An Exact Dynamic Stiffness Element Using a Higher Order Shear Deformation Theory for Free Vibration Analysis of Composite Plate Assemblies

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Abstract

An exact dynamic stiffness method based on higher order shear deformation theory is developed for the first time using symbolic computation in order to carry out free vibration analysis of composite plate assemblies. Hamilton's principle is applied to derive the governing differential equations of motion and natural boundary conditions. Then by imposing the geometric boundary conditions in algebraic form the dynamic stiffness matrix is developed. The Wittrick-Williams algorithm is used as solution technique to compute the natural frequencies and mode shapes for a range of laminated composite plates and stepped panels. The effects of significant parameters such as thickness ratio, orthotropy ratio, step ratio, number of layers, lay-up and stacking sequence and boundary conditions on the natural frequencies and mode shapes are critically examined and discussed. The accuracy of the method is demonstrated by comparing results with those available in the literature.

Keywords: Dynamic Stiffness Method, Composite Plates, Free Vibration, Stepped Panels, Wittrick-Williams algorithm.

1. Introduction

During the last three decades thin-walled composite structures have played very important roles in aerospace, automotive, marine and civil engineering design, amongst many others. The use of advanced composite materials allows structures to be much stiffer and stronger and yet much lighter. When these materials are combined with cutting-edges manufacturing technologies, they provide design engineers a competitive edge over conventional design with metallic construction. For this reason, research in the static and dynamic behavior of composite structures has continued to grow. In particular, free vibration analysis of assemblies of composite plates has received wide attention over the years. The research is further stimulated by the fact that many practical structural components can be modelled adequately as thin or thick metallic or composite plates. One method of analysis, other than the conventional finite element method (FEM) for this type of structures is that of the dynamic stiffness method (DSM) (see

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[1]). Application of this method involves developing the dynamic stiffness (DS) matrix for each individual element in the structure and then assembling them into a global DS matrix for subsequent free vibration analysis. This method is, in many ways, analogous to the conventional finite element method (see [2]). The main difference between the two methods is that the FEM discretizes a structural element based on assumed shape functions to derive the mass and stiffness matrices separately, whereas the DSM uses a single element matrix containing both mass and stiffness proprieties, which are derived from the exact frequency-dependent shape functions obtained from the solution of the governing differential equations of the element in free vibration. The assembly procedure for the two methods is essentially the same, but the solution techniques are different in that the FEM generally leads to a linear eingenvalue problem in sharp contrast to the non-linear (transcendental) eigen-solution encountered in the DSM, which is generally solved by applying the well-established algorithm of Wittrick and Williams [3]. For structures consisting of beam elements there is no restriction on the application of the DSM and there are some well known software based on the method to analyze plane or space frames [4]. Another important difference between the FEM and the DSM is that, the number of natural frequencies that can be computed using the FEM is restricted to the number of chosen degrees of freedom of the structure and the accuracy of results diminishes with higher order modes. This can be a serious limitation in modal analysis. By contrast, the DSM has no such limitation and any number of natural frequencies can be computed to any desired accuracy using the DS matrix without the need to increase the number of elements to achieve higher accuracy. Moreover, when fast iterative matrix solvers are used, the DSM will be much more efficient than the FEM. With regard to plate elements the DSM gives exact results because the equations of motion are solved in Lèvy-type closed form to obtain the element properties and no other approximation is made en route during the analysis. Wittrick and Williams [5] are known to be the first who attempted the extension of DSM to plate elements. Their pioneering formulation is interesting and relies on extensive use of complex algebra. In 1972, Williams [6] presented two computer programs, GASVIP and VIPAL to compute the natural frequencies, based on DSM. Essentially GASVIP was used to set up the overall stiffness matrix for the structure, and VIPAL demonstrated the use of substructuring. A couple of years later, Wittrick and Williams reported the computer code VIPASA [5] for free vibration analysis of prismatic plate assemblies, which was a significant development at the time. VIPASA code allowed free vibration analysis of isotropic or anisotropic plates and had many additional features. The complex stiffnesses described in [7] were incorporated, as well as allowances for eccentric connections between the component plates were accounted for, but more importantly, the code used a powerful algorithm as solution technique, developed by Wittrick and Williams [3] to compute natural frequencies of plated structures. The algorithm is robust and it ensures that no natural frequencies of the structure are missed. (A brief discussion of the Wittrick-Williams algorithm is presented in section (2.5)). In 1983, Williams and Anderson [8] showed modifications to the eigenvalue algorithm described in [3]. They made use of Lagrangian multipliers to apply point constraints at any location of plate edges. Each sinusoidal mode of the freely vibrating plate in the longitudinal direction was included within the dynamic stiffness matrix. These modifications formed the basis for the enhanced computer code VICON (VIpasa with CONstraints) which was a significant improvement, (see [9]), over the previous code. However, the analysis capability

of VICON was based on classical plate theory (CPT), and particularly for composite plates, attention was focused on symmetric laminates. A later version of the code included plates on Winkler foundations [10]. Next, a major enhancement of the program took place in the early nineties when the optimum design features were added and the new program VICONOPT (VICON with OPTimization) [11, 12] was born. Anderson and Kennedy [13] incorporated the effect of the shear deformation into VICONOPT few years later using a numerical approach. The general purpose application of VICONOPT was further enhanced by them [13] to allow for analysis of angle-ply laminates. An interesting historical review of the DSM procedure for plates can be found in [14]. It should be noted that DSM has been extensively researched by Banerjee [1, 15, 16, 17, 18, 19], amongst others for modal analysis of structures idealized by beam elements based on Euler-Bernoulli, Timoshenko and associated coupled beam theories. The extension of the DSM to plate elements is no-doubt difficult, but indeed, essential to model complex structures. Following the earlier research on DS theories of isotropic and composite plates, Boscolo and Banerjee advanced the state of the art on these topics by including the effects of shear deformation and rotatory inertia and thereby providing a detailed modal analysis procedure through the application of symbolic computation and Matlab [20, 21, 22, 23]. They used the first order shear deformation theory (FSDT) for which the introduction of a user specified shear corrector factor was necessary. The current paper is partly motivated by these earlier developments and the most important contribution made by the authors here is the inclusion of higher order shear deformation theory (HSDT), for the first time, when developing the DS matrix for laminated composite plates. This useful extension is of considerable theoretical and computational complexity as will be shown later. The research is particularly relevant when analysing thick composite plates for their free vibration characteristics. It should be recognised that Reddy and co-authors [24, 25, 26] have used HSDT in a different context in free vibration analysis of composite plates without resorting to the development of the DSM. From a historical prospective HSDT, can be traced back to third order plate bending theory originally proposed by Vlasov [27] in the late fifties. His theory was substantiated and extended to laminated composite plates many years later by Reddy [24] using a variational approach. This is sometimes referred to as Vlasov-Reddy theory (VRT). Further improvements of this theory can be found in the work of Jemielita [28, 29]. During the last two decades, a variable kinematics 2D model approach with hierarchical capabilities, particularly for laminated composite beams, plates and shells, has been proposed by Carrera *et al.* for mechanical [30, 31, 32, 33, 34] and multifield [35, 36, 37] problems. Inclusion of HSDT in the DSM framework will enable free vibration analysis of plates with moderate to high thickness to width ratio, in an accurate and computationally efficient manner. One of the great advantages of using HSDT as opposed to FSDT is that the former accounts for the effects of the shear deformation in a judicious manner without using a fictitious (and often controversial) shear correction or shape factor that is prevalent in the latter. The usefulness of HSDT becomes apparent when analysing composite structures, particularly of thicker dimensions, because fiber reinforced composites have generally very low shear modulii. Both the in-plane and out-of-plane free vibration analyses are considered in this paper. Extensive results which include validation and assessment of the effects of significant parameters such as the thickness to width (or length) ratio, orthotropy ratio, step ratio, number of layers, stacking sequence and boundary conditions, have been computed and discussed. The paper finished with some concluding remarks.

2. Theoretical formulation

2.1. Displacement field and governing differential equations

In the derivation that follows, the hypotheses of straightness and normality of a transverse normal after deformation are assumed to be no longer valid for the displacement field which is now considered to be a cubic function in the thickness coordinate; and hence the use of higher order shear deformation theory (HSDT). This is in sharp contrast to earlier formulations based on CPT and FSDT. For a composite plate, the kinematics of deformation of a transverse normal using both first order and higher order shear deformation are schematically shown in Fig. 1. The laminate is assumed to be composed of N_l layers so that the theory is sufficiently general. The integer k is used as a superscript denoting the layer number where the numbering starts from the bottom. After imposing the transverse shear stress homogeneous conditions [38, 39] at the top/bottom surface of the plate, the displacements field is given below in the usual form:

$$\begin{aligned} u\,(x,y,z,t) &= u_0\,(x,y,t) + z\,\phi_x\,(x,y,t) - z^3\,\frac{4}{3\,h^2}\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\ v\,(x,y,z,t) &= v_0\,(x,y,t) + z\,\phi_y\,(x,y,t) - z^3\,\frac{4}{3\,h^2}\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \\ w\,(x,y,z,t) &= w_0\,(x,y,t) \end{aligned} \tag{1}$$

where u, v, w are general displacements within the plate in the x, y, and z directions, respectively, whereas u_0, v_0, w_0 are the corresponding displacements of the reference surface (mid-plane Ω). Hamilton principle is now applied. The variational statement is:

$$\sum_{k=1}^{N_l} \int_{t_1}^{t_2} \delta\,\mathcal{L}^k\,dt = 0 \tag{2}$$

where $\mathcal{L}^k$ is the Lagrangian for the kth layer of the composite plate. The first variation can be expressed as:

$$\delta\mathcal{L}^k = \delta T^k - \delta U^k \tag{3}$$

where δU^k is the virtual strain energy, δT^k is the virtual kinetic energy, and assume the following form:

$$\begin{aligned} \delta U^k &= \int_{\Omega^k}\int_{z^k} \left(\boldsymbol{\delta\varepsilon}^{k^T}\,\boldsymbol{\sigma}^k\right)\,d\Omega^k\,dz \\ \delta T^k &= \int_{\Omega^k}\int_{z^k} \left(\rho^k\,\delta\boldsymbol{\eta}^T\,\ddot{\boldsymbol{\eta}}\right)\,d\Omega^k\,dz \end{aligned} \tag{4}$$

the stresses ($\boldsymbol{\sigma}$), the strains ($\boldsymbol{\varepsilon}$) and the displacements ($\boldsymbol{\eta}$) vectors are expressed as follows:

$$\begin{aligned} \boldsymbol{\sigma} &= \left\{\begin{array}{ccccc} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} & \sigma_{xz} & \sigma_{yz} \end{array}\right\}^T \\ \boldsymbol{\varepsilon} &= \left\{\begin{array}{ccccc} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} & \gamma_{xz} & \gamma_{yz} \end{array}\right\}^T \\ \boldsymbol{\eta} &= \left\{\begin{array}{ccc} u & v & w \end{array}\right\}^T \end{aligned} \tag{5}$$

ρ^k denotes mass density while an over dot denotes differentiation with respect to time. The subscript T signifies an array transposition and δ the variational operator. Constitutive and geometrical relationships (deformation) are respectively defined as:

$$\boldsymbol{\sigma}^k = \tilde{\boldsymbol{C}}^k \, \boldsymbol{\varepsilon}^k, \qquad\qquad \boldsymbol{\varepsilon} = \boldsymbol{\mathcal{D}}\, \boldsymbol{\eta} \tag{6}$$

where $\tilde{\boldsymbol{C}}^k$ is the plane stress constitutive matrix and $\boldsymbol{\mathcal{D}}$ is the differential matrix (see Appendix A for details). Substituting Eq. (6) into the Eq. (4) and imposing the condition in Eq. (2), the equations of motion are obtained after extensive algebraic manipulation as:

$$\begin{aligned}
\delta u_0 :\quad & A_{11}\, u_{0,xx} + A_{12}\, v_{0,yx} + A_{16}\,(u_{0,yx} + v_{0,xx}) + B_{11}\,\phi_{x,xx} + B_{12}\,\phi_{y,yx} + B_{16}\,(\phi_{x,yx} + \phi_{y,xx}) + E_{11}\,\mathrm{c}_2\,\phi_{x,xx} \\
& + E_{11}\,\mathrm{c}_2\, w_{0,xxx} + E_{12}\,\mathrm{c}_2\,\phi_{y,yx} + E_{12}\,\mathrm{c}_2\, w_{0,yyx} + E_{16}\,\mathrm{c}_2\,\phi_{x,yx} + E_{16}\,\mathrm{c}_2\,\phi_{y,xx} + 2\,E_{16}\,\mathrm{c}_2\, w_{0,xyx} + A_{16}\, u_{0,xy} \\
& + A_{26}\, v_{0,yy} + A_{66}\,(u_{0,yy} + v_{0,xy}) + B_{16}\,\phi_{x,xy} + B_{26}\,\phi_{y,yy} + B_{66}\,(\phi_{x,yy} + \phi_{y,xy}) + E_{12}\,\mathrm{c}_2\,\left(\phi_{x,xy} + w_{0,xxy}\right) \\
& + E_{26}\,\mathrm{c}_2\,\left(\phi_{y,yy} + w_{0,yyy}\right) + E_{66}\,\mathrm{c}_2\,\left(\phi_{x,yy} + \phi_{y,xy} + 2\, w_{0,xyy}\right) = I_0\,\ddot{u}_0 + I_1\,\ddot{\phi}_x + I_3\,\mathrm{c}_2\,\ddot{\phi}_x + I_3\,\mathrm{c}_2\,\ddot{w}_{0,x}
\end{aligned}$$

$$\begin{aligned}
\delta v_0 :\quad & A_{16}\, u_{0,xx} + A_{26}\, v_{0,yx} + A_{66}\,(u_{0,yx} + v_{0,xx}) + B_{16}\,\phi_{x,xx} + B_{26}\,\phi_{y,yx} + B_{66}\,(\phi_{x,yx} + \phi_{y,xx}) + E_{16}\,\mathrm{c}_2\,\phi_{x,xx} \\
& + E_{16}\,\mathrm{c}_2\, w_{0,xxx} + E_{26}\,\mathrm{c}_2\,\phi_{y,yx} + E_{26}\,\mathrm{c}_2\, w_{0,yyx} + E_{66}\,\mathrm{c}_2\,\phi_{x,yx} + E_{66}\,\mathrm{c}_2\,\phi_{y,xx} + 2\,E_{66}\,\mathrm{c}_2\, w_{0,xyx} + A_{12}\, u_{0,xy} \\
& + A_{22}\, v_{0,yy} + A_{26}\,(u_{0,yy} + v_{0,xy}) + B_{12}\,\phi_{x,xy} + B_{22}\,\phi_{y,yy} + B_{26}\,(\phi_{x,yy} + \phi_{y,xy}) + E_{12}\,\mathrm{c}_2\,\left(\phi_{x,xy} + w_{0,xxy}\right) \\
& + E_{22}\,\mathrm{c}_2\,\left(\phi_{y,yy} + w_{0,yyy}\right) + E_{26}\,\mathrm{c}_2\,\left(\phi_{x,yy} + \phi_{y,xy} + 2\, w_{0,xyy}\right) = I_0\,\ddot{v}_0 + I_1\,\ddot{\phi}_y + I_3\,\mathrm{c}_2\,\ddot{\phi}_y + I_3\,\mathrm{c}_2\,\ddot{w}_{0,y}
\end{aligned}$$

$$\begin{aligned}
\delta w_0 :\quad & A_{44}\,(\phi_{y,y} + w_{0,yy}) + A_{45}\,(\phi_{x,y} + w_{0,xy}) + D_{44}\,\mathrm{c}_1\,(\phi_{y,y} + w_{0,yy}) + D_{45}\,\mathrm{c}_1\,(\phi_{x,y} + w_{0,xy}) \\
& + A_{45}\,(\phi_{y,x} + w_{0,xy}) + A_{55}\,(\phi_{x,x} + w_{0,xx}) + D_{45}\,\mathrm{c}_1\,(\phi_{y,x} + w_{0,xy}) + D_{55}\,\mathrm{c}_1\,(\phi_{x,x} + w_{0,xx}) \\
& + D_{44}\,\mathrm{c}_1\,(\phi_{y,y} + w_{0,yy}) + D_{45}\,\mathrm{c}_1\,(\phi_{x,y} + w_{0,xy}) + F_{44}\,\mathrm{c}_1^2\,(\phi_{y,y} + w_{0,yy}) + F_{45}\,\mathrm{c}_1^2\,(\phi_{x,y} + w_{0,xy}) \\
& + D_{45}\,\mathrm{c}_1\,(\phi_{y,x} + w_{0,xy}) + D_{55}\,\mathrm{c}_1\,(\phi_{x,x} + w_{0,xx}) + F_{45}\,\mathrm{c}_1^2\,(\phi_{y,x} + w_{0,xy}) + F_{55}\,\mathrm{c}_1^2\,(\phi_{x,x} + w_{0,xx}) \\
& - E_{11}\,\mathrm{c}_2\, u_{0,xxx} - E_{12}\,\mathrm{c}_2\, v_{0,xxy} - E_{16}\,\mathrm{c}_2\,(u_{0,xxy} + v_{0,xxx}) - F_{11}\,\mathrm{c}_2\,\phi_{x,xxx} - F_{12}\,\mathrm{c}_2\,\phi_{y,xxy} \\
& - F_{16}\,\mathrm{c}_2\,(\phi_{x,xxy} + \phi_{y,xxx}) - H_{11}\,\mathrm{c}_2^2\,(\phi_{x,xxx} + w_{0,xxxx}) - H_{12}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + w_{0,xxyy}) \\
& - H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + \phi_{y,xxx} + 2w_{0,xxxy}) - 2\,E_{16}\,\mathrm{c}_2\, u_{0,xxy} - 2\,E_{26}\,\mathrm{c}_2\, v_{0,xyy} - 2\,E_{66}\,\mathrm{c}_2\,(u_{0,xyy} + v_{0,xxy}) \\
& - 2\,F_{16}\,\mathrm{c}_2\,\phi_{x,xxy} - 2\,F_{26}\,\mathrm{c}_2\,\phi_{y,xyy} - 2\,F_{66}\,\mathrm{c}_2\,(\phi_{x,xyy} + \phi_{y,xxy}) - 2\,H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + w_{0,xxxy}) \\
& - 2\,H_{26}\,\mathrm{c}_2^2\,(\phi_{y,xyy} + w_{0,xyyy}) - 2\,H_{66}\,\mathrm{c}_2^2\,(\phi_{x,xyy} + \phi_{y,xxy} + 2\,w_{0,xxyy}) - E_{12}\,\mathrm{c}_2\, u_{0,xyy} - E_{22}\,\mathrm{c}_2\, v_{0,yyy} \\
& - E_{26}\,\mathrm{c}_2\,(u_{0,yyy} + v_{0,xyy}) - F_{12}\,\mathrm{c}_2\,\phi_{x,xyy} - F_{22}\,\mathrm{c}_2\,\phi_{y,yyy} - F_{26}\,\mathrm{c}_2\,(\phi_{x,yyy} + \phi_{y,xyy}) \\
& - H_{12}\,\mathrm{c}_2^2\,(\phi_{x,xyy} + w_{0,xxyy}) - H_{22}\,\mathrm{c}_2^2\,(\phi_{y,yyy} + w_{0,yyyy}) - 2\,H_{26}\,\mathrm{c}_2^2\,(\phi_{x,yyy} + \phi_{y,xyy} + 2\,w_{0,xyyy}) \\
& = I_0\,\ddot{w}_0 - I_6\,\mathrm{c}_2^2\,(\ddot{w}_{0,xx} + \ddot{w}_{0,yy}) - I_3\,\mathrm{c}_2^2\,(\ddot{u}_{0,x} + \ddot{u}_{0,y}) - (I_4 + I_6)\;\mathrm{c}_2^2\left(\ddot{\phi}_{x,x} + \ddot{\phi}_{y,y}\right)
\end{aligned}$$

$$
\begin{aligned}
\delta\phi_x: \quad & B_{11}\, u_{0,xx} + B_{12}\, v_{0,yx} + B_{16}\, \left(u_{0,yx} + v_{0,xx}\right) + D_{11}\, \phi_{x,xx} + D_{12}\, \phi_{y,xy} + D_{16}\, \left(\phi_{x,yx} + \phi_{y,xx}\right) \\
& + F_{11}\, \mathrm{c}_2\, \left(\phi_{x,xx} + w_{0,xxx}\right) + F_{12}\, \mathrm{c}_2 \left(\phi_{y,yx} + w_{0,yyx}\right) + F_{16}\, \mathrm{c}_2 \left(\phi_{x,yx} + \phi_{y,xx} + 2\, w_{0,xyx}\right) \\
& + B_{16}\, u_{0,xy} + B_{26}\, v_{0,yy} + B_{66}\, \left(u_{0,yy} + v_{0,xy}\right) + D_{16}\, \phi_{x,xy} + D_{26}\, \phi_{y,yy} + D_{66}\, \left(\phi_{x,yy} + \phi_{y,xy}\right) \\
& + F_{16}\, \mathrm{c}_2\, \left(\phi_{x,xy} + w_{0,xxy}\right) + F_{26}\, \mathrm{c}_2 \left(\phi_{y,yy} + w_{0,yyy}\right) + F_{66}\, \mathrm{c}_2 \left(\phi_{x,yy} + \phi_{y,xy} + 2\, w_{0,xyy}\right) \\
& + E_{11}\, \mathrm{c}_2\, u_{0,xx} + E_{12}\, \mathrm{c}_2\, v_{0,yx} + E_{16}\, \mathrm{c}_2 \left(u_{0,yx} + v_{0,xx}\right) + F_{11}\, \mathrm{c}_2\, \phi_{x,xx} + F_{12}\, \mathrm{c}_2\, \phi_{y,xy} + F_{16}\, \mathrm{c}_2 \left(\phi_{x,yx} + \phi_{y,xx}\right) \\
& + H_{11}\, \mathrm{c}_2^2 \left(\phi_{x,xx} + w_{0,xxx}\right) + H_{12}\, \mathrm{c}_2^2 \left(\phi_{y,yx} + w_{0,yyx}\right) + H_{16}\, \mathrm{c}_2^2 \left(\phi_{x,yx} + \phi_{y,xx} + 2\, w_{0,xyx}\right) \\
& + E_{16}\, \mathrm{c}_2\, u_{0,xy} + E_{26}\, \mathrm{c}_2\, v_{0,yy} + E_{66}\, \mathrm{c}_2 \left(u_{0,yy} + v_{0,xy}\right) + F_{16}\, \mathrm{c}_2\, \phi_{x,xy} + F_{26}\, \mathrm{c}_2\, \phi_{y,yy} + F_{66}\, \mathrm{c}_2 \left(\phi_{x,yy} + \phi_{y,xy}\right) \\
& + H_{16}\, \mathrm{c}_2^2 \left(\phi_{x,xy} + w_{0,xxy}\right) + H_{26}\, \mathrm{c}_2^2 \left(\phi_{y,yy} + w_{0,yyy}\right) + H_{66}\, \mathrm{c}_2^2 \left(\phi_{x,yy} + \phi_{y,xy} + 2\, w_{0,xyy}\right) \\
& - A_{45} \left(\phi_y + 2\, w_{0,y}\right) - A_{55} \left(\phi_x + 2\, w_{0,x}\right) - 2\, D_{45}\, \mathrm{c}_1 \left(\phi_y + 2\, w_{0,y}\right) - 2\, D_{55}\, \mathrm{c}_1 \left(\phi_x + 2\, w_{0,x}\right) \\
& - F_{45}\, \mathrm{c}_1^2 \left(\phi_y + 2\, w_{0,y}\right) - F_{55}\, \mathrm{c}_1^2 \left(\phi_x + 2\, w_{0,x}\right) = \left(I_1 + \mathrm{c}_2\, I_3\right)\, \ddot{u}_0 + \left(I_2 + 2\, \mathrm{c}_2\, I_4 + \mathrm{c}_2^2\, I_6\right)\, \ddot{\phi}_x + \left(I_4 + \mathrm{c}_2^2\, I_6\right)\, \ddot{w}_{0,x}
\end{aligned}
$$

