \varepsilon_3}(\omega, 11) = 1, \quad d_{\varepsilon_1 - \varepsilon_3}(\omega, 12) = -1, \quad d_{\varepsilon_1 - \varepsilon_3}(\omega, 13) = 1$$

are the only non-zero values of $d_\alpha(\omega, k)$ for $1 \leqslant k \leqslant n$ and $\alpha \in \Phi$, and therefore $\deg(\omega) = 1$. We have that

$$d_{\varepsilon_2 - \varepsilon_3}(\omega', 5) = 1, \quad d_{\varepsilon_1 - \varepsilon_3}(\omega', 12) = 1, \quad d_{\varepsilon_1 - \varepsilon_3}(\omega', 13) = 1$$

are the only non-zero values of $d_\alpha(\omega', k)$ for $1 \leqslant k \leqslant n$ and $\alpha \in \Phi$, and therefore $\deg(\omega') = 3$. □

Remark. For μ an e-regular point, and ω an admissible path from $\odot$ to μ, there exist $2^{\ell(\mu)}$ paths ω' such that $\omega' \sim \omega$. □

We say that a path, ω, is an *alcove-wall path* if (i) $\deg(\omega_{\leqslant k}) = 0$ for all $1 \leqslant k \leqslant n$ and (ii) every step lies either on a wall or in an alcove. It is clear that any alcove-wall path is admissible.

Definition 1.10. Let $\mu \in \phi(\Pi_n)$ be an e-regular point and let ω^μ denote our (fixed choice of) path from $\odot$ to μ. For $\lambda \in \phi(\Pi_n)$, we define

$$m_{\omega^\mu}(\lambda) = \sum_{\omega \in \mathrm{Path}(\lambda, \omega^\mu)} t^{\deg(\omega)}.$$

□

Given ω an admissible path of length n, we let $\mathfrak{f}_k$ denote the facet containing the point $\omega(k)$ for $1 \leqslant k \leqslant n$.

Definition 1.11. We fix an admissible path ω from $\odot$ to μ of length n. For $1 \leqslant k \leqslant n$, we let

$$A_+(\omega, k) = \{(\gamma, m_k e) \mid \omega(k) \in h_{\gamma, m_k e}\} \setminus \{(\gamma, m_{k-1} e) \mid \omega(k-1) \in h_{\gamma, m_{k-1} e}\},$$
$$A_-(\omega, k) = \{(\gamma, m_{k-1} e) \mid \omega(k-1) \in h_{\gamma, m'_{k-1} e}\} \setminus \{(\gamma, m_k e) \mid \omega(k) \in h_{\gamma, m'_k e}\}.$$

The orthogonality condition on the admissible path ensures that for $1 \leqslant k \leqslant n$, the set $A_+(\omega, k)$ (respectively $A_-(\omega, k)$) either consists of one element, denoted $\alpha_+(\omega, k)$ (respectively $\alpha_-(\omega, k)$) or is empty. □

Remark. The $\alpha_+(\omega, k)$ (respectively $\alpha_-(\omega, k)$) record the steps in ω which are on to (respectively off of) hyperplanes in the geometry. $\square$

Definition 1.12. Let ω be an admissible path from $\odot$ to μ of length n. For $1 \leqslant k \leqslant n$, we set A_k to be the alcove, minimal in the length ordering, such that $\langle \lambda + \rho, \alpha_+(\omega, i)^\vee \rangle > 0$ for all $\lambda \in A_+(\omega, k)$ and all $0 \leqslant i \leqslant k$ such that $A_+(\omega, k) \neq \emptyset$. We define the *alcove-series* of ω to be the ordered set whose elements are given by the alcoves A_k for $0 \leqslant k \leqslant n$ recorded without repeats and in increasing order. $\square$

Example 1.13. Consider a geometry of type $\widehat{A}_2$ with $\rho = (8, 4, 2)$ and $n = 13$. The path ω^γ in Figure 2 is an alcove-wall path. We let $\overline{\omega}^\gamma$ denote the alcove-wall path

$$(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_1).$$

Both paths pass through (the same) alcoves of length $0, 1, 2, 3$, which we denote by $\mathfrak{a}(i)$ for $i = 0, 1, 2, 3$. We have that

$$A_k^{\omega^\gamma} = \begin{cases} \{\mathfrak{a}(0)\} & \text{for } k = 0, 1, 2, \\ \{\mathfrak{a}(1)\} & \text{for } k = 3, 4, 5, 6, 7, 8, 9, \\ \{\mathfrak{a}(2)\} & \text{for } k = 10, 11, \\ \{\mathfrak{a}(3)\} & \text{for } k = 12, 13; \end{cases} \qquad A_k^{\overline{\omega}^\gamma} = \begin{cases} \{\mathfrak{a}(0)\} & \text{for } k = 0, 1, 2, \\ \{\mathfrak{a}(1)\} & \text{for } k = 3, 4, 5, 6, 7, 8, \\ \{\mathfrak{a}(2)\} & \text{for } k = 9, \\ \{\mathfrak{a}(3)\} & \text{for } k = 10, 11, 12, 13; \end{cases}$$

and so the alcove series in both cases is given by $\{\mathfrak{a}(0), \mathfrak{a}(1), \mathfrak{a}(2), \mathfrak{a}(3)\}$. $\square$

We let $\mathfrak{A}$ denote the set of all alcoves in E_r. We let $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ denote alcoves in our geometry and let $\mathfrak{a}_0, \ldots, \mathfrak{a}_{\ell(\mu)}$ denote the alcove series of an admissible path from $\odot$ to μ. We define maps

$$n_{\mathfrak{a}_{\leqslant i}} : \mathfrak{A} \to \mathbb{N}_0[t] \quad m_{\mathfrak{a}_{\leqslant i}} : \mathfrak{A} \to \mathbb{N}_0[t] \quad e_{\mathfrak{a}_{\leqslant i}} : \mathfrak{A} \to \mathbb{N}_0[t + t^{-1}],$$

where t is a formal parameter, as follows. We have abused notation by writing $\mathfrak{a}_{\leqslant i}$ to remind the reader that at each point these maps are defined with respect to the set of alcoves $\{\mathfrak{a}_j \mid j \leqslant i\}$. We set

$$n_{\mathfrak{a}_{\leqslant i}}(\mathfrak{a}_i) = 1, \quad m_{\mathfrak{a}_{\leqslant i}}(\mathfrak{a}_i) = 1, \quad e_{\mathfrak{a}_{\leqslant i}}(\mathfrak{a}_i) = 1.$$

We define

$$n_{\mathfrak{a}_{\leqslant i}}(\mathfrak{b}) = 0, \quad m_{\mathfrak{a}_{\leqslant i}}(\mathfrak{b}) = 0, \quad e_{\mathfrak{a}_{\leqslant i}}(\mathfrak{b}) = 0$$

whenever $\ell(\mathfrak{b}) \not\leqslant \ell(\mathfrak{a}_i)$. For each adjacent pair of alcoves $\mathfrak{a}_i$ and $\mathfrak{a}_{i+1}$, we let s_i denote the reflection in the hyperplane passing through $\overline{\mathfrak{a}}_i \cap \overline{\mathfrak{a}}_{i+1}$. The closure, $\overline{\mathfrak{b}}$, of any alcove $\mathfrak{b}$ has one wall which is in the W^e-orbit of s_i, and we shall write $s_i \cdot \mathfrak{b}$ for the image of $\mathfrak{b}$ in that wall. Then, with $m_{\mathfrak{a}_{\leqslant i}}$ known, we set

$$m_{\mathfrak{a}_{\leqslant i+1}}(s_i \cdot \mathfrak{b}) = \begin{cases} m_{\mathfrak{a}_{\leqslant i}}(\mathfrak{b}) + t^{-1} m_{\mathfrak{a}_{\leqslant i}}(s_i \cdot \mathfrak{b}), & \ell(s_i \cdot \mathfrak{b}) > \ell(\mathfrak{b}), \\ m_{\mathfrak{a}_{\leqslant i}}(\mathfrak{b}) + t m_{\mathfrak{a}_{\leqslant i}}(s_i \cdot \mathfrak{b}), & \ell(s_i \cdot \mathfrak{b}) < \ell(\mathfrak{b}). \end{cases} \tag{1.1}$$

We refer to this procedure as the *cancellation-free Soergel algorithm*.

Proposition 1.14. Given $e > h$, suppose that μ and λ belong to alcoves $\mathfrak{a}$ and $\mathfrak{b}$ respectively, and furthermore that $\mu \in W^e \cdot \lambda$. We let $\mathfrak{a}_0$ denote the fundamental alcove and $\mathfrak{a}_0, \ldots, \mathfrak{a}_{\ell(\mu)} = \mathfrak{a}$ denote the alcove series of an admissible path ω^μ. We have that $m_{\omega^\mu}(\lambda) = m_{\mathfrak{a}_{\leqslant \ell(\mu)}}(\mathfrak{b})$. $\square$

Proof. For $1 \leqslant i \leqslant \ell(\mu)$, note that the ith hyperplane $\overline{\mathfrak{a}}_i \cap \overline{\mathfrak{a}}_{i+1}$ is the hyperplane given by the ith non-trivial $\alpha_+(m, k)$. This gives the required bijection between paths (obtained from ω^μ by a series of reflections through the $h_{\alpha_+(m,k)}$ for $1 \leqslant k \leqslant n$) and terms in Soergel's cancellation-free algorithm (given by a sequence of alcoves, which are determined by the alcove walls $\overline{\mathfrak{a}}_i \cap \overline{\mathfrak{a}}_{i+1}$ through which we reflect).

The $\alpha_+(\omega^\mu, k)$ and $\alpha_-(\omega^\mu, k')$ for $1 \leqslant k < k' \leqslant n$ come in pairs (whenever we step on to a hyperplane, we must step off of it at some later point). For a pair $1 \leqslant k < k' \leqslant n$, the hyperplanes $h_{\alpha_+(\omega^\mu, k'')}$ and $h_{\alpha_-(\omega^\mu, k'')}$ for $k < k'' < k'$ are orthogonal to $h_{\alpha_+(m,k)}$.

Fix two points $\lambda, \lambda + \varepsilon_i \in E_r$ and suppose that $\lambda \in h_{\alpha, me}$. Assume that $\lambda + \varepsilon_i$ belongs to $E_r^+(\alpha, me)$ or $E_r^-(\alpha, me)$. Let $h_{\beta, m'e}$ denote a hyperplane orthogonal to $h_{\alpha, me}$ and $s_{\beta, m'e}$ denote the reflection through this hyperplane. It is clear that $s_{\beta, me} \cdot (\lambda + \varepsilon_i)$ still belongs to either $E_r^+(\alpha, me)$ or $E_r^-(\alpha, me)$, respectively. (Compare this with the definition of the degree of a path, Definition 1.6.) Note that in general, this would not be true for non-orthogonal hyperplanes.

Therefore the contribution $d_{\alpha_+(m,k)}(\omega, k')$ to the degree given by the step at point $k' - 1$, is the same as if it were taken at point $k + 1$. Thus we can assume that $k' = k + 1$, in other words that our path is an alcove-wall path. Folding up an alcove-wall path, ω^μ, so that it terminates at λ corresponds to tracing one of the terms in the Soergel cancellation-free algorithm, as follows:

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(i) When the path steps from alcove $\mathfrak{b}$ onto the wall $\overline{\mathfrak{b}} \cap \overline{s_i \cdot \mathfrak{b}}$ and through to the alcove $s_i \cdot \mathfrak{b}$, the degree of the path does not change on alcoves (as $-1 + 1 = 0$), as illustrated in the second and third diagrams in Figure 6. This is equivalent to the first term in each of the two cases of equation 1.1.

(ii) When the path steps from alcove $s_i \cdot \mathfrak{b}$ onto the wall $\overline{\mathfrak{b}} \cap \overline{s_i \cdot \mathfrak{b}}$ and then returns to the alcove $s_i \cdot \mathfrak{b}$, the degree either increases or decreases by one, as seen in the first and fourth diagrams in Figure 6, respectively. This is equivalent to the second term in the two cases of equation 1.1.

