Buckling of Composite Plate Assemblies using Higher Order Shear Deformation Theory - An Exact Method of Solution

F. A. Fazzolari[1,*], J. R. Banerjee[2], M. Boscolo[3]

City University London, Northampton Square, London EC1V 0HB, UK

Abstract

An exact dynamic stiffness element based on higher order shear deformation theory and extensive use of symbolic algebra is developed for the first time to carry out a buckling analysis of composite plate assemblies. The principle of minimum potential energy is applied to derive the governing differential equations and natural boundary conditions. Then by imposing the geometric boundary conditions in algebraic form the dynamic stiffness matrix, which includes contributions from both stiffness and initial pre-stress terms, is developed. The Wittrick-Williams algorithm is used as solution technique to compute the critical buckling loads and mode shapes for a range of laminated composite plates including stiffened plates. The effects of significant parameters such as thickness-to-length ratio, orthotropy ratio, number of layers, lay-up and stacking sequence and boundary conditions on the critical buckling loads and mode shapes are investigated. The accuracy of the method is demonstrated by comparing results whenever possible with those available in the literature.

Keywords: Dynamic Stiffness Method, Composite Plates, Buckling, Stiffened Plates, Wittrick-Williams algorithm.

1. Introduction

Aerospace structures are generally made up of thin-walled structures such as plates and shells. Such structures often experience severe loading conditions. A certain load, referred to as critical load when applied, the structure suddenly changes its equilibrium configuration. This phenomenon is generally referred to as buckling instability. The topic is a major design consideration and has continued to be an important area of research because it represents one of the main reasons for aircraft and other structural failures. Several methodologies have been developed over the years to solve the problem. A simplified approach to calculate the ith critical load, is to consider the critical load as the load at which more than one infinitesimally adjacent equilibrium configurations exist that can be identified with the ith bifurcation point (Euler's method) [1]. In a linearized structural stability analysis, the determination of the critical load leads to a linear eigenvalues problem. The bifurcation method can be successfully used particularly for plates, when the critical equilibrium configuration shows a slight geometry change as the critical

*Corresponding author: Tel:+44(0)2070408483, Fax:+44(0)2070408566
Email address: Fiorenzo.Fazzolari.1@city.ac.uk (F. A. Fazzolari)
[1]Ph.D. Candidate, School of Engineering and Mathematical Sciences
[2]Professor, School of Engineering and Mathematical Sciences
[3]Research Fellow, School of Engineering and Mathematical Sciences

buckling load is reached. However, as explained by Leissa [2], linearized stability analysis is meaningful, if and only if, the initial in-plane loading does not produce an out-of-plane deformation. Furthermore, there are many cases in which Euler's method may fail, in particular when thin-walled structures like shells exhibit the snap-buckling phenomenon. In such cases, the most general approach, based on the solution of the complete equilibrium and stability equations [3, 4] is preferred.

Amongst a wide class of methodologies employed to analyze the elastic stability of advanced composite structures, the dynamic stiffness method (DSM) is probably the most accurate and computationally efficient option. The DSM based on Lèvy-type closed form solution for plates [5] is indeed an exact approach to the solution procedure. Wittick [6] laid the groundwork of the DSM for plates. The basic assumption in this work is that the deformation of any component plate varies sinusoidally in the longitudinal direction. Using this assumption, a stiffness matrix may be derived that relates the amplitudes of the edge forces and moments to the corresponding edge displacements and rotations for a single component plate. For the exact DSM, this stiffness matrix is derived directly from the equations of equilibrium that describe the buckling behavior of the plate. Essentially, Wittrick [6] developed an exact stiffness matrix for a single isotropic, long flat plate subject to uniform axial compression. His analysis basically used classical plate theory (CPT). Wittrick and Curzon [7] later extended this analysis to account for the spatial phase difference between the perturbation forces and displacements which occur at the edges of the plate during buckling due to the presence of in-plane shear loading. This phase difference was accounted for by defining the magnitude of these quantities using complex quantities. Wittrick [8] then extended his analysis further to consider flat isotropic plates under any general state of stress that remains uniform in the longitudinal direction (i.e., combinations of bi-axial direct stress and in-plane shear). A method very similar to that described in [6] was also presented by Smith in [9] for the bending, buckling, and vibration of plate-beam structures. Following these developments, Williams [10] presented two computer programs, GASVIP and VIPAL to compute the natural frequencies and initial buckling stress of prismatic plate assemblies subjected to uniform longitudinal stress or uniform longitudinal compression, respectively. GASVIP was used to set up the overall stiffness matrix for the structure, and VIPAL demonstrated the use of substructuring. Next, Wittrick and Williams [11] reported on the VIPASA computer code for the buckling and vibration analyses of prismatic plate assemblies. This code allowed for analysis of isotropic or anisotropic plates using a general state of stress (including in-plane shear). The complex stiffnesses described in [12] were incorporated in VIPASA, as well as allowances were made for eccentric connections between component plates. This code also implemented an algorithm, referred to as the Wittrick-Williams algorithm [13] for determining any buckling load for any given wavelength. The development of this algorithm was necessary because the complex stiffnesses described above are transcendental functions of the load factor and half wavelength of the buckling modes of the structure which make a determinant plot cumbersome and unfeasible. Viswanathan and Tamekuni [14, 15] presented an exact FSM based upon CPT for the elastic stability analysis of composite stiffened structures subjected to biaxial in-plane loads. The structure was idealized as an assemblage of laminated plate elements (flat or curved) and beam elements. Tamekuni, and Baker extended this analysis in [16] considering long curved plates subject to any general state of stress, together with in-plane shear loads.