$$
\begin{aligned}
\delta\phi_y: \quad & B_{16}\, u_{0,xx} + B_{26}\, v_{0,yx} + B_{66}\, \left(u_{0,yx} + v_{0,xx}\right) + D_{16}\, \phi_{x,xx} + D_{26}\, \phi_{y,xy} + D_{66}\, \left(\phi_{x,yx} + \phi_{y,xx}\right) \\
& + F_{16}\, \mathrm{c}_2\, \left(\phi_{x,xx} + w_{0,xxx}\right) + F_{26}\, \mathrm{c}_2 \left(\phi_{y,yx} + w_{0,yyx}\right) + F_{66}\, \mathrm{c}_2 \left(\phi_{x,yx} + \phi_{y,xx} + 2\, w_{0,xyx}\right) \\
& + B_{12}\, u_{0,xy} + B_{22}\, v_{0,yy} + B_{26}\, \left(u_{0,yy} + v_{0,xy}\right) + D_{12}\, \phi_{x,xy} + D_{22}\, \phi_{y,yy} + D_{26}\, \left(\phi_{x,yy} + \phi_{y,xy}\right) \\
& + F_{12}\, \mathrm{c}_2\, \left(\phi_{x,xy} + w_{0,xxy}\right) + F_{22}\, \mathrm{c}_2 \left(\phi_{y,yy} + w_{0,yyy}\right) + F_{26}\, \mathrm{c}_2 \left(\phi_{x,yy} + \phi_{y,xy} + 2\, w_{0,xyy}\right) \\
& + E_{16}\, \mathrm{c}_2\, u_{0,xx} + E_{26}\, \mathrm{c}_2\, v_{0,yx} + E_{66}\, \mathrm{c}_2 \left(u_{0,yx} + v_{0,xx}\right) + F_{16}\, \mathrm{c}_2\, \phi_{x,xx} + F_{26}\, \mathrm{c}_2\, \phi_{y,xy} + F_{66}\, \mathrm{c}_2 \left(\phi_{x,yx} + \phi_{y,xx}\right) \\
& + H_{16}\, \mathrm{c}_2^2 \left(\phi_{x,xx} + w_{0,xxx}\right) + H_{26}\, \mathrm{c}_2^2 \left(\phi_{y,yx} + w_{0,yyx}\right) + H_{66}\, \mathrm{c}_2^2 \left(\phi_{x,yx} + \phi_{y,xx} + 2\, w_{0,xyx}\right) \\
& + E_{12}\, \mathrm{c}_2\, u_{0,xy} + E_{22}\, \mathrm{c}_2\, v_{0,yy} + E_{26}\, \mathrm{c}_2 \left(u_{0,yy} + v_{0,xy}\right) + F_{12}\, \mathrm{c}_2\, \phi_{x,xy} + F_{22}\, \mathrm{c}_2\, \phi_{y,yy} + F_{26}\, \mathrm{c}_2 \left(\phi_{x,yy} + \phi_{y,xy}\right) \\
& + H_{12}\, \mathrm{c}_2^2 \left(\phi_{x,xy} + w_{0,xxy}\right) + H_{22}\, \mathrm{c}_2^2 \left(\phi_{y,yy} + w_{0,yyy}\right) + H_{26}\, \mathrm{c}_2^2 \left(\phi_{x,yy} + \phi_{y,xy} + 2\, w_{0,xyy}\right) \\
& - A_{44} \left(\phi_y + 2\, w_{0,y}\right) - A_{45} \left(\phi_x + 2\, w_{0,x}\right) - 2\, D_{44}\, \mathrm{c}_1 \left(\phi_y + 2\, w_{0,y}\right) - 2\, D_{45}\, \mathrm{c}_1 \left(\phi_x + 2\, w_{0,x}\right) \\
& - F_{44}\, \mathrm{c}_1^2 \left(\phi_y + 2\, w_{0,y}\right) - F_{45}\, \mathrm{c}_1^2 \left(\phi_x + 2\, w_{0,x}\right) = \left(I_1 + \mathrm{c}_2\, I_3\right)\, \ddot{v}_0 + \left(I_2 + 2\, \mathrm{c}_2\, I_4 + \mathrm{c}_2^2\, I_6\right)\, \ddot{\phi}_y + \left(I_4 + \mathrm{c}_2^2\, I_6\right)\, \ddot{w}_{0,y}
\end{aligned}
$$

(7)

The natural boundary conditions are:

$$
\begin{aligned}
\delta u_0: \quad \mathcal{N}_{xx} = {} & A_{11}\, u_{0,x} + B_{11}\, \phi_{x,x} + E_{11}\, \mathrm{c}_2\, \phi_{x,x} + E_{11}\, \mathrm{c}_2\, w_{0,xx} + A_{12}\, v_{0,y} + B_{12}\, \phi_{y,y} + E_{12}\, \mathrm{c}_2\, \phi_{y,y} + E_{12}\, \mathrm{c}_2\, w_{0,yy} \\
& + A_{16}\, u_{0,y} + A_{16}\, v_{0,x} + B_{16}\, \phi_{x,y} + B_{16}\, \phi_{y,x} + E_{16}\, \mathrm{c}_2\, \phi_{x,y} + E_{16}\, \mathrm{c}_2\, \phi_{y,x} + 2\, E_{16}\, \mathrm{c}_2\, w_{0,xy}
\end{aligned}
$$

$$
\begin{aligned}
\delta v_0: \quad \mathcal{N}_{xy} = {} & A_{16}\, u_{0,x} + B_{16}\, \phi_{x,x} + E_{16}\, \mathrm{c}_2\, \phi_{x,x} + E_{16}\, \mathrm{c}_2\, w_{0,xx} + A_{26}\, v_{0,y} + B_{26}\, \phi_{y,y} + E_{26}\, \mathrm{c}_2\, \phi_{y,y} + E_{26}\, \mathrm{c}_2\, w_{0,yy} \\
& + A_{66}\, u_{0,y} + A_{66}\, v_{0,x} + B_{66}\, \phi_{x,y} + E_{66}\, \mathrm{c}_2\, \phi_{y,x} + E_{66}\, \mathrm{c}_2\, \phi_{x,y} + E_{66}\, \mathrm{c}_2\, \phi_{y,x} + 2\, E_{66}\, \mathrm{c}_2\, w_{0,xy}
\end{aligned}
$$

$$\delta w_0: \quad \mathcal{Q}_x = H_{11}\,c_2^2\,\phi_{x,xx} + H_{11}\,c_2^2\,w_{0,xxx} + E_{11}\,c_2\,u_{0,xx} + F_{11}\,c_2\,\phi_{x,xx} + E_{12}\,c_2\,v_{0,yx} + F_{12}\,c_2\,\phi_{y,yx}$$
$$+ H_{12}\,c_2^2\,\phi_{y,yx} + H_{12}\,c_2^2\,w_{0,yyx} + 2\,E_{16}\,c_2\,u_{0,xy} + 2\,F_{16}\,c_2\,\phi_{x,xy} + 2\,H_{16}c_2^2\,\phi_{x,xy} + E_{16}\,c_2\,u_{0,yx}$$
$$+ E_{16}\,c_2\,v_{0,xx} + F_{16}\,c_2\,\phi_{x,yx} + H_{16}\,c_2^2\,\phi_{x,yx} + H_{16}\,c_2^2\,\phi_{y,xx} + 2\,H_{16}\,c_2^2\,w_{0,xxy} + 2\,E_{26}\,c_2\,v_{0,yy}$$
$$+ 2\,F_{26}\,c_2\,\phi_{y,yy} + 2\,H_{26}c_2^2\,w_{0,yyy} + 4\,H_{66}c_2^2\,w_{0,xyy} + 2\,H_{26}\,c_2^2\,\phi_{x,yy} + 2\,H_{26}\,c_2^2\,\phi_{y,xy} + 2\,E_{66}\,c_2\,u_{0,yy}$$
$$+ 2\,E_{66}\,c_2\,v_{0,xy} + 2\,F_{66}\,c_2\,\phi_{x,yy} + 2\,F_{66}\,c_2\,\phi_{y,xy} - 2\,D_{45}\,c_1\,\phi_y - 2\,D_{45}\,c_1\,w_{0,y} - F_{45}\,c_1^2\,\phi_y$$
$$- F_{45}\,c_1^2\,w_{0,y} - A_{55}\,\phi_x - A_{55}\,w_{0,x} - D_{55}\,c_1\,\phi_x - 2\,c_1\,w_{0,x} - F_{55}\,c_1^2\,\phi_x - F_{55}\,c_1^2\,w_{0,x}$$

$$\delta\phi_x: \quad \mathcal{M}_{xx} = D_{11}\,\phi_{x,x} + H_{11}\,c_2^2\,\phi_{x,x} + H_{11}\,c_2^2\,w_{0,xx} + B_{11}\,u_{0,x} + E_{11}\,c_2\,u_{0,x} + 2\,F_{11}\,c_2\,\phi_{x,x} + F_{11}\,c_2\,w_{0,xx}$$
$$+ F_{11}\,c_2\,w_{0,xx} + B_{12}\,v_{0,y} + D_{12}\,\phi_{y,y} + F_{12}\,c_2\,\phi_{y,y} + F_{12}\,c_2\,w_{0,yy} + E_{12}\,c_2\,v_{0,y} + F_{12}\,c_2\,\phi_{y,y} + H_{12}\,c_2^2\,\phi_{y,y}$$
$$+ H_{12}\,c_2^2\,w_{0,yy} + B_{16}\,u_{0,y} + B_{16}\,v_{0,x} + D_{16}\,\phi_{x,y} + D_{16}\,\phi_{y,x} + F_{16}\,c_2\,\phi_{x,y} + F_{16}\,c_2\,\phi_{y,x} + 2\,F_{16}\,c_2\,w_{0,xy}$$
$$+ E_{16}\,c_2\,u_{0,y} + E_{16}\,c_2\,v_{0,x} + F_{16}\,c_2\,\phi_{x,y} + F_{16}\,c_2\,\phi_{y,x} + H_{16}\,c_2^2\,\phi_{x,y} + H_{16}\,c_2^2\,\phi_{y,x} + 2\,H_{16}\,c_2^2\,w_{0,xy}$$

$$\delta\phi_y: \quad \mathcal{M}_{xy} = D_{16}\,\phi_{x,x} + H_{16}\,c_2^2\,\phi_{x,x} + H_{16}\,c_2^2\,w_{0,xx} + B_{16}\,u_{0,x} + E_{16}\,c_2\,u_{0,x} + 2\,F_{16}\,c_2\,\phi_{x,x} + F_{16}\,c_2\,w_{0,xx}$$
$$+ F_{16}\,c_2\,w_{0,xx} + B_{26}\,v_{0,y} + D_{12}\,\phi_{y,y} + F_{26}\,c_2\,\phi_{y,y} + F_{26}\,c_2\,w_{0,yy} + E_{26}\,c_2\,v_{0,y} + F_{26}\,c_2\,\phi_{y,y} + H_{26}\,c_2^2\,\phi_{y,y}$$
$$+ H_{26}\,c_2^2\,w_{0,yy} + B_{66}\,u_{0,y} + B_{66}\,v_{0,x} + D_{66}\,\phi_{x,y} + D_{66}\,\phi_{y,x} + F_{66}\,c_2\,\phi_{x,y} + F_{66}\,c_2\,\phi_{y,x} + 2\,F_{66}\,c_2\,w_{0,xy}$$
$$+ E_{66}\,c_2\,u_{0,y} + E_{66}\,c_2\,v_{0,x} + F_{66}\,c_2\,\phi_{x,y} + F_{66}\,c_2\,\phi_{y,x} + H_{66}\,c_2^2\,\phi_{x,y} + H_{66}\,c_2^2\,\phi_{y,x} + 2\,H_{66}\,c_2^2\,w_{0,xy}$$

$$\delta w_{0,x}: \quad \mathcal{P}_{xx} = H_{11}\,c_2^2\,\phi_{x,x} + H_{11}\,c_2^2\,w_{0,xx} + E_{11}\,c_2\,u_{0,x} + F_{11}\,c_2\,\phi_{x,x} + E_{12}\,c_2\,v_{0,y} + F_{12}\,c_2\,\phi_{y,y} + H_{12}\,c_2^2\,\phi_{y,y}$$
$$+ H_{12}\,c_2^2\,w_{0,yy} + E_{16}\,c_2\,u_{0,y} + E_{16}\,c_2\,v_{0,x} + F_{16}\,c_2\,\phi_{x,y} + F_{16}\,c_2\,\phi_{y,x} + H_{16}\,c_2^2\,\phi_{x,y} + H_{16}\,c_2^2\,\phi_{y,x}$$
$$+ 2\,H_{16}\,c_2^2\,w_{0,xy} \tag{8}$$

where the suffix after the comma denotes the partial derivative with respect to that variable, and

$$\begin{aligned}(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) &= \sum_{k=1}^{N_l} \int_{z^k} \tilde{C}_{ij}^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right)\, dz \\ (I_0, I_1, I_2, I_3, I_4, I_6) &= \sum_{k=1}^{N_l} \int_{z^k} \rho^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right)\, dz\end{aligned} \tag{9}$$

are laminate stiffnesses and inertia terms, respectively with i and j varying form 1 to 6.

2.2. Dynamic stiffness formulation

Once the equations of motion and the natural boundary conditions, i.e., Eqs. (7) and (8) are obtained, the classical method to carry out exact free vibration analysis of a plate consists of (i) solving the system of differential equations in Navier or Lèvy type closed form in an exact manner, (ii) applying particular boundary conditions on the edges and finally (iii) obtaining the frequency equation by eliminating the integration constants [40, 41, 42, 43]. This method, although extremely useful for analysing an individual plate, it lacks generality and cannot be easily applied to complex structures for which researchers usually resort to approximate methods such as the FEM. In this respect, the dynamic stiffness method (DSM), which is analogous to FEM, but is more powerful as it always retains the exactness of the solution even when it is applied to complex structures. The dynamic stiffness matrix of a structural element used in the DSM has many other advantages. It can be offset and/or rotated and assembled in a global DS matrix in the same way as the FEM. This global DS matrix contains implicitly all the exact natural frequencies of the structure which can be computed by using the well established algorithm of Wittrick and Williams [3].

A general procedure to develop the dynamic stiffness matrix of a structural element can be summarized as follows:

(i) Seek a closed form analytical solution of the governing differential equations of motion of the structural element undergoing free vibration.

(ii) Apply a number of general boundary conditions in algebraic forms that are equal to twice the number of integration constants; these are usually nodal displacements and forces.

(iii) Eliminate the constants by relating the amplitudes of the harmonically varying nodal forces to those of the corresponding displacements which essentially generates the frequency-dependent dynamic stiffness matrix, providing the force-displacement relationship between nodes.

Referring to the equations of motions given by Eqs. (7), an exact solution can be sought in Lèvy's form for symmetric, cross ply laminates. For such laminates $\boldsymbol{B} = \boldsymbol{E} = 0$, and $\tilde{C}^k_{16} = \tilde{C}^k_{26} = \tilde{C}^k_{45} = 0$ and the out-of-plane displacements are uncoupled from the in-plane ones.

2.3. Lèvy-type closed form exact solution and DS formulation

The solution of Eqs.(7) is sought as:

$$
\begin{aligned}
u^0(x,y,t) &= \sum_{m=1}^{\infty} U_m(x)\,\mathrm{e}^{i\omega t}\sin(\alpha\, y), \qquad v^0(x,y,t) = \sum_{m=1}^{\infty} V_m(x)\,\mathrm{e}^{i\omega t}\cos(\alpha\, y),\\
w^0(x,y,t) &= \sum_{m=1}^{\infty} W_m(x)\,\mathrm{e}^{i\omega t}\sin(\alpha\, y), \qquad \phi_x(x,y,t) = \sum_{m=1}^{\infty} \Phi_{x_m}(x)\,\mathrm{e}^{i\omega t}\sin(\alpha\, y),\\
\phi_y(x,y,t) &= \sum_{m=1}^{\infty} \Phi_{y_m}(x)\,\mathrm{e}^{i\omega t}\cos(\alpha\, y)
\end{aligned} \tag{10}
$$

where ω is the unknown circular frequency, $\alpha = \frac{m\pi}{L}$ and $m = 1, 2, \ldots, \infty$. This is the so-called Lèvy's solution which assumes that two the opposite sides of the plate are simply supported (S-S), i.e. $w = \phi_x = 0$ at $y = 0$ and $y = L$. Substituting Eq. (10) into Eqs. (7) a set of five ordinary differential equations that are uncoupled between in-plane and out-of-plane deformations, is obtained which can be written in two different matrix forms as follows:

$$
\begin{bmatrix} \mathcal{L}_{p_{11}} & \mathcal{L}_{p_{12}} \\ \mathcal{L}_{p_{21}} & \mathcal{L}_{p_{22}} \end{bmatrix} \begin{bmatrix} U_m \\ V_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \mathcal{L}_{o_{11}} & \mathcal{L}_{o_{12}} & \mathcal{L}_{o_{13}} \\ \mathcal{L}_{o_{21}} & \mathcal{L}_{o_{22}} & \mathcal{L}_{o_{23}} \\ \mathcal{L}_{o_{31}} & \mathcal{L}_{o_{32}} & \mathcal{L}_{o_{33}} \end{bmatrix} \begin{bmatrix} W_m \\ \Phi_x \\ \Phi_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}
$$

where $\mathcal{L}_{p_{ij}}$ $(i,j = 1,2)$ and $\mathcal{L}_{o_{ij}}$ $(i,j = 1,2,3)$ are differential operators. For the in-plane free vibrations case, $\mathcal{L}_{p_{ij}}$ $(i, j = 1, 2)$ are given by:

$$\begin{aligned}
\mathcal{L}_{p_{11}} &= (A_{66}\,\alpha^2 - I_0\,\omega^2) - A_{11}\,\mathcal{D}_x^2 \\
\mathcal{L}_{p_{12}} &= (A_{12} + A_{66})\,\alpha\,\mathcal{D}_x \\
\mathcal{L}_{p_{21}} &= (A_{12} + A_{66})\,\alpha\,\mathcal{D}_x \\
\mathcal{L}_{p_{22}} &= (-A_{22}\,\alpha^2 + I_0\,\omega^2) + A_{66}\,\mathcal{D}_x^2
\end{aligned} \tag{12}$$

For the out-of-plane case, $\mathcal{L}_{o_{ij}}$ $(i, j = 1, 2, 3)$ are given by:

$$\begin{aligned}
\mathcal{L}_{o_{11}} = &\left(-\alpha^2\,(A_{44} + 2\,c_2\,D_{44} + c_2^2\,F_{44} + \alpha^2\,c_1^2\,H_{22}) + (I_0 + \alpha^2\,c_1^2\,I_6)\,\omega^2\right) + (A_{55} + 2\,c_1\,D_{55} + c_1^2\,F_{55} \\
&+ 2\,\alpha^2\,c_1^2\,H_{12} + 4\,\alpha^2\,c_1^2\,H_{66} - c_1^2\,I_6\omega^2)\,\mathcal{D}_x^2 + (-c_1^2\,H_{11})\,\mathcal{D}_x^4
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{o_{12}} = &A_{55} + 2\,c_1\,D_{55} + \alpha^2\,c_1\,F_{12} + c_1^2\,F_{55} + 2\,\alpha^2\,c_1\,F_{66} + \alpha^2\,c_1^2\,H_{12} + 2\,\alpha^2\,c_1^2\,H_{66} - c_1\,I_4\,\omega^2 - c_1^2\,I_6\,\omega^2)\,\mathcal{D}_x \\
&+ (-c_1\,F_{11} - c_1^2\,H_{11})\,\mathcal{D}_x^3
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{o_{13}} = &-\alpha\,(A_{44} + c_2\,(2\,D_{44} + c_2\,F_{44}) + \alpha^2\,c_1\,(F_{22} + c_1\,H_{22}) - c_1\,(I_4 + c_1\,I_6)\,\omega^2) + (\alpha\,c_1\,F_{12} + 2\,\alpha\,c_1\,F_{66} \\
&+ \alpha\,c_1^2\,H_{12} + 2\,\alpha\,c_1^2\,H_{66})\mathcal{D}_x^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{o_{21}} = &(-A_{55} - c_2\,(2\,D_{55} + c_2\,F_{55}) - \alpha^2\,c_1\,(F_{12} + 2\,F_{66} + c_1\,H_{12} + 2\,c_1\,H_{66}) + c_1\,(I_4 + c_1\,I_6)\,\omega^2)\,\mathcal{D}_x \\
&+ c_1\,(F_{11} + c_1\,H_{11})\,\mathcal{D}_x^3
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{o_{22}} = &(-A_{55} - c_2\,(2\,D_{55} + c_2\,F_{55}) - \alpha^2\,(D_{66} + 2\,c_1\,F_{66} + c_1^2\,H_{66}) + (I_2 + 2c_1\,I_4 + c_1^2\,I_6)\,\omega^2) + (D_{11} \\
&+ 2\,c_1\,F_{11} + c_1^2\,H_{11})\mathcal{D}_x^2
\end{aligned}$$

$$\mathcal{L}_{o_{23}} = (-\alpha\,D_{12} - \alpha\,D_{66} - 2\,\alpha\,c_1\,F_{12} - 2\,\alpha\,c_1\,F_{66} - \alpha\,c_1^2\,H_{12} - \alpha\,c_1^2\,H_{66})\,\mathcal{D}_x$$

$$\begin{aligned}
\mathcal{L}_{o_{31}} = &-\alpha\,(A_{44} + c_2\,(2\,D_{44} + c_2\,F_{44}) + \alpha^2\,c_1\,(F_{22} + c_1\,H_{22}) - c_1\,(I_4 + c_1\,I_6)\,\omega^2) + \alpha\,c_1\,(F_{12} + 2\,F_{66} \\
&+ c_1\,H_{12} + 2\,c_1\,H_{66})\,\mathcal{D}_x^2
\end{aligned}$$

$$\mathcal{L}_{o_{32}} = \alpha \left(D_{12} + D_{66} + c_1 \left(2\,F_{12} + 2\,F_{66} + c_1 \left(H_{12} + H_{66}\right)\right)\right)\mathcal{D}_x$$

$$\mathcal{L}_{o_{33}} = \left(-A_{44} - c_2 \left(2\,D_{44} + c_2\,F_{44}\right) - \alpha^2 \left(D_{22} + c_1 \left(2\,F_{22} + c_1\,H_{22}\right)\right) + \left(I_2 + c_1 \left(2\,I_4 + c_1\,I_6\right)\right)\omega^2\right) + \left(D_{66} + c_1 \left(2\,F_{66} + c_1\,H_{66}\right)\right)\mathcal{D}_x^2 \tag{13}$$

where $\mathcal{D}_x = \frac{\mathrm{d}}{\mathrm{dx}}$, $c_1 = -\frac{4}{3\,h^2}$ and $c_2 = -\frac{4}{h^2}$ and A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} have already been defined in Eq. (9). Expanding the determinant of the matrices in Eq. (11) the following differential equations for the in-plane and out-of-plane cases are respectively obtained as follows:

$$\left(b_1\,\mathcal{D}_x^4 + b_2\,\mathcal{D}_x^2 + b_3\right)\Xi = 0, \qquad \left(a_1\,\mathcal{D}_x^8 + a_2\,\mathcal{D}_x^6 + a_3\,\mathcal{D}_x^4 + a_4\,\mathcal{D}_x^2 + a_5\right)\Psi = 0 \tag{14}$$

where

$$\Xi = U_m,\, V_m, \qquad \Psi = W_m,\, \Phi_{y_m},\, \Phi_{x_m} \tag{15}$$

Using a trial solution e^{λ} in Eq. (14) yields the following auxiliary equations for the two cases:

$$b_1\,\lambda_p^4 + b_2\,\lambda_p^2 + b_3 = 0, \qquad a_1\,\lambda_o^8 + a_2\,\lambda_o^6 + a_3\,\lambda_o^4 + a_4\,\lambda_o^2 + a_5 = 0 \tag{16}$$