For ease in the above, we have tacitly assumed that we never simultaneously step off of a hyperplane and on to another hyperplane in the same step (as in ω^γ in Example 1.13). In general, this is not the case (as in $\overline{\omega}^\gamma$ in Example 1.13). Our ignoring of this is justified as the Soergel-degree is given by summing over the Soergel-degrees of the steps from passing through these separate facets (note that in Definition 1.6, the contributions of the d_α for $\alpha \in R^+$ to the sum are independent). ∎

In a similar fashion to above, we assume that $n_{\mathfrak{a}_i}$ has been calculated inductively, and we set

$$n'_{\mathfrak{a}_{\leqslant i+1}}(s_i \cdot \mathfrak{b}) = \begin{cases} n_{\mathfrak{a}_{\leqslant i}}(\mathfrak{b}) + t^{-1} n_{\mathfrak{a}_{\leqslant i}}(s_i \cdot \mathfrak{b}), & \ell(s_i \cdot \mathfrak{b}) > \ell(\mathfrak{b}); \\ n_{\mathfrak{a}_{\leqslant i}}(\mathfrak{b}) + t n_{\mathfrak{a}_{\leqslant i}}(s_i \cdot \mathfrak{b}), & \ell(s_i \cdot \mathfrak{b}) < \ell(\mathfrak{b}); \end{cases}$$

and

$$n_{\mathfrak{a}_{\leqslant i+1}}(\mathfrak{b}) = n'_{\mathfrak{a}_{\leqslant i+1}}(\mathfrak{b}) - \sum_{\{\mathfrak{d} | \ell(\mathfrak{d}) < \ell(\mathfrak{a}_{i+1})\}} (n'_{\mathfrak{a}_{\leqslant i+1}}(\mathfrak{d})|_{t=0}) n_{\mathfrak{d}}(\mathfrak{b}).$$

We refer to this procedure as the *Soergel algorithm*.

Remark. We have that $n'_{a_{\leqslant i+1}}(\mathfrak{b}) = m_{a_{\leqslant i+1}}(\mathfrak{b})$ for $i = 0, 1$. However this is not true for $i \geqslant 2$; this is because the inductive definition of the map $n'_{\mathfrak{a}_{\leqslant i+1}}$ is given in terms of $n_{\mathfrak{a}_{\leqslant i}}$ (not in terms of $n'_{\leqslant i}$) and therefore there are cancellations. □

Finally, with $e_{\mathfrak{a}_{\leqslant i}}$ known by induction, we set

$$e_{\mathfrak{a}_{\leqslant i+1}}(s_i \cdot \mathfrak{c}) = (t + t^{-1}) e_{\mathfrak{a}_{\leqslant i}}(s_i \cdot \mathfrak{c}) + e_{\mathfrak{a}_{\leqslant i}}(\mathfrak{c}) + (n'_{\mathfrak{a}_{\leqslant i+1}}(s_i \cdot \mathfrak{c})|_{t=0})$$

if $\ell(s_i \cdot \mathfrak{c}) > \ell(\mathfrak{c})$, and $e_{\mathfrak{a}_{\leqslant i+1}}(s_i \cdot \mathfrak{c}) = 0$ otherwise. We refer to this procedure as the *character algorithm*.

Remark. It is shown in [Soe97] that $n_{\mathfrak{a}_{\leqslant \ell(\mu)}}(\mathfrak{b})$ is equal to the associated Kazhdan–Lusztig polynomial and is therefore independent of the alcove series (or equivalently, path) taken.

Assume that $\mathfrak{a}_0, \ldots, \mathfrak{a}_{\ell(\mu)}$ is the alcove series of our path ω^μ from $\odot$ to μ. We let $\mathfrak{b}$ denote an alcove in E_r and $\lambda \in W^e \cdot \mu$ denote a point in $\mathfrak{b}$. From now on we denote $e_{\omega^\mu}(\lambda) = e_{\mathfrak{a}_{\leqslant \ell(\mu)}}$ and $n_\mu(\lambda) = n_{\mathfrak{a}_{\leqslant \ell(\mu)}}$. We have chosen this notation to emphasise that the character algorithm is dependent on the choice of path, whereas the Soergel algorithm is not. □

Example 1.15. Let $e = 4$ and $\rho = (4, 2)$ and consider the root system of type $\widehat{A}_1$. The projection onto the space E_1 can be pictured as a horizontal copy of the real line, and $\phi(\Pi)$ can be pictured as the integral points along this line. In order to make the paths as clear as possible, we picture them as paths connecting points in $\mathbb{N}\{\varepsilon_1, \varepsilon_2\} \subset E_2$ as depicted in Figure 7. To obtain the projection of such a path onto E_1, one can simply flatten the paths in the obvious fashion. As pointed out in [PRH14, Pla13], these can be regarded as walks on Pascal's triangle. Let $\mu = (0, 11)$ and let ω^μ denote the path

$$(\varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2).$$

For $\lambda = (4, 7)$, there are two elements $\omega, \omega' \in \text{Path}_{11}(\lambda, \omega^\mu)$, depicted in Figure 7. The former is of degree 2 and the latter of degree 0. In the former case, $d_{\varepsilon_1 - \varepsilon_2}(\omega, 3) = 1$, $d_{\varepsilon_1 - \varepsilon_2}(\omega, 7) = 1$. In the latter case, $d_{\varepsilon_1 - \varepsilon_2}(\omega', 7) = 1$ $d_{\varepsilon_1 - \varepsilon_2}(\omega', 10) = -1$.

Let $\nu = (5, 6)$; there are two elements of $\omega'', \omega''' \in \text{Path}_{11}(\nu, \omega^\mu)$, depicted in Figure 8, of degree 3 and degree 1 respectively. In the former case, $d_{\varepsilon_1 - \varepsilon_2}(\omega'', 3) = 1$, $d_{\varepsilon_1 - \varepsilon_2}(\omega'', 7) = 1$ and $d_{\varepsilon_1 - \varepsilon_2}(\omega'', 11) = 1$. In the latter case, $d_{\varepsilon_1 - \varepsilon_2}(\omega''', 7) = 1$.

Figure 9 records the result of running the (cancellation-free) Soergel algorithm along the path ω^μ. Notice that the algorithm produces $m_{\omega^\mu}(\lambda) = t^2 + 1$ and $n_\mu(\lambda) = t^2$; similarly $m_{\omega^\mu}(\nu) = t^3 + t$ and $n_\mu(\nu) = t^3$. We have that $e_{\omega^\mu}(\lambda) = 1$ and $e_{\omega^\mu}(\nu) = 0$. □
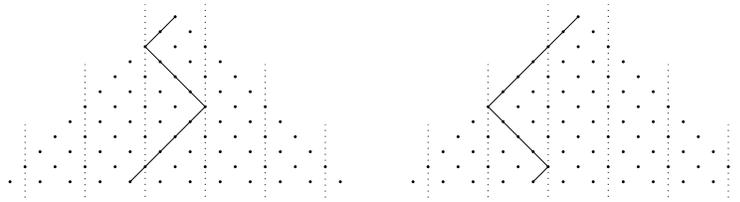
Fig. 7: Two paths $\omega, \omega' \in \text{Path}((4,7), \omega^{(0,11)})$.

Fig. 8: Two paths $\omega'', \omega''' \in \text{Path}((6,5), \omega^{(0,11)})$.

alcove	$\mathfrak{a}_{3'}$	$\mathfrak{a}_{2'}$	$\mathfrak{a}_{1'}$	$\mathfrak{a}_0$	$\mathfrak{a}_1$	$\mathfrak{a}_2$
				1		
			1	t		
		1	t	t^2	t	
	1	t	t^2+1	t^3+t	t^2	t

Fig. 9: This table records the result of running the (cancellation-free) Soergel algorithm along the path ω^μ. The alcoves are labelled by their length and primed (respectively unprimed) if they correspond to an alcove to the left (respectively right) of the origin in the diagrams in Figures 7 and 8.

Proposition 1.16. Let λ, μ denote points belonging to alcoves in E_r. Fix an admissible path ω^μ. Let ν vary over all points such that $\text{Path}(\nu, \omega^\mu) \neq \emptyset$ and $\text{Path}(\lambda, \omega^\nu) \neq \emptyset$. We have that

$$m_{\omega^\mu}(\lambda) = \sum_{\substack{\text{Path}(\nu,\omega^\mu)\neq\emptyset \\ \text{Path}(\lambda,\omega^\nu)\neq\emptyset}} n_\nu(\lambda) e_{\omega^\mu}(\nu).$$

$\square$

Proof. Assume that $\mathfrak{a}_0, \ldots, \mathfrak{a}_{\ell(\mu)}$ is the alcove series of the path ω^μ from $\odot$ to μ. A subpattern in Soergel's algorithm is removed if $(n'_{\mathfrak{a}_{\leqslant i+1}}(\mathfrak{d})|_{t=0}) \neq 0$ for some alcove $\mathfrak{d}$. The subpatterns removed in the n-algorithm are (of course) not removed by the m-algorithm; the e-algorithm will keep track of the leading terms in these subpatterns. The leading term of the subpattern will remain constant unless it is reflected through a hyperplane through the lower closure of the alcove, in which case we multiply the subpattern by $(t + t^{-1})$. This is particularly clear from the alcove-wall path definition of Soergel's algorithm (see also the singular combinatorics for Soergel's algorithm developed in [RH06]). The result then follows from the definitions. ∎

Example 1.17. Let $l = 3$, $n = 13$, $e = 8$, $\rho = (8,4,2)$ and consider the root system of type $\widehat{A}_2$. Take $\alpha = (4,6,3)$, $\beta = (5,6,2)$ and $\gamma = (4,9,0)$. Let ω^γ be the alcove-wall path depicted in Figure 2 in the introduction. The set of elements in $\text{Path}(-, \omega^\gamma)$, together with their degrees, is depicted across Figures 3 and 4. Figures 10 and 11 depict the four steps of running Soegel's algorithm along ω^γ.

Under the Soergel procedure, we remove the subpattern labelled by the zero in the alcove containing the point β. The 'new zero' is recorded by the character algorithm. We have that

$$m_{\omega^\gamma}(\lambda) = e_{\omega^\gamma}(\gamma) n_\gamma(\lambda) + e_{\omega^\gamma}(\beta) n_\beta(\lambda)$$

for any point $\lambda \in \phi(\Pi_n)$. Here $e_{\omega^\gamma}(\beta) = t^0$ and $e_{\omega^\gamma}(\gamma) = t^0$ and $e_{\omega^\gamma}(\lambda) = 0$ otherwise. This rewriting process is depicted in Figure 11.

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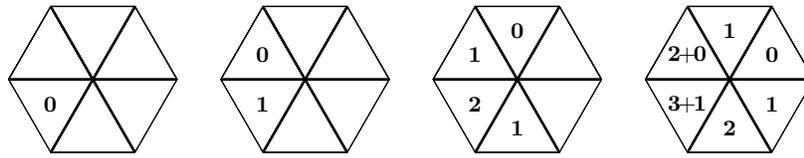
Fig. 10: The first four steps of running Soergel's cancellation-free algorithm along ω^γ. We have recorded the powers of the polynomials only (for example, $2 + 0$ should be read as $t^2 + t^0$).
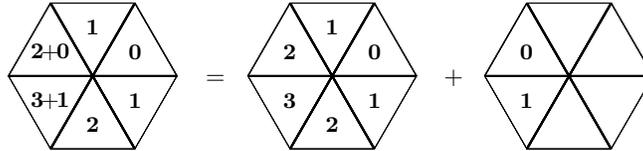
Fig. 11: Rewriting the $m_{\omega^\gamma}(\lambda)$ in terms of $n_\mu(\lambda)$ and $e_{\omega^\gamma}(\mu)$.

□

Example 1.18. Let $l = 3$, $n = 21$, $e = 6$, $\rho = (6, 4, 2)$ and consider the root system of type $\widehat{A}_2$. We leave it as an exercise for the reader to show that

$$m_{\omega^{(4,17,0)}}(\lambda) = n_{(4,17,0)}(\lambda) + n_{(15,4,2)}(\lambda) + (t + t^{-1})n_{(6,9,0)}(\lambda)$$

for any $\lambda \in \phi(\Pi_n)$. This is the smallest example where we find a path in negative degree. In this case,

$$e_{\omega^{(4,17,0)}}(6, 9, 0) = (t + t^{-1}), \quad e_{\omega^{(4,17,0)}}(15, 4, 2) = t^0, \quad e_{\omega^{(4,17,0)}}(4, 17, 0) = t^0.$$

□

1.5 Algebras with Soergel-path bases

Fix $e > h$. We shall now define a family of algebras whose representation theory is governed by paths in Euclidean space and show that the decomposition numbers of such an algebra are given by Soergel's algorithm. Our proof is based on Kleshchev and Nash's algorithm for computing decomposition numbers (see [KN10]).