Anisotropic material properties were also allowed. This analysis utilized complex stiffnesses as described in [12]. The works described in [9, 16, 13] are more or less similar. The differences are discussed in [11]. Williams and Anderson [17] presented modifications to the eigenvalue algorithm described in [13]. Further modifications presented in [17] allowed the buckling mode corresponding to a general loading to be represented as a series of sinusoidal modes in combination with Lagrangian multipliers to apply point constraints at any location on edges. These modifications formed the basis for the computer code VICON (VIpasa with CONstraints) described in [18]. However, the analysis capability of VICON was limited to plates analyzed using CPT. Anderson and Kennedy [19] incorporated a first order shear deformation plate theory (FSDT) into VICONOPT. A numerical approach to obtain exact plate stiffnesses that include the effects of transverse shear deformation was presented by them in [19]. It is worth noting that DSM has been extensively researched by Banerjee [20, 21, 22, 23, 24, 25], amongst a few others for modal analysis of structures idealized by beam elements based on Euler/Bernoulli, Timoshenko and associated coupled beam theories. The current paper is partly motivated by these earlier investigations and the most important contribution made by the authors here is the inclusion of the higher order shear deformation theory (HSDT) and the use of a systematic symbolic procedure, for the first time, when developing the DS matrix for laminated composite plates for buckling analysis. This useful extension is of considerable theoretical and computational complexity as will be shown later. The research is particularly relevant when analysing thick composite plates for their buckling characteristics. It should be recognised that Reddy and co-authors [26, 27, 28] have used HSDT for composite plates in a different context without resorting to the development of the DSM. From a historical prospective HSDT, can be essentially traced back to third order plate bending theory originally proposed by Vlasov [29] in the late fifties. His theory was substantiated and extended to laminated composite plates many years later by Reddy [26] using a variational approach. This is sometimes referred to as Vlasov-Reddy theory (VRT). Further improvements of this theory can be found in the work of Jemielita [30, 31]. Recently, for the analysis of anisotropic plates and shells, an advanced hierarchical trigonometric Ritz formulation (HTRF) based on refined variable kinematics 2D and quasi-3D plate/shell theories has been proposed by Fazzolari and Carrera for mechanical [32, 33, 34, 35, 36, 37] and multifield [38] problems. Inclusion of HSDT in the DSM framework will enable buckling analysis of plates with moderate to high thickness-to-width ratio, in an accurate and computationally efficient manner. The usefulness of HSDT becomes apparent when analysing composite structures idealized by plates, particularly of thicker dimension, because fiber reinforced composites have generally very low shear modulii. Extensive results which include validation and assessment of the effects on critical buckling load of significant parameters such as the thickness to width (or length) ratio, orthotropy ratio, number of layers, stacking sequence and boundary conditions, have been obtained, examined and discussed.

2. Theoretical formulation

2.1. Displacement field and governing differential equations

In the derivation that follows, the hypotheses of straightness and normality of a transverse normal after deformation are assumed to be no longer valid for the displacement field which is now considered to be a cubic function in the thickness coordinate, and hence the use of higher order shear deformation theory (HSDT). This development is in sharp contrast to earlier developments based on CPT and FSDT and no doubt a significant step forward. The deformation pattern through thickness of the plate is shown in Fig. 1. A laminated composite plate composed of N_l layers is considered in order to make the theory sufficiently general. The integer k is used as a superscript denoting the layer number which starts from the bottom of the plate. The kinematics of deformation of a transverse normal using both first order and higher order shear deformation are shown in Fig. 1. After imposing the transverse shear stress homogeneous conditions [39, 40] at the top/bottom surface of the plate, the displacements field are given below in the usual form:

$$\begin{aligned} u\left(x,y,z,t\right) &= u_0\left(x,y,t\right) + z\,\phi_x\left(x,y,t\right) + c_1\,z^3\left(\phi_x\left(x,y,t\right) + \frac{\partial w_0\left(x,y,t\right)}{\partial x}\right) \\ v\left(x,y,z,t\right) &= v_0\left(x,y,t\right) + z\,\phi_y\left(x,y,t\right) + c_1\,z^3\left(\phi_y\left(x,y,t\right) + \frac{\partial w_0\left(x,y,t\right)}{\partial y}\right) \\ w\left(x,y,z,t\right) &= w_0\left(x,y,t\right) \end{aligned} \tag{1}$$

where u, v, w are the plate displacement components of the displacement vector,

$$\boldsymbol{\eta} = \left\{ \begin{array}{ccc} u & v & w \end{array} \right\}^T \tag{2}$$

$c_1 = -\frac{4}{3\,h^2}$ whereas u_0, v_0, w_0 are the displacement components defined on the plate middle surface Ω in the directions x, y and z. The principle of minimum potential energy is now applied. The variational statement at multilayer level is:

$$\sum_{k=1}^{N_l} \delta\,\Pi^k = 0 \tag{3}$$

where Π^k is the total potential energy for the kth layer of the composite plate. The first variation can be expressed as:

$$\delta\Pi^k = \delta U^k + \delta V^k \tag{4}$$

where δU^k is the virtual potential strain energy, δV^k is the virtual potential energy due to external loadings, and assume the following form:

$$\delta U^k = \int_{\Omega^k}\int_{z^k}\left(\boldsymbol{\delta\varepsilon}^{k^T}\,\boldsymbol{\sigma}^k\right)\,d\Omega^k\,dz, \quad \delta V^k = \int_{\Omega^k}\int_{z^k}\left(\delta\varepsilon^{nl}_{xx}\,\tilde{\sigma}_{x0} + \delta\varepsilon^{nl}_{yy}\,\tilde{\sigma}_{y0}\right)\,d\Omega^k\,dz \tag{5}$$

the stresses, $\boldsymbol{\sigma}$ and the strains, $\boldsymbol{\varepsilon}$ vectors are expressed as follows:

$$\boldsymbol{\sigma} = \left\{ \begin{array}{ccccc} \sigma_{xx} & \sigma_{yy} & \tau_{xy} & \tau_{xz} & \tau_{yz} \end{array} \right\}^T, \quad \boldsymbol{\varepsilon} = \left\{ \begin{array}{ccccc} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} & \gamma_{xz} & \gamma_{yz} \end{array} \right\}^T \tag{6}$$

$\tilde{\sigma}_{x0}$ and $\tilde{\sigma}_{y0}$ denote the in-plane initial stresses. The non-linear strains ε^{nl}_{xx} and ε^{nl}_{yy} are approximated with the Von Karman's non-linearity:

$$\varepsilon^{nl}_{xx} = \frac{1}{2}\left(w,x\right)^2 \qquad \varepsilon^{nl}_{yy} = \frac{1}{2}\left(w,y\right)^2 \tag{7}$$

The subscript T signifies an array transposition and δ the variational operator. Constitutive and geometrical relationships are defined respectively as:

$$\boldsymbol{\sigma}^k = \tilde{\boldsymbol{C}}^k\,\boldsymbol{\varepsilon}^k \qquad\qquad \boldsymbol{\varepsilon} = \boldsymbol{\mathcal{D}}\,\boldsymbol{\eta} \tag{8}$$

where $\tilde{\boldsymbol{C}}^k$ is the plane stress constitutive matrix and $\boldsymbol{\mathcal{D}}$ is the differential matrix (see Appendix A for details). Substituting Eq. (8) into the Eq. (5) and imposing the condition in Eq. (3), the equations of motion are obtained after extensive algebraic manipulation as:

$$\begin{aligned}
\delta u_0 :\quad & A_{11}\, u_{0,xx} + A_{12}\, v_{0,yx} + A_{16}\,(u_{0,yx} + v_{0,xx}) + B_{11}\,\phi_{x,xx} + B_{12}\,\phi_{y,yx} + B_{16}\,(\phi_{x,yx} + \phi_{y,xx}) + E_{11}\,\mathrm{c}_2\,\phi_{x,xx} \\
& + E_{11}\,\mathrm{c}_2\, w_{0,xxx} + E_{12}\,\mathrm{c}_2\,\phi_{y,yx} + E_{12}\,\mathrm{c}_2\, w_{0,yyx} + E_{16}\,\mathrm{c}_2\,\phi_{x,yx} + E_{16}\,\mathrm{c}_2\,\phi_{y,xx} + 2\,E_{16}\,\mathrm{c}_2\, w_{0,xyx} + A_{16}\, u_{0,xy} \\
& + A_{26}\, v_{0,yy} + A_{66}\,(u_{0,yy} + v_{0,xy}) + B_{16}\,\phi_{x,xy} + B_{26}\,\phi_{y,yy} + B_{66}\,(\phi_{x,yy} + \phi_{y,xy}) + E_{12}\,\mathrm{c}_2\,(\phi_{x,xy} + w_{0,xxy}) \\
& + E_{26}\,\mathrm{c}_2\,(\phi_{y,yy} + w_{0,yyy}) + E_{66}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) = 0
\end{aligned}$$

$$\begin{aligned}
\delta v_0 :\quad & A_{16}\, u_{0,xx} + A_{26}\, v_{0,yx} + A_{66}\,(u_{0,yx} + v_{0,xx}) + B_{16}\,\phi_{x,xx} + B_{26}\,\phi_{y,yx} + B_{66}\,(\phi_{x,yx} + \phi_{y,xx}) + E_{16}\,\mathrm{c}_2\,\phi_{x,xx} \\
& + E_{16}\,\mathrm{c}_2\, w_{0,xxx} + E_{26}\,\mathrm{c}_2\,\phi_{y,yx} + E_{26}\,\mathrm{c}_2\, w_{0,yyx} + E_{66}\,\mathrm{c}_2\,\phi_{x,yx} + E_{66}\,\mathrm{c}_2\,\phi_{y,xx} + 2\,E_{66}\,\mathrm{c}_2\, w_{0,xyx} + A_{12}\, u_{0,xy} \\
& + A_{22}\, v_{0,yy} + A_{26}\,(u_{0,yy} + v_{0,xy}) + B_{12}\,\phi_{x,xy} + B_{22}\,\phi_{y,yy} + B_{26}\,(\phi_{x,yy} + \phi_{y,xy}) + E_{12}\,\mathrm{c}_2\,(\phi_{x,xy} + w_{0,xxy}) \\
& + E_{22}\,\mathrm{c}_2\,(\phi_{y,yy} + w_{0,yyy}) + E_{26}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) = 0
\end{aligned}$$