Substituting $\mu_p = \lambda_p^2$ and $\mu_o = \lambda_o^2$, the fourth and eighth order polynomials of Eqs. (16) become

$$b_1\,\mu_p^2 + b_2\,\mu_p + b_3 = 0, \qquad a_1\,\mu_o^4 + a_2\,\mu_o^3 + a_3\,\mu_o^2 + a_4\,\mu_o + a_5 = 0 \tag{17}$$

The two roots for the in-plane case i.e. the quadratic equation of the left are give by:

$$\begin{aligned} \mu_{p1} &= \frac{-b_2 + \sqrt{b_2^2 - 4\,b_1\,b_3}}{2\,b_1} \\ \mu_{p2} &= \frac{-b_2 - \sqrt{b_2^2 - 4\,b_1\,b_3}}{2\,b_1} \end{aligned} \tag{18}$$

whereas the four roots for the out of plane case, i.e. the quartic equation on the right by:

$$\begin{aligned} \mu_{o1} &= -s_1 - \frac{1}{2}\sqrt{-s_5 + s_2 - \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} - \frac{1}{2}\sqrt{s_9} \\ \mu_{o2} &= -s_1 + \frac{1}{2}\sqrt{-s_5 + s_2 - \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} - \frac{1}{2}\sqrt{s_9} \\ \mu_{o3} &= -s_1 - \frac{1}{2}\sqrt{-s_5 + s_2 + \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} + \frac{1}{2}\sqrt{s_9} \\ \mu_{o4} &= -s_1 + \frac{1}{2}\sqrt{-s_5 + s_2 + \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} + \frac{1}{2}\sqrt{s_9} \end{aligned} \tag{19}$$

where

$$\begin{aligned} & s_1 = \frac{a_1}{4\,a_2}, \qquad s_2 = -\frac{4\,a_3}{3\,a_1^2} + \frac{a_2^2}{2\,a_1^2}, \qquad s_3 = 2\,a_3 - 9\,a_2\,a_3\,a_4 - 72\,a_2^2\,a_5 + 27\,a_2^2\,a_5 + 27\,a_1\,a_4^2, \\ & s_4 = a_3^2 - 3\,a_2\,a_4 + 12\,a_1\,a_5, \qquad s_5 = \frac{1}{a_1}\left(\frac{s_3 + \sqrt{s_3 - 4\,s_4}}{32}\right)^{\frac{1}{3}}, \qquad s_6 = \sqrt[3]{4}\,s_4, \qquad s_7 = \sqrt[3]{32}\,s_5\,a_1, \\ & s_8 = \left(\frac{a_2}{a_1}\right)^3 \frac{4\,a_2\,a_3}{a_1^2} - \frac{8\,a_4}{a_1}, \qquad s_9 = s_5 + \frac{s_2}{2} + \frac{s_6}{3\,s_7\,a_1} \end{aligned} \tag{20}$$

The explicit form of the coefficients a_j $(j = 1, 2, 3, 4, 5)$ and b_j $(j = 1, 2, 3)$ can be found in Appendix B. Note that when computing μ_{pj} $(j = 1, 2)$ and μ_{oj} $(j = 1, 2, 3, 4)$, some roots may turn out to be complex, but the amplitude of the displacements $U_m(x)$, $V_m(x)$, $W_m(x)$, $\Phi_{x_m}(x)$, $\Phi_{y_m}(x)$ will always be real, whilst the associated coefficients can be complex. As complex roots occur in conjugate pairs, the associated coefficients will also occur in conjugate pairs. The solution for out-of-plane and in-plane free vibration can thus be written as:

$$
\begin{aligned}
W_m(x) &= A_1\,\mathrm{e}^{+\mu_{o1}x} + A_2\,\mathrm{e}^{-\mu_{o1}x} + A_3\,\mathrm{e}^{+\mu_{o2}x} + A_4\,\mathrm{e}^{-\mu_{o2}x} \\
&\quad + A_5\,\mathrm{e}^{+\mu_{o3}x} + A_6\,\mathrm{e}^{-\mu_{o3}x} + A_7\,\mathrm{e}^{+\mu_{o4}x} + A_8\,\mathrm{e}^{-\mu_{o4}x} \\[2ex]
\Phi_{x_m}(x) &= B_1\,\mathrm{e}^{+\mu_{o1}x} + B_2\,\mathrm{e}^{-\mu_{o1}x} + B_3\,\mathrm{e}^{+\mu_{o2}x} + B_4\,\mathrm{e}^{-\mu_{o2}x} \\
&\quad + B_5\,\mathrm{e}^{+\mu_{o3}x} + B_6\,\mathrm{e}^{-\mu_{o3}x} + B_7\,\mathrm{e}^{+\mu_{o4}x} + B_8\,\mathrm{e}^{-\mu_{o4}x} \\[2ex]
\Phi_{y_m}(x) &= C_1\,\mathrm{e}^{+\mu_{o1}x} + C_2\,\mathrm{e}^{-\mu_{o1}x} + C_3\,\mathrm{e}^{+\mu_{o2}x} + C_4\,\mathrm{e}^{-\mu_{o2}x} \\
&\quad + C_5\,\mathrm{e}^{+\mu_{o3}x} + C_6\,\mathrm{e}^{-\mu_{o3}x} + C_7\,\mathrm{e}^{+\mu_{o4}x} + C_8\,\mathrm{e}^{-\mu_{o4}x} \\[2ex]
U_m(x) &= D_1\,\mathrm{e}^{+\mu_{p1}x} + D_2\,\mathrm{e}^{-\mu_{p1}x} + D_3\,\mathrm{e}^{+\mu_{p2}x} + D_4\,\mathrm{e}^{-\mu_{p2}x} \\[2ex]
V_m(x) &= E_1\,\mathrm{e}^{+\mu_{p1}x} + E_2\,\mathrm{e}^{-\mu_{p1}x} + E_3\,\mathrm{e}^{+\mu_{p2}x} + E_4\,\mathrm{e}^{-\mu_{p2}x}
\end{aligned}
\tag{21}
$$

where $A_1 - A_8$, $B_1 - B_8$, $C_1 - C_8$, $D_1 - D_4$, $E_1 - E_4$ are integration constants. For both in-plane and out-of-plane cases, the constants are not all independent. Thus a set of four independent constants, for the in-plane case, and a set of eight independent constants, for the out-of-plane case, can be chosen and then related to the others. Constants $E_1 - E_4$ for in-plane case, and $B_1 - B_8$ for out-of-plane case are respectively chosen here to be independent. By substituting Eqs. (21) into (11) the following relationships can be obtained for the in-plane case:

$$
\begin{aligned}
D_1 &= \beta_1 E_1, \qquad & D_2 &= -\beta_1 E_2 \\
D_3 &= \beta_2 E_3, \qquad & D_4 &= -\beta_2 E_4
\end{aligned}
\tag{22}
$$

Likewise, for the out-of-plane case

$$
\begin{aligned}
A_1 &= \delta_1 B_1, & A_2 &= -\delta_1 B_2, & C_1 &= \gamma_1 B_1, & C_2 &= -\gamma_1 B_2 \\
A_3 &= \delta_2 B_3, & A_4 &= -\delta_2 B_4, & C_3 &= \gamma_2 B_3, & C_4 &= -\gamma_2 B_4 \\
A_5 &= \delta_3 B_5, & A_6 &= -\delta_3 B_6, & C_5 &= \gamma_3 B_5, & C_6 &= -\gamma_3 B_6 \\
A_7 &= \delta_4 B_7, & A_8 &= -\delta_4 B_8, & C_7 &= \gamma_4 B_7, & C_8 &= -\gamma_4 B_8
\end{aligned}
\tag{23}
$$

where

$$\beta_j = \frac{I_0\,\omega^2 - A_{66}\,\alpha^2 + A_{11}\,\mu_{pj}^2}{(A_{12} + A_{66})\,\alpha\mu_{pj}}$$

$$\begin{aligned}\delta_i = &-(-\alpha^2\,(D_{12} + D_{66} + c_1\,(2\,D_{12} + 2\,D_{66} + c_1\,(H_{12} + H_{66})))^2\,\mu_{oi}^2 + (-A_{55} - c_2\,(2D_{55} + c_2\,D_{55})\\ &- \alpha^2\,(D_{66} + c_1\,(2\,D_{66} + c_1\,H_{66})) + (I_2 + c_1\,(2\,I_4 + c_1\,I_6))\,\omega^2 + (D_{11} + c_1\,(2\,D_{11} + c_1\,H_{11}))\,\mu_{oi}^2)\,(A_{44}\\ &+ c_2\,(2\,D_{44} + c_2\,D_{44}) + \alpha^2\,(D_{22} + 2\,c_1\,D_{22} + c_1^2\,H_{22}) - (I_2 + 2\,c_1\,I_4 + c_1^2\,I_6)\,\omega^2 - (D_{66} + 2\,c_1\,D_{66}\,c_1^2\,H_{66})\,\mu_{oi}^2))/\\ &(-\mu_{oi}\,(A_{44} + c_2\,(2\,D_{44} + c_2\,D_{44}) + \alpha^2\,(D_{22} + 2\,c_1\,D_{22} + c_1^2\,H_{22}) - (I_2 + 2\,c_1\,I_4 + c_1^2\,I_6)\,\omega^2\\ &- (D_{66} + 2\,c_1\,D_{66} + c_1^2\,H_{66})\,\mu_{oi}^2)(A_{55} + 2\,c_2\,D_{55} + c_2^2\,D_{55} + c_1\,(\alpha^2\,(D_{12} + 2\,D_{66} + c_1\,H_{12} + 2\,c_1\,H_{66}) - (I_4\\ &+ c_1\,I_6)\,\omega^2 - (D_{11} + c_1\,H_{11})\,\mu_{oi}^2)) + \alpha^2\,(D_{12} + D_{66} +\ c_1\,(2\,D_{12} + 2\,D_{66} + c_1\,(H_{12} + H_{66})))\,\mu_{oi}\,(A_{44} + 2\,c_2\,D_{44}\\ &+ c_2^2\,D_{44} + c_1\,(\alpha^2\,(D_{22} + c_1\,H_{22}) - (I_4 + c_1\,I_6)\,\omega^2 - (D_{12} + 2\,D_{66} + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_{oi}^2)))\end{aligned}$$

$$\begin{aligned}\gamma_i = &-\frac{1}{(\alpha\,(D_{12} + D_{66} + c_1\,(2\,F_{12} + 2\,F_{66} + c_1\,(H_{12} + H_{66})))\,\mu_{oi})}(A_{55} + 2\,c_2\,D_{55} + \alpha^2\,D_{66} + c_2^2\,F_{55}\\ &+ 2\,\alpha^2\,c_1\,F_{66} + \alpha^2\,c_1^2\,H_{66} - I_2\,\omega^2 - 2\,c_1\,I_4\,\omega^2 - c_1^2\,I_6\,\omega^2 - D_{11}\,\mu_{oi}^2 - 2\,c_1\,F_{11}\,\mu_{oi}^2 - c_1^2\,H_{11}\,\mu_{oi}^2 - (\alpha^2\,(D_{12}\\ &+ D_{66} + c_1\,(2\,F_{12} + 2\,F_{66} + c_1\,(H_{12} + H_{66})))^2\,\mu_{oi}^3(-A_{55} - c_2\,(2\,D_{55} + c_2\,F_{55}) + c_1\,(-\alpha^2\,(F_{12} + 2\,F_{66}\\ &+ c_1\,(H_{12} + 2\,H_{66})) + (I_4 + c_1\,I_6)\,\omega^2 + (F_{11} + c_1\,H_{11})\,\mu_{oi}^2)))/(-\mu_{oi}\,(A_{44} + c_2\,(2\,D_{44} + c_2\,F_{44}) + \alpha^2\,(D_{22}\\ &+ c_1\,(2\,F_{22} + c_1\,H_{22})) - (I_2 + c_1\,(2\,I_4 + c_1\,I_6))\,\omega^2 - (D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\mu_{oi}^2)\,(A_{55} + 2\,c_2\,D_{55} + c_2^2\,F_{55}\\ &+ c_1\,(\alpha^2\,(F_{12} + 2\,F_{66} + c_1\,(H_{12} + 2\,H_{66})) - (I_4 + c_1\,I_6)\,\omega^2 - (F_{11} + c_1\,H_{11})\,\mu_{oi}^2)) + \alpha^2\,(D_{12} + D_{66} + c_1\,(2\,F_{12}\\ &+ 2\,F_{66} + c_1\,(H_{12} + H_{66})))\,\mu_{oi}\,(A_{44} + 2\,c_2\,D_{44} + c_2^2\,F_{44} + c_1\,(\alpha^2\,(F_{22} + c_1\,H_{22}) - (I_4 + c_1\,I_6)\,\omega^2 - (F_{12}\\ &+ 2\,F_{66} + c_1\,(H_{12} + 2\,H_{66}))\,\mu_{oi}^2))) - (\mu_{oi}\,(A_{55} + c_2\,(2\,D_{55} + c_2\,F_{55}) + \alpha^2\,(D_{66} + c_1\,(2\,F_{66} + c_1 H_{66})) - (I_2\\ &+ c_1\,(2\,I_4 + c_1\,I_6))\,\omega^2 - (D_{11} + c_1\,(2\,F_{11} + c_1\,H_{11}))\,\mu_{oi}^2)\,(A_{44} + c_2\,(2\,D_{44} + c_2\,F_{44}) + \alpha^2\,(D_{22} + c_1\,(2\,F_{22}\\ &+ c_1\,H_{22})) - (I_2 + c_1\,(2\,I_4 + c_1\,I_6))\,\omega^2 - (D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_{oi}^2)(-A_{55} - c_2\,(2\,D_{55} + c_2\,F_{55})\\ &+ c_1(-\alpha^2\,(F_{12} + 2\,F_{66} + c_1\,(H_{12} + 2\,H_{66})) + (I_4 + c_1\,I_6)\,\omega^2 + (F_{11} + c_1\,H_{11})\,\mu_{oi}^2)))/(-\mu_{oi}\,(A_{44} + c_2\,(2\,D_{44}\\ &+ c_2\,F_{44}) + \alpha^2\,(D_{22} + c_1\,(2\,F_{22} + c_1\,H_{22})) - (I_2 + c_1\,(2\,I_4 + c_1\,I_6))\,\omega^2 - (D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_{oi}^2)\,(A_{55}\\ &+ 2\,c_2\,D_{55} + c_2^2\,F_{55} + c_1\,(\alpha^2\,(F_{12} + 2\,F_{66} + c_1\,(H_{12} + 2\,H_{66})) - (I_4 + c_1\,I_6)\,\omega^2 - (F_{11} + c_1\,H_{11})\,\mu_{oi}^2))\\ &+ \alpha^2\,(D_{12} + D_{66} + c_1(2F_{12} + 2\,F_{66} + c_1\,(H_{12} + H_{66})))\,\mu_{oi}\,(A_{44} + 2\,c_2\,D_{44} + c_2^2\,F_{44} + c_1\,(\alpha^2\,(F_{22} + c_1\,H_{22})\\ &- (I_4 + c_1\,I_6)\,\omega^2\,(F_{12} + 2\,F_{66} + c_1\,(H_{12} + 2\,H_{66}))\,\mu_{oi}^2))))\end{aligned} \tag{24}$$

with $j = 1, 2$ and $i = 1, 2, 3, 4$. The procedure leading to Eqs. (22), (23) and (24) must be undertaken with sufficient care, because if wrong set of constants are chosen from Eq. (21) to obtain the relationship connecting other sets of constant, numerical instability can occur. When Eqs. (22) and (23) are substituted into Eqs. (21) a solution in terms of only eight integration constants for the out-of-plane case and only four for the in-plane case can be respectively formulated. Thus

$$W_m\,(x) = B_1\,\delta_1\,\mathrm{e}^{+\mu_{o1}\,x} - B_2\,\delta_1\,\mathrm{e}^{-\mu_{o1}\,x} + B_3\,\delta_2\,\mathrm{e}^{+\mu_{o2}\,x} - B_4\,\delta_2\,\mathrm{e}^{-\mu_{o2}\,x}$$
$$+ B_5\,\delta_3\,\mathrm{e}^{+\mu_{o3}\,x} - B_6\,\delta_3\,\mathrm{e}^{-\mu_{o3}\,x} + B_7\,\delta_4\,\mathrm{e}^{+\mu_{o4}\,x} - B_8\,\delta_4\,\mathrm{e}^{-\mu_{o4}\,x}$$

$$\Phi_{x_m}\,(x) = B_1\;\mathrm{e}^{+\mu_{o1}\,x} + B_2\;\mathrm{e}^{-\mu_{o1}\,x} + B_3\;\mathrm{e}^{+\mu_{o2}\,x} + B_4\;\mathrm{e}^{-\mu_{o2}\,x}$$
$$+ B_5\;\mathrm{e}^{+\mu_{o3}\,x} + B_6\;\mathrm{e}^{-\mu_{o3}\,x} + B_7\;\mathrm{e}^{+\mu_{o4}\,x} + B_8\;\mathrm{e}^{-\mu_{o4}\,x}$$

$$\Phi_{y_m}\,(x) = B_1\,\gamma_1\,\mathrm{e}^{+\mu_{o1}\,x} - B_2\,\gamma_1\,\mathrm{e}^{-\mu_{o1}\,x} + B_3\,\gamma_2\,\mathrm{e}^{+\mu_{o2}\,x} - B_4\,\gamma_2\,\mathrm{e}^{-\mu_{o2}\,x} \tag{25}$$
$$+ B_5\,\gamma_3\,\mathrm{e}^{+\mu_{o3}\,x} - B_6\,\gamma_3\,\mathrm{e}^{-\mu_{o3}\,x} + B_7\,\gamma_4\,\mathrm{e}^{+\mu_{o4}\,x} - B_8\,\gamma_4\,\mathrm{e}^{-\mu_{o4}\,x}$$

$$U_m\,(x) = E_1\,\beta_1\,\mathrm{e}^{+\mu_{p1}\,x} - E_2\,\beta_1\,\mathrm{e}^{-\mu_{p1}\,x} + E_3\,\beta_2\,\mathrm{e}^{+\mu_{p2}\,x} - E_4\,\beta_2\,\mathrm{e}^{-\mu_{p2}\,x}$$

$$V_m\,(x) = E_1\;\mathrm{e}^{+\mu_{p1}\,x} + E_2\;\mathrm{e}^{-\mu_{p1}\,x} + E_3\;\mathrm{e}^{+\mu_{p2}\,x} + E_4\;\mathrm{e}^{-\mu_{p2}\,x}$$