Definition 1.19. Let $A_n(\rho, e)$ denote a graded cellular algebra with a highest weight theory with respect to some poset Λ_n. Let

$$\Lambda_n \hookrightarrow \phi(\Pi_n) \subset E_r,$$

where E_r is equipped with the action of the affine Weyl group, W^e, associated to a root system Φ, and ϕ is a projection from the set of paths in E_s for some $s > r$ as in Section 1.3. We say that the algebra $A_n(\rho, e)$ has a *Soergel-path basis* with respect to Φ if (i) there exists a degree preserving bijective map

$$\omega : \mathcal{T}(\lambda, \mu) \to \mathrm{Path}(\lambda, \omega^\mu)$$

such that $\omega(\mathsf{T}^\mu) = \omega^\mu$ is admissible for all e-regular $\mu \in \Lambda_n$, and (ii) if $\lambda \in \Lambda_n$, and $\mu \in W^e \cdot \lambda$ with $\ell(\mu) \leqslant \ell(\lambda)$, then $\mu \in \Lambda_n$. □

Proposition 1.20. Let $A_n(\rho, e)$ denote an algebra with a Soergel-path basis and suppose that $d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$ for all $\lambda \neq \mu \in \Lambda_n$. Then the following hold:

(i) we have $\mathrm{Dim}_t(\Delta_\mu(\lambda)) = m_{\omega^\mu}(\lambda) \in \mathbb{N}_0[t, t^{-1}]$ and $\mathrm{Dim}_t(L_\mu(\lambda)) = e_{\omega^\mu}(\lambda) \in \mathbb{N}_0[t + t^{-1}]$;

(ii) if $\mathrm{Dim}_t(\Delta_\mu(\lambda)) = 0$, then $d_{\lambda\mu}(t) = 0$;

(iii) we have $\mathrm{Dim}_t(\Delta_\mu(\mu)) = \mathrm{Dim}_t(L_\mu(\mu)) = 1$;

(iv) if $\mathrm{Path}(\lambda, \omega^\mu) = \emptyset$, then $\mathrm{Dim}_t(\Delta_\mu(\lambda)) = 0$;

(v) if $\mathrm{Path}(\lambda, \omega^\mu) = \emptyset$, then $\mathrm{Dim}_t(L_\mu(\lambda)) = 0$;

(vi) we have that

$$\text{Dim}_t(\Delta_\mu(\lambda)) = \sum_{\substack{\nu \neq \mu \\ \text{Path}(\nu, \omega^\mu) \neq \emptyset \\ \text{Path}(\lambda, \omega^\nu) \neq \emptyset}} d_{\lambda\nu}(t)\text{Dim}_t(L_\mu(\nu)) + d_{\lambda\mu}(t).$$

□

Proof. Part (i) is clear by Proposition 1.3, (iii) is a restatement of the condition that ω^μ is the only path in $\text{Path}(\mu, \omega^\mu)$. A necessary condition for $\text{Dim}_t(\text{Hom}(P(\mu), \Delta(\lambda)) \neq 0$ is that $\Delta_\mu(\lambda) \neq 0$, therefore (ii) follows.

Part (iv) is by definition, and part (v) follows from the cellular structure. Finally, (vi) follows from $(i), (iii), (v)$ and our assumption that $d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$ for $\lambda \neq \mu$. ∎

A point $\mu \in \Lambda_n$ is e-regular if and only if all points in the orbit $W^e \cdot \mu$ are also e-regular. If $A_n(\rho, e)$ is an algebra with a Soergel-path basis, then Proposition 1.4 implies that the linkage classes (under the ρ-shifted W^e-action) decompose into blocks for the algebra $A_n(\rho, e)$. We therefore say that a block of $A_n(\rho, e)$ is an *e-regular block* if it belongs to an e-regular linkage class. We say that a standard, simple, or projective module is e-regular if it belongs to an e-regular block.

Theorem 1.21. Let $A_n(\rho, e)$ denote an algebra with a Soergel-path basis of type Φ. Suppose that $d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$ for all $\lambda, \mu \in \Lambda_n$ such that $\lambda \neq \mu$. The graded decomposition numbers of an e-regular block of $A_n(\rho, e)$ are given by the Soergel algorithm

$$d_{\lambda\mu}(t) = n_\mu(\lambda)$$

and the characters of the e-regular simple modules are given by the character algorithm

$$\text{Dim}_t(L_\mu(\lambda)) = e_{\omega^\mu}(\lambda).$$

□

Proof. By Proposition 1.20 (ii), we may assume $\text{Path}(\lambda, \omega^\mu) \neq \emptyset$. We now calculate $d_{\lambda\mu}(t)$ and $\text{Dim}_t(L_\mu(\lambda))$ by induction on the length ordering on alcoves. Induction begins when $\ell(\mu, \lambda) = 0$, hence $\mu = \lambda$, and we have $d_{\mu\mu}(t) = 1$ by Proposition 1.20 (iii) and $\text{Dim}_t(L_\mu(\mu)) = e_{\omega^\mu}(\mu) = 1$.

Let $\ell(\mu, \lambda) \geqslant 1$. By induction, we know $d_{\lambda\nu}(t)$ and $\text{Dim}_t(L_\mu(\nu))$ for points $\nu \in \Lambda_n$ such that $\ell(\mu, \nu), \ell(\lambda, \nu) < \ell(\mu, \lambda)$. By Proposition 1.20 (vi) we have

$$\text{Dim}_t(L_\mu(\lambda)) + d_{\lambda\mu}(t) = \text{Dim}_t(\Delta_\mu(\lambda)) - \sum_{\substack{\nu \neq \mu, \, \nu \neq \lambda \\ \text{Path}(\nu, \omega^\mu) \neq \emptyset \\ \text{Path}(\lambda, \omega^\nu) \neq \emptyset}} d_{\lambda\nu}(t)\text{Dim}_t(L_\mu(\nu)).$$

By induction and Proposition 1.20 (i), the right-hand side is equal to

$$m_{\omega^\mu}(\lambda) - \sum_{\substack{\nu \neq \mu, \, \nu \neq \lambda \\ \text{Path}(\nu, \omega^\mu) \neq \emptyset \\ \text{Path}(\lambda, \omega^\nu) \neq \emptyset}} n_\nu(\lambda)e_{\omega^\mu}(\nu).$$

We know that this final sum is equal to $e_\mu(\lambda)n_\lambda(\lambda) + n_\mu(\lambda)e_\mu(\mu)$ by Proposition 1.16. Our base case for induction showed that $n_\lambda(\lambda) = 1 = e_\mu(\mu)$, therefore

$$m_{\omega^\mu}(\lambda) - \sum_{\substack{\nu \neq \mu, \, \nu \neq \lambda \\ \text{Path}(\nu, \omega^\mu) \neq \emptyset \\ \text{Path}(\lambda, \omega^\nu) \neq \emptyset}} n_\nu(\lambda)e_{\omega^\mu}(\nu) = e_{\omega^\mu}(\lambda) + n_\mu(\lambda).$$

Recall that $\text{Dim}_t(L_\mu(\lambda)) \in \mathbb{N}_0[t + t^{-1}]$ and $d_{\lambda\mu} \in t\mathbb{N}_0[t]$. Therefore there is a unique solution to the equality (see [KN10, Section 4.1: Basic Algorithm] for a general form, or [Soe97] for the interpretation in terms of Kazhdan–Lusztig theory) given by

$$\text{Dim}_t(L_\mu(\lambda)) = e_{\omega^\mu}(\lambda), \quad d_{\lambda\mu}(t) = n_\mu(\lambda). \qquad ∎$$

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Corollary 1.22. Let $A_n(\rho, e)$ denote an algebra with a Soergel-path basis and suppose $d_{\lambda\mu}(t) \in t\mathbb{N}_0(t)$ for $\lambda \neq \mu$. Let $\lambda, \lambda' \in \mathfrak{a}$ and $\mu, \mu' \in \mathfrak{b}$ for some alcoves $\mathfrak{a}, \mathfrak{b}$ and suppose that $\mu \in W^e \cdot \lambda$ and $\mu' \in W^e \cdot \lambda'$. Then

$$d_{\lambda\mu}(t) = d_{\lambda'\mu'}(t).$$

$\square$

Proof. This follows as Soergel's algorithm is well-defined on alcoves.   ∎

2   The diagrammatic Cherednik algebra

In this section we recall the definition of the diagrammatic Cherednik algebras (reduced steadied quotients of weighted KLR algebras in Webster's terminology) constructed in [Web].

2.1   Combinatorial preliminaries

Fix integers $\ell, n \in \mathbb{Z}_{\geqslant 0}$, $g \in \mathbb{R}_{>0}$ and $e \in \{3, 4, \dots\} \cup \{\infty\}$. We define a *weighting* $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{R}^\ell$ to be any ℓ-tuple such that $\theta_i - \theta_j$ is not an integer multiple of g for $1 \leqslant i < j \leqslant \ell$. Let κ denote an *e-multicharge* $\kappa = (\kappa_1, \dots, \kappa_\ell) \in (\mathbb{Z}/e\mathbb{Z})^\ell$.

Definition 2.1. We say that $\theta \in \mathbb{R}^\ell$ is a *well-separated* (respectively FLOTW) weighting if $|\theta_j - \theta_i| > ng$ (respectively $|\theta_i - \theta_j| < g$) for all $1 \leqslant i < j \leqslant \ell$. $\square$

Definition 2.2. An ℓ-*multipartition* $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of n is an ℓ-tuple of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$. We will denote the set of ℓ-multipartitions of n by $\mathscr{P}_n^\ell$. $\square$

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}_n^\ell$. The *Young diagram* $[\lambda]$ is defined to be the set

$$\{(r, c, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, \ell\} \mid c \leqslant \lambda_r^{(m)}\}.$$

We refer to elements of $[\lambda]$ as nodes (of $[\lambda]$ or λ). We define the *residue* of a node $(r, c, m) \in [\lambda]$ to be $\kappa_m + c - r$ (mod e).

We define the *Russian array* as follows. For each $1 \leqslant m \leqslant \ell$, we place a point on the real line at θ_m and consider the region bounded by half-lines at angles $3\pi/4$ and $\pi/4$. We tile the resulting quadrant with a lattice of squares, each with diagonal of length $2g$. For each node of $[\lambda]$ we draw a box in the Russian array. We place the first node of component m at θ_m on the real line, with rows going northwest from this node, and columns going northeast. The diagram is tilted ever-so-slightly in the clockwise direction so that the top vertex of the box (r, c, m) (that is, the box in the rth row and cth column of the mth component of $[\lambda]$) has x-coordinate $\theta_m + g(r - c) + (r + c)\epsilon$.

Here the tilt ϵ is chosen so that $n\epsilon$ is absolutely small with respect to g (so that $\epsilon \ll g/n$) and with respect to the weighting (so that g does not divide any number in the interval $|\theta_i - \theta_j| + (-n\epsilon, +n\epsilon)$ for $1 \leqslant i < j \leqslant \ell$). With these assumptions firmly in place, we will drop any mention of ϵ when speaking of the ghost distance, $g \in \mathbb{R}_{>0}$, or the weighting, $\theta \in \mathbb{R}^\ell$.

We define a *loading*, $\mathbf{i}$, to be an element of $(\mathbb{R} \times (\mathbb{Z}/e\mathbb{Z}))^n$ such that no real number occurs with multiplicity greater than one. Given a multipartition $\lambda \in \mathscr{P}_n^\ell$ we have an associated loading, $\mathbf{i}_\lambda$, given by the projection of the top vertex of each box $(r, c, m) \in [\lambda]$ to its x-coordinate $\mathbf{i}_{(r,c,m)} \in \mathbb{R}$, and attaching to each point the residue $\kappa_m + c - r$ (mod e) of this node.

We let D_λ denote the underlying ordered subset of $\mathbb{R}$ given by the points of the loading. Given $a \in D_\lambda$, we abuse notation and let a denote the corresponding node of λ (that is, the node whose top vertex projects onto x-coordinate $a \in \mathbb{R}$). The *residue sequence* of λ is given by reading the residues of the nodes of λ according to the ordering given by D_λ.

Example 2.3. Let $\ell = 2$, $g = 1$, $\epsilon = 1/100$, and $\theta = (0, 0.5)$. The bipartition $((2, 1), (1^3))$ has Young diagram and corresponding loading $\mathbf{i}_\lambda$ given in Figure 12. The residue sequence of λ is $(\kappa_1 + 1, \kappa_1, \kappa_2, \kappa_1 - 1, \kappa_2 - 1, \kappa_2 - 2)$, and the ordered set D_λ is $\{-0.97, 0.02, 0.52, 1.03, 1.53, 2.54\}$. The node $x = -0.97$ in λ can be identified with the node in the first row and second column of the first component of λ.