$$\begin{aligned}
\delta w_0 :\quad & A_{44}\,(\phi_{y,y} + w_{0,yy}) + A_{45}\,(\phi_{x,y} + w_{0,xy}) + D_{44}\,\mathrm{c}_1\,(\phi_{y,y} + w_{0,yy}) + D_{45}\,\mathrm{c}_1\,(\phi_{x,y} + w_{0,xy}) \\
& + A_{45}\,(\phi_{y,x} + w_{0,xy}) + A_{55}\,(\phi_{x,x} + w_{0,xx}) + D_{45}\,\mathrm{c}_1\,(\phi_{y,x} + w_{0,xy}) + D_{55}\,\mathrm{c}_1\,(\phi_{x,x} + w_{0,xx}) \\
& + D_{44}\,\mathrm{c}_1\,(\phi_{y,y} + w_{0,yy}) + D_{45}\,\mathrm{c}_1\,(\phi_{x,y} + w_{0,xy}) + F_{44}\,\mathrm{c}_1^2\,(\phi_{y,y} + w_{0,yy}) + F_{45}\,\mathrm{c}_1^2\,(\phi_{x,y} + w_{0,xy}) \\
& + D_{45}\,\mathrm{c}_1\,(\phi_{y,x} + w_{0,xy}) + D_{55}\,\mathrm{c}_1\,(\phi_{x,x} + w_{0,xx}) + F_{45}\,\mathrm{c}_1^2\,(\phi_{y,x} + w_{0,xy}) + F_{55}\,\mathrm{c}_1^2\,(\phi_{x,x} + w_{0,xx}) \\
& - E_{11}\,\mathrm{c}_2\, u_{0,xxx} - E_{12}\,\mathrm{c}_2\, v_{0,xxy} - E_{16}\,\mathrm{c}_2\,(u_{0,xxy} + v_{0,xxx}) - F_{11}\,\mathrm{c}_2\,\phi_{x,xxx} - F_{12}\,\mathrm{c}_2\,\phi_{y,xxy} \\
& - F_{16}\,\mathrm{c}_2\,(\phi_{x,xxy} + \phi_{y,xxx}) - H_{11}\,\mathrm{c}_2^2\,(\phi_{x,xxx} + w_{0,xxxx}) - H_{12}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + w_{0,xxyy}) \\
& - H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + \phi_{y,xxx} + 2w_{0,xxxy}) - 2\,E_{16}\,\mathrm{c}_2\, u_{0,xxy} - 2\,E_{26}\,\mathrm{c}_2\, v_{0,xyy} - 2\,E_{66}\,\mathrm{c}_2\,(u_{0,xyy} + v_{0,xxy}) \\
& - 2\,F_{16}\,\mathrm{c}_2\,\phi_{x,xxy} - 2\,F_{26}\,\mathrm{c}_2\,\phi_{y,xyy} - 2\,F_{66}\,\mathrm{c}_2\,(\phi_{x,xyy} + \phi_{y,xxy}) - 2\,H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + w_{0,xxxy}) \\
& - 2\,H_{26}\,\mathrm{c}_2^2\,(\phi_{y,xyy} + w_{0,xyyy}) - 2\,H_{66}\,\mathrm{c}_2^2\,(\phi_{x,xyy} + \phi_{y,xxy} + 2\,w_{0,xxyy}) - E_{12}\,\mathrm{c}_2\, u_{0,xyy} - E_{22}\,\mathrm{c}_2\, v_{0,yyy} \\
& - E_{26}\,\mathrm{c}_2\,(u_{0,yyy} + v_{0,xyy}) - F_{12}\,\mathrm{c}_2\,\phi_{x,xyy} - F_{22}\,\mathrm{c}_2\,\phi_{y,yyy} - F_{26}\,\mathrm{c}_2\,(\phi_{x,yyy} + \phi_{y,xyy}) \\
& - H_{12}\,\mathrm{c}_2^2\,(\phi_{x,xyy} + w_{0,xxyy}) - H_{22}\,\mathrm{c}_2^2\,(\phi_{y,yyy} + w_{0,yyyy}) - 2\,H_{26}\,\mathrm{c}_2^2\,(\phi_{x,yyy} + \phi_{y,xyy} + 2\,w_{0,xyyy}) \\
& = \tilde{N}_{x_0}\, w_{0,xx} + \tilde{N}_{y_0}\, w_{0,yy}
\end{aligned}$$