The expressions for forces and moments can also be found in the same way by substituting Eqs. (25) into Eqs. (8). In this way

$$N_{xx}\,(x,y) = \Big(\mathrm{e}^{\mu_{p1}x}\,(E_1 + E_2\,\mathrm{e}^{-2\,\mu_{p1}\,x})\,(-A_{12}\,\alpha + A_{11}\,\mu_{p1}\,\beta_1) +$$
$$\mathrm{e}^{\mu_{p2}x}\,(E_3 + E_4\,\mathrm{e}^{-2\,\mu_{p2}\,x})\,(-A_{12}\,\alpha + A_{11}\,\mu_{p1}\,\beta_2)\Big)\;\sin(\alpha y) = \mathcal{N}_{xx}\;\sin(\alpha\,y)$$

$$N_{xy}\,(x,y) = \Big(\mathrm{e}^{\mu_{p1}x}\,(E_1 - E_2\,\mathrm{e}^{-2\,\mu_{p1}\,x})\,(A_{66}\,(\mu_{p1} + \alpha\,\beta_1)) +$$
$$\mathrm{e}^{\mu_{p2}x}\,(E_3 - E_4\,\mathrm{e}^{-2\,\mu_{p2}\,x})\,(A_{66}\,(\mu_{p2} + \alpha\,\beta_2))\Big)\;\cos(\alpha\,y) = \mathcal{N}_{xy}\;\cos(\alpha\,y)$$

$$Q_x\,(x,y) = \Big(\mathrm{e}^{\mu_{o1}\,x}\,(B_1 + B_2\,\mathrm{e}^{-2\,\mu_{o1}\,x})(A_{55} + A_{55}\,\delta_1\,\mu_{o1} + 2\,c_2\,(D_{55} + D_{55}\,\delta_1\,\mu_{o1}) + c_2^2\,(F_{55} + \delta_1\,F_{55}\,\mu_{o1})$$
$$+ c_1\,(\alpha\,\gamma_1\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_{o1} - \mu_{o1}^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_1\,H_{11}\,\mu_{o1}) + \alpha^2\,(2\,F_{66}$$
$$+ 2\,c_1\,H_{66} + c_1\,\delta_1\,H_{12}\,\mu_{o1} + 4\,c_1\,\delta_1\,H_{66}\,\mu_{o1}))) +$$
$$\mathrm{e}^{\mu_{o2}\,x}(B_3 + B_4\,\mathrm{e}^{-2\,\mu_{o2}\,x})(A_{55} + A_{55}\,\delta_2\,\mu_{o2} + 2\,c_2\,(D_{55} + D_{55}\,\delta_2\,\mu_{o2}) + c_2^2\,(F_{55} + \delta_2\,F_{55}\,\mu_{o2})$$
$$+ c_1\,(\alpha\,\gamma_2\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_{o2} - \mu_{o2}^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_2\,H_{11}\,\mu_{o2}) + \alpha^2\,(2\,F_{66}$$
$$+ 2\,c_1\,H_{66} + c_1\,\delta_2\,H_{12}\,\mu_{o2} + 4\,c_1\,\delta_1\,H_{66}\,\mu_{o2}))) +$$
$$\mathrm{e}^{\mu_{o3}\,x}(B_5 + B_6\mathrm{e}^{-2\,\mu_{o3}\,x})(A_{55} + A_{55}\,\delta_3\,\mu_{o3} + 2\,c_2\,(D_{55} + D_{55}\,\delta_3\,\mu_{o3}) + c_2^2\,(F_{55} + \delta_3\,F_{55}\,\mu_{o3})$$
$$+ c_1\,(\alpha\,\gamma_3\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_{o3} - \mu_{o3}^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_3\,H_{11}\,\mu_{o3}) + \alpha^2\,(2\,F_{66}$$
$$+ 2\,c_1\,H_{66} + c_1\,\delta_3\,H_{12}\,\mu_{o3} + 4\,c_1\,\delta_3\,H_{66}\,\mu_{o3}))) +$$
$$\mathrm{e}^{\mu_{o4}\,x}(B_3 + B_4\,\mathrm{e}^{-2\,\mu_{o4}\,x})(A_{55} + A_{55}\,\delta_4\,\mu_{o4} + 2\,c_2\,(D_{55} + D_{55}\,\delta_4\,\mu_{o4}) + c_2^2\,(F_{55} + \delta_4\,F_{55}\,\mu_{o4})$$
$$+ c_1\,(\alpha\,\gamma_4\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_{o4} - \mu_{o4}^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_4\,H_{11}\,\mu_{o4}) + \alpha^2\,(2\,F_{66}$$
$$+ 2\,c_1\,H_{66} + c_1\,\delta_4\,H_{12}\,\mu_{o4} + 4\,c_1\,\delta_4\,H_{66}\,\mu_{o4})))\Big)\;\sin(\alpha\,y) = \mathcal{Q}_x\;\sin(\alpha\,y)$$

$$
\begin{aligned}
M_{xx}\,(x,y) = \Big(&\mathrm{e}^{\mu_{o1}\,x}(B_1 + B_2\,\mathrm{e}^{-2\,\mu_{o1}\,x})(\alpha^2\,c_1\,\delta_1\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_1\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_{o1}\,(D_{11} \\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_1\,F_{11}\,\mu_{o1} + c_1\,\delta_1\,H_{11}\,\mu_{o1})))+ \\
&\mathrm{e}^{\mu_{o2}\,x}(B_3 + B_4\,\mathrm{e}^{-2\,\mu_{o2}\,x})(\alpha^2\,c_1\,\delta_2\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_2\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_{o2}\,(D_{11} \\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_2\,F_{11}\,\mu_{o2} + c_1\,\delta_2\,H_{11}\,\mu_{o2})))+ \\
&\mathrm{e}^{\mu_{o3}\,x}(B_5 + B_5\,\mathrm{e}^{-2\,\mu_{o3}\,x})(\alpha^2\,c_1\,\delta_3\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_3\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_{o3}\,(D_{11} \\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_3\,F_{11}\,\mu_{o3} + c_1\,\delta_3\,H_{11}\,\mu_{o3})))+ \\
&\mathrm{e}^{\mu_{o4}\,x}\,(B_7 + B_8\,\mathrm{e}^{-2\,\mu_{o4}\,x})(\alpha^2\,c_1\,\delta_4\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_4\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_{o4}\,(D_{11} \\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_4\,F_{11}\,\mu_{o4} + c_1\,\delta_4\,H_{11}\,\mu_{o4})))\Big)\,\sin(\alpha\,y) = \mathcal{M}_{xx}\,\sin(\alpha\,y)
\end{aligned}
$$

$$
\begin{aligned}
M_{xy}\,(x,y) = \Big(&\mathrm{e}^{\mu_{o1}x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_{o1}x})(\gamma_1\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_{o1} + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_1\,F_{66}\,\mu_{o1} + 2\,c_1\,\delta_1\,H_{66}\,\mu_{o1})))+ \\
&\mathrm{e}^{\mu_{o2}x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_{o2}x})(\gamma_2\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_{o2} + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_2\,F_{66}\,\mu_{o1} + 2\,c_1\,\delta_2\,H_{66}\,\mu_{o2})))+ \\
&\mathrm{e}^{\mu_{o3}x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_{o3}x})(\gamma_3\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_{o3} + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_3\,F_{66}\,\mu_{o3} + 2\,c_1\,\delta_3\,H_{66}\,\mu_{o2})))+ \\
&\mathrm{e}^{\mu_{o4}x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_{o4}x})(\gamma_4\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_{o4} + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_4\,F_{66}\,\mu_{o1} + 2\,c_1\,\delta_4\,H_{66}\,\mu_{o4})))\Big)\,\cos(\alpha\,y) = \mathcal{M}_{xy}\,\cos(\alpha\,y)
\end{aligned}
$$

$$
\begin{aligned}
P_{xx}\,(x,y) = \Big(&\mathrm{e}^{\mu_{o1}\,x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_{o1}\,x})\,(\alpha^2\,c_1\,\delta_1\,H_{12} + \alpha\,\gamma_1\,(F_{12} + c_1\,H_{12}) - \mu_{o1}\,(F_{11} + c_1\,H_{11}\,(1 + \delta_1\,\mu_{o1})))+ \\
&\mathrm{e}^{\mu_{o2}\,x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_{o2}\,x})\,(\alpha^2\,c_1\,\delta_2\,H_{12} + \alpha\,\gamma_2\,(F_{12} + c_1\,H_{12}) - \mu_{o2}\,(F_{11} + c_1\,H_{11}\,(1 + \delta_2\,\mu_{o2})))+ \\
&\mathrm{e}^{\mu_{o3}\,x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_{o3}\,x})\,(\alpha^2\,c_1\,\delta_3\,H_{12} + \alpha\,\gamma_3\,(F_{12} + c_1\,H_{12}) - \mu_{o3}\,(F_{11} + c_1\,H_{11}\,(1 + \delta_3\,\mu_{o3})))+ \\
&\mathrm{e}^{\mu_{o4}\,x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_{o4}\,x})\,(\alpha^2\,c_1\,\delta_4\,H_{12} + \alpha\,\gamma_4\,(F_{12} + c_1\,H_{12}) - \mu_{o4}\,(F_{11} + c_1\,H_{11}\,(1 + \delta_4\,\mu_{o4})))\Big) \\
&\sin(\alpha\,y) = \mathcal{P}_{xx}\,\sin(\alpha\,y)
\end{aligned}
\tag{26}
$$

At this point, zero boundary conditions are generally imposed to eliminate the constants in the classical method in order to establish the frequency equation for a single plate element. By contrast, the development of the dynamic stiffness matrix entails imposition of general boundary conditions in algebraic form. Thus in order to develop the two dynamic stiffness matrices for in-plane and out-of-plane cases (which will be subsequently combined), the following boundary conditions are applied next.

In-plane case:

$$
\begin{aligned}
x = 0 \; &: \qquad U_m = U_{m_1} \, , \, V = V_{m_1} \\
x = b \; &: \qquad U_m = U_{m_2} \, , \, V = V_{m_2} \\
\\
x = 0 \; &: \qquad \mathcal{N}_{xx} = -\mathcal{N}_{xx_1} \, , \, \mathcal{N}_{xy} = -\mathcal{N}_{xy_1} \\
x = b \; &: \qquad \mathcal{N}_{xx} = \;\; \mathcal{N}_{xx_2} \, , \, \mathcal{N}_{xy} = \;\; \mathcal{N}_{xy_2}
\end{aligned}
\tag{27}
$$

Out-of-plane case:

$$
\begin{aligned}
x = 0 \; &: \qquad W_m = W_{m_1} \, , \, \Phi_{x_m} = \Phi_{x_1} \, , \, \Phi_{y_m} = \Phi_{y_1} \, , \, W_{m_{,x}} = W_{m1_{,x}} \\
x = b \; &: \qquad W_m = W_{m_2} \, , \, \Phi_{x_m} = \Phi_{x_2} \, , \, \Phi_{y_m} = \Phi_{y_2} \, , \, W_{m_{,x}} = W_{m2_{,x}} \\
\\
x = 0 \; &: \qquad \mathcal{Q}_x = -\mathcal{Q}_{x_1} \, , \, \mathcal{M}_{xx} = -\mathcal{M}_{xx_1} \, , \, \mathcal{M}_{xy} = -\mathcal{M}_{xy_1} \, , \, \mathcal{P}_{xx} = -\mathcal{P}_{xx_1} \\
x = b \; &: \qquad \mathcal{Q}_x = \;\; \mathcal{Q}_{x_2} \, , \, \mathcal{P}_{xx} = \;\; \mathcal{P}_{xx_2} \, , \, \mathcal{M}_{xy} = \;\; \mathcal{M}_{xy_2} \, , \, \mathcal{P}_{xx} = \;\; \mathcal{P}_{xx_2}
\end{aligned}
\tag{28}
$$

By substituting Eqs. (27), (28) into Eq.(25), the following matrix relationship is obtained:

$$
\begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix}
=
\begin{bmatrix}
\beta_1 & -\beta_1 & \beta_2 & -\beta_2 \\
1 & 1 & 1 & 1 \\
\beta_1 \, \mathrm{e}^{b\,\mu_{p1}} & -\beta_1 \, \mathrm{e}^{-b\,\mu_{p1}} & \beta_2 \, \mathrm{e}^{b\,\mu_{p2}} & -\beta_2 \, \mathrm{e}^{-b\,\mu_{p2}} \\
\mathrm{e}^{b\,\mu_{p1}} & \mathrm{e}^{-b\,\mu_{p1}} & \mathrm{e}^{b\,\mu_{p2}} & \mathrm{e}^{-b\,\mu_{p2}}
\end{bmatrix}
\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}
\tag{29}
$$

and

$$
\begin{bmatrix} W_1 \\ \Phi_{x_1} \\ \Phi_{y_1} \\ W_{1_{,x}} \\ W_2 \\ \Phi_{x_2} \\ \Phi_{y_2} \\ W_{2_{,x}} \end{bmatrix}
=
\begin{bmatrix}
\delta_1 & -\delta_1 & \delta_2 & -\delta_2 & \delta_3 & -\delta_3 & \delta_4 & -\delta_4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\gamma_1 & -\gamma_1 & \gamma_2 & -\gamma_2 & \gamma_3 & -\gamma_3 & \gamma_4 & -\gamma_4 \\
f_1 & -f_1 & f_2 & -f_2 & f_3 & -f_3 & f_4 & -f_4 \\
\delta_1 \, \mathrm{e}^{b\,\mu_{o1}} & -\delta_1 \, \mathrm{e}^{-b\,\mu_{o1}} & \delta_2 \, \mathrm{e}^{b\,\mu_{o2}} & -\delta_2 \, \mathrm{e}^{-b\,\mu_{o2}} & \delta_3 \, \mathrm{e}^{b\,\mu_{o3}} & -\delta_3 \, \mathrm{e}^{-b\,\mu_{o3}} & \delta_4 \, \mathrm{e}^{b\,\mu_{o4}} & -\delta_4 \, \mathrm{e}^{-b\,\mu_{o4}} \\
\mathrm{e}^{b\,\mu_{o1}} & -\mathrm{e}^{-b\,\mu_{o1}} & \mathrm{e}^{b\,\mu_{o2}} & -\mathrm{e}^{-b\,\mu_{o2}} & \mathrm{e}^{b\,\mu_{o3}} & -\mathrm{e}^{-b\,\mu_{o3}} & \mathrm{e}^{b\,\mu_{o4}} & -\mathrm{e}^{-b\,\mu_{o4}} \\
\gamma_1 \, \mathrm{e}^{b\,\mu_{o1}} & -\gamma_1 \, \mathrm{e}^{-b\,\mu_{o1}} & \gamma_2 \, \mathrm{e}^{b\,\mu_{o2}} & -\gamma_2 \, \mathrm{e}^{-b\,\mu_{o2}} & \gamma_3 \, \mathrm{e}^{b\,\mu_{o3}} & -\gamma_3 \, \mathrm{e}^{-b\,\mu_{o3}} & \gamma_4 \, \mathrm{e}^{b\,\mu_{o4}} & -\gamma_4 \, \mathrm{e}^{-b\,\mu_{o4}} \\
f_1 \, \mathrm{e}^{b\,\mu_{o1}} & -f_1 \, \mathrm{e}^{-b\,\mu_{o1}} & f_2 \, \mathrm{e}^{b\,\mu_{o2}} & -f_2 \, \mathrm{e}^{-b\,\mu_{o2}} & f_3 \, \mathrm{e}^{b\,\mu_{o3}} & -f_3 \, \mathrm{e}^{-b\,\mu_{o3}} & f_4 \, \mathrm{e}^{b\,\mu_{o4}} & -f_4 \, \mathrm{e}^{-b\,\mu_{o4}}
\end{bmatrix}
\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix}
\tag{30}
$$

where

$$
f_i = \delta_i \, \mu_{oi}, \quad \text{with} \;\; i = 1, 2, 3, 4
$$

Equations (29) and (30) can be written as

$$
\boldsymbol{\delta}_p = \boldsymbol{A}_p \, \boldsymbol{C}_p, \quad \boldsymbol{\delta}_o = \boldsymbol{A}_o \, \boldsymbol{C}_o
\tag{31}
$$

By applying the same procedure for forces and moments, i.e. substituting Eqs. (27), (28) into Eq. (26) the following matrix relationship is obtained:

$$
\begin{bmatrix} \mathcal{N}_{xx_1} \\ \mathcal{N}_{xy_1} \\ \mathcal{N}_{xx_2} \\ \mathcal{N}_{xy_2} \end{bmatrix}
=
\begin{bmatrix}
\mathrm{t}_1 & \mathrm{t}_1 & \mathrm{t}_2 & \mathrm{t}_2 \\
-\mathrm{g}_1 & \mathrm{g}_1 & -\mathrm{g}_2 & \mathrm{g}_2 \\
-\mathrm{e}^{b\,\mu_{p1}} \, \mathrm{t}_1 & -\mathrm{e}^{-b\,\mu_{p1}} \, \mathrm{t}_1 & -\mathrm{e}^{b\,\mu_{p2}} \, \mathrm{t}_2 & -\mathrm{e}^{-b\,\mu_{p2}} \, \mathrm{t}_2 \\
\mathrm{e}^{b\,\mu_{p1}} \, \mathrm{g}_1 & -\mathrm{e}^{-b\,\mu_{p1}} \, \mathrm{g}_1 & \mathrm{e}^{b\,\mu_{p2}} \, \mathrm{g}_2 & -\mathrm{e}^{-b\,\mu_{p2}} \, \mathrm{g}_2
\end{bmatrix}
\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}
\tag{32}
$$

where

$$\mathrm{t}_i = \mathrm{A}_{12}\,\alpha - \mathrm{A}_{11}\,\mu_{pi}\,\beta_i, \qquad \mathrm{g}_i = \mathrm{A}_{66}\,(\mu_{pi} + \alpha\,\beta_i) \qquad \text{with} \quad i = 1, 2 \tag{33}$$

and

$$\begin{bmatrix} \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{P}_{xx_1} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \\ \mathcal{M}_{xx_2} \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{Q}_4 \\ \mathcal{T}_1 & -\mathcal{T}_1 & \mathcal{T}_2 & -\mathcal{T}_2 & \mathcal{T}_3 & -\mathcal{T}_3 & \mathcal{T}_4 & -\mathcal{T}_4 \\ -\mathcal{I}_1 & -\mathcal{I}_1 & -\mathcal{I}_2 & -\mathcal{I}_2 & -\mathcal{I}_3 & -\mathcal{I}_3 & -\mathcal{I}_4 & -\mathcal{I}_4 \\ \mathcal{L}_1 & -\mathcal{L}_1 & \mathcal{L}_2 & -\mathcal{L}_2 & \mathcal{L}_3 & -\mathcal{L}_3 & \mathcal{Y}_L & -\mathcal{L}_4 \\ \mathcal{Q}_1\,\mathrm{e}^{b\,\mu_{o1}} & -\mathcal{Q}_1\,\mathrm{e}^{-b\,\mu_{o1}} & \mathcal{Q}_2\,\mathrm{e}^{b\,\mu_{o2}} & -\mathcal{Q}_2\,\mathrm{e}^{-b\,\mu_{o2}} & \mathcal{Q}_3\,\mathrm{e}^{b\,\mu_{o3}} & -\mathcal{Q}_3\,\mathrm{e}^{-b\,\mu_{o3}} & \mathcal{Q}_4\,\mathrm{e}^{b\,\mu_{o4}} & -\mathcal{Q}_4\,\mathrm{e}^{-b\,\mu_{o4}} \\ -\mathcal{T}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{T}_1\mathrm{e}^{-b\,\mu_{o1}} & -\mathcal{T}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{T}_2\mathrm{e}^{-b\,\mu_{o2}} & -\mathcal{T}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{T}_3\mathrm{e}^{-b\,\mu_{o3}} & -\mathcal{T}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{T}_4\mathrm{e}^{-b\,\mu_{o4}} \\ \mathcal{I}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{I}_1\mathrm{e}^{-b\,\mu_{o1}} & \mathcal{I}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{I}_2\mathrm{e}^{-b\,\mu_{o2}} & \mathcal{I}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{I}_3\mathrm{e}^{-b\,\mu_{o3}} & \mathcal{I}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{I}_4\mathrm{e}^{-b\,\mu_{o4}} \\ -\mathcal{L}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{L}_1\mathrm{e}^{-b\,\mu_{o1}} & -\mathcal{L}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{L}_2\mathrm{e}^{-b\,\mu_{o2}} & -\mathcal{L}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{L}_3\mathrm{e}^{-b\,\mu_{o3}} & -\mathcal{L}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{L}_4\mathrm{e}^{-b\,\mu_{o4}} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix} \tag{34}$$

where

$$\mathcal{Q}_i = -A_{55}(1+\delta_i\mu_{o1}) - 2c_2(D_{55} + D_{55}\delta_i\mu_{oi}) - c_2^2(F_{55} + \delta_i F_{55}\mu_{oi}) - c_1(\alpha\gamma_i(F_{12} + 2F_{66} + c_1 H_{12} + 2c_1 H_{66})\mu_{oi} - \mu_{oi}^2(F_{11} + c_1 H_{11} + c_1\delta_i H_{11}\mu_{oi}) + \alpha^2(2F_{66} + 2c_1 H_{66} + c_1\delta_i H_{12}\mu_{oi} + 4c_1\delta_i H_{66}\mu_{oi}))$$

$$\mathcal{T}_i = \alpha^2 c_1\delta_i(F_{12} + c_1 H_{12}) + \alpha\gamma_i(D_{12} + c_1(2F_{12} + c_1 H_{12})) - \mu_{oi}(D_{11} + c_1(2F_{11} + c_1 H_{11} + \delta_i F_{11}\mu_{oi} + c_1\delta_i H_{11}\mu_{oi}))$$

$$\mathcal{I}_i = \gamma_1(D_{66} + c_1(2F_{66} + c_1 H_{66}))\mu_{oi} - \alpha(D_{66} + c_1(2F_{66} + c_1 H_{66} + 2\delta_i F_{66}\mu_{oi} + 2c_1\delta_i H_{66}\mu_{oi}))$$

$$\mathcal{L}_i = c_1(\alpha^2 c_1\delta_i H_{12} + \alpha\gamma_i(F_{12} + c_1 H_{12}) - \mu_{oi}(F_{11} + c_1 H_{11}(1+\delta_i\mu_{oi}))) \qquad \text{with} \quad i = 1, 2, 3, 4 \tag{35}$$

Equations (32) and (34) can be written as

$$\boldsymbol{F}_p = \boldsymbol{R}_p\,\boldsymbol{C}_p; \quad \boldsymbol{F}_o = \boldsymbol{R}_o\,\boldsymbol{C}_o \tag{36}$$

By eliminating the constants vectors $\boldsymbol{C}_p$ and $\boldsymbol{C}_o$ the two dynamic stiffness matrices for the in-plane and out-of-plane cases are respectively formulated as follows:

$$\boldsymbol{K}_p = \boldsymbol{R}_p\,\boldsymbol{A}_p^{-1}; \qquad \boldsymbol{K}_o = \boldsymbol{R}_o\,\boldsymbol{A}_o^{-1} \tag{37}$$

i.e.

$$\boldsymbol{K}_p = \begin{bmatrix} s_{nn} & s_{nl} & f_{nn} & f_{nl} \\ & s_{ll} & -f_{nl} & f_{ll} \\ & & s_{nn} & -s_{nl} \\ & \text{Sym} & & s_{tl} \end{bmatrix}, \quad \boldsymbol{K}_o = \begin{bmatrix} s_{qq} & s_{qm} & s_{qt} & s_{qh} & f_{qq} & f_{qm} & f_{qt} & f_{qh} \\ & s_{mm} & s_{mt} & s_{mh} & -f_{qm} & f_{mm} & f_{mt} & f_{mh} \\ & & s_{tt} & s_{th} & f_{qt} & -f_{mt} & f_{tt} & f_{th} \\ & & & s_{hh} & -f_{qh} & f_{mh} & -f_{th} & f_{hh} \\ & & \text{Sym} & & s_{qq} & -s_{qm} & s_{qt} & -s_{qh} \\ & & & & & s_{mm} & -s_{mt} & s_{mh} \\ & & & & & & s_{tt} & -s_{th} \\ & & & & & & & s_{hh} \end{bmatrix} \tag{38}$$

Finally the in-plane DS matrix $\boldsymbol{K}_p$ and the out-of-plane DS matrix $\boldsymbol{K}_o$ are combined together, to give the complete dynamic stiffness matrix as:

$$\boldsymbol{F} = \boldsymbol{K}\,\boldsymbol{\delta} \tag{39}$$

or more explicitly

$$\begin{bmatrix} \mathcal{N}_{xx_1} \\ \mathcal{N}_{xy_1} \\ \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{P}_{xx_1} \\ \mathcal{N}_{xx_2} \\ \mathcal{N}_{xy_2} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \\ \mathcal{P}_{xx_2} \end{bmatrix} = \begin{bmatrix} s_{nn} & s_{nl} & 0 & 0 & 0 & 0 & f_{nn} & f_{nl} & 0 & 0 & 0 & 0 \\ & s_{ll} & 0 & 0 & 0 & 0 & -f_{nl} & f_{ll} & 0 & 0 & 0 & 0 \\ & & s_{qq} & s_{qm} & s_{qt} & s_{qh} & 0 & 0 & f_{qq} & f_{qm} & f_{qt} & f_{qh} \\ & & & s_{mm} & s_{mt} & s_{mh} & 0 & 0 & -f_{qm} & f_{mm} & f_{mt} & f_{mh} \\ & & & & s_{tt} & s_{th} & 0 & 0 & f_{qt} & -f_{mt} & f_{tt} & f_{th} \\ & & & & & s_{hh} & 0 & 0 & -f_{qh} & f_{mh} & -f_{th} & f_{hh} \\ & & & & & & s_{nn} & -s_{nl} & 0 & 0 & 0 & 0 \\ & & & & & & & s_{ll} & 0 & 0 & 0 & 0 \\ & & & & \text{Sym} & & & & s_{qq} & -s_{qm} & s_{qt} & -s_{qh} \\ & & & & & & & & & s_{mm} & -s_{mt} & s_{mh} \\ & & & & & & & & & & s_{tt} & -s_{th} \\ & & & & & & & & & & & s_{hh} \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ W_1 \\ \Phi_{x_1} \\ \Phi_{y_1} \\ W_{1,x} \\ U_2 \\ V_2 \\ W_2 \\ \Phi_{x_2} \\ \Phi_{y_2} \\ W_{2,x} \end{bmatrix} \tag{40}$$

The above dynamic stiffness matrix will now be used in conjunction with the Wittrick-Williams algorithm [3] to analyze assemblies of composite plates to investigate their free vibration characteristics based on HSDT. Explicit expressions for each element of the DS matrix were obtained via symbolic computation, but they are far too extensive and voluminous to report. The correctness of these expressions was further checked by implementing them in a Matlab program and then carrying out a wide rage of numerical simulations.