Definition 2.4. We refer to an unordered multiset $\mathcal{R}$ of n elements from $(\mathbb{Z}/e\mathbb{Z})$ as a residue set of cardinality n. We let $\mathscr{P}_n^\ell(\mathcal{R})$ denote the subset of $\mathscr{P}_n^\ell$ whose residue set is equal to $\mathcal{R}$. $\square$
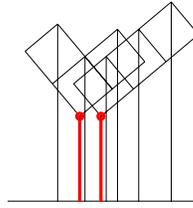
Fig. 12: The diagram and loading of the bipartition $((2,1),(1^3))$ for $\ell = 2$, $g = 1$, $\theta = (0, 0.5)$.

□

Example 2.5. Let $n = 3$, $\ell = 2$, $e = 4$, $g = 2$, $\kappa = (0, 2)$, and $\theta = (0, 1)$. The set $\mathscr{P}_3^2(\{0, 1, 2\})$ consists of 4 bipartitions, namely $(\varnothing, (1^3)), ((1), (1^2)), ((2), (1))$, and $((3), \varnothing)$. We record the diagrams corresponding to these bipartitions in Figure 13; in the cases where one of the components is empty, we record where it would be, for perspective.
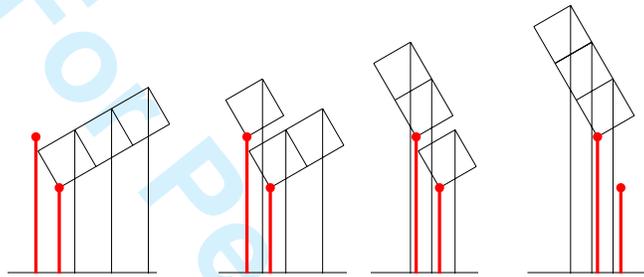
Fig. 13: The loadings of the bipartitions of 3 with residue $\{0, 1, 2\}$ for $\theta = (0, 1)$, $g = 2$.

The respective sets D_μ for the bipartitions $(\varnothing, (1^3)), ((1), (1^2)), ((2), (1))$, and $((3), \varnothing)$, are as follows:

$$\{1 + 2\epsilon, 3 + 3\epsilon, 5 + 4\epsilon\}, \{0 + 2\epsilon, 1 + 2\epsilon, 3 + 3\epsilon\}, \{-2 + 3\epsilon, 0 + 2\epsilon, 1 + 2\epsilon\}, \{-4 + 4\epsilon, -2 + 3\epsilon, 0 + 2\epsilon\}.$$

□

Definition 2.6. Let $\lambda, \mu \in \mathscr{P}_n^\ell$. A λ-tableau of weight μ is a bijective map $\mathsf{T} : [\lambda] \to D_\mu$ which respects residues. In other words, we fill a given node (r, c, m) of the diagram $[\lambda]$ with a real number d from D_μ (without multiplicities) so that the residue attached to the real number d in the loading $\mathbf{i}_\mu$ is equal to $\kappa_m + c - r$ (mod e). □

Definition 2.7. A λ-tableau, T, of shape λ and weight μ is said to be semistandard if

- $\mathsf{T}(1, 1, m) > \theta_m$,
- $\mathsf{T}(r, c, m) > \mathsf{T}(r - 1, c, m) + g$,
- $\mathsf{T}(r, c, m) > \mathsf{T}(r, c - 1, m) - g$.

We denote the set of all semistandard tableaux of shape λ and weight μ by $\mathrm{SStd}(\lambda, \mu)$. Given $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$, we write $\mathrm{Shape}(\mathsf{T}) = \lambda$. □

Remark 2.8. In this paper, we only consider examples of multipartitions in which each component is a hook. This means that when drawing diagrams in the Russian convention, no two nodes have the same x-coordinate for $\epsilon = 0$, therefore we omit ϵ from our tableaux and weightings without introducing ambiguity.

□

Definition 2.9. Let $\mathbf{i}$ and $\mathbf{j}$ denote two loadings of size n. We say that $\mathbf{i}$ dominates $\mathbf{j}$ if for every real number $a \in \mathbb{R}$ and every $r \in \mathbb{Z}/e\mathbb{Z}$, we have that

$$|\{(x, r) \in \mathbf{i} \mid x < a\}| \geqslant |\{(x, r) \in \mathbf{j} \mid x < a\}|.$$

Given $\lambda, \mu \in \mathscr{P}_n^\ell$, $\theta \in \mathbb{R}^\ell$, we say that λ θ-dominates μ (and write $\mu \trianglelefteq_\theta \lambda$) if $\mathbf{i}_\lambda$ dominates $\mathbf{i}_\mu$. □

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Example 2.10. We have the following two important examples of dominance orders. Let $n = 3$ and $\ell = 2$ and take (θ_1, θ_2) so that (i) $0 < \theta_2 - \theta_1 < g$ (ii) $\theta_2 - \theta_1 > ng$. The dominance order on a $\mathscr{P}_n^l(\mathcal{R})$ is given by intersecting the posets in Figure 14 with the set of bipartitions with residue class $\mathcal{R}$. The leftmost poset in Figure 14 will be of the most interest to us in this paper. □
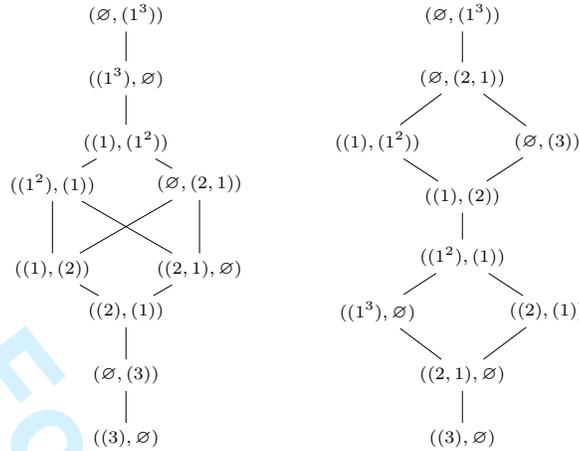
Fig. 14: The Hasse diagrams of the posets corresponding to the FLOTW and well-separated weightings.

Example 2.11. We continue Example 2.5 with $n = 3$, $\ell = 2$, $e = 4$, $g = 2$, $\kappa = (0, 2)$ and $\theta = (0, 1)$. In this case, the dominance order on bipartitions of residue $\{0, 1, 2\}$ is given by reading the diagrams in Figure 13 from left to right in ascending order. In other words

$$(\varnothing, (1^3)) \lhd_\theta ((1), (1^2)) \lhd_\theta ((2), (1)) \lhd_\theta ((3), \varnothing).$$

Recall the loadings of these bipartitions from Example 2.5. Recall that we let $\epsilon \to 0$ for ease of notation. Figure 15 lists all three semistandard tableaux of shape λ and weight μ (for $\mu \neq \lambda$) for λ, μ in this residue class. □
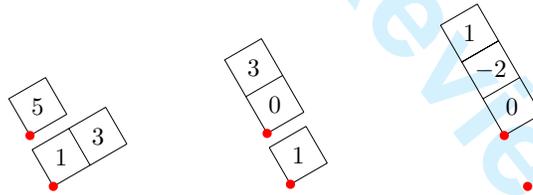
Fig. 15: The semistandard tableaux $\mathsf{T} \in \mathrm{SStd}(((1), (1^2)), (\varnothing, (1^3)))$, $\mathsf{U} \in \mathrm{SStd}(((2), (1)), ((1), (1^2)))$, and $\mathsf{V} \in \mathrm{SStd}(((3), \varnothing), ((2), (1)))$, respectively.

Remark. We have that $\mathscr{P}_n^\ell = \cup_{\mathcal{R}} \mathscr{P}_n^\ell(\mathcal{R})$ is a disjoint decomposition of the set $\mathscr{P}_n^\ell$; notice that all of the above combinatorics respects this decomposition. □

2.2   The diagrammatic Cherednik algebra

Recall that we have fixed $\ell, n \in \mathbb{Z}_{>0}$, $g \in \mathbb{R}_{>0}$ and $e \in \{3, 4, \dots\} \cup \{\infty\}$. Given any weighting $\theta = (\theta_1, \dots, \theta_\ell)$ and $\kappa = (\kappa_1, \dots, \kappa_\ell)$ an e-multicharge, we will define what we refer to as the diagrammatic Cherednik algebra, $A(n, \theta, \kappa)$.

This is an example of one of many finite dimensional algebras (reduced steadied quotients of weighted KLR algebras in Webster's terminology) constructed in [Web], whose module categories are equivalent, over the complex field, to category $\mathcal{O}$ for the rational cyclotomic Cherednik algebra [Web, Theorem 2.3 and 3.9].

Definition 2.12. We define a θ-*diagram* of type $G(\ell, 1, n)$ to be a *frame* $\mathbb{R} \times [0, 1]$ with distinguished black points on the northern and southern boundaries given by the loadings $\mathbf{i}_\mu$ and $\mathbf{i}_\lambda$ for some $\lambda, \mu \in \mathscr{P}_n^\ell(\mathcal{R})$ and a

collection of curves each of which starts at a northern point and ends at a southern point of the same residue, i say (we refer to this as a *black i-strand*). We further require that each curve has a mapping diffeomorphically to $[0,1]$ via the projection to the y-axis. Each curve is allowed to carry any number of dots. We draw

- a dashed line g units to the left of each strand, which we call a *ghost i-strand* or *i-ghost*;

- vertical red lines at $\theta_m \in \mathbb{R}$ each of which carries a residue κ_m for $1 \leqslant m \leqslant \ell$ which we call a *red κ_m-strand*.

We now require that there are no triple points or tangencies involving any combination of strands, ghosts or red lines and no dots lie on crossings. We consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies, double points and dots on crossings. □

Remark. Note that our diagrams do not distinguish between 'over' and 'under' crossings. □

Definition 2.13 (Definition 4.1 [Web])**.** The *diagrammatic Cherednik algebra*, $A(n, \theta, \kappa)$, is the $\mathbb{C}$-algebra spanned by all θ-diagrams modulo the following local relations (here a local relation means one that can be applied on a small region of the diagram).

(2.1) Any diagram may be deformed isotopically; that is, by a continuous deformation of the diagram which at no point introduces or removes any crossings of strands (black, ghost, or red).

(2.2) For $i \neq j$ we have that dots pass through crossings.
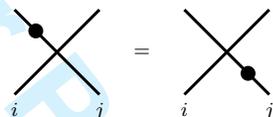
(2.3) For two like-labelled strands we get an error term.

(2.4) For double crossings of black strands, we have the following.

(2.5) If $j \neq i - 1$, then we can pass ghosts through black strands.
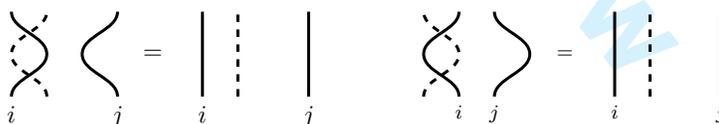
(2.6) On the other hand, in the case where $j = i - 1$, we have the following.

(2.7) We also have the relation below, obtained by symmetry.
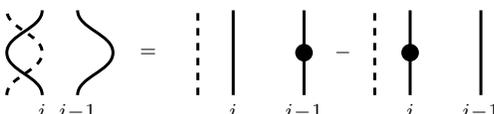
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(2.8) Strands can move through crossings of black strands freely.

Similarly, this holds for triple points involving ghosts, except for the following relations when $j = i - 1$.

(2.9)

(2.10)

In the diagrams with crossings in (2.9) and (2.10), we say that the black (respectively ghost) strand bypasses the crossing of ghost strands (respectively black strands). The ghost strands may pass through red strands freely. For $i \neq j$, the black i-strands may pass through red j-strands freely. If the red and black strands have the same label, a dot is added to the black strand when straightening. Diagrammatically, these relations are given by

(2.11)

and their mirror images. All black crossings and dots can pass through red strands, with a correction term.

(2.12)

(2.13)

(2.14)

Finally, we have the following non-local idempotent relation.

(2.15) Any idempotent where the strands can be broken into two groups separated by a blank space of size $> g$ (so no ghost from the right-hand group can be left of a strand in the left group and *vice versa*) with all double-red strands in the right-hand group is referred to as unsteady and set to be equal to zero.

□

Remark. We remark that the θ-diagrams are clearly dependent on our choice of the parameters $e, \ell, n \in \mathbb{N}$ and $g \in \mathbb{R}_{>0}$, as well as $\theta \in \mathbb{R}^\ell$ and $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$. The dependence of $A(n, \theta, \kappa)$ on e and ℓ is subsumed by the dependence on κ. We have chosen to not emphasise the dependence of $A(n, \theta, \kappa)$ on $g \in \mathbb{R}_{>0}$ because this 'ghost' variable can always be scaled appropriately so that the definition of the algebra depends only on $\theta \in \mathbb{R}^\ell$.