$$\begin{aligned}
\delta \phi_x :\quad & B_{11}\, u_{0,xx} + B_{12}\, v_{0,yx} + B_{16}\,(u_{0,yx} + v_{0,xx}) + D_{11}\,\phi_{x,xx} + D_{12}\,\phi_{y,xy} + D_{16}\,(\phi_{x,yx} + \phi_{y,xx}) \\
& + F_{11}\,\mathrm{c}_2\,(\phi_{x,xx} + w_{0,xxx}) + F_{12}\,\mathrm{c}_2\,(\phi_{y,yx} + w_{0,yyx}) + F_{16}\,\mathrm{c}_2\,(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}) \\
& + B_{16}\, u_{0,xy} + B_{26}\, v_{0,yy} + B_{66}\,(u_{0,yy} + v_{0,xy}) + D_{16}\,\phi_{x,xy} + D_{26}\,\phi_{y,yy} + D_{66}\,(\phi_{x,yy} + \phi_{y,xy}) \\
& + F_{16}\,\mathrm{c}_2\,(\phi_{x,xy} + w_{0,xxy}) + F_{26}\,\mathrm{c}_2\,(\phi_{y,yy} + w_{0,yyy}) + F_{66}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) \\
& + E_{11}\,\mathrm{c}_2\, u_{0,xx} + E_{12}\,\mathrm{c}_2\, v_{0,yx} + E_{16}\,\mathrm{c}_2\,(u_{0,yx} + v_{0,xx}) + F_{11}\,\mathrm{c}_2\,\phi_{x,xx} + F_{12}\,\mathrm{c}_2\,\phi_{y,xy} + F_{16}\,\mathrm{c}_2\,(\phi_{x,yx} + \phi_{y,xx}) \\
& + H_{11}\,\mathrm{c}_2^2\,(\phi_{x,xx} + w_{0,xxx}) + H_{12}\,\mathrm{c}_2^2\,(\phi_{y,yx} + w_{0,yyx}) + H_{16}\,\mathrm{c}_2^2\,(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}) \\
& + E_{16}\,\mathrm{c}_2\, u_{0,xy} + E_{26}\,\mathrm{c}_2\, v_{0,yy} + E_{66}\,\mathrm{c}_2\,(u_{0,yy} + v_{0,xy}) + F_{16}\,\mathrm{c}_2\,\phi_{x,xy} + F_{26}\,\mathrm{c}_2\,\phi_{y,yy} + F_{66}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy}) \\
& + H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xy} + w_{0,xxy}) + H_{26}\,\mathrm{c}_2^2\,(\phi_{y,yy} + w_{0,yyy}) + H_{66}\,\mathrm{c}_2^2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) \\
& - A_{45}\,(\phi_y + 2\,w_{0,y}) - A_{55}\,(\phi_x + 2\,w_{0,x}) - 2\,D_{45}\,\mathrm{c}_1\,(\phi_y + 2\,w_{0,y}) - 2\,D_{55}\,\mathrm{c}_1\,(\phi_x + 2\,w_{0,x}) \\
& - F_{45}\,\mathrm{c}_1^2\,(\phi_y + 2\,w_{0,y}) - F_{55}\,\mathrm{c}_1^2\,(\phi_x + 2\,w_{0,x}) = 0
\end{aligned}$$

(9)

$$
\begin{aligned}
\delta\phi_y: \quad & B_{16}\,u_{0,xx} + B_{26}\,v_{0,yx} + B_{66}\,\left(u_{0,yx} + v_{0,xx}\right) + D_{16}\,\phi_{x,xx} + D_{26}\,\phi_{y,xy} + D_{66}\,\left(\phi_{x,yx} + \phi_{y,xx}\right) \\
& + F_{16}\,\mathrm{c}_2\,\left(\phi_{x,xx} + w_{0,xxx}\right) + F_{26}\,\mathrm{c}_2\left(\phi_{y,yx} + w_{0,yyx}\right) + F_{66}\,\mathrm{c}_2\left(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}\right) \\
& + B_{12}\,u_{0,xy} + B_{22}\,v_{0,yy} + B_{26}\,\left(u_{0,yy} + v_{0,xy}\right) + D_{12}\,\phi_{x,xy} + D_{22}\,\phi_{y,yy} + D_{26}\,\left(\phi_{x,yy} + \phi_{y,xy}\right) \\
& + F_{12}\,\mathrm{c}_2\,\left(\phi_{x,xy} + w_{0,xxy}\right) + F_{22}\,\mathrm{c}_2\left(\phi_{y,yy} + w_{0,yyy}\right) + F_{26}\,\mathrm{c}_2\left(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}\right) \\
& + E_{16}\,\mathrm{c}_2\,u_{0,xx} + E_{26}\,\mathrm{c}_2\,v_{0,yx} + E_{66}\,\mathrm{c}_2\left(u_{0,yx} + v_{0,xx}\right) + F_{16}\,\mathrm{c}_2\,\phi_{x,xx} + F_{26}\,\mathrm{c}_2\,\phi_{y,xy} + F_{66}\,\mathrm{c}_2\left(\phi_{x,yx} + \phi_{y,xx}\right) \\
& + H_{16}\,\mathrm{c}_2^2\left(\phi_{x,xx} + w_{0,xxx}\right) + H_{26}\,\mathrm{c}_2^2\left(\phi_{y,yx} + w_{0,yyx}\right) + H_{66}\,\mathrm{c}_2^2\left(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}\right) \\
& + E_{12}\,\mathrm{c}_2\,u_{0,xy} + E_{22}\,\mathrm{c}_2\,v_{0,yy} + E_{26}\,\mathrm{c}_2\left(u_{0,yy} + v_{0,xy}\right) + F_{12}\,\mathrm{c}_2\,\phi_{x,xy} + F_{22}\,\mathrm{c}_2\,\phi_{y,yy} + F_{26}\,\mathrm{c}_2\left(\phi_{x,yy} + \phi_{y,xy}\right) \\
& + H_{12}\,\mathrm{c}_2^2\left(\phi_{x,xy} + w_{0,xxy}\right) + H_{22}\,\mathrm{c}_2^2\left(\phi_{y,yy} + w_{0,yyy}\right) + H_{26}\,\mathrm{c}_2^2\left(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}\right) \\
& - A_{44}\left(\phi_y + 2\,w_{0,y}\right) - A_{45}\left(\phi_x + 2\,w_{0,x}\right) - 2\,D_{44}\,\mathrm{c}_1\left(\phi_y + 2\,w_{0,y}\right) - 2\,D_{45}\,\mathrm{c}_1\left(\phi_x + 2\,w_{0,x}\right) \\
& - F_{44}\,\mathrm{c}_1^2\left(\phi_y + 2\,w_{0,y}\right) - F_{45}\,\mathrm{c}_1^2\left(\phi_x + 2\,w_{0,x}\right) = 0
\end{aligned}
$$