2.4. Assembly procedure, boundary conditions and similarities with FEM

Once the DS matrix of a laminate element has been developed, it can be rotated and/or offset if required and thus can be assembled to form the global DS matrix of the final structure. The assembly procedure is schematically shown in Fig. 2 which is similar to that of FEM. Although like the FEM, a mesh is required in the DSM, it should be noted that the latter is mesh independent in the sense that additional elements are required only when there is a change in the geometry of the structure. A single DS laminate element is enough to compute any number of its natural frequencies to any desired accuracy, which, of course, is impossible in the FEM. However, for the type of structures under consideration DS plate elements do not have point nodes, but have line nodes instead. In this particular case, no change in geometry along the longitudinal direction is admitted. This is in addition to the assumed simple support boundary conditions on two opposite sides, inherent in DSM for plate elements at present. The other two sides of the plate can have any boundary conditions. The application of the boundary conditions of the global dynamic stiffness matrix involves the use of the so-called penalty method. This consists of adding a large stiffness to the appropriate leading diagonal term which corresponds to the degree of freedom of the node that needs to be suppressed. It is thus possible to apply free (F), simple support (S) and clamped (C) boundary conditions on the structure by penalizing the appropriate degrees of freedom. Clearly for simple support boundary condition, V, W and Φ_y are penalized. On the other hand, for clamped boundary condition U, V, W, Φ_y, Φ_x, W,x will have to be penalized. Of course for the free-edge boundary condition no penalty will be applied. Because of the similarities between DSM and FEM, DS elements can be implemented in FEM codes and thus the accuracy of results can be enhanced very considerably.

2.5. Application of the Wittrick-Williams Algorithm

In order to compute the natural frequencies of a structure by using the DSM, an efficient way to solve the eigenvalue-problem is to apply the Wittrick-Williams algorithm [3] which has featured in literally hundreds of papers. For the sake of completeness the procedure is briefly summarized as follows.
First the global dynamic stiffness matrix of the final structure K^* is computed for an arbitrarily chosen trial frequency ω^*. Next, by applying the usual form of Gauss elimination the global stiffness matrix, is transformed into its upper triangular $K^{*\triangle}$ form. The number of negative terms on the leading diagonal of $K^{*\triangle}$ is now defined as the sign count $s(K^*)$ which forms the fundamental basis of the algorithm. In its simplest form, the algorithm states that j, the number of natural frequencies (ω) of a structures that lie below an arbitrarily chosen trial frequency (ω^*) is given by:

$$j = j_0 + s(K^*) \tag{41}$$

where j_0 is the number of natural frequencies of all single elements within the structure which are still lower than the trial frequency (ω^*) when their nodes are fully clamped. It is necessary to account for this clamped-clamped frequencies because exact free vibration analysis using DSM allows an infinity number of natural frequencies to be accounted for when all the nodes of the structures are fully clamped, i.e. in the overall formulation $\boldsymbol{K}\,\boldsymbol{\delta} = 0$, these natural frequencies correspond to $\boldsymbol{\delta} = 0$ modes. Thus j_0 is an integral part of the algorithm and not really a peripheral issue. However, unless exceptionally high frequencies are needed, j_0 is usually zero and the dominant term of the algorithm is the sign-count $s(K^*)$, of Eq. (41). One way of avoiding the computation of j_0 is to split the structure into sufficient number of elements so that the clamped-clamped natural frequencies of an individual element in the structure are never exceeded. Once $s(K^*)$ and j_0 of Eq. (41) are known, any suitable method, for example, bi-section technique, can be devised to bracket any natural frequency within any desired accuracy. The mode shapes are routinely computed by using standard eigenvector recovery procedure in which the global dynamic stiffness matrix is computed at the natural frequency and the force vector is set to zero whilst deleting one row of the DS matrix and giving one of the nodal displacement component an arbitrarily chosen value and then determining the rest of the displacements in terms of the chosen one.

3. Results ans Discussion

The first set of results was obtained to validate the dynamic stiffness theory using HSDT presented in this paper. For the fundamental natural frequency, Table 1 shows representative results in non-dimensional form for a cross-ply composite square plate simply supported on all edges using the present theory along side the published results from literature. Of particular significance, is the inclusion of the 3D elasticity solution and numerical results using ANSYS which show close agreement with the results obtained by the present theory. Note that the ANSYS results were obtained by using SHELL181 element. Results in Table 1 cover a broad range of laminate lay-ups and stacking sequences. It is evident that the DS theory using HSDT predicts natural frequencies of composite plate in an accurate manner. The maximum error incurred when compared to 3D elasticity solution is 4.54% for an artificially large value of the orthotropic ratio $E_1/E_2 = 40$. For realistic orthotropic ratios, the error is expected to be much less. (Note that for carbon-epoxy and glass-epoxy composite structures the ratio E_1/E_2 is around 10.) The next set of results was obtained to examine the effects of the thickness to length ratio and the orthotropic ratio on the first four natural frequencies of the square plate, simply supported on all edges, but with stacking sequence $[0°/90°/0°/90°/\bar{0}°]_s$. The results using the current DSM based on HSDT are shown in Table 2 together with the ones obtained by using the DSM program (based on FSDT) developed by Boscolo and Banerjee [22, 23]. Some interesting observations can be made from these results. Clearly, the difference in natural frequencies when using the more accurate HSDT and as opposed to relatively less accurate FSDT, increases when the plate becomes progressively thicker, as expected. One of the anomalies in using FSDT arises from the difficulty to select the shear corrector factor (χ), which is generally introduced on an ad-hoc basis in an attempt to account for the correct shear stress distribution which in reality is not uniform through the cross section. Strictly speaking, the FSDT can never achieve zero shear stress distribution at the free boundaries. Thus there is an element of uncertainty in choosing the shear corrector factor and different authors have used different values (see Mindlin [44] , Reissner [45]). The problem of choosing the shear corrector factor is even more troublesome for composites. However, this factor is taken to be 5/6 (see [45]) in the FSDT results shown in Table 2. By contrast the HSDT results based on refined displacement field do not rely on such fictitious (and quite often arbitrarily chosen) shear correction factor because the HSDT intrinsically account for the parabolic shear stresses distribution. To confirm the predictable accuracy of the current method, 3D elasticity solution has been used for comparison purposes. Both the influence of the thickness-to-length ratio and the orthotropic ratio on results are also shown in Table 2. The next set of results are focused on the effect of boundary conditions. For two representative values of thickness ratio (b/h), the results in Table 3 show the effects of the boundary conditions on the first four natural frequencies of the above plate. It should be noted that FSDT results are also included in the table. Clearly, when the plate is simply supported on to opposite sides and clamped on the other two sides, the natural frequencies assume higher values as expected. For this case, the maximum error encountered is in the third natural frequency when using FSDT instead of the more accurate HSDT. The absolute values of these errors are around 7.5% and 4.2% when the thickness ratios are 5 and 10 respectively, as can be seen in Table 3. It also evident from the results that on occasions, the FSDT results are lower than the HSDT ones. The reason for this can be attributed to the fact that the choice of the shear correction factor (which is non-existent and unnecessary in HSDT), influences the FSDT results in some unpredictable way. Such discrepancies are not uncommon and can be found in the literature. In order to demonstrate the applicability of the theory to an assembly of composite plates, a stepped panel which is schematically shown in Fig. 3 has also been analysed. As in previous cases, the results were obtained in non-dimensional form and with particular reference to Fig. 3. The ratio b_1/b, b_2/b, b_3/b are taken to be 1/5, 1/20, 1/10, respectively which are representative from a practical standpoint. The results where obtained for different boundary condition, and for a wide range of thickness ratios between the stiffened plate and parent plate (t_2/t_1) ranging from 2 to 6. The first ten natural frequencies using the present theory with S-C-S-C and S-F-S-F boundary conditions, are shown in Table 4 for different values of b/h. The results shown are exact and cannot be found in the existing literature, neither can they be obtained in an exact sense using other methods. The following comments about these results are relevant. Understandably, the natural frequencies are higher for S-C-S-C boundary conditions compared to S-F-S-F ones, as expected, but more importantly, for thick plates, e.g. $b/h = 2$, increasing t_2/t_1, decreases all the natural frequencies significantly. By contrast, for relatively thin plates with $b/h = 10$ the natural frequencies increase with increasing t_2/t_1 ratio for this particular problem. The reason for this can be attributed to the fact that for higher b/h ratio the effect of mass of the stiffened plate appears to be more pronounced than its stiffness, yielding lower natural frequencies as a consequence. The final set of results was obtained to demonstrate the mode shapes of the composite plate and the stepped panel using HSDT based DSM. In Figs. 4 to 6, a direct comparison of the first, fifth and ninth modes between the simple cross-ply laminated composite plate and the stepped panel has been made for the boundary conditions S-C-S-C, when the step ratio $t_2/t_1 = 2$, whilst the overall dimensions for the two configurations are kept the same. These figures reveal some interesting features. For the fundamental mode, see Fig. 4, there is hardly any difference in the natural frequency and mode shape between the simple plate and stepped panel. This is in sharp contrast to the fifth and ninth modes shown in Figs. 5 and 6 respectively, where some differences in the natural frequencies and mode shapes are prevalent. It is clear from these two figures that significant alteration in the mode shapes is possible when required as a result of using stepped panel. The corresponding results for S-F-S-F boundary condition are shown in Figs. 7, 8 and 9. Figure 7 shows that the fundamental natural frequency changes significantly, but the mode shape follows more or less the same pattern. The sixth and ninth modes shown in Figs. 8 and 9, fortuitously reveal the same picture as the fundamental one shown in Fig. 7. It is interesting to note that for S-C-S-C and S-F-S-F boundary conditions, results and trends are markedly different. These observations are important when solving frequency attenuation problems to avoid certain undesirable natural frequencies and mode shapes of complex composite structures.

4. Concluding Remarks

An exact dynamic stiffness theory for composite plate elements using higher order shear deformation theory is developed for the first time in this paper using Hamiltonian mechanics and symbolic algebra. The theory is implemented in a computer program to carry out free vibration analysis of composite structures modelled as plate assemblies. The proposed theory is a significant refinement over recently developed dynamic stiffness method using classical and first order shear deformation plate theories. The developed DSM model is particularly useful when analyzing thick composite plates with moderate to high orthotropic ratios for which the FEM may become unreliable, particularly at high frequencies. A detailed parametric study has been carried out by varying significant plate parameters and boundary conditions. The results have been critically examined and the theory has been assessed using existing theories and in particular three-dimensional mathematical theory of elasticity. A stepped composite plates has also been analyzed for its dynamic behavior. Based on the computed results the following comments can be made:

- The proposed exact dynamic stiffness composite plate element based on HSDT is shown to be more accurate in terms of results and computational efficiency when compared with FEM in free vibration analysis of composite plate assemblies.

- The theory provides a significant refinement over FSDT element, particularly when thick plates with high orthotropic ratios are analyzed.

- The boundary conditions do not seem to affect the error incurred using FSDT as opposed to more accurate HSDT.

- The dynamic behavior of stepped composite plates are very different from those of simple composite plates depending on the boundary conditions, but significant alteration in mode shapes is possible by using stepped panels. This could be useful in solving frequency attenuation problems.

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Tables

Table 1: Dimensionless fundamental natural frequency parameter $\hat{\omega} = \frac{\omega\, b}{h}\sqrt{\frac{\rho}{E_2}}$, for a cross-ply square composite plate simply supported at all edges with $a/h = 5$, E_1/E_2 = open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Stacking sequence	Models	E_1/E_2					
		3		20		40	
$[0°/\bar{90}°]_s$	3D-Elasticity [46]	6.6185	$\Delta^{\dagger}_{3D}\%$	9.5603	$\Delta_{3D}\%$	10.7515	$\Delta_{3D}\%$
	ANSYS‡	6.5638	(−0.83)	9.2574	(−3.17)	10.221	(−4.93)
Classical Lèvy's solution	Reddy [46]	6.5527	(−0.99)	9.2348	(−3.40)	10.2631	(−4.54)
DSM	HSDT	6.5527	(−0.99)	9.2349	(−3.40)	10.2632	(−4.54)
$[0°/90°/\bar{0}°]_s$	3D-Elasticity [46]	6.6468	$\Delta_{3D}\%$	9.948	$\Delta_{3D}\%$	11.3435	$\Delta_{3D}\%$
	ANSYS‡	6.5780	(−1.04)	9.7363	(−2.13)	11.051	(−2.58)
Classical Lèvy's solution	Reddy [46]	6.5850	(−0.93)	9.8413	(−1.07)	11.2617	(−0.72)
DSM	HSDT	6.5850	(−0.93)	9.8413	(−1.07)	11.2617	(−0.72)
$[0°/90°/0°/90°/\bar{0}°]_s$	3D-Elasticity [46]	6.66	$\Delta_{3D}\%$	10.1368	$\Delta_{3D}\%$	11.6698	$\Delta_{3D}\%$
	ANSYS‡	6.5879	(−1.08)	9.9986	(−1.36)	11.4926	(−5.74)
Classical Lèvy's solution	Reddy [46]	6.5959	(−0.96)	10.0598	(−0.76)	11.6198	(−0.43)
DSM	HSDT	6.5959	(−0.96)	10.0599	(−0.76)	11.620	(−0.43)

† $\Delta_{3D}\% = \frac{\hat{\omega} - \hat{\omega}_{3D}}{\hat{\omega}_{3D}} \times 100$.

‡ FEM mesh 50 × 50 elements.

Table 2: Dimensionless fundamental natural frequency parameter $\hat{\omega} = \frac{\omega\, b}{h}\sqrt{\frac{\rho}{E_2}}$, of a cross-ply square composite plate, simply supported at all edges with stacking sequence $[0°/90°/0°/90°/\bar{0}°]_s$, b/h = open, E_1/E_2 = open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

E_1/E_2	Models	b/h							
		2		5		10		100	
3	HSDT	4.5542	$\Delta^{\dagger}_{HT}\%$	6.5974	$\Delta_{HT}\%$	7.2559	$\Delta_{HT}\%$	7.5327	$\Delta_{HT}\%$
	$FSDT^{*}_{\chi=\frac{5}{6}}$	4.5375	(−0.36)	6.5955	(−0.03)	7.2556	(−0.004)	7.5327	(0.00)
10	HSDT	5.1766	$\Delta_{HT}\%$	8.5341	$\Delta_{HT}\%$	9.9576	$\Delta_{HT}\%$	10.6416	$\Delta_{HT}\%$
	$FSDT_{\chi=\frac{5}{6}}$	5.1355	(−0.79)	8.5227	(−0.13)	9.9532	(−0.04)	10.6416	(0.00)
20	HSDT	5.5412	$\Delta_{HT}\%$	10.0646	$\Delta_{HT}\%$	12.5357	$\Delta_{HT}\%$	13.9312	$\Delta_{HT}\%$
	$FSDT_{\chi=\frac{5}{6}}$	5.4572	(−1.52)	10.0415	(−0.23)	12.5229	(−0.10)	13.9312	(0.00)
30	HSDT	5.7410	$\Delta_{HT}\%$	10.9927	$\Delta_{HT}\%$	14.3872	$\Delta_{HT}\%$	16.5764	$\Delta_{HT}\%$
	$FSDT_{\chi=\frac{5}{6}}$	5.6065	(−2.34)	10.9605	(−0.29)	14.3663	(−0.15)	16.5764	(0.00)
40	HSDT	5.8815	$\Delta_{HT}\%$	11.6200	$\Delta_{HT}\%$	15.8222	$\Delta_{HT}\%$	18.8499	$\Delta_{HT}\%$
	$FSDT_{\chi=\frac{5}{6}}$	5.6925	(−3.21)	11.5788	(−0.35)	15.7943	(−0.17)	18.8499	(0.00)

† $\Delta_{HT}\% = \frac{\hat{\omega}_{FSDT} - \hat{\omega}_{HSDT}}{\hat{\omega}_{HSDT}} \times 100$.

∗ χ Shear Corrector Factor.

Table 3: Dimensionless natural frequency parameter $\hat{\omega} = \frac{\omega\, b}{h}\sqrt{\frac{\rho}{E_2}}$, of a cross-ply square composite plate, with stacking sequence $[0°/90°/0°/90°/\bar{0}°]_s$, $b/h = 5$, $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$.

b/h	Mode		S-S-S-S				S-S-S-F				S-S-S-C		
5		m n	HSDT	FSDT‡	$\Delta^{\dagger}_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$
	1	1 1	11.620	11.579	(−0.35)	1 1	7.442	7.652	(3.13)	1 1	12.538	12.027	(−4.08)
	2	2 1	20.326	20.916	(2.90)	1 2	15.292	14.976	(−2.07)	2 1	20.853	21.157	(1.46)
	3	1 2	22.742	21.547	(−5.25)	2 1	18.264	19.028	(4.18)	1 2	24.001	21.660	(−9.75)
	4	2 2	28.227	27.706	(−1.85)	2 2	22.745	23.027	(1.24)	2 2	29.250	27.792	(−4.98)
10		m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$
	1	1 1	15.822	15.794	(−0.17)	1 1	9.622	9.740	(1.23)	1 1	18.524	18.116	(−2.20)
	2	2 1	31.972	32.736	(2.39)	1 2	21.486	21.314	(−0.80)	2 1	33.342	33.862	(1.56)
	3	1 2	37.075	36.189	(−2.39)	2 1	29.238	30.126	(3.03)	1 2	39.463	37.479	(−5.03)
	4	2 2	46.480	46.315	(−0.35)	2 2	35.421	35.988	(1.60)	2 2	48.365	47.285	(−2.23)
			S-F-S-F				S-C-S-F				S-C-S-C		
5		m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$
	1	1 1	7.263	7.489	(3.11)	1 1	8.348	8.463	(1.38)	1 1	13.715	12.688	(−7.49)
	2	2 1	7.909	8.073	(2.07)	1 2	16.105	15.039	(−6.62)	2 1	21.553	21.518	(−0.16)
	3	1 2	18.113	18.916	(4.43)	2 1	18.620	19.342	(3.88)	1 2	25.310	21.725	(−14.2)
	4	2 2	18.113	19.330	(6.72)	2 2	23.310	23.071	(−1.03)	2 2	30.335	27.843	(−8.21)
10		m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$	m n	HSDT	FSDT	$\Delta_{\mathrm{HT}}\%$
	1	1 1	9.394	9.516	(1.29)	1 1	10.764	10.842	(0.72)	1 1	21.438	20.547	(−4.16)
	2	1 2	10.248	10.352	(1.01)	1 2	23.977	23.177	(−3.34)	2 1	34.970	35.149	(0.51)
	3	1 3	29.017	28.638	(−1.30)	2 1	29.637	30.497	(2.90)	1 2	41.668	38.546	(−7.49)
	4	2 1	29.054	29.956	(3.10)	2 2	36.952	37.075	(0.33)	2 2	50.153	48.107	(−4.08)

† $\Delta_{\mathrm{HT}}\% = \frac{\hat{\omega}_{\mathrm{FSDT}} - \hat{\omega}_{\mathrm{HSDT}}}{\hat{\omega}_{\mathrm{HSDT}}} \times 100$.

‡ Shear Corrector Factor $\chi = 5/6$.