Finally, for the reader's convenience, we make note of the choices of notation we have used, and how they match those in [Web, Definition 4.1]. There, Webster defines the algebra over a ring, S. He notes that one interesting case (giving rise to a nice grading) is when S is a field, and his parameters $h, z_1, \ldots, z_\ell$ are all zero.

This is the setting we work in. Furthermore, Webster has a parameter k in his presentation, which we have set to -1. Finally, the strands in Webster's diagrams have ghosts appearing to the left or right of their associated strands, depending on whether g (which he denotes by $-\kappa$ in the version of his paper we cite) is negative or positive, respectively. Here, we have allowed only positive g, (or equivalently negative κ) so as to reduce some confusion.   □

2.3 The grading on the diagrammatic Cherednik algebra

This algebra is graded as follows:

  ○ dots have degree 2;

  ○ the crossing of two strands has degree 0, unless they have the same label, in which case it has degree -2;

  ○ the crossing of a black strand with label i and a ghost has degree 1 if the ghost has label $i-1$ and 0 otherwise;

  ○ the crossing of a black strand with a red strand has degree 0, unless they have the same label, in which case it has degree 1.

In other words,

$$\deg \; \bullet_i = 2 \qquad \deg \underset{i \quad j}{\times} = -2\delta_{i,j} \qquad \deg \underset{i \quad j}{\times} = \delta_{j,i+1} \qquad \deg \underset{i \quad j}{\times} = \delta_{j,i-1}$$

$$\deg \underset{i \quad j}{\times} = \delta_{i,j} \qquad \deg \underset{i \quad j}{\times} = \delta_{j,i}.$$

2.4 Representation theory of the diagrammatic Cherednik algebra

Given any $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$, we have a θ-diagram B_T consisting of a *frame* in which the n black strands each connecting a northern and southern distinguished point are drawn so that they trace out the bijection determined by T in such a way that we use the minimal number of crossings without creating any bigons between pairs of strands or strands and ghosts. This diagram is not unique up to isotopy (since we have not specified how to resolve triple points), but we can choose one such diagram arbitrarily.

Given a pair of semistandard tableaux of the same shape $(\mathsf{S}, \mathsf{T}) \in \mathrm{SStd}(\lambda, \mu) \times \mathrm{SStd}(\lambda, \nu)$, we have a diagram $C_{\mathsf{S},\mathsf{T}} = B_\mathsf{S} B_\mathsf{T}^*$ where B_T^* is the diagram obtained from B_T by flipping it through the horizontal axis. Notice that there is a unique element $\mathsf{T}^\lambda \in \mathrm{SStd}(\lambda, \lambda)$ and the corresponding basis element $C_{\mathsf{T}^\lambda, \mathsf{T}^\lambda}$ is the idempotent in which all black strands are vertical. A degree function on tableaux is defined in [Web, Defintion 2.13]; for our purposes it is enough to note that $\deg(\mathsf{T}) = \deg(B_\mathsf{T})$ as we shall always work with the θ-diagrams directly.

Theorem 2.14 ([Web, Section 2.6]). The algebra $A(n, \theta, \kappa)$ is a graded cellular algebra with a highest weight theory. The cellular basis is given by

$$\mathcal{C} = \{C_{\mathsf{S},\mathsf{T}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}(\lambda, \nu), \lambda, \mu, \nu \in \mathscr{P}_n^\ell\}$$

with respect to the θ-dominance order on the set $\mathscr{P}_n^\ell$ and the anti-isomorphism given by flipping a diagram through the horizontal axis.   □

Remark. Notice that the basis of $A(n, \theta, \kappa)$ also respects the decomposition of $\mathscr{P}_n^\ell$ by residue sets. Given a residue set $\mathcal{R}$, we let $A_\mathcal{R}(n, \theta, \kappa)$ denote the subalgebra of $A(n, \theta, \kappa)$ with basis given by all θ-diagrams indexed by multipartitions $\lambda, \mu, \nu \in \mathscr{P}_n^\ell(\mathcal{R})$.   □

Theorem 2.15 ([Web], Theorem 6.2). Over $\mathbb{C}$, the (basic algebra of the) diagrammatic Cherednik algebra $A(n, \theta, \kappa)$ is Koszul. Over $\mathbb{C}$, we therefore have that the graded decomposition numbers $d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$ for $\lambda \neq \mu \in \mathscr{P}_n^\ell$.   □

Example 2.16. We continue Example 2.5 with $n = 3$, $\ell = 2$, $e = 4$, $g = 2$, and $\kappa = (0, 2)$ and $\theta = (0, 1)$. Consider the block with residue $\mathcal{R} = \{0, 1, 2\}$.

The loadings of the four distinct $\lambda, \mu \in \mathscr{P}_n^\ell(\mathcal{R})$ are depicted in Figure 13. The three distinct semistandard tableaux S of shape λ and weight μ for $\lambda \neq \mu$ are depicted in Figure 15 and the corresponding basis elements B_S

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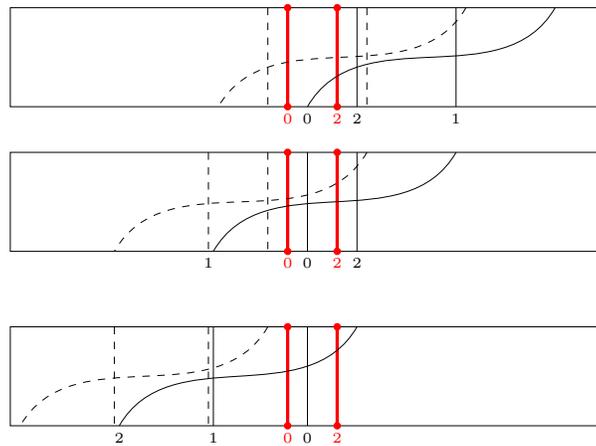
Fig. 16: The basis elements corresponding to the tableaux T, U, and V in $\mathrm{SStd}(((1),(1^2)),(\varnothing,(1^3)))$, $\mathrm{SStd}(((2),(1)),((1),(1^2)))$, and $\mathrm{SStd}(((3),\varnothing),((2),(1)))$, respectively (see Figure 15).
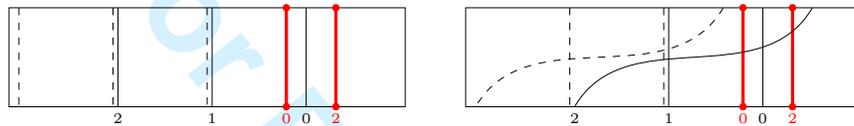
Fig. 17: Basis of the standard module $\Delta((3),\varnothing)$. The frames of the diagrams have been truncated in the horizontal axis so as to fit next to one another.

are depicted in Figure 16. For each $\lambda \in \mathscr{P}_n^\ell(\mathcal{R})$, there is an element $\mathsf{T}^\lambda \in \mathrm{SStd}(\lambda,\lambda)$ (the trivial bijection from $\mathbf{i}_\lambda$ to itself) and a corresponding element B_{T^λ} (for each λ the diagram B_{T^λ} consists of 3 vertical black strands with x-coordinates given by $\mathbf{i}_\lambda$). The full 7-dimensional algebra is given by taking the pairs of flipped elements C_{ST} for S and T of the same shape.

The left standard module $\Delta((3),\varnothing)$ is two-dimensional with basis as depicted in Figure 17. The former diagram contains no crossings, and so is of degree 0. In the latter diagram (corresponding to the tableau $\mathsf{V} \in \mathrm{SStd}(((3),\varnothing),((2),(1))))$, the crossing of the black strand of residue 2 with the red strand of residue 2 has degree 1 (and is the only crossing of non-zero degree); therefore this diagram is of degree 1. We shall now show that $[\Delta((3),\varnothing) : L((2),(1))] = t$. To see this, it suffices to check that the latter diagram in Figure 17 belongs to the radical of $\Delta((3),\varnothing)$.

By the definition of the bilinear form on $\Delta((3),\varnothing)$, it is enough to show that $(B_\mathsf{V})^*(B_\mathsf{V}) = 0$. We apply relation (2.11) followed by relations (2.5) and (2.11) to the product $B_\mathsf{V}^* B_\mathsf{V}$ to obtain the rightmost diagram as in Figure 18.
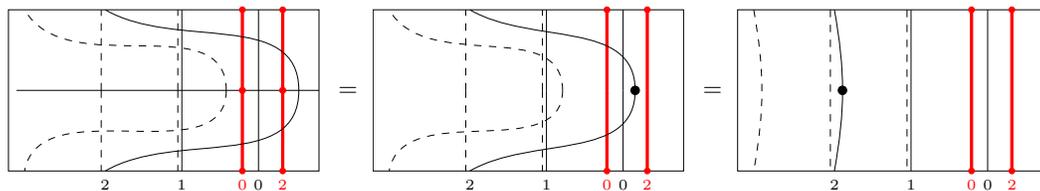
Fig. 18: The product $(B_\mathsf{V})^*(B_\mathsf{V})$. The first equality follows from relations (2.11); the second follows from relations (2.5) and (2.11).

We now apply relation (2.7) to the rightmost diagram in Figure 18, thus obtaining two terms, as depicted in Figure 19. The first term is zero because the black 2-strand can be pulled $> |g|$ units to the left (by isotopy, as in relation (2.1)) so that the centre of the diagram is an unsteady idempotent (and so is 0, by relation (2.15)). We can then repeat the argument above for the second term, and hence rewrite it as a sum of two terms: the

former (respectively latter) term is unsteady (respectively is unsteady after applying relation (2.11)). Therefore the product is zero, as required.
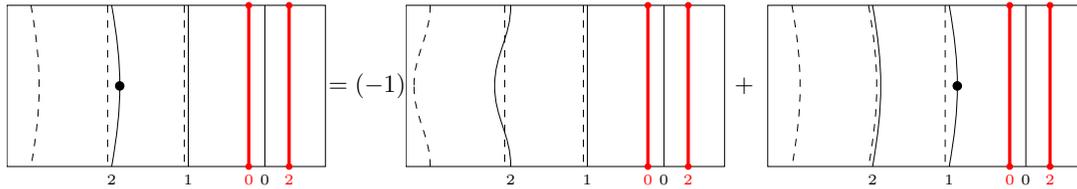
Fig. 19: The two terms obtained by applying relation (2.7) to the rightmost diagram in Figure 18.

One can treat the other standard modules in a similar fashion and hence compute the entire decomposition matrix of this block (depicted in Figure 20). Alternatively, one can calculate the decomposition numbers using the graded characters of standard modules and appealing to Proposition 1.3.

$$
\begin{array}{c|cccc}
(\varnothing, (1^3)) & 1 & & & \\
((1), (1^2)) & t & 1 & & \\
((2), (1)) & \cdot & t & 1 & \\
((3), \varnothing) & \cdot & \cdot & t & 1
\end{array}
$$

Fig. 20: The graded decomposition matrix for the algebra $A_{\mathcal{R}}(n, \theta, \kappa)$ as in Example 2.5.

$\square$

3   The quiver Temperley–Lieb algebra of type $G(\ell, 1, n)$

In this section we shall define the quiver Temperley–Lieb algebras of type $G(\ell, 1, n)$ and prove that these algebras possess Soergel-path bases of type $\widehat{A}_{\ell-1}$. We hence calculate the graded decomposition numbers of the e-regular blocks of these algebras. We then consider the level 2 case is greater detail; we calculate the full submodule structure of the standard modules of an e-regular block of such an algebra.

Fix $\ell, n, e \in \mathbb{N}$ and $g \in \mathbb{R}_{>0}$. Recall from Definition 2.1 that $\theta \in \mathbb{R}^\ell$ is said to be a FLOTW weighting if $0 < |\theta_j - \theta_i| < g$ for all $1 \leqslant i < j \leqslant \ell$. We say that an e-multicharge, $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$, is *adjacency-free* if $\kappa_i \notin \{\kappa_j - 1, \kappa_j, \kappa_j + 1\}$ for $1 \leqslant i < j \leqslant \ell$. Note that a tacit assumption for $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ to be adjacency-free is that $2\ell \leqslant e$.

We let π denote the set of all ℓ-multipartitions of n all of whose components have at most one column. We refer to this as the set of one-column multipartitions. We let

$$
e_\pi = \sum_{\lambda \notin \pi} C_{\mathsf{T}^\lambda \mathsf{T}^\lambda}.
$$

We shall see in the proof of Proposition 3.3 below that any such choice of FLOTW weighting, $\theta \in \mathbb{R}^\ell$, guarantees that π is a saturated subset of $\mathscr{P}_n^\ell$ with respect to the θ-dominance order.