The natural boundary conditions are:

$$
\begin{aligned}
\delta u_0: \quad \mathcal{N}_{xx} = {} & A_{11}\,u_{0,x} + B_{11}\,\phi_{x,x} + E_{11}\,\mathrm{c}_2\,\phi_{x,x} + E_{11}\,\mathrm{c}_2\,w_{0,xx} + A_{12}\,v_{0,y} + B_{12}\,\phi_{y,y} + E_{12}\,\mathrm{c}_2\,\phi_{y,y} + E_{12}\,\mathrm{c}_2\,w_{0,yy} \\
& + A_{16}\,u_{0,y} + A_{16}\,v_{0,x} + B_{16}\,\phi_{x,y} + B_{16}\,\phi_{y,x} + E_{16}\,\mathrm{c}_2\,\phi_{x,y} + E_{16}\,\mathrm{c}_2\,\phi_{y,x} + 2\,E_{16}\,\mathrm{c}_2\,w_{0,xy}
\end{aligned}
$$

$$
\begin{aligned}
\delta v_0: \quad \mathcal{N}_{xy} = {} & A_{16}\,u_{0,x} + B_{16}\,\phi_{x,x} + E_{16}\,\mathrm{c}_2\,\phi_{x,x} + E_{16}\,\mathrm{c}_2\,w_{0,xx} + A_{26}\,v_{0,y} + B_{26}\,\phi_{y,y} + E_{26}\,\mathrm{c}_2\,\phi_{y,y} + E_{26}\,\mathrm{c}_2\,w_{0,yy} \\
& + A_{66}\,u_{0,y} + A_{66}\,v_{0,x} + B_{66}\,\phi_{x,y} + E_{66}\,\mathrm{c}_2\,\phi_{y,x} + E_{66}\,\mathrm{c}_2\,\phi_{x,y} + E_{66}\,\mathrm{c}_2\,\phi_{y,x} + 2\,E_{66}\,\mathrm{c}_2\,w_{0,xy}
\end{aligned}
$$

$$
\begin{aligned}
\delta w_0: \quad \mathcal{Q}_{x} = {} & H_{11}\,\mathrm{c}_2^2\,\phi_{x,xx} + H_{11}\,\mathrm{c}_2^2\,w_{0,xxx} + E_{11}\,\mathrm{c}_2\,u_{0,xx} + F_{11}\,\mathrm{c}_2\,\phi_{x,xx} + E_{12}\,\mathrm{c}_2\,v_{0,yx} + F_{12}\,\mathrm{c}_2\,\phi_{y,yx} \\
& + H_{12}\,\mathrm{c}_2^2\,\phi_{y,yx} + H_{12}\,\mathrm{c}_2^2\,w_{0,yyx} + 2\,E_{16}\,\mathrm{c}_2\,u_{0,xy} + 2\,F_{16}\,\mathrm{c}_2\,\phi_{x,xy} + 2\,H_{16}\mathrm{c}_2^2\,\phi_{x,xy} + E_{16}\,\mathrm{c}_2\,u_{0,yx} \\
& + E_{16}\,\mathrm{c}_2\,v_{0,xx} + F_{16}\,\mathrm{c}_2\,\phi_{x,yx} + H_{16}\,\mathrm{c}_2^2\,\phi_{x,yx} + H_{16}\,\mathrm{c}_2^2\,\phi_{y,xx} + 2\,H_{16}\,\mathrm{c}_2^2\,w_{0,xxy} + 2\,E_{26}\,\mathrm{c}_2\,v_{0,yy} \\
& + 2\,F_{26}\,\mathrm{c}_2\,\phi_{y,yy} + 2\,H_{26}\mathrm{c}_2^2\,w_{0,yyy} + 4\,H_{66}\mathrm{c}_2^2\,w_{0,xyy} + 2\,H_{26}\,\mathrm{c}_2^2\,\phi_{x,yy} + 2\,H_{26}\,\mathrm{c}_2^2\,\phi_{y,xy} + 2\,E_{66}\,\mathrm{c}_2\,u_{0,yy} \\
& + 2\,E_{66}\,\mathrm{c}_2\,v_{0,xy} + 2\,F_{66}\,\mathrm{c}_2\,\phi_{x,yy} + 2\,F_{66}\,\mathrm{c}_2\,\phi_{y,xy} - 2\,D_{45}\,\mathrm{c}_1\,\phi_y - 2\,D_{45}\,\mathrm{c}_1\,w_{0,y} - F_{45}\,\mathrm{c}_1^2\,\phi_y \\
& - F_{45}\,\mathrm{c}_1^2\,w_{0,y} - A_{55}\,\phi_x - A_{55}\,w_{0,x} - D_{55}\,\mathrm{c}_1\,\phi_x - 2\,\mathrm{c}_1\,w_{0,x} - F_{55}\,\mathrm{c}_1^2\,\phi_x - F_{55}\,\mathrm{c}_1^2\,w_{0,x}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_{xx} = {} & D_{11}\,\phi_{x,x} + H_{11}\,\mathrm{c}_2^2\,\phi_{x,x} + H_{11}\,\mathrm{c}_2^2\,w_{0,xx} + B_{11}\,u_{0,x} + E_{11}\,\mathrm{c}_2\,u_{0,x} + 2\,F_{11}\,\mathrm{c}_2\,\phi_{x,x} + F_{11}\,\mathrm{c}_2\,w_{0,xx} \\
& + F_{11}\,\mathrm{c}_2\,w_{0,xx} + B_{12}\,v_{0,y} + D_{12}\,\phi_{y,y} + F_{12}\,\mathrm{c}_2\,\phi_{y,y} + F_{12}\,\mathrm{c}_2\,w_{0,yy} + E_{12}\,\mathrm{c}_2\,v_{0,y} + F_{12}\,\mathrm{c}_2\,\phi_{y,y} + H_{12}\,\mathrm{c}_2^2\,\phi_{y,y} \\
& + H_{12}\,\mathrm{c}_2^2\,w_{0,yy} + B_{16}\,u_{0,y} + B_{16}\,v_{0,x} + D_{16}\,\phi_{x,y} + D_{16}\,\phi_{y,x} + F_{16}\,\mathrm{c}_2\,\phi_{x,y} + F_{16}\,\mathrm{c}_2\,\phi_{y,x} + 2\,F_{16}\,\mathrm{c}_2\,w_{0,xy} \\
& + E_{16}\,\mathrm{c}_2\,u_{0,y} + E_{16}\,\mathrm{c}_2\,v_{0,x} + F_{16}\,\mathrm{c}_2\,\phi_{x,y} + F_{16}\,\mathrm{c}_2\,\phi_{y,x} + H_{16}\,\mathrm{c}_2^2\,\phi_{x,y} + H_{16}\,\mathrm{c}_2^2\,\phi_{y,x} + 2\,H_{16}\,\mathrm{c}_2^2\,w_{0,xy}
\end{aligned}
$$