Table 4: Dimensionless natural frequency parameter $\hat{\omega} = \frac{\omega\, b}{h}\sqrt{\frac{\rho}{E_2}}$, of a simply supported cross-ply square composite plate, using a HSDT for stacking sequence $[0°/90°/90°/0°]$, $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$.

t_2/t_1			S-C-S-C						S-F-S-F				
		b/h	2		10		50	b/h	2		10		50
	Mode	m n		m n		m n		m n		m n		m n	
	1	1 1	6.097	1 1	18.473	1 1	30.046	1 1	3.403	1 1	7.244	1 1	8.002
	2	2 1	9.227	2 1	30.000	2 1	41.947	1 2	3.946	1 2	8.217	1 2	9.586
	3	1 2	12.115	1 2	37.404	3 1	71.210	2 1	7.616	2 1	23.303	2 1	30.356
	4	3 1	13.372	2 2	44.296	1 2	77.876	1 3	7.736	2 2	23.444	2 2	30.863
2	5	2 2	14.055	3 1	47.505	2 2	84.531	2 2	8.301	1 3	27.773	1 3	35.785
	6	1 3	16.099	1 3	56.002	3 2	102.922	2 3	9.803	2 3	36.539	2 3	49.778
	7	1 4	16.920	3 2	57.595	4 1	115.004	1 4	9.886	3 1	42.134	3 1	62.796
	8	3 2	17.025	2 3	61.073	4 2	136.311	3 1	12.160	3 2	42.253	3 2	64.236
	9	1 5	17.289	4 1	66.859	1 3	149.614	2 4	12.173	3 3	51.799	3 3	82.335
	10	4 1	18.240	3 3	71.819	2 3	153.671	3 2	12.957	1 4	52.953	1 4	86.661
	Mode	m n		m n		m n		m n		m n		m n	
	1	1 1	5.922	1 1	18.783	1 1	30.145	1 1	2.705	1 1	10.633	1 1	15.947
	2	1 2	7.725	1 2	33.663	2 1	71.096	1 2	3.762	1 2	11.327	1 2	16.435
	3	2 1	8.750	2 1	34.431	1 2	71.258	2 1	5.371	1 3	26.287	1 3	36.993
	4	2 2	8.750	2 2	44.351	2 2	97.426	1 3	6.107	2 1	27.064	2 1	48.923
4	5	1 3	9.403	1 3	47.389	1 3	124.620	3 1	8.544	2 2	28.594	2 2	50.370
	6	3 1	11.022	3 1	53.139	3 1	135.590	1 4	8.952	2 3	37.628	2 3	71.117
	7	2 3	13.713	2 3	56.002	2 3	141.442	1 5	8.952	3 1	45.638	3 1	85.170
	8	4 1	14.039	3 2	60.239	3 2	148.635	2 2	9.824	3 2	46.783	3 2	85.414
	9	1 4	14.076	3 3	69.624	3 3	186.566	2 3	9.824	1 4	49.484	1 4	85.546
	10	1 5	15.088	1 4	71.520	4 1	208.478	2 4	11.007	3 3	54.682	2 4	106.295
	Mode	m n		m n		m n		m n		m n		m n	
	1	1 1	4.444	1 1	19.266	1 1	34.798	1 1	1.948	1 1	11.953	1 1	29.947
	2	1 2	4.444	1 2	29.658	1 2	69.167	2 1	4.277	1 2	13.722	1 2	29.948
	3	2 1	5.869	2 1	36.346	2 1	96.522	1 2	6.748	1 3	23.874	1 3	42.819
	4	2 2	5.869	2 2	42.039	1 3	110.864	2 2	7.103	2 1	28.197	2 1	72.821
6	5	3 1	8.500	1 3	42.082	2 2	114.264	3 1	7.197	2 2	30.669	2 2	72.821
	6	1 3	8.913	1 4	44.622	2 3	144.387	3 2	8.510	2 3	36.323	1 4	103.419
	7	1 4	10.091	2 3	51.451	3 1	175.783	1 3	9.027	1 4	40.146	1 5	103.524
	8	3 2	10.263	3 1	56.107	3 2	184.041	1 4	9.027	1 5	44.438	1 6	109.667
	9	2 3	10.393	1 5	58.325	1 4	199.545	2 3	9.457	3 1	46.155	2 3	114.436
	10	1 5	10.405	1 6	59.233	3 3	211.700	2 4	9.457	2 4	47.365	2 4	114.437

Figures

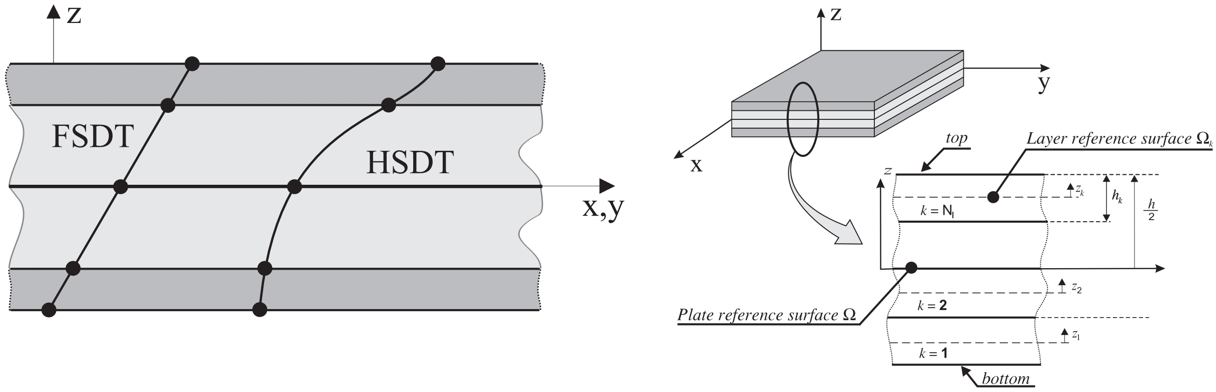


Figure 1: Kinematic descriptions of displacements using FSDT and HSDT for a multilayered plate

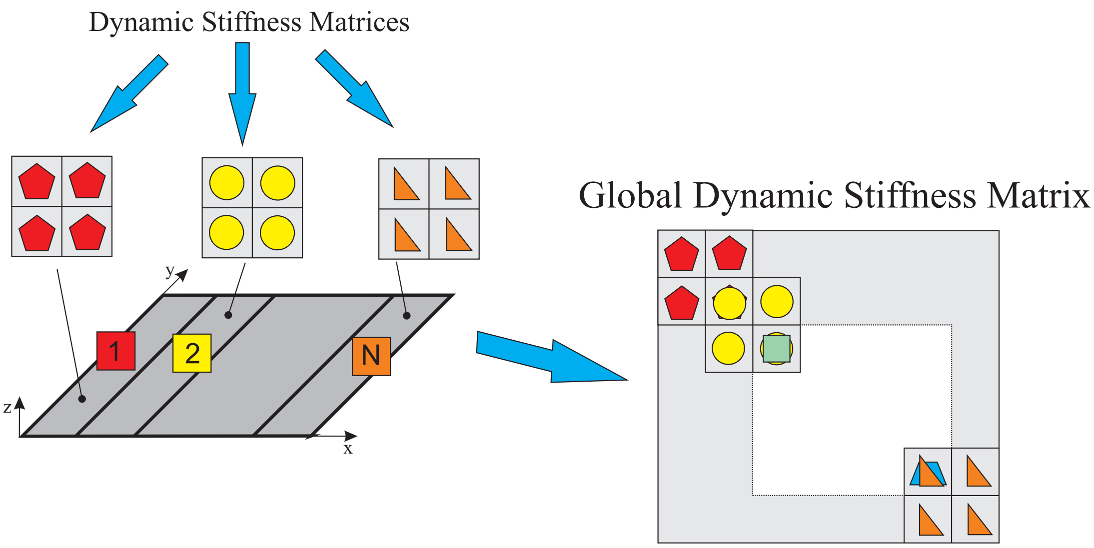


Figure 2: Direct assembly of dynamic stiffness elements.

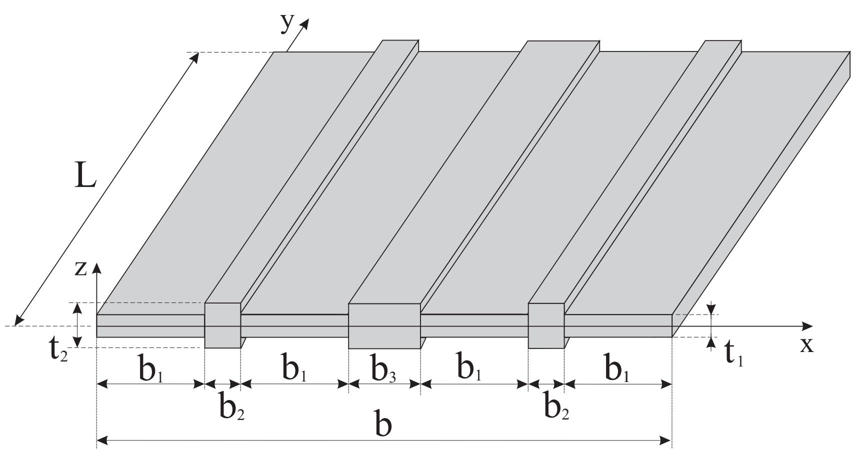
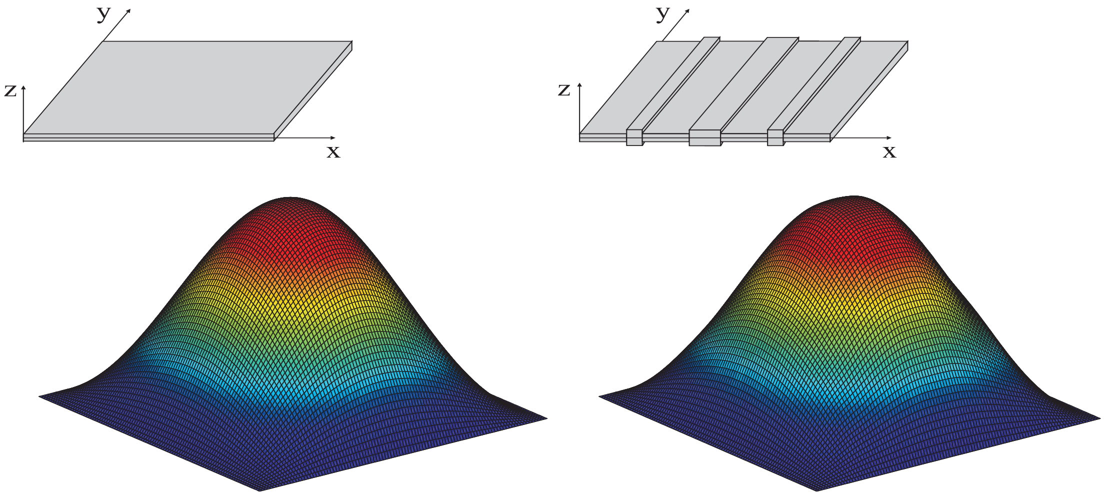
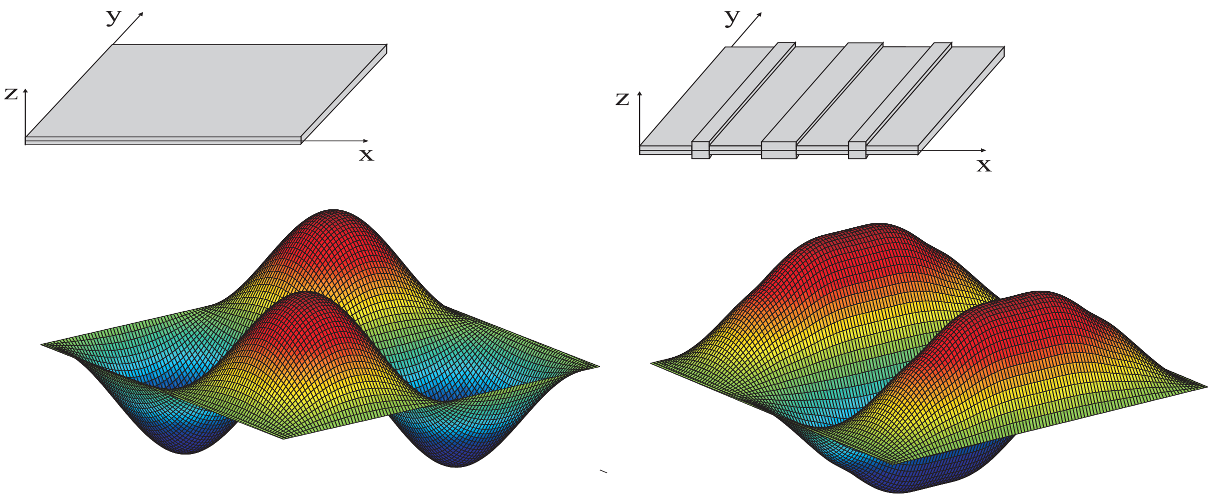


Figure 3: A stepped composite plate



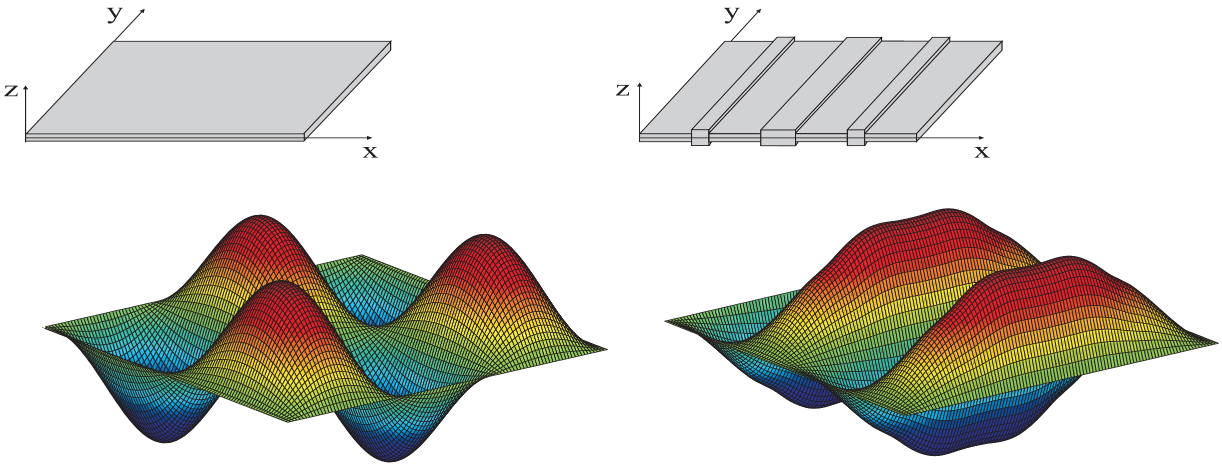
(a) 1^{st} Mode $\hat{\omega} = 18.886$ (b) 1^{st} Mode $\hat{\omega} = 18.473$

Figure 4: Fundamental natural frequency and mode shape of a simple and stepped composite plate, with boundary condition S-C-S-C and stacking sequence $[0°/90°]_s$, $\frac{b}{t_1} = 10$, $\frac{t_2}{t_1} = 2$.



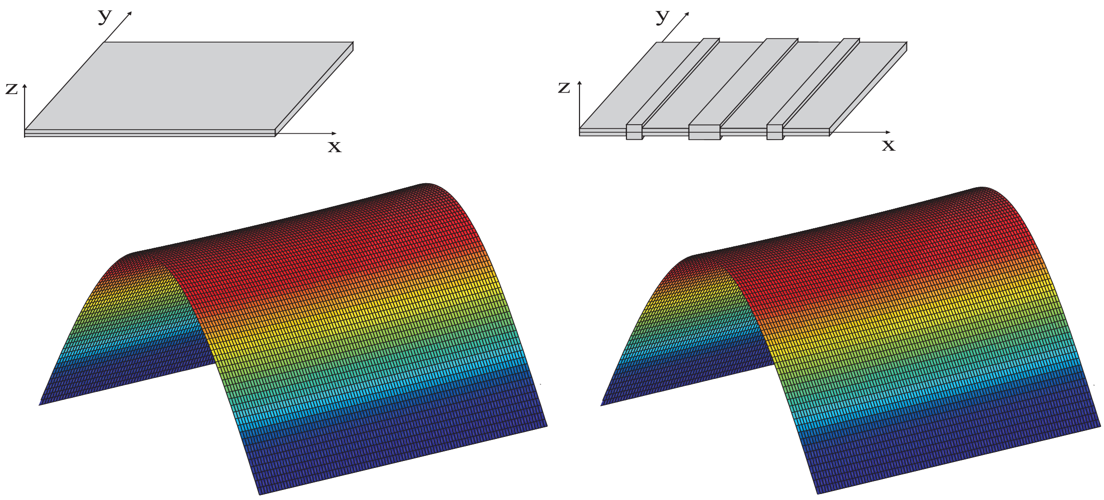
(a) 5^{th} Mode $\hat{\omega} = 43.286$ (b) 5^{th} Mode $\hat{\omega} = 47.505$

Figure 5: Fifth natural frequency and mode shape of a simple and stepped composite plate, with boundary condition S-C-S-C and stacking sequence $[0°/90°]_s$, $\frac{b}{t_1} = 10$, $\frac{t_2}{t_1} = 2$.



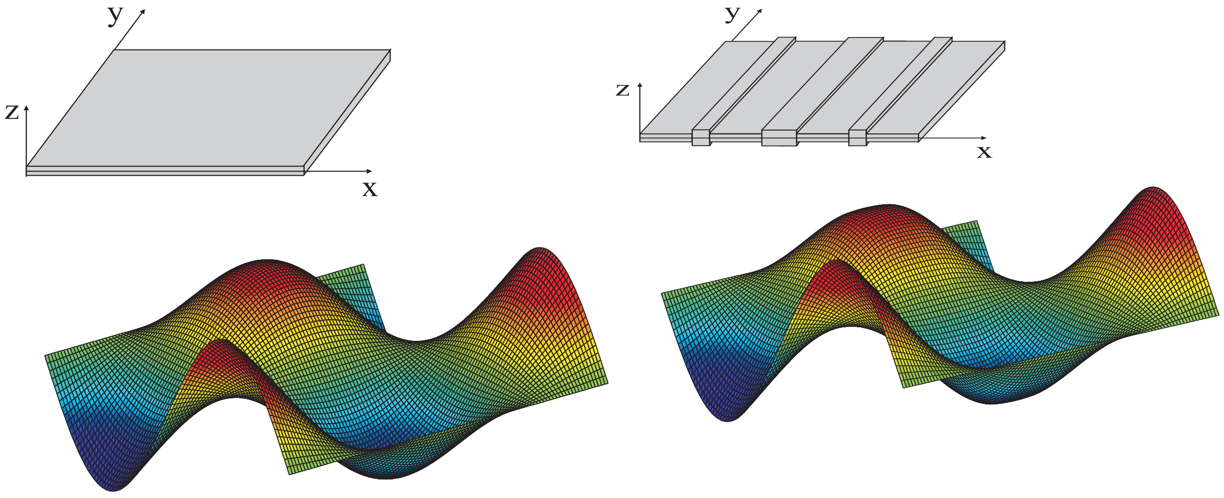
(a) 9^{th} Mode $\hat{\omega} = 64.577$ (b) 9^{th} Mode $\hat{\omega} = 66.859$

Figure 6: Ninth natural frequency and mode shape of a simple and stepped composite plate, with boundary condition S-C-S-C and stacking sequence $[0°/90°]_s$, $\frac{b}{t_1} = 10$, $\frac{t_2}{t_1} = 2$.



(a) 1^{st} Mode $\hat{\omega} = 5.447$ (b) 1^{st} Mode $\hat{\omega} = 7.244$

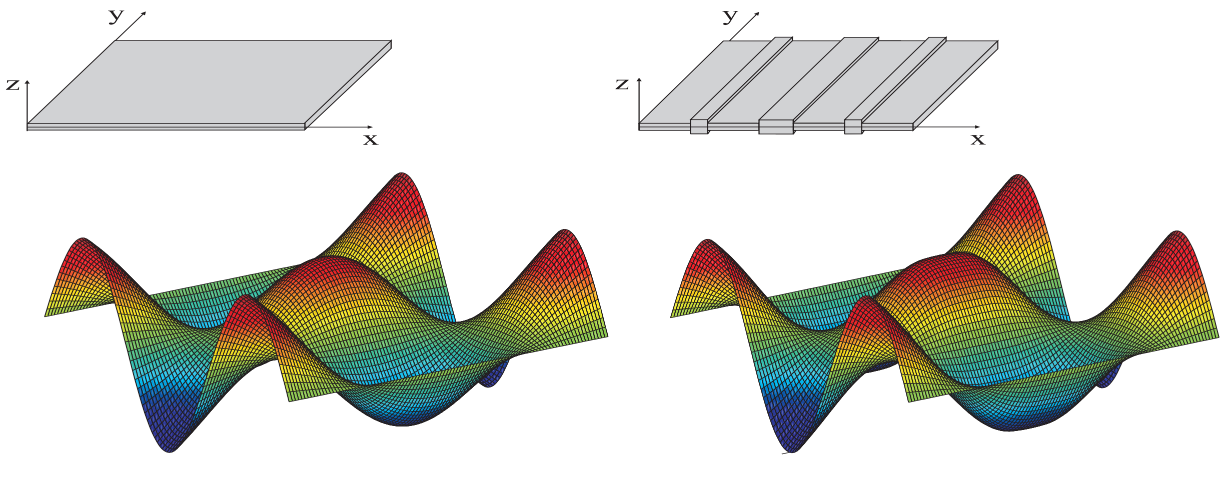
Figure 7: Fundamental natural frequency and mode shape of a simple and stepped composite plate, with boundary condition S-F-S-F and stacking sequence $[0°/90°]_s$, $\frac{b}{t_1} = 10$, $\frac{t_2}{t_1} = 2$.



(a) 6^{th} Mode $\hat{\omega} = 33.783$

(b) 6^{th} Mode $\hat{\omega} = 36.539$

Figure 8: Sixth natural frequency and mode shape of a simple and stepped composite plate, with boundary condition S-F-S-F and stacking sequence $[0°/90°]_s$, $\frac{b}{t_1} = 10$, $\frac{t_2}{t_1} = 2$



(a) 9^{th} Mode $\hat{\omega} = 47.997$

(b) 9^{th} Mode $\hat{\omega} = 52.953$

Figure 9: Ninth natural frequency and mode shape of a simple and stepped composite plate, with boundary condition S-F-S-F and stacking sequence $[0°/90°]_s$, $\frac{b}{t_1} = 10$, $\frac{t_2}{t_1} = 2$

Appendix A. Laminate Geometric and Constitutive Equations

The geometric relation for a lamina in the local or lamina reference system can be written as:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x & 0 & 0 \\ 0 & \mathcal{D}_y & 0 \\ \mathcal{D}_y & \mathcal{D}_x & 0 \\ 0 & \mathcal{D}_z & \mathcal{D}_y \\ \mathcal{D}_z & 0 & \mathcal{D}_x \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{A.1}$$

and in terms of the functional degrees of freedom:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x & 0 & \left(c_1\, z^3\right) \mathcal{D}_{xx} & \left(z + c_1\, z^3\right) \mathcal{D}_x & 0 \\ 0 & \mathcal{D}_y & \left(c_1\, z^3\right) \mathcal{D}_{yy} & 0 & \left(z + c_1\, z^3\right) \mathcal{D}_y \\ \mathcal{D}_y & \mathcal{D}_x & \left(c_1\, z^3\, \mathcal{D}_{xy} + c_1\, z^3\, \mathcal{D}_{yx}\right) & \left(z + c_1\, z^3\right) \mathcal{D}_y & \left(z + c_1\, z^3\right) \mathcal{D}_x \\ 0 & 0 & \left(1 + 3\, c_1\, z^2\right) \mathcal{D}_y & 0 & \left(1 + 3\, c_1\, z^2\right) \\ 0 & 0 & \left(1 + 3\, c_1\, z^2\right) \mathcal{D}_x & \left(1 + 3\, c_1\, z^2\right) & 0 \end{bmatrix} \begin{bmatrix} u^0 \\ v^0 \\ w^0 \\ \phi_x \\ \phi_y \end{bmatrix} \tag{A.2}$$

where $\mathcal{D}_x$ and $\mathcal{D}_y$ are the derivatives in x and y respectively and $c_1 = -\frac{4}{3\,h^2}$. The constitutive equations in the lamina reference system can be written, in terms of reduced stiffness coefficients, as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 & 0 & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & 0 & 0 & 0 \\ 0 & 0 & \tilde{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \tilde{C}_{44} & 0 \\ 0 & 0 & 0 & 0 & \tilde{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} \tag{A.3}$$

where the $\tilde{C}_{ij}$ are expressed in terms of stiffness coefficients C_{ij}, as:

$$\begin{aligned} &\tilde{C}_{11} = C_{11} - \frac{C_{13}^2}{C_{33}} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \qquad \tilde{C}_{12} = C_{12} - \frac{C_{13}\,C_{23}}{C_{33}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \qquad \tilde{C}_{22} = C_{22} - \frac{C_{23}^2}{C_{33}} = \frac{E_2}{1-\nu_{12}\nu_{21}} \\ &\tilde{C}_{44} = C_{44} = G_{23}, \qquad \tilde{C}_{55} = C_{55} = G_{13} \qquad \tilde{C}_{66} = C_{66} = G_{12} \end{aligned} \tag{A.4}$$

where E_1 is the elastic modulus in the fibre direction, E_2 the elastic modulus in perpendicular to the fibre, ν_{12} and $\nu_{21} = \nu_{12}\, E_2/E_1$ the Poisson's ratios, $G_{12} = G_{13}$ and G_{23} the shear modulus of each single orthotropic lamina. If the lamina is placed at an angle θ in the laminate or global reference system, the equation need to be transformed as follows:

$$\begin{aligned} \overline{C}_{11} =& \tilde{C}_{11}\mathcal{C}^4 + 2(\tilde{C}_{12} + 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{22}\mathcal{S}^4 \\ \overline{C}_{12} =& (\tilde{C}_{11} + \tilde{C}_{22} - 4\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{12}(\mathcal{S}^4 + \mathcal{C}^4) \\ \overline{C}_{16} =& (\tilde{C}_{11} - \tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}\mathcal{C}^3 + (\tilde{C}_{12} - \tilde{C}_{22} + 2\tilde{C}_{66})\mathcal{S}^3 \\ \overline{C}_{22} =& \tilde{C}_{11}\mathcal{S}^4 + 2(\tilde{C}_{12} + 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{22}\mathcal{C}^4 \\ \overline{C}_{26} =& (\tilde{C}_{11} - \tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}^3\mathcal{C} + (\tilde{C}_{12} - \tilde{C}_{22} + 2\tilde{C}_{66})\mathcal{S}\mathcal{C}^3 \\ \overline{C}_{66} =& (\tilde{C}_{11} + \tilde{C}_{22} - 2\tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{66}(\mathcal{S}^4 + \mathcal{C}^4) \\ \overline{C}_{44} =& \tilde{C}_{44}\mathcal{C}^2 + \tilde{C}_{55}\mathcal{S}^2 \\ \overline{C}_{55} =& \tilde{C}_{44}\mathcal{S}^2 + \tilde{C}_{55}\mathcal{C}^2 \\ \overline{C}_{45} =& (\tilde{C}_{55} - \tilde{C}_{44})\mathcal{C}\mathcal{S} \end{aligned} \tag{A.5}$$

where $\mathcal{C} = \cos(\theta)$ and $\mathcal{S} = \sin(\theta)$. This leads to the constitutive equation for the k-th lamina in the laminate or global reference system:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} & 0 & 0 \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} & 0 & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{A.6}$$