Lemma 3.1. Let κ be an adjacency-free e-multicharge. Given two choices of FLOTW weighting, $\theta^{(1)}, \theta^{(2)} \in \mathbb{R}^\ell$, the algebras $A(n, \theta^{(1)}, \kappa)$ and $A(n, \theta^{(2)}, \kappa)$ are isomorphic. $\square$

Proof. Given κ an adjacency-free e-multicharge, we have that for any two choices $\theta^{(1)}$ and $\theta^{(2)}$ of FLOTW weighting, the combinatorics of tableaux are identical. This is because (i) the x-coordinates of the $\theta^{(1)}$ and $\theta^{(2)}$ loadings of any given multipartition differ only by moving the nodes of a given component by less than $|g|$ units to the left or right and (ii) the restriction on κ implies that the distance between any pair of nodes of adjacent residue is at least $|g|$. This results in a bijection between the cellular bases of the algebras $A(n, \theta^{(1)}, \kappa)$ and $A(n, \theta^{(2)}, \kappa)$. The basis elements identified under this bijection may be obtained from one another by isotopy. Therefore this is an isomorphism of algebras, via relation (2.1) of Section 2.2.   ∎

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By the previous lemma, it is enough to consider a preferred choice of FLOTW weighting. For the remainder of the paper, given $\mathrm{TL}_n(\kappa)$ with $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$, we shall fix $g = \ell$ and $\theta = (0, 1, 2, \ldots, \ell - 1)$ (this choice is easily seen to satisfy the conditions above), and adopt the conventions of Remark 2.8. With this choice made, the loadings of multipartitions in π have a very simple form. Namely, given any $\lambda \in \pi$ of the form

$$\lambda = (1^{\lambda_1}, 1^{\lambda_2}, \ldots, 1^{\lambda_\ell})$$

(with $\sum_{i=1}^l \lambda_i = n$) the loading $\mathbf{i}_\lambda$ has x-coordinates given by the set

$$\{(i - 1) + j\ell \mid \lambda_i \neq 0 \text{ and } 1 \leqslant j \leqslant \lambda_i\}.$$

Given θ as above, we refer to the θ-dominance order on multipartitions as the FLOTW dominance order.

Definition 3.2. Fix an adjacency-free e-multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ and a FLOTW weighting $\theta \in \mathbb{R}^\ell$. We define the *quiver Temperley–Lieb algebra of type* $G(\ell, 1, n)$, denoted $\mathrm{TL}_n(\kappa)$, to be the algebra

$$\mathrm{TL}_n(\kappa) = A(n, \theta, \kappa)/(A(n, \theta, \kappa)e_\pi A(n, \theta, \kappa)).$$

□

Proposition 3.3. The quiver Temperley–Lieb algebra of type $G(\ell, 1, n)$ is a graded cellular algebra with a highest weight theory. The cellular basis is given by

$$\{C_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}(\lambda, \nu), \lambda, \mu, \nu \in \pi\},$$

with respect to the FLOTW dominance order on the set of one-column multipartitions, π. We have that $d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$ for $\lambda \neq \mu$ elements of π.   □

Proof. We shall show that the set π is saturated in the θ-dominance order for $\theta \in \mathbb{R}^\ell$ a FLOTW weighting. In other words, given any $\lambda \in \pi$ and $\mu \lhd_\theta \lambda$, we have that $\mu \in \pi$. This will imply that

$$\langle C_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}(\lambda, \nu), \lambda, \mu, \nu \notin \pi\rangle_\mathbb{C}$$

is an ideal of $A(n, \theta, \kappa)$ (the ideal generated by e_π, in fact) and the resulting quotient has the desired basis (by conditions (2) and (3) of Definition 1.1 and Theorem 2.14). The graded decomposition numbers (as well as dimensions of higher extension groups) are preserved under this quotient; this follows by the arguments of [Don98, Appendix: Lemmas A3.1 and A3.3] (for the ungraded case) as the ideal is generated by a (degree zero) idempotent.

Recall that $0 < |\theta_j - \theta_i| < g$ for $1 \leqslant i < j \leqslant \ell$. This implies that if we add a box to the second column of any component $\lambda^{(m)}$ (that is, add a node $(1, 2, m)$ for some m), this box has x-coordinate strictly less than any box in the first column of any component, and thus the resulting multipartition is more dominant in the θ-dominance ordering. Therefore the set of one-column multipartitions is saturated.   ■

Definition 3.4. Let λ be a one-column multipartition $(1^{\lambda_1}, 1^{\lambda_2}, \ldots, 1^{\lambda_\ell})$. A node of λ is *removable* if it can be removed from the diagram of λ to leave the diagram of a (one-column) multipartition, while a node not in the diagram of λ is an *addable* node of λ if it can be added to the diagram of λ to give the diagram of a one-column multipartition.

If the node has residue $r \in \mathbb{Z}/e\mathbb{Z}$, we say that the node is *r-removable* or *r-addable*. Given $\lambda \in \pi$ and $r \in \mathbb{Z}/e\mathbb{Z}$, we let $\mathrm{Add}(\lambda, r)$ denote the set of $1 \leqslant j \leqslant \ell$ such that there is an r-addable node in the jth component of λ.   □

In the previous section, we refrained from defining the degree of a general tableau. This was because of the technicalities in defining *addable* and *removable* nodes for such tableaux (see [Web, Section 2.2]). These difficulties do not appear for tableaux corresponding to one-column multipartitions.

Definition 3.5. Let λ and μ be two one-column ℓ-multipartitions of n. Let $\square$ denote a removable node of $[\lambda]$ of residue $r \in \mathbb{Z}/e\mathbb{Z}$. We set

$$d_\lambda(\square) = \big|\{\text{addable } r\text{-nodes of } \lambda \text{ to the right of } \square\}\big|$$
$$- \big|\{\text{removable } r\text{-nodes of } \lambda \text{ to the right of } \square\}\big|.$$

Given $1 \leqslant k \leqslant n$ and $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$, we let T_k denote the node of $[\lambda]$ containing the entry $D_\mu(k)$ and we let $\mathsf{T}_{\leqslant k}$ denote the tableau consisting of the nodes with entries in $D_\mu\{1, 2, \ldots, k\}$. We define the *degree* of the tableau T recursively, setting $\deg(\mathsf{T}) = 0$ when T is the unique $\varnothing$-tableau. We set

$$\deg(\mathsf{T}) = d_\lambda(\mathsf{T}_n) + \deg(\mathsf{T}_{<n}).$$

□

Example 3.6. Let $e = 4$, $\ell = 2$, $n = 7$, and $\kappa = (0, 2)$. By tableau-linkage, it is clear that any residue class decomposes as a sum of blocks of $\mathrm{TL}_n(\kappa)$. Fix the residue class to be $\{0, 0, 1, 2, 2, 3, 3\}$. The one-column multipartitions with these residues for our given value e-multicharge are

$$\{((1^7), (0)), ((1^6), (1)), ((1^3), (1^4)), ((1^2), (1^5))\}. \tag{3.1}$$

Formally, the loading of the multipartition $\lambda = ((1^7), (0))$ is

$$D_\lambda = \{0 + \epsilon, 2 + 2\epsilon, 4 + 3\epsilon, 6 + 4\epsilon, 8 + 5\epsilon, 10 + 6\epsilon, 12 + 7\epsilon\}.$$

With the conventions of Remark 2.8 in place, the loadings of the multipartitions in equation (3.1) are given by

$$(0, 2, 4, 6, 8, 10, 12), \quad (0, 1, 2, 4, 6, 8, 10), \quad (0, 1, 2, 3, 4, 5, 7), \quad (0, 1, 2, 3, 5, 7, 9),$$

respectively. The semistandard tableaux of shape $(1^3, 1^4)$ are given in Figure 21, along with their degrees. For example, the nodes in the rightmost diagram are of degree 0 except for those containing the integers 4 and 12, which are of degree 1. Therefore the rightmost tableau has degree 2.
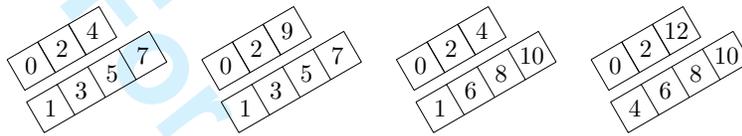
Fig. 21: These semistandard tableaux are of weights $((1^3), (1^4))$, $((1^2), (1^5))$, $((1^6), (1))$ and $((1^7), \varnothing)$ respectively. The tableaux are of degrees 0, 1, 1, and 2 respectively.

$\square$

3.1 The geometry

Let n, ℓ, e be non-negative integers such that $e \geqslant 2\ell$ and let $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ be an adjacency-free e-multicharge. We identify $\lambda \in \pi$ with a point in E_ℓ via the map $(1^{\lambda_1}, \ldots, 1^{\lambda_\ell}) \mapsto \sum_i \lambda_i \varepsilon_i$. We then let $E_{\ell-1}$ denote the quotient space of E_ℓ by the relation $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$. Let $\Phi_{\ell-1}$ be a root system of type $A_{\ell-1}$ with simple roots

$$\{\varepsilon_i - \varepsilon_j : 1 \leqslant i < j \leqslant \ell\},$$

and let W_ℓ^e denote the corresponding affine Weyl group, generated by the affine reflections $s_{i,j,me}$ with $1 \leqslant i < j \leqslant \ell$ and $m \in \mathbb{Z}$ and which acts on $E_{\ell-1}$ via

$$s_{i,j,me}(x) = x - (\langle x, \varepsilon_i - \varepsilon_j \rangle - me)(\varepsilon_i - \varepsilon_j).$$

Let $\rho = (e - \kappa_1, \ldots, e - \kappa_\ell)$. Given an element $w \in W_\ell^e$, we set

$$w \cdot_\rho x = w(x + \rho) - \rho.$$

We say that $\lambda \in \pi$ is an e-regular multipartition if it is identified with an e-regular point in $E_{\ell-1}$. Note that condition (ii) of Definition 1.19 is satisfied for this embedding of Λ_n in Euclidean space.

Lemma 3.7. Given $\lambda \in \pi$, we have that

$$\langle \lambda + \rho, \varepsilon_i - \varepsilon_j \rangle = me$$

for some $m \in \mathbb{Z}$, if and only if the addable nodes in the ith and jth components of the multipartition λ have the same residue. $\square$

Proof. To see this, note that both statements are equivalent to

$$(\lambda_i + e - \kappa_i) \equiv (\lambda_j + e - \kappa_j) \pmod{e}. \qquad \blacksquare$$

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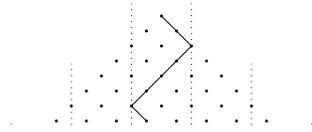
Fig. 22: The path $\omega(\mathsf{T}) \in \mathrm{Path}((3,4), \omega^{(7,0)})$.

Definition 3.8. Given $\lambda, \mu \in \pi$ and $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$, we define the *component word* $R(\mathsf{T})$ of T to be given by reading the entries of the tableau in numerical order and recording the components in which they appear. We define the path $\omega(\mathsf{T})$ to be the associated path in the alcove geometry.   □

Example 3.9. Let $e = 4$, $\ell = 2$, $n = 7$, and $\kappa = (0, 2)$. Let $\mathsf{T} \in \mathrm{SStd}(((1^3),(1^4)),((1^7),\varnothing))$ denote the rightmost tableau depicted in Figure 21. The component word, $R(\mathsf{T})$, is $(1, 1, 2, 2, 2, 2, 1)$. The path $\omega(\mathsf{T})$ is pictured in Figure 22.   □

Given the unique $\mathsf{T}^\mu \in \mathrm{SStd}(\mu, \mu)$, it is clear that $\omega(\mathsf{T}^\mu) = \omega^\mu$ is the path corresponding to the word $w : \{1, \ldots, n\} \to \{1, \ldots, \ell\}$ given by

$$w(1) = \min\{i \mid \mu_i \neq 0\}$$

and for $i > 1$,

$$w(i) = (w(i-1) + j) \text{ modulo } \ell$$

where $j \geqslant 1$ is minimal such that $\langle \omega^\mu(i-1) + \rho, \varepsilon_{w(i-1)+j} \rangle < \mu_{w(i-1)+j}$ where the subscripts are also read modulo ℓ.