$$
\begin{aligned}
\delta\phi_y: \quad \mathcal{M}_{xy} = {} & D_{16}\,\phi_{x,x} + H_{16}\,\mathrm{c}_2^2\,\phi_{x,x} + H_{16}\,\mathrm{c}_2^2\,w_{0,xx} + B_{16}\,u_{0,x} + E_{16}\,\mathrm{c}_2\,u_{0,x} + 2\,F_{16}\,\mathrm{c}_2\,\phi_{x,x} + F_{16}\,\mathrm{c}_2\,w_{0,xx} \\
& + F_{16}\,\mathrm{c}_2\,w_{0,xx} + B_{26}\,v_{0,y} + D_{12}\,\phi_{y,y} + F_{26}\,\mathrm{c}_2\,\phi_{y,y} + F_{26}\,\mathrm{c}_2\,w_{0,yy} + E_{26}\,\mathrm{c}_2\,v_{0,y} + F_{26}\,\mathrm{c}_2\,\phi_{y,y} + H_{26}\,\mathrm{c}_2^2\,\phi_{y,y} \\
& + H_{26}\,\mathrm{c}_2^2\,w_{0,yy} + B_{66}\,u_{0,y} + B_{66}\,v_{0,x} + D_{66}\,\phi_{x,y} + D_{66}\,\phi_{y,x} + F_{66}\,\mathrm{c}_2\,\phi_{x,y} + F_{66}\,\mathrm{c}_2\,\phi_{y,x} + 2\,F_{66}\,\mathrm{c}_2\,w_{0,xy} \\
& + E_{66}\,\mathrm{c}_2\,u_{0,y} + E_{66}\,\mathrm{c}_2\,v_{0,x} + F_{66}\,\mathrm{c}_2\,\phi_{x,y} + F_{66}\,\mathrm{c}_2\,\phi_{y,x} + H_{66}\,\mathrm{c}_2^2\,\phi_{x,y} + H_{66}\,\mathrm{c}_2^2\,\phi_{y,x} + 2\,H_{66}\,\mathrm{c}_2^2\,w_{0,xy}
\end{aligned}
\tag{10}
$$

$$\delta\phi_x : \quad \delta w_{0,x} : \quad \mathcal{P}_{xx} = H_{11}\,c_2^2\,\phi_{x,x} + H_{11}\,c_2^2\,w_{0,xx} + E_{11}\,c_2\,u_{0,x} + F_{11}\,c_2\,\phi_{x,x} + E_{12}\,c_2\,v_{0,y} + F_{12}\,c_2\,\phi_{y,y} + H_{12}\,c_2^2\,\phi_{y,y} + H_{12}\,c_2^2\,w_{0,yy} + E_{16}\,c_2\,u_{0,y} + E_{16}\,c_2\,v_{0,x} + F_{16}\,c_2\,\phi_{x,y} + F_{16}\,c_2\,\phi_{y,x} + H_{16}\,c_2^2\,\phi_{x,y} + H_{16}\,c_2^2\,\phi_{y,x} + 2\,H_{16}\,c_2^2\,w_{0,xy}$$

where the suffix after the comma denotes the partial derivative with respect to that variable and

$$\begin{aligned}(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) &= \sum_{k=1}^{N_l} \int_{z^k} \tilde{C}_{ij}^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right) dz \\ (I_0, I_1, I_2, I_3, I_4, I_6) &= \sum_{k=1}^{N_l} \int_{z^k} \rho^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right) dz\end{aligned} \tag{11}$$

are laminate stiffnesses and rotatory inertial terms, respectively with i and j varying form 1 to 6. The in-plane loadings can be defined as $\tilde{N}_{x_0} = \lambda\, N_{x_0}$ and $\tilde{N}_{y_0} = \lambda\, N_{y_0}$, where N_{x_0}, N_{y_0} are the initial in-plane loadings and λ is a scalar load factor, c_1 has already been defined (see Eq. (1)) and $c_2 = -\frac{4}{h^2}$.