Appendix B. Polynomial Coefficients

The polynomial coefficients for out-of-plane and in-plane cases are following defined:

$$a_1 = c_1^2\,(F_{11}^2 - D_{11}\,H_{11})(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))$$

$$\begin{aligned}
a_2 = {} & -c_1^2\,(A_{44} + \alpha^2\,D_{22})\,F_{11}^2 + A_{55}\,D_{11}\,(D_{66} + c_1\,(2F_{66} + c_1\,H_{66})) + c_2^2\,(D_{11}\,D_{66}\,F_{55} + 2\,c_1\,D_{11}\,F_{55}\,F_{66} \\
& + c_1^2\,(-F_{11}^2\,F_{44} + D_{11}\,F_{44}\,H_{11} + D_{11}\,F_{55}\,H_{66})) + 2\,c_2\,(-c_1^2\,D_{44}\,F_{11}^2 + D_{11}\,(c_1^2\,D_{44}\,H_{11} + D_{55}\,(D_{66} \\
& + 2\,c_1\,F_{66} + c_1^2\,H_{66}))) + c_1^2\,(A_{44}\,D_{11}\,H_{11} - \alpha^2\,(D_{12}^2\,H_{11} - 2\,D_{12}\,(F_{11}\,(F_{12} + 2\,F_{66} + c_1\,H_{12} + 2\,c_1\,H_{66}) \\
& - H_{11}\,(D_{66} + c_1\,(F_{12} - c_1\,H_{66}))) + c_1\,(F_{11}^2\,(2\,F_{22} + c_1\,H_{22}) - 2\,F_{11}\,(F_{12}^2 - 4\,F_{66}^2 + c_1\,(F_{12} + 2\,F_{66})\,H_{12} \\
& + 2\,D_{66}\,(H_{12} + 2\,H_{66})) + H_{11}\,(c_1\,(F_{12} + 2\,F_{66})^2 + 4\,D_{66}\,(F_{12} - c_1\,H_{66}))) + D_{11}\,(F_{12}^2 + 4\,F_{66}^2 - D_{22}\,H_{11} \\
& - 2\,D_{66}\,(H_{12} + 2\,H_{66}) + 2\,F_{12}\,(2\,F_{66} + c_1\,(H_{12} + 2\,H_{66})) + c_1\,(-2\,F_{22}\,H_{11} + c_1\,(H_{12}^2 - H_{11}\,H_{22} + 2\,H_{12} \\
& H_{66})))) - (H_{11}\,(D_{11} + D_{66} + 2\,c_1\,F_{66} + c_1^2\,H_{66})\,I_2 + 2\,c_1\,D_{11} H_{11}\,I_4 - 2\,F_{11}\,(D_{66} + c_1\,(2F_{66} + c_1\,H_{66}))\,I_4 \\
& + D_{11}\,(D_{66} + c_1\,(2\,F_{66} + c_1\,(H_{11} + H_{66})))\,I_6 - F_{11}^2\,(I_2 + c_1\,(2\,I_4 + c_1\,I_6)))\,\omega^2)
\end{aligned}$$

$$\begin{aligned}
a_3 = {} & -A_{44}\;A_{55}\;D_{11} - 2\;A_{55}\;c_2\;D_{11}\;D_{44} - 2\,A_{44}\,c_2\,D_{11}\,D_{55} + 2\,\alpha^2\,c_2\,D_{12}^2\,D_{55} - 2\,\alpha^2\,c_2\,D_{11}\,D_{22}\,D_{55} - 4\,c_2^2\,D_{11}\,D_{44}\,D_{55} \\
& - A_{44}\,\alpha^2\,D_{11}\,D_{66} - 2\,\alpha^2\,c_2\,D_{11}\,D_{44}\,D_{66} + 4\,\alpha^2\,c_2\,D_{12}\,D_{55}\,D_{66} - 2\,A_{44}\,\alpha^2\,c_1\,D_{12}\,F_{11} - 4\,\alpha^2\,c_1\,c_2\,D_{12}\,D_{44}\,F_{11} - 4\,A_{44} \\
& \alpha^2\,c_1\,D_{66}\,F_{11} - 8\,\alpha^2\,c_1\,c_2\,D_{44}\,D_{66}\,F_{11} + 2\,A_{44}\,\alpha^2\,c_1\,D_{11}\,F_{12} + 4\,\alpha^2\,c_1\,c_2\,D_{11}\,D_{44}\,F_{12} + 4\,\alpha^2\,c_1\,c_2\,D_{12}\,D_{55}\,F_{12} + 8\,\alpha^2 \\
& c_1\,c_2\,D_{55}\,D_{66}\,F_{12} + 2\,A_{44}\,\alpha^2\,c_1^2\,F_{11}\,F_{12} + 2\,\alpha^4\,c_1^2\,D_{22}\,F_{11}\,F_{12} + 4\,\alpha^2\,c_1^2\,c_2\,D_{44}\,F_{11}\,F_{12} - 2\,\alpha^4\,c_1^2\,D_{12}\,F_{12}^2 + 2\,\alpha^2\,c_1^2\,c_2 \\
& D_{55}\,F_{12}^2 - 4\,\alpha^4\,c_1^3\,F_{12}^3 - 4\,\alpha^2\,c_1\,c_2\,D_{11}\,D_{55}\,F_{22} - 2\,\alpha^4\,c_1^2\,D_{12}\,F_{11}\,F_{22} - 2\,\alpha^4\,c_1^2\,D_{66}\,F_{11}\,F_{22} + 2\,\alpha^4\,c_1^2\,D_{11}\,F_{12}\,F_{22} + 4\,\alpha^4 \\
& c_1^3\,F_{11}\,F_{12}\,F_{22} - A_{55}\,c_2^2\,D_{11}\,F_{44} - 2\,c_2^3\,D_{11}\,D_{55}\,F_{44} - \alpha^2\,c_2^2\,D_{11}\,D_{66}\,F_{44} - 2\,\alpha^2\,c_1\,c_2^2\,D_{12}\,F_{11}\,F_{44} - 4\,\alpha^2\,c_1\,c_2^2\,D_{66}\,F_{11}\,F_{44} \\
& + 2\,\alpha^2\,c_1\,c_2^2\,D_{11}\,F_{12}\,F_{44} + 2\,\alpha^2\,c_1^2\,c_2^2\,F_{11}\,F_{12}\,F_{44} - A_{44}\,c_2^2\,D_{11}\,F_{55} + \alpha^2\,c_2^2\,D_{12}^2\,F_{55} - \alpha^2\,c_2^2\,D_{11}\,D_{22}\,F_{55} - 2\,c_2^3\,D_{11}\,D_{44}\,F_{55} \\
& + 2\,\alpha^2\,c_2^2\,D_{12}\,D_{66}\,F_{55} + 2\,\alpha^2\,c_1\,c_2^2\,D_{12}\,F_{12}\,F_{55} + 4\,\alpha^2\,c_1\,c_2^2\,D_{66}\,F_{12}\,F_{55} + \alpha^2\,c_1^2\,c_2^2\,F_{12}^2\,F_{55} - 2\,\alpha^2\,c_1\,c_2^2\,D_{11}\,F_{22}\,F_{55} - c_2^4 \\
& D_{11}\,F_{44}\,F_{55} + 2\,A_{44}\,\alpha^2\,c_1\,D_{11}\,F_{66} + 4\,\alpha^2\,c_1\,c_2\,D_{11}\,D_{44}\,F_{66} + 4\,A_{44}\,\alpha^2\,c_1^2\,F_{11}\,F_{66} + 4\,\alpha^4\,c_1^2\,D_{22}\,F_{11}\,F_{66} + 8\,\alpha^2\,c_1^2\,c_2\,D_{44}\,F_{11} \\
& F_{66} - 8\,\alpha^4\,c_1^2\,D_{12}\,F_{12}\,F_{66} + 8\,\alpha^2\,c_1^2\,c_2\,D_{55}\,F_{12}\,F_{66} - 16\,\alpha^4\,c_1^3\,F_{12}^2\,F_{66} + 4\,\alpha^4\,c_1^2\,D_{11}\,F_{22}\,F_{66} + 12\,\alpha^4\,c_1^3\,F_{11}\,F_{22}\,F_{66} + 2\,\alpha^2\,c_1 \\
& c_2^2\,D_{11}\,F_{44}\,F_{66} + 4\,\alpha^2\,c_1^2\,c_2^2\,F_{11}\,F_{44}\,F_{66} + 4\,\alpha^2\,c_1^2\,c_2^2\,F_{12}\,F_{55}\,F_{66} - 8\,\alpha^4\,c_1^2\,D_{12}\,F_{66}^2 + 8\,\alpha^2\,c_1^2\,c_2\,D_{55}\,F_{66}^2 - 16\,\alpha^4\,c_1^3\,F_{12}\,F_{66}^2 \\
& + 4\,\alpha^2\,c_1^2\,c_2^2\,F_{55}\,F_{66}^2 - 2\,A_{44}\,\alpha^2\,c_1^2\,D_{12}\,H_{11} - 4\,\alpha^2\,c_1^2\,c_2\,D_{12}\,D_{44}\,H_{11} - 4\,A_{44}\,\alpha^2\,c_1^2\,D_{66}\,H_{11} - \alpha^4\,c_1^2\,D_{22}\,D_{66}\,H_{11} - 8\,\alpha^2\,c_1^2\,c_2 \\
& D_{44}\,D_{66}\,H_{11} + 2\,\alpha^4\,c_1^3\,D_{22}\,F_{12}\,H_{11} - 2\,\alpha^4\,c_1^3\,D_{12}\,F_{22}\,H_{11} - 4\,\alpha^4\,c_1^3\,D_{66}\,F_{22}\,H_{11} + 2\,\alpha^4\,c_1^4\,F_{12}\,F_{22}\,H_{11} - 2\,\alpha^2\,c_1^2\,c_2^2\,D_{12}\,F_{44}\,H_{11} \\
& - 4\,\alpha^2\,c_1^2\,c_2^2\,D_{66}\,F_{44}\,H_{11} + 2\,\alpha^4\,c_1^3\,D_{22}\,F_{66}\,H_{11} + 4\,\alpha^4\,c_1^4\,F_{22}\,F_{66}\,H_{11} + 2\,\alpha^4\,c_1^2\,D_{12}^2\,H_{12} - 2\,\alpha^4\,c_1^2\,D_{11}\,D_{22}\,H_{12} + 4\,\alpha^4\,c_1^2\,D_{12} \\
& D_{66}\,H_{12} - 2\,\alpha^4\,c_1^3\,D_{22}\,F_{11}\,H_{12} + 4\,\alpha^4\,c_1^3\,D_{12}\,F_{12}\,H_{12} + 8\,\alpha^4\,c_1^3\,D_{66}\,F_{12}\,H_{12} - 2\,\alpha^4\,c_1^4\,F_{12}^2\,H_{12} - 2\,\alpha^4\,c_1^3\,D_{11}\,F_{22}\,H_{12} - 2\,\alpha^4\,c_1^4 \\
& F_{11}\,F_{22}\,H_{12} - 8\,\alpha^4\,c_1^4\,F_{12}\,F_{66}\,H_{12} - 8\,\alpha^4\,c_1^4\,F_{66}^2\,H_{12} + 2\,\alpha^4\,c_1^4\,D_{12}\,H_{12}^2 + 4\,\alpha^4\,c_1^4\,D_{66}\,H_{12}^2 - 2\,\alpha^2\,c_1^2\,c_2\,D_{11}\,D_{55}\,H_{22} - \alpha^4\,c_1^2\,D_{11} \\
& D_{66}\,H_{22} - 2\,\alpha^4\,c_1^3\,D_{12}\,F_{11}\,H_{22} - 4\,\alpha^4\,c_1^3\,D_{66}\,F_{11}\,H_{22} + 2\,\alpha^4\,c_1^3\,D_{11}\,F_{12}\,H_{22} + 2\,\alpha^4\,c_1^4\,F_{11}\,F_{12}\,H_{22} - \alpha^2\,c_1^2\,c_2^2\,D_{11}\,F_{55}\,H_{22} + 2 \\
& \alpha^4\,c_1^3\,D_{11}\,F_{66}\,H_{22} + 4\,\alpha^4\,c_1^4\,F_{11}\,F_{66}\,H_{22} - 2\,\alpha^4\,c_1^4\,D_{12}\,H_{11}\,H_{22} - 4\,\alpha^4\,c_1^4\,D_{66}\,H_{11}\,H_{22} - A_{44}\,\alpha^2\,c_1^2\,D_{11}\,H_{66} + 4\,\alpha^4\,c_1^2\,D_{12}^2\,H_{66} \\
& - 4\,\alpha^4\,c_1^2\,D_{11}\,D_{22}\,H_{66} - 2\,\alpha^2\,c_1^2\,c_2\,D_{11}\,D_{44}\,H_{66} - 4\,\alpha^2\,c_1^2\,c_2\,D_{12}\,D_{55}\,H_{66} + 8\,\alpha^4\,c_1^2\,D_{12}\,D_{66}\,H_{66} - 8\,\alpha^2\,c_1^2\,c_2\,D_{55}\,D_{66}\,H_{66} - 4 \\
& \alpha^4\,c_1^3\,D_{22}\,F_{11}\,H_{66} + 8\,\alpha^4\,c_1^3\,D_{12}\,F_{12}\,H_{66} + 16\,\alpha^4\,c_1^3\,D_{66}\,F_{12}\,H_{66} - 4\,\alpha^4\,c_1^3\,D_{11}\,F_{22}\,H_{66} - 2\,\alpha^4\,c_1^4\,F_{11}\,F_{22}\,H_{66} - \alpha^2\,c_1^2\,c_2^2\,D_{11}\,F_{44} \\
& H_{66} - 2\,\alpha^2\,c_1^2\,c_2^2\,D_{12}\,F_{55}\,H_{66} - 4\,\alpha^2\,c_1^2\,c_2^2\,D_{66}\,F_{55}\,H_{66} - \alpha^4\,c_1^4\,D_{22}\,H_{11}\,H_{66} + 4\,\alpha^4\,c_1^4\,D_{12}\,H_{12}\,H_{66} + 8\,\alpha^4\,c_1^4\,D_{66}\,H_{12}\,H_{66} - \alpha^4 \\
& c_1^4\,D_{11}\,H_{22}\,H_{66} + A_{55}\,\alpha^2(D_{12}^2 - D_{11}(D_{22} + c_1(2\,F_{22} + c_1\,H_{22})) + 2\,D_{12}(D_{66} + c_1(F_{12} - c_1\,H_{66})) + c_1(c_1(F_{12} + 2\,F_{66})^2 \\
& + 4\,D_{66}(F_{12} - c_1\,H_{66}))) + (D_{66}(A_{55} + c_2(2\,D_{55} + c_2\,F_{55}))\,I_2 + 2\,c_1(D_{66}\,F_{11}\,I_0 + (A_{55} + c_2(2\,D_{55} + c_2\,F_{55}))\,F_{66}\,I_2) + \\
& c_1^2(D_{66}\,H_{11}\,I_0 - \alpha^2(F_{12} + 2\,F_{66})^2\,I_2 + A_{44}\,H_{11}\,I_2 + \alpha^2\,D_{22}\,H_{11}\,I_2 + 2\,c_2\,D_{44}\,H_{11}\,I_2 + \alpha^2\,D_{66}\,H_{11}\,I_2 + c_2^2\,F_{44}\,H_{11}\,I_2 + 2\,\alpha^2 \\
& D_{66}\,H_{12}\,I_2 + A_{55}\,H_{66}\,I_2 + 2\,c_2\,D_{55}\,H_{66}\,I_2 + 4\,\alpha^2\,D_{66}\,H_{66}\,I_2 + c_2^2\,F_{55}\,H_{66}\,I_2 + 2\,\alpha^2\,D_{12}\,F_{12}\,I_4 + 4\,\alpha^2\,D_{12}\,F_{66}\,I_4 + 2\,F_{11}(2\,F_{66} \\
& (I_0 - \alpha^2\,I_2) - (A_{44} + 2\,c_2\,D_{44} + c_2^2\,F_{44})\,I_4 + \alpha^2(-F_{12}\,I_2 + (D_{12} - D_{22} + D_{66})\,I_4)) - \alpha^2\,D_{12}(D_{12} + 2\,D_{66})\,I_6) + D_{11} \\
& ((A_{55} + c_2(2\,D_{55} + c_2\,F_{55}))\,I_2 + 2\,c_1(F_{66}\,I_0 + (A_{55} + c_2(2\,D_{55} + c_2\,F_{55}))\,I_4) + \alpha^2\,c_1^4(H_{22} + H_{66})\,I_6 + D_{66}(I_0 + \alpha^2\,c_1^2\,I_6) \\
& + 2\,\alpha^2\,c_1^3(H_{12}\,I_4 + 2\,H_{66}\,I_4 - (F_{12} - F_{22} + F_{66})\,I_6) + c_1^2(H_{66}(I_0 + 4\,\alpha^2\,I_2) + (A_{44} + A_{55} + c_2(2\,D_{44} + 2\,D_{55} + c_2(F_{44} \\
& + F_{55})))\,I_6 + \alpha^2(2\,H_{12}\,I_2 - 2(F_{12} + 2\,F_{66})\,I_4 + D_{22}\,I_6))) + 2\,c_1^3(-4\,\alpha^2\,F_{66}^2\,I_4 + F_{66}(H_{11}(I_0 - \alpha^2\,I_2) - 6\,\alpha^2\,F_{11}\,I_4) + \alpha^2 \\
& (F_{22}\,H_{11}\,I_2 + F_{12}^2\,I_4 + (D_{12} + 2\,D_{66})(H_{11} + H_{12} + 2\,H_{66})\,I_4 - F_{12}((H_{11} + H_{12} + 2\,H_{66})\,I_2 + (D_{12} + 2\,D_{66})\,I_6)) + F_{11} \\
& (H_{66}(I_0 + 2\,\alpha^2\,I_2) + \alpha^2(H_{12}\,I_2 - 2(F_{12} + F_{22})\,I_4 + (D_{12} + 2\,D_{66})\,I_6))) + c_1^4(-\alpha^2(H_{12}^2\,I_2 + 2\,H_{12}\,H_{66}\,I_2 - 2(F_{11} + F_{12} \\
& + 2\,F_{66})\,H_{12}\,I_4 + 2\,F_{11}(H_{22} - H_{66})\,I_4 + ((F_{12} + 2\,F_{66})(2\,F_{11} + F_{12} + 2\,F_{66}) - 2(D_{12} + 2\,D_{66})\,H_{66})\,I_6) + H_{11}(H_{66}(I_0 \\
& + \alpha^2\,I_2) + \alpha^2(H_{22}\,I_2 - 2(F_{12} + 2\,F_{66})\,I_4 + 2(D_{12} + 2\,D_{66})\,I_6))))\,\omega^2 - c_1^2(-I_4(2\,F_{11}\,I_2 + (D_{66} + c_1(4\,F_{11} + 2\,F_{66} + c_1 \\
& H_{66}))\,I_4) + ((D_{11} + D_{66} + 2\,c_1\,F_{66} + c_1^2\,H_{66})\,I_2 + 2\,c_1(D_{11} - c_1\,F_{11})\,I_4)\,I_6 + c_1^2\,D_{11}\,I_6^2 + H_{11}\,I_2(I_2 + c_1(2\,I_4 + c_1\,I_6)))\,\omega^4
\end{aligned} \tag{B.1}$$