Example 3.10. As in the introduction, let $\ell = 3$, $n = 13$, $e = 8$, $\kappa = (0, 4, 6)$. For $\mu = (5, 6, 2)$ the component word of T^μ is

$$(1, 2, 3, 1, 2, 3, 1, 2, 1, 2, 1, 2, 2).$$

□

Proposition 3.11. Given an e-regular $\mu \in \pi$, we fix the path $\omega(\mathsf{T}^\mu)$ as above. We have that ω defines a bijective map

$$\omega : \mathrm{SStd}(\lambda, \mu) \to \mathrm{Path}(\lambda, \omega^\mu).$$

□

Proof. The map ω is clearly an injective map, it remains to show that both sets have the same size. The sets $\mathrm{SStd}(\mu, \mu)$ and $\mathrm{Path}(\mu, \omega^\mu)$ each possess a unique element T^μ, respectively ω^μ. For $1 \leqslant k \leqslant n$, let $r(k)$ denote the residue of the node T_k^μ and let $t(k)$ denote the component of the ℓ-multipartition in which this node is added. For $1 \leqslant k \leqslant n$, it follows by Lemma 3.7 that

$$\mathrm{Add}(\mathrm{Shape}(\mathsf{T}_{\leqslant k-1}^\mu), r(k)) = \{i \mid \omega^\mu(k) \in h_{\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e} \text{ for some } m_{ik} \in \mathbb{Z}\}.$$

We let d_k denote the cardinality of this set.

We construct both $\mathsf{T} \in \mathrm{SStd}(-, \mu)$ and $\omega \in \mathrm{Path}(-, \omega^\mu)$ step-by-step; in the former case, by adding one node at a time to the tableau and in the latter case by taking one step at a time in the geometry.

The number of choices to be made at the kth point in the tableau is equal to d_k, for $1 \leqslant k \leqslant n$. Therefore the number of tableaux of weight μ is equal to $d_1 d_2 \ldots d_n$. On the other hand, in the notation of Section 1.3, any path $\omega \in \mathrm{Path}(-, \omega^\mu)$ may be written as

$$\omega = s_{\varepsilon_{i(1)} - \varepsilon_{t(1)}, m_{i1}e}^1 \cdots s_{\varepsilon_{i(n)} - \varepsilon_{t(n)}, m_{in}e}^n \omega^\mu$$

for $i(k) \in \mathrm{Add}(\mathrm{Shape}(\mathsf{T}_{\leqslant k-1}^\mu), r(k))$ and $m_{ik} \in \mathbb{Z}$ (of course, if $d_k = 1$, the reflection is necessarily trivial). The number of such paths is equal to the number of distinct possible series of reflections, $d_1 \ldots d_n$. The result follows.   ■

Corollary 3.12. If $\lambda, \mu \in \pi$, label simple modules in the same $\mathrm{TL}_n(\kappa)$-block, this implies that their images in $E_{\ell-1}$ are in the same $W_{\ell-1}^e$-orbit under the ρ-shifted action.   □

Proof. This follows from Proposition 1.4, as it is easy to see that the equivalence classes of the relation generated by $\lambda \sim \mu$ if $\mathrm{Path}(\lambda, \omega^\mu) \neq \emptyset$ are the same as the $W_{\ell-1}^e$-orbits. ∎

Lemma 3.13. Let $\lambda = (1^{\lambda_1}, \ldots, 1^{\lambda_\ell}) \in \pi$ be such that $\lambda_i > \lambda_j$ for some $1 \leqslant i, j \leqslant \ell$ and suppose that the residues of the addable nodes in ith and jth components of λ are equal to $r \in (\mathbb{Z}/e\mathbb{Z})$.

Then $\lambda \in E_{\ell-1}$ lies on a hyperplane of the form $x_i - x_j = m_{ij}e$ for some $m_{ij} \in \mathbb{Z}$. We have that $(\lambda + \varepsilon_i) \in E_{\ell-1}^+(\varepsilon_i - \varepsilon_j, m_{ij}e)$ and $(\lambda + \varepsilon_j) \in E_{\ell-1}^-(\varepsilon_i - \varepsilon_j, m_{ij}e)$. □

Proof. We have seen that λ lies on a hyperplane by Lemma 3.7. We have assumed that $\lambda_i > \lambda_j$, and so

$$\langle \lambda + \rho + \varepsilon_i, \varepsilon_i - \varepsilon_j \rangle > \langle \lambda + \rho, \varepsilon_i - \varepsilon_j \rangle$$

$$\langle \lambda + \rho + \varepsilon_j, \varepsilon_i - \varepsilon_j \rangle < \langle \lambda + \rho, \varepsilon_i - \varepsilon_j \rangle$$

as required. ∎

Proposition 3.14. The map $\omega : \mathrm{SStd}(\lambda, \mu) \to \mathrm{Path}(\lambda, \omega^\mu)$ is degree preserving. □

Proof. We fix a tableau $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$ and let $\omega := \omega(\mathsf{T})$ denote the corresponding element of $\mathrm{Path}(\lambda, \omega^\mu)$. For $1 \leqslant k \leqslant n$, we truncate to consider the path of length $k - 1$ (respectively tableau with $k - 1$ nodes), $\omega_{\leqslant k-1}$ (respectively $\mathsf{T}_{\leqslant k-1}$) and identify this with the multipartition $\mathrm{Shape}(\mathsf{T}_{\leqslant k-1}) \in \pi$.

Let r_k denote the residue of the addable node T_k and let $t(k)$ denote the component in which this node is added. By the definition of the Soergel-degree, we are interested in the cases where $1 \leqslant i \leqslant \ell$ is such that

(i) $\omega(k-1) \in h_{\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e}$ and $\omega(k) \in E_{\ell-1}^-(\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e)$ for some $m_{ik} \in \mathbb{Z}$,

(ii) $\omega(k-1) \in E_{\ell-1}^+(\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e)$ and $\omega(k) \in h_{\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e}$ for some $m_{ik} \in \mathbb{Z}$.

By Lemma 3.13, the $1 \leqslant i \leqslant \ell$ above label the components of

(i) the r_k-addable nodes of $\mathsf{T}_{\leqslant k-1}$ to the right of T_k,

(ii) the $(r_k - 1)$-addable nodes of $\mathsf{T}_{\leqslant k-1}$ to the right of T_k.

We observe that, because of the condition $\kappa_i \notin \{\kappa_j, \kappa_j + 1\}$ for $i \neq j$, the set of $1 \leqslant i \leqslant \ell$ which label $(r_k - 1)$-addable nodes of $\mathsf{T}_{\leqslant k}$ to the right of T_k is equal to the set of r_k-removable nodes of $\mathsf{T}_{\leqslant k-1}$ to the right of T_k. Therefore the result follows. ∎

Proposition 3.15. Given an e-regular $\mu \in \mathscr{P}_n^\ell$, the path ω^μ is admissible. □

Proof. It is clear that $\deg(\omega_{\leqslant k}) = 0$ for $1 \leqslant k \leqslant n$. Now assume that $\omega^\mu(k)$ lies on two (or more) distinct hyperplanes $x_i - x_j = m_1e$ and $x_{i'} - x_{j'} = m_2e$ for some $1 \leqslant k \leqslant n$ and $m_1, m_2 \in \mathbb{Z}$. We will show that i, j, i', j' are necessarily distinct, and so the hyperplanes are orthogonal.

To prove the claim, we recall our description of ω^μ. Let r_k denote the residue of the addable node T_k and let $t(k)$ denote the component in which this node is added. It is clear that the result holds for $k = 0$, we proceed by induction. For $1 \leqslant k \leqslant n$, assume $\omega^\mu(k)$ lies on the hyperplane $h_{\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e}$ for some $m_{ik} \in \mathbb{Z}$. Our assumption on κ ensures that $\kappa_{t(k)} \neq \kappa_j, \kappa_j \pm 1$ for any $1 \leqslant j \leqslant \ell$. This implies that if $\langle \omega^\mu(k) + \rho, \varepsilon_{t(k)} - \varepsilon_j \rangle \equiv 0 \pmod{e}$ for any $1 \leqslant j \leqslant \ell$, then $\langle \omega^\mu(k) + \rho, \varepsilon_j \rangle = \langle \omega^\mu + \rho, \varepsilon_j \rangle$. Our assumption that μ is e-regular implies that there is a maximum of one such value of $1 \leqslant j \leqslant \ell$. The result follows. ∎

This allows us to obtain the main theorem as stated in the introduction to this paper.

Theorem 3.16. Let $n \in \mathbb{N}$ and $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ be an adjacency-free e-multicharge. The algebra $\mathrm{TL}_n(\kappa)$ has a Soergel-path basis of type $\widehat{A}_{\ell-1}$. The graded decomposition numbers of an e-regular block are given by Soergel's algorithm

$$d_{\lambda\mu}(t) = n_\mu(\lambda),$$

and the characters of the e-regular simple modules are given by the character algorithm

$$\mathrm{Dim}_t(L_\mu(\lambda)) = e_{\omega^\mu}(\lambda).$$

□

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Proof. This follows from Theorems 1.21 and 2.15 and Propositions 3.3, 3.11, 3.14, and 3.15. ∎

We also observe the following stability in the decomposition numbers as n tends to infinity. Fix $n, \ell \in \mathbb{N}$. Given λ a one-column multipartition of n and $i \geqslant 0$, we let $\lambda + (1^i, \dots, 1^i)$ denote the one-column multipartition of $n + i\ell$ obtained by adding i boxes to every component of λ. This defines an injective map from multipartitions of n to multipartitions of $n' = n + i\ell$. These points may be identified with points in the hyperplanes $\varepsilon_1 + \cdots + \varepsilon_\ell = n$ and $\varepsilon_1 + \cdots + \varepsilon_\ell = n + i\ell$ of $E_{\ell-1}$, respectively. We identify points in these two hyperplanes via the projection in the direction $\varepsilon_1 + \cdots + \varepsilon_\ell$.

Theorem 3.17. The decomposition numbers of $\mathrm{TL}_n(\kappa)$ are stable as n tends to infinity. To be more precise,

$$d_{\lambda\mu}(t) = d_{\lambda+(1^i,\dots,1^i),\mu+(1^i,\dots,1^i)}(t)$$

for $i \geqslant 0$. □

Proof. Given $\omega \in \mathrm{Path}(\lambda, \omega^\mu)$ we let $\omega' \in \mathrm{Path}(\lambda + (1^i, \dots, 1^i), \omega^{\mu+(1^i,\dots,1^i)})$ denote the concatenated path

$$(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\ell)^i \circ \omega.$$

It is clear that this map is a degree preserving bijection. The result follows. ∎

Recall that a one-column multipartition is said to be e-regular if it corresponds to an e-regular point in Euclidean space under the embedding of Subsection 3.1. The following corollary is immediate from the definition of the quiver Temperley–Lieb algebras as saturated quotients of the diagrammatic Cherednik algebras.

Corollary 3.18. Fix an adjacency-free e-multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$ and a FLOTW weighting, $\theta \in \mathbb{R}^\ell$. Let λ, μ denote a pair of e-regular one-column multipartitions. The graded decomposition numbers for $A(n, \theta, \kappa)$ are

$$d_{\lambda\mu}(t) = n_\mu(\lambda),$$

where $n_\mu(\lambda)$ is the associated affine Kazhdan–Lusztig polynomial of type $\widehat{A}_{\ell-1}$. These decomposition numbers are stable in the limit as n tends to infinity as in Theorem 3.17 above. □

Under the equivalence of [Web, Theorem A] we immediately obtain the corollary in the introduction to the paper.

The Soergel-path basis contains a vast amount of information concerning the representation theory of the quiver Temperley–Lieb algebras of type $G(\ell, 1, n)$. We have already seen that it provides a new interpretation for Soergel's algorithm for computing the decomposition numbers of $\mathrm{TL}_n(\kappa)$. In the next section we shall consider the $\ell = 2$ case, calculate the full submodule structure of the standard modules of $\mathrm{TL}_n(\kappa)$ for $\kappa \in (\mathbb{Z}/e\mathbb{Z})^2$, and show that the algebra is positively graded.

3.2   Remarks on the generalised blob algebras

We have already remarked that our approach to the algebras $\mathrm{TL}_n(\kappa)$ is heavily inspired by the combinatorics of [MW03]. In [MW03], the generalised blob algebra, which we denote $b_n(\kappa)$, is defined as a certain quotient of the Ariki–Koike algebra of type $G(\ell, 1, n)$ with e-multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^\ell$.

In [MW03], it is conjectured that the decomposition numbers of the generalised blob algebras are given by the same Kazhdan–Lusztig polynomials as those considered here. Our algebra is a quotient of the diagrammatic Cherednik algebra, whereas the generalised blob algebra is the corresponding quotient of the Ariki–Koike algebra. For a fixed weighting θ, the standard/Specht modules of these algebras have the same labelling set; however, there is no known cellular basis for the Ariki–Koike algebra with respect to the θ-dominance order (except when θ is well-separated, see [GJ11]) and hence no way to relate the representation theories of the generalised blob and Ariki–Koike algebras via an analogue of Proposition 3.3. Moreover, the resulting quotient algebra would not be amenable to our methods as it does not possess a Soergel-path basis (for example, for $\ell = 2$ the blob algebra is not positively graded, [Pla13]). However, we present the following conjecture.