$$
a_4 = -2\,\alpha^4\,c_2\,D_{12}^2\,D_{44} + 2\,\alpha^4\,c_2\,D_{11}\,D_{22}\,D_{44} + 8\,\alpha^2\,c_2^2\,D_{12}\,D_{44}\,D_{55} - 4\,\alpha^4\,c_2\,D_{12}\,D_{44}\,D_{66} + 2\,\alpha^4\,c_2\,D_{22}\,D_{55}\,D_{66} + 16\,\alpha^2\,c_2^2\,D_{44}
D_{55}\,D_{66} + 2\,A_{44}\,A_{55}\,\alpha^2(D_{12} + 2\,D_{66}) + 4\,A_{55}\,\alpha^2\,c_2\,D_{44}(D_{12} + 2\,D_{66}) + 4\,A_{44}\,\alpha^2\,c_2\,D_{55}(D_{12} + 2\,D_{66}) + 4\,\alpha^4\,c_1\,c_2\,D_{22}\,D_{44}
F_{11} - 4\,\alpha^4\,c_1\,c_2\,D_{12}\,D_{44}\,F_{12} - 4\,\alpha^4\,c_1\,c_2\,D_{22}\,D_{55}\,F_{12} - 8\,\alpha^4\,c_1\,c_2\,D_{44}\,D_{66}\,F_{12} - \alpha^6\,c_1^2\,D_{22}\,F_{12}^2 - 2\,\alpha^4\,c_1^2\,c_2\,D_{44}\,F_{12}^2 + 4\,\alpha^4\,c_1\,c_2
D_{12}\,D_{55}\,F_{22} + 8\,\alpha^4\,c_1\,c_2\,D_{55}\,D_{66}\,F_{22} + 2\,\alpha^6\,c_1^2\,D_{12}\,F_{12}\,F_{22} - 4\,\alpha^4\,c_1^2\,c_2\,D_{55}\,F_{12}\,F_{22} + 2\,\alpha^6\,c_1^3\,F_{12}^2\,F_{22} - \alpha^6\,c_1^2\,D_{11}\,F_{22}^2 - 2\,\alpha^6
c_1^3\,F_{11}\,F_{22}^2 - \alpha^4\,c_2^2\,D_{12}^2\,F_{44} + \alpha^4\,c_2^2\,D_{11}\,D_{22}\,F_{44} + 4\,\alpha^2\,c_2^3\,D_{12}\,D_{55}\,F_{44} - 2\,\alpha^4\,c_2^2\,D_{12}\,D_{66}\,F_{44} + 8\,\alpha^2\,c_2^3\,D_{55}\,D_{66}\,F_{44} + 2\,A_{55}\,\alpha^2\,c_2^2
(D_{12} + 2\,D_{66})\,F_{44} + 2\,\alpha^4\,c_1\,c_2^2\,D_{22}\,F_{11}\,F_{44} - 2\,\alpha^4\,c_1\,c_2^2\,D_{12}\,F_{12}\,F_{44} - 4\,\alpha^4\,c_1\,c_2^2\,D_{66}\,F_{12}\,F_{44} - \alpha^4\,c_1^2\,c_2^2\,F_{12}^2\,F_{44} + 4\,\alpha^2\,c_2^3\,D_{12}\,D_{44}
F_{55} + \alpha^4\,c_2^2\,D_{22}\,D_{66}\,F_{55} + 8\,\alpha^2\,c_2^3\,D_{44}\,D_{66}\,F_{55} + 2\,A_{44}\,\alpha^2\,c_2^2(D_{12} + 2\,D_{66})\,F_{55} - 2\,\alpha^4\,c_1\,c_2^2\,D_{22}\,F_{12}\,F_{55} + 2\,\alpha^4\,c_1\,c_2^2\,D_{12}\,F_{22}\,F_{55}
+ 4\,\alpha^4\,c_1\,c_2^2\,D_{66}\,F_{22}\,F_{55} - 2\,\alpha^4\,c_1^2\,c_2^2\,F_{12}\,F_{22}\,F_{55} + 2\,\alpha^2\,c_2^4\,D_{12}\,F_{44}\,F_{55} + 4\,\alpha^2\,c_2^4\,D_{66}\,F_{44}\,F_{55} - 4\,\alpha^4\,c_1\,c_2\,D_{22}\,D_{55}\,F_{66} - 4\,\alpha^6\,c_1^2
D_{22}\,F_{12}\,F_{66} - 8\,\alpha^4\,c_1^2\,c_2\,D_{44}\,F_{12}\,F_{66} + 4\,\alpha^6\,c_1^2\,D_{12}\,F_{22}\,F_{66} - 8\,\alpha^4\,c_1^2\,c_2\,D_{55}\,F_{22}\,F_{66} - 4\,\alpha^4\,c_1^2\,c_2^2\,F_{12}\,F_{44}\,F_{66} - 2\,\alpha^4\,c_1\,c_2^2\,D_{22}\,F_{55}
F_{66} - 4\,\alpha^4\,c_1^2\,c_2^2\,F_{22}\,F_{55}\,F_{66} - 4\,\alpha^6\,c_1^2\,D_{22}\,F_{66}^2 - 8\,\alpha^4\,c_1^2\,c_2\,D_{44}\,F_{66}^2 - 8\,\alpha^6\,c_1^3\,F_{22}\,F_{66}^2 - 4\,\alpha^4\,c_1^2\,c_2^2\,F_{44}\,F_{66}^2 + 2\,\alpha^4\,c_1^2\,c_2\,D_{22}\,D_{44}\,H_{11}
- \alpha^6\,c_1^4\,F_{22}^2\,H_{11} + \alpha^4\,c_1^2\,c_2^2\,D_{22}\,F_{44}\,H_{11} + 2\,\alpha^6\,c_1^2\,D_{22}\,D_{66}\,H_{12} - 2\,\alpha^6\,c_1^3\,D_{22}\,F_{12}\,H_{12} + 2\,\alpha^6\,c_1^3\,D_{12}\,F_{22}\,H_{12} + 4\,\alpha^6\,c_1^3\,D_{66}\,F_{22}\,H_{12}
+ 2\,\alpha^6\,c_1^4\,F_{12}\,F_{22}\,H_{12} + 4\,\alpha^6\,c_1^4\,F_{22}\,F_{66}\,H_{12} - \alpha^6\,c_1^4\,D_{22}\,H_{12}^2 - \alpha^6\,c_1^2\,D_{12}^2\,H_{22} + \alpha^6\,c_1^2\,D_{11}\,D_{22}\,H_{22} + 4\,\alpha^4\,c_1^2\,c_2\,D_{12}\,D_{55}\,H_{22} - 2\,\alpha^6
c_1^2\,D_{12}\,D_{66}\,H_{22} + 8\,\alpha^4\,c_1^2\,c_2\,D_{55}\,D_{66}\,H_{22} + 2\,\alpha^6\,c_1^3\,D_{22}\,F_{11}\,H_{22} - 2\,\alpha^6\,c_1^3\,D_{12}\,F_{12}\,H_{22} - 4\,\alpha^6\,c_1^3\,D_{66}\,F_{12}\,H_{22} - \alpha^6\,c_1^4\,F_{12}^2\,H_{22} + 2\,\alpha^4
c_1^2\,c_2^2\,D_{12}\,F_{55}\,H_{22} + 4\,\alpha^4\,c_1^2\,c_2^2\,D_{66}\,F_{55}\,H_{22} - 4\,\alpha^6\,c_1^4\,F_{12}\,F_{66}\,H_{22} - 4\,\alpha^6\,c_1^4\,F_{66}^2\,H_{22} + \alpha^6\,c_1^4\,D_{22}\,H_{11}\,H_{22} + 4\,\alpha^4\,c_1^2\,c_2\,D_{12}\,D_{44}\,H_{66} +
2\,\alpha^4\,c_1^2\,c_2\,D_{22}\,D_{55}\,H_{66} + 4\,\alpha^6\,c_1^2\,D_{22}\,D_{66}\,H_{66} + 8\,\alpha^4\,c_1^2\,c_2\,D_{44}\,D_{66}\,H_{66} - 4\,\alpha^6\,c_1^3\,D_{22}\,F_{12}\,H_{66} + 4\,\alpha^6\,c_1^3\,D_{12}\,F_{22}\,H_{66} + 8\,\alpha^6\,c_1^3\,D_{66}
F_{22}\,H_{66} + 2\,\alpha^4\,c_1^2\,c_2^2\,D_{12}\,F_{44}\,H_{66} + 4\,\alpha^4\,c_1^2\,c_2^2\,D_{66}\,F_{44}\,H_{66} + \alpha^4\,c_1^2\,c_2^2\,D_{22}\,F_{55}\,H_{66} - 2\,\alpha^6\,c_1^4\,D_{22}\,H_{12}\,H_{66} + 2\,\alpha^6\,c_1^4\,D_{12}\,H_{22}\,H_{66} +
4\,\alpha^6\,c_1^4\,D_{66}\,H_{22}\,H_{66} - A_{44}\,\alpha^4(D_{12}^2 - D_{11}\,D_{22} + c_1(-2\,D_{22}\,F_{11} + 4\,D_{66}\,F_{12} + c_1(F_{12} + 2\,F_{66})^2 - c_1\,D_{22}\,H_{11} - 4\,c_1\,D_{66}\,H_{66}) +
2\,D_{12}(D_{66} + c_1(F_{12} - c_1\,H_{66}))) + A_{55}\,\alpha^4(2\,c_1(F_{22}(D_{12} + 2\,D_{66} - c_1(F_{12} + 2\,F_{66})) + c_1(D_{12} + 2\,D_{66})\,H_{22}) + D_{22}(D_{66} +
c_1(-2(F_{12} + F_{66}) + c_1\,H_{66}))) + (-(A_{55}(D_{66} + c_1(2\,F_{66} + c_1\,H_{66})) + 2\,c_2(D_{11}\,D_{44} + D_{55}\,D_{66} + c_1(2\,D_{44}\,F_{11} + 2\,D_{55}\,F_{66}
+ c_1\,D_{44}\,H_{11} + c_1\,D_{55}\,H_{66})) + c_2^2(D_{11}\,F_{44} + D_{66}\,F_{55} + c_1(2\,F_{11}\,F_{44} + 2\,F_{55}\,F_{66} + c_1\,F_{44}\,H_{11} + c_1\,F_{55}\,H_{66})))\,I_0 - c_2(2\,D_{44}
+ c_2\,F_{44})(A_{55} + c_2(2\,D_{55} + c_2\,F_{55}))\,I_2 - \alpha^2(-D_{12}^2\,I_0 - 2\,D_{12}\,D_{66}\,I_0 + 2\,c_1\,D_{22}\,F_{11}\,I_0 - 4\,c_1\,D_{66}\,F_{12}\,I_0 - 4\,c_1^2\,F_{12}^2\,I_0 + 4\,c_1^2
F_{11}\,F_{22}\,I_0 - 8\,c_1^2\,F_{12}\,F_{66}\,I_0 + c_1^2\,D_{22}\,H_{11}\,I_0 + 2\,c_1^3\,F_{22}\,H_{11}\,I_0 - 2\,c_1^2\,D_{66}\,H_{12}\,I_0 - 4\,c_1^3\,F_{12}\,H_{12}\,I_0 - 4\,c_1^3\,F_{66}\,H_{12}\,I_0 - c_1^4\,H_{12}^2\,I_0
+ 2\,c_1^3\,F_{11}\,H_{22}\,I_0 + c_1^4\,H_{11}\,H_{22}\,I_0 + D_{11}(D_{22} + c_1(2\,F_{22} + c_1\,H_{22}))\,I_0 - 4\,c_1^3\,F_{12}\,H_{66}\,I_0 - 2\,c_1^4\,H_{12}\,H_{66}\,I_0 - 2\,c_1\,D_{12} 2\,F_{12}
+ 2\,F_{66} + c_1(H_{12} + H_{66}))\,I_0 + A_{55}\,D_{22}\,I_2 + 2\,c_2\,D_{22}\,D_{55}\,I_2 + A_{55}\,D_{66}\,I_2 + 2\,c_2\,D_{44}\,D_{66}\,I_2 + 2\,c_2\,D_{55}\,D_{66}\,I_2 + 4\,c_1\,c_2\,D_{44}\,F_{11}
I_2 - 2\,A_{55}\,c_1\,F_{12}\,I_2 - 4\,c_1\,c_2\,D_{44}\,F_{12}\,I_2 - 4\,c_1\,c_2\,D_{55}\,F_{12}\,I_2 + 2\,A_{55}\,c_1\,F_{22}\,I_2 + 4\,c_1\,c_2\,D_{55}\,F_{22}\,I_2 + c_2^2\,D_{66}\,F_{44}\,I_2 + 2\,c_1\,c_2^2\,F_{11}
F_{44}\,I_2 - 2\,c_1\,c_2^2\,F_{12}\,F_{44}\,I_2 + c_2\,D_{11}(2\,D_{44} + c_2\,F_{44})\,I_2 + c_2^2\,D_{22}\,F_{55}\,I_2 + c_2^2\,D_{66}\,F_{55}\,I_2 - 2\,c_1\,c_2^2\,F_{12}\,F_{55}\,I_2 + 2\,c_1\,c_2^2\,F_{22}\,F_{55}\,I_2
- 2\,A_{55}\,c_1\,F_{66}\,I_2 - 4\,c_1\,c_2\,D_{44}\,F_{66}\,I_2 - 4\,c_1\,c_2\,D_{55}\,F_{66}\,I_2 - 2\,c_1\,c_2^2\,F_{44}\,F_{66}\,I_2 - 2\,c_1\,c_2^2\,F_{55}\,F_{66}\,I_2 + 2\,c_1^2\,c_2\,D_{44}\,H_{11}\,I_2 + c_1^2\,c_2^2\,F_{44}
H_{11}\,I_2 + A_{55}\,c_1^2\,H_{22}\,I_2 + 2\,c_1^2\,c_2\,D_{55}H_{22}\,I_2 + c_1^2\,c_2^2\,F_{55}\,H_{22}\,I_2 + A_{55}\,c_1^2\,H_{66}\,I_2 + 2\,c_1^2\,c_2\,D_{44}\,H_{66}\,I_2 + 2\,c_1^2\,c_2\,D_{55}\,H_{66}\,I_2 + c_1^2\,c_2^2
F_{44}\,H_{66}\,I_2 + c_1^2\,c_2^2\,F_{55}\,H_{66}\,I_2 + 4\,A_{55}\,c_1\,D_{66}\,I_4 + 8\,c_1\,c_2\,D_{44}\,D_{66}\,I_4 + 8\,c_1\,c_2\,D_{55}\,D_{66}\,I_4 - 2\,A_{55}\,c_1^2\,F_{12}\,I_4 - 4\,c_1^2\,c_2\,D_{44}\,F_{12}\,I_4 -
4\,c_1^2\,c_2\,D_{55}\,F_{12}\,I_4 + 4\,c_1\,c_2^2\,D_{66}\,F_{44}\,I_4 - 2\,c_1^2\,c_2^2\,F_{12}\,F_{44}\,I_4 + 4\,c_1\,c_2^2\,D_{66}\,F_{55}\,I_4 - 2\,c_1^2\,c_2^2\,F_{12}\,F_{55}\,I_4 + 2\,c_1\,D_{12}(A_{55} + c_2(2\,D_{44} + 2
D_{55} + c_2(F_{44} + F_{55})))\,I_4 - 4\,A_{55}\,c_1^2\,F_{66}\,I_4 - 8\,c_1^2\,c_2\,D_{44}\,F_{66}\,I_4 - 8\,c_1^2\,c_2\,D_{55}\,F_{66}\,I_4 - 4\,c_1^2\,c_2^2\,F_{44}\,F_{66}\,I_4 - 4\,c_1^2\,c_2^2\,F_{55}\,F_{66}\,I_4 + 2
c_1^2(D_{12} + 2\,D_{66})(A_{55} + c_2(2\,D_{44} + 2\,D_{55} + c_2(F_{44} + F_{55})))\,I_6) - A_{44}((A_{55} + 2\,c_2\,D_{55} + \alpha^2\,D_{66} + c_2^2\,F_{55})\,I_2 + D_{11}(I_0 + \alpha^2
I_2) + 2\,c_1(F_{11}(I_0 + \alpha^2\,I_2) + \alpha^2(-(F_{12} + F_{66})\,I_2 + (D_{12} + 2\,D_{66})\,I_4)) + c_1^2(H_{11}(I_0 + \alpha^2\,I_2) + \alpha^2(H_{66}\,I_2 - 2(F_{12} + 2\,F_{66})
I_4 + 2(D_{12} + 2\,D_{66})\,I_6))) - \alpha^4\,c_1^2(-4\,F_{66}^2\,I_2 + 2\,D_{22}\,H_{12}\,I_2 + 2\,D_{66}\,H_{12}\,I_2 - c_1^2\,H_{12}^2\,I_2 + D_{11}\,H_{22}\,I_2 + D_{66}\,H_{22}\,I_2 + 2\,c_1\,F_{11}
H_{22}\,I_2 - 2\,c_1\,F_{66}\,H_{22}\,I_2 + c_1^2\,H_{11}\,H_{22}\,I_2 + 4\,D_{22}\,H_{66}\,I_2 + 4\,D_{66}\,H_{66}\,I_2 - 2\,c_1^2\,H_{12}\,H_{66}\,I_2 + c_1^2\,H_{22}\,H_{66}\,I_2 + 4\,D_{12}\,F_{66}\,I_4 - 4\,D_{22}
F_{66}\,I_4 - 8\,c_1\,F_{66}^2\,I_4 + 2\,c_1\,D_{12}\,H_{12}\,I_4 + 2\,c_1\,D_{22}\,H_{12}\,I_4 + 4\,c_1\,D_{66}\,H_{12}\,I_4 + 4\,c_1^2\,F_{66}\,H_{12}\,I_4 + 2\,c_1\,D_{12}\,H_{22}\,I_4 + 4\,c_1\,D_{66}\,H_{22}\,I_4
- 4\,c_1^2\,F_{66}\,H_{22}\,I_4 + 4\,c_1\,D_{12}\,H_{66}\,I_4 + 4\,c_1\,D_{22}\,H_{66}\,I_4 + 8\,c_1\,D_{66}\,H_{66}\,I_4 + (-D_{12}^2 + D_{11}\,D_{22} + 2\,D_{12}(-D_{66} + c_1^2(H_{22} + H_{66}))
+ 4\,c_1^2(-F_{66}^2 + D_{66}(H_{22} + H_{66})) + D_{22}(D_{66} + c_1(2\,F_{11} - 2\,F_{66} + c_1(H_{11} + H_{66}))))\,I_6 + 2\,F_{12}(-(F_{22} + 2\,F_{66} + c_1(H_{12} + H_{22}
+ 2\,H_{66}))\,I_2 + (D_{12} - D_{22} + c_1(-2\,F_{22} + c_1\,H_{12} - c_1\,H_{22}))\,I_4 - c_1(D_{12} + D_{22} + 2\,D_{66} + c_1\,F_{22} + 2\,c_1\,F_{66})\,I_6) - F_{12}^2(I_2 +
c_1(-2\,I_4 + c_1\,I_6)) + 2\,F_{22}((-D_{11} + D_{12} + D_{66})\,I_4 + c_1^2(-H_{11} + H_{12} + H_{66})\,I_4 + c_1(H_{12}\,I_2 + 2\,H_{66}\,I_2 - 2\,F_{11}\,I_4 + D_{12}\,I_6
+ 2\,D_{66}\,I_6) - 2\,F_{66}(I_2 + c_1(3\,I_4 + c_1\,I_6)))))\,\omega^2 + (D_{11}\,I_0\,I_2 + D_{66}\,I_0\,I_2 + 2\,c_1\,F_{11}\,I_0\,I_2 + 2\,c_1\,F_{66}\,I_0\,I_2 + c_1^2\,H_{11}\,I_0\,I_2 + c_1^2
H_{66}\,I_0\,I_2 + A_{55}\,I_2^2 + 2\,c_2\,D_{55}\,I_2^2 + c_2^2\,F_{55}\,I_2^2 + 2\,\alpha^2\,c_1^2\,H_{12}\,I_2^2 + 4\,\alpha^2\,c_1^2\,H_{66}\,I_2^2 + 2\,c_1\,D_{11}\,I_0\,I_4 + 2\,c_1\,D_{66}\,I_0\,I_4 + 4\,c_1^2\,F_{11}\,I_0\,I_4
+ 4\,c_1^2\,F_{66}\,I_0\,I_4 + 2\,c_1^3\,H_{11}\,I_0\,I_4 + 2\,c_1^3\,H_{66}\,I_0\,I_4 + 2\,A_{55}\,c_1\,I_2\,I_4 + 4\,c_1\,c_2\,D_{55}\,I_2\,I_4 - 4\,\alpha^2\,c_1^2\,F_{12}\,I_2\,I_4 + 2\,c_1\,c_2^2\,F_{55}\,I_2\,I_4 - 8\,\alpha^2
c_1^2\,F_{66}\,I_2\,I_4 + 4\,\alpha^2\,c_1^3\,H_{12}\,I_2\,I_4 + 8\,\alpha^2\,c_1^3\,H_{66}\,I_2\,I_4 - A_{44}\,c_1^2\,I_4^2 - \alpha^2\,c_1^2\,D_{11}\,I_4^2 + 2\,\alpha^2\,c_1^2\,D_{12}\,I_4^2 - \alpha^2\,c_1^2\,D_{22}\,I_4^2 - 2\,c_1^2\,c_2\,D_{44}\,I_4^2
+ 2\,\alpha^2\,c_1^2\,D_{66}\,I_4^2 - 2\,\alpha^2\,c_1^3\,F_{11}\,I_4^2 - 4\,\alpha^2\,c_1^3\,F_{12}\,I_4^2 - 2\,\alpha^2\,c_1^3\,F_{22}\,I_4^2 - c_1^2\,c_2^2\,F_{44}\,I_4^2 - 12\,\alpha^2\,c_1^3\,F_{66}\,I_4^2 - \alpha^2\,c_1^4\,H_{11}\,I_4^2 + 2\,\alpha^2\,c_1^4\,H_{12}\,I_4^2
- \alpha^2\,c_1^4\,H_{22}\,I_4^2 + 2\,\alpha^2\,c_1^4\,H_{66}\,I_4^2 + c_1^2(D_{66}\,I_0 + 2\,c_1\,F_{11}\,I_0 + 2\,c_1\,F_{66}\,I_0 + c_1^2\,H_{11}\,I_0 + c_1^2\,H_{66}\,I_0 + (A_{44} + A_{55} + \alpha^2\,D_{22} + 2\,c_2
(D_{44} + D_{55}))\,I_2 + 2\,\alpha^2\,D_{66}\,I_2 + 2\,\alpha^2\,c_1\,F_{11}\,I_2 - 4\,\alpha^2\,c_1\,F_{12}\,I_2 + 2\,\alpha^2\,c_1\,F_{22}\,I_2 + c_2^2\,F_{44}\,I_2 + c_2^2\,F_{55}\,I_2 - 4\,\alpha^2\,c_1\,F_{66}\,I_2 + \alpha^2\,c_1^2
H_{11}\,I_2 + \alpha^2\,c_1^2\,H_{22}\,I_2 + 2\,\alpha^2\,c_1^2\,H_{66}\,I_2 + D_{11}(I_0 + \alpha^2\,I_2) + 4\,\alpha^2\,c_1(D_{12} + 2\,D_{66} - c_1(F_{12} + 2\,F_{66}))\,I_4)\,I_6 + 2\,\alpha^2\,c_1^4(D_{12} + 2
D_{66})\,I_6^2)\,\omega^4 + c_1^2(I_4^2 - I_2\,I_6)(I_2 + c_1(2\,I_4 + c_1\,I_6))\,\omega^6
$$

$$
a_5 = -(A_{55} + 2\,c_2\,D_{55} + c_2^2\,F_{55} + \alpha^2(D_{66} + 2\,c_1\,F_{66} + c_1^2\,H_{66}) - (I_2 + 2\,c_1\,I_4 + c_1^2\,I_6)\,\omega^2)(\alpha^4(A_{44}\,D_{22} + 2\,c_2\,D_{22}\,D_{44} + c_2^2\,D_{22}
F_{44} + \alpha^2\,c_1^2(-F_{22}^2 + D_{22}\,H_{22})) - (c_2(2\,D_{44} + c_2\,F_{44})\,I_0 + A_{44}(I_0 + \alpha^2\,I_2) + \alpha^2((D_{22} + 2\,c_1\,F_{22} + c_1^2\,H_{22})\,I_0 + c_2(2\,D_{44} +
c_2\,F_{44})\,I_2) + \alpha^4\,c_1^2(H_{22}\,I_2 - 2\,F_{22}\,I_4 + D_{22}\,I_6))\,\omega^2 + (\alpha^2\,c_1^2(-I_4^2 + I_2\,I_6) + I_0(I_2 + c_1(2\,I_4 + c_1\,I_6)))\,\omega^4)
$$

$$
\begin{aligned}
b_1 &= -A_{11}\,A_{66} \\
b_2 &= -A_{12}^2\,\alpha^2 + A_{11}\,A_{22}\,\alpha^2 - 2\,A_{12}\,A_{66}\,\alpha^2 - A_{11}\,I_0\,\omega^2 - A_{66}\,I_0\,\omega^2 \\
b_3 &= -A_{22}\,A_{66}\,\alpha^4 + A_{22}\,\alpha^2\,I_0\,\omega^2 + A_{66}\,\alpha^2\,I_0\,\omega^2 - I_0^2\,\omega^4
\end{aligned}
\qquad \text{(B.2)}
$$