Conjecture 3.19. The algebras $b_n(\kappa)$ and $\mathrm{TL}_n(\kappa)$ are Morita equivalent. □

3.3   The quiver Temperley–Lieb algebras of level two

For $\ell = 2$, the structure of the standard modules for $\mathrm{TL}_n(\kappa)$ labelled by e-regular points is particularly simple. The proofs in this section are lightly sketched, but augmented with illustrative examples.

We remark that the submodule lattices obtained here are identical to those computed for the blob algebra in [MW00]. This provides further evidence that the quiver Temperley–Lieb algebras are (graded) Morita equivalent to the generalised blob algebras.

Let $\mathfrak{a}_i$ denote the alcove of length $i = \ell(\mathfrak{a}_i)$ to the right of the origin and $\mathfrak{a}_{i'}$ denote the alcove of length $i = \ell(\mathfrak{a}_{i'})$ to the left of the origin, as depicted in the examples below. Fix a point λ_0 in the alcove containing the origin. We let λ_i and $\lambda_{i'}$ denote the points in alcoves $\mathfrak{a}_i$ and $\mathfrak{a}_{i'}$ which are in the same orbit as λ_0.

Proposition 3.20. For $\kappa \in (\mathbb{Z}/e\mathbb{Z})^2$, the algebra $\mathrm{TL}_n(\kappa)$ is positively graded. We have that

$$d_{\mu,\nu}(t) = n_\nu(\mu) = \begin{cases} t^{j-i} & \text{for } i < j \\ 0 & \text{otherwise.} \end{cases}$$

where $\mu \in \{\lambda_i, \lambda_{i'}\}$ and $\nu \in \{\lambda_j, \lambda_{j'}\}$.

$\square$

Proof. Positivity follows as our paths start at $\odot$ and the root system is of rank 1. The closed form for these Kazhdan–Lusztig polynomials is well-known (see for example the introduction to [MW03]).   ∎

Remark. The algebra $\mathrm{TL}_n(\kappa)$ is not positively graded for $\ell \geqslant 3$, as seen in Example 1.18. $\square$

Given any pair $\mu \in \{\lambda_i, \lambda_{i'}\}$ and $\nu \in \{\lambda_j, \lambda_{j'}\}$ with $i < j$, there exists a unique element of $\mathrm{Path}(\mu, \omega^\nu)$ of maximal degree equal to $j - i$. This is the unique path, $\omega(\mathsf{T})$, terminating at μ which may be obtained from the path ω^μ using the maximum number of reflections in the hyperplanes $\overline{\mathfrak{a}}_0 \cap \overline{\mathfrak{a}}_1$ and $\overline{\mathfrak{a}}_0 \cap \overline{\mathfrak{a}}_{1'}$. We let 1_μ^ν denote the element B_T for $\omega(\mathsf{T})$ the unique maximal path in $\mathrm{Path}(\nu, \mu)$.

Example 3.21. Let $n = 11$, $e = 4$ and $\kappa = (0, 2)$. Some of maximal and non-maximal paths are given in Figures 23 and 24.
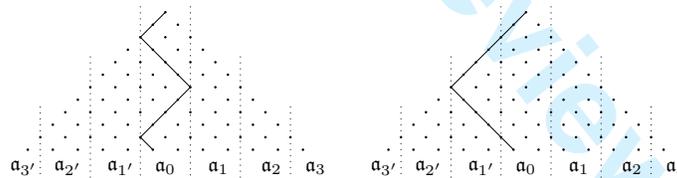
Fig. 23: The two elements of $\mathrm{Path}(\lambda_0, \omega^{\lambda_{3'}})$. The former (respectively latter) is maximal of degree 3 (respectively non-maximal of degree 1).
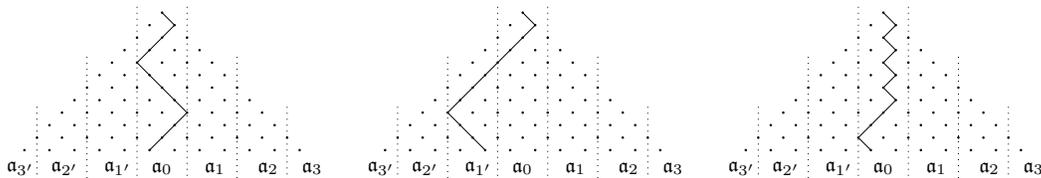
Fig. 24: Some examples of maximal paths (of degree 2, 1, and 1 respectively).

The elements 1_μ^ν corresponding to the paths in Figure 24 are depicted in Figure 25.

$\square$

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$1_0^{2'} =$

$1_{1'}^{2'} =$

$1_0^{1'} =$
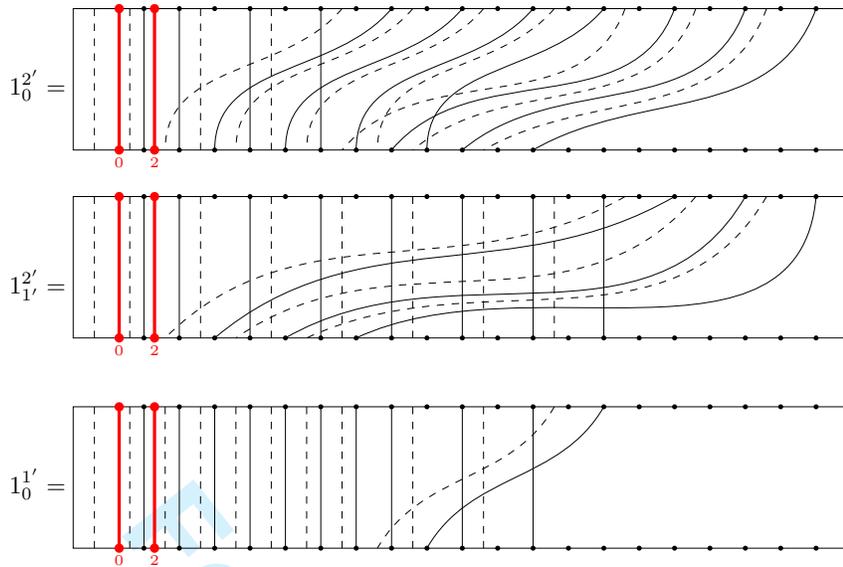
Fig. 25: The elements $1_0^{2'}$, $1_0^{1'}$, and $1_{1'}^{2'}$ corresponding to the paths in Figure 24.

Theorem 3.22. If $\ell = 2$, the full submodule structure of the $\mathrm{TL}_n(\kappa)$-modules $\Delta(\lambda_i)$ and $\Delta(\lambda_{i'})$ are given by the strong Alperin diagrams (in the sense of [Alp80]) below.
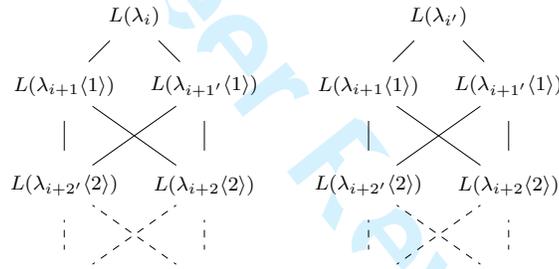
Letting $\mu \in \{\lambda_i, \lambda_{i'}\}$ and $\nu \in \{\lambda_j, \lambda_{j'}\}$, we therefore have that

$$\mathrm{Dim}_t(\mathrm{Hom}_{\mathrm{TL}_n(\kappa)}(\Delta(\nu), \Delta(\mu))) = t^{j-i},$$

for $i < j$ (in which case this homomorphism is injective) and the dimension is 0 otherwise. □

Proof. Given any pair $\mu \in \{\lambda_i, \lambda_{i'}\}$ and $\nu \in \{\lambda_j, \lambda_{j'}\}$ with $i < j$, we have seen that if $\omega(\mathsf{T}) \in \mathrm{Path}(\mu, \nu)$ is maximal, then it labels a decomposition number $d_{\mu\nu}(t) = t^{j-i}$. Therefore $1_\mu^\nu = B_\mathsf{T}$ generates a simple composition factor $L(\nu)\langle j - i \rangle$ of the standard module $\Delta(\mu)$. We shall show that

$$1_{\lambda_{i+1}}^\nu \circ 1_\mu^{\lambda_{i+1}} = c 1_\mu^\nu = 1_{\lambda_{i+1'}}^\nu \circ 1_\mu^{\lambda_{i+1'}} \tag{3.2}$$

for $c = \pm 1$ and $j = i + 2$, and the result will follow. First, notice that $\deg(1_\mu^\nu) = j - i$ and this is the unique basis element of $\Delta_\nu(\mu)$ of this degree. By comparing degrees, we deduce that equation (3.2) holds for some $c \in \mathbb{C}$. It remains to show that $c = \pm 1$ (note that, for the result to hold, it is enough to show that $c \neq 0$).

It is clear that the lefthand side of Equation 3.2 is a diagram with distinguished black points on northern and southern boundaries given by the loadings corresponding to the partitions ν and μ respectively (with $j = i + 2$). If the bijection traced out by the strands (after concatenation) uses the minimal number of crossings, then we are done.

Suppose that we are not in the case above, then we must apply the relations to the concatenated diagram to obtain a diagram of the form $c 1_\mu^\nu$ for some $c \in \mathbb{C}$. Then the concatenated diagram has a number of 'extra

crossings' of strands of the same residue (that is, crossings which do not appear in 1_μ^ν). The rightmost of these crossings involves a pair of strands of residue r, say. This crossing is bypassed by the ghost of the strand of residue $r - 1$ immediately to its right (for an example, see Figure 26). Applying relation (2.10), the product can be written as a sum of two terms: one is zero modulo more dominant terms, the other differs from the original diagram only where we have untied the distinguished crossing (for an example, see Figure 26).

Now suppose that the resulting diagram is not equal to 1_μ^ν, in which case it has a rightmost 'extra crossing' of residue $r + 1$. Now consider the ghost of the leftmost of the two strands we untied in the previous step; the ghost of this strand bypasses the rightmost 'extra crossing'. Repeating the above argument for all the crossings, we obtain the result. ■
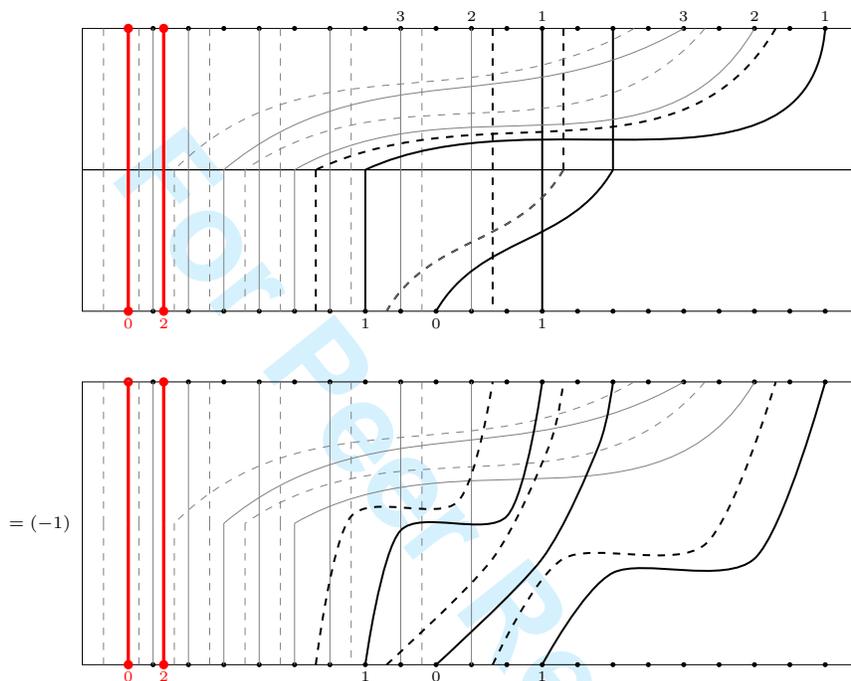
Fig. 26: The top diagram is obtained by concatenation of the diagram $1_{1'}^{2'}$ above $1_0^{1'}$. The lower diagram is obtained by applying relation (2.10) to the product $1_{1'}^{2'} \circ 1_0^{1'}$. We move the ghost 0 strand through the crossing pair of black strands of residue 1 (we do not record the diagram which is zero modulo more dominant terms).
We have made emphasised the strands to which we are applying relation (2.10) and we have recorded their residues along the southern edge of the frames. Along the northern edge of the frame of the top diagram, we have recorded the residues of the 3 extra crossings of like-labelled pairs.

Acknowledgements

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