2.2. Dynamic stiffness formulation

Once the equations of motion and the natural boundary conditions, i.e., Eqs. (9) and (10) above are obtained, the classical method to carry out an exact buckling analysis of a plate consists of (i) solving the system of differential equations in Navier or Lèvy-type closed form in an exact manner, (ii) applying particular boundary conditions on the edges and finally (iii) obtaining the stability equation by eliminating the integration constants [41, 42, 43, 44]. This method, although extremely useful for analysing an individual plate, it lacks generality and cannot be easily applied to complex structures assembled from plates for which researchers usually resort to approximate methods such as the FEM. In this respect, the dynamic stiffness method (DSM), which is, in many ways, analogous to FEM has no such limitations and importantly it always retains the exactness of the solution even when applied to complex structures. This is because once the dynamic stiffness matrix of a structural element is obtained from the exact solution of the governing differential equations and it can be offset and/or rotated and assembled in a global DS matrix in the same way as the FEM. This global DS matrix thus contains implicitly all the exact critical buckling loads of the structure which can be computed by using the well established algorithm of Wittrick-Williams [13].

A general procedure to develop the dynamic stiffness matrix of a structural element is generally summarized as follows:

(i) Seek a closed form analytical solution of the governing differential equations of the structural element.

(ii) Apply a number of general boundary conditions in algebraic forms that are equal to twice the number of integration constants; these are usually nodal displacements and forces.

(iii) Eliminate the integration constants by relating the amplitudes of the harmonically varying nodal forces to those of the corresponding displacements which essentially generates the dynamic stiffness matrix, providing the force-displacement relationship at the nodes of the structural element.

Referring to the equations of motion Eqs.(9), an exact solution can be found in Lèvy's form for symmetric, cross ply laminates. For such laminates $\boldsymbol{B} = \boldsymbol{E} = 0$, and $\tilde{C}_{16}^k = \tilde{C}_{26}^k = \tilde{C}_{45}^k = 0$ and the out-of-plane displacements are uncoupled from the in-plane ones.

2.3. Lèvy-type closed form exact solution and DS formulation

The solution of Eqs. (9) related to the out-of-plane displacements is sought as:

$$
\begin{aligned}
&w^0(x,y,t)=\sum_{m=1}^{\infty} W_m(x)\,\mathrm{e}^{i\omega t}\sin(\alpha\,y), \qquad \phi_x(x,y,t)=\sum_{m=1}^{\infty} \Phi_{x_m}(x)\,\mathrm{e}^{i\omega t}\sin(\alpha\,y),\\
&\phi_y(x,y,t)=\sum_{m=1}^{\infty} \Phi_{y_m}(x)\,\mathrm{e}^{i\omega t}\cos(\alpha\,y)
\end{aligned}
\tag{12}
$$

where ω is the unknown circular or angular frequency, $\alpha=\frac{m\,\pi}{L}$ and $m=1,2,\ldots,\infty$. Equation (12) is the so-called Lèvy's solution which assumes that two opposite sides of the plate are simply supported (S-S), i.e. $w=\phi_x=0$ at $y=0$ and $y=L$. Substituting Eq. (12) into Eqs. (9) a set of three ordinary differential equations is derived which can be written in matrix form as follows:

$$
\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13}\\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23}\\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}
\begin{bmatrix} W_m\\ \Phi_x\\ \Phi_y \end{bmatrix}
=\begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}
\tag{13}
$$

where $\mathcal{L}_{ij}$ $(i,j=1,2,3)$ are differential operators and given by:

$$
\begin{aligned}
\mathcal{L}_{11}&=-c_1^2\mathcal{D}_x^4H_{11}+\mathcal{D}_x^2\left(A_{55}+2c_2D_{55}+c_2^2F_{55}+2\alpha^2c_1^2H_{12}+4\alpha^2c_1^2H_{66}+\lambda N_{x0}\right)-\alpha^2\Big(A_{44}+2c_2D_{44}\\
&\quad+c_2^2F_{44}+\alpha^2c_1^2H_{22}+\lambda N_{y0}\Big)\\
\mathcal{L}_{12}&=\mathcal{D}_x^3\left(-c_1F_{11}-c_1^2H_{11}\right)+\mathcal{D}_x\left(A_{55}+2c_2D_{55}+\alpha^2c_1F_{12}+c_2^2F_{55}+2\alpha^2c_1F_{66}+\alpha^2c_1^2H_{12}+2\alpha^2c_1^2H_{66}\right)\\
&\quad+\left(-c_1\,F_{11}-c_1^2\,H_{11}\right)\mathcal{D}_x^3\\
\mathcal{L}_{13}&=-\alpha\left(A_{44}+c_2(2D_{44}+c_2F_{44})+\alpha^2c_1(F_{22}+c_1H_{22})\right)+\mathcal{D}_x^2\left(\alpha c_1F_{12}+2\alpha c_1F_{66}+\alpha c_1^2H_{12}+2\alpha c_1^2H_{66}\right)\\
\mathcal{L}_{21}&=c_1\mathcal{D}_x^3(F_{11}+c_1H_{11})+\mathcal{D}_x\left(-A_{55}-c_2(2D_{55}+c_2F_{55})-\alpha^2c_1(F_{12}+2F_{66}+c_1H_{12}+2c_1H_{66})\right)\\
\mathcal{L}_{22}&=-A_{55}-c_2(2D_{55}+c_2F_{55})+\mathcal{D}_x^2\left(D_{11}+2c_1F_{11}+c_1^2H_{11}\right)-\alpha^2\left(D_{66}+2c_1F_{66}+c_1^2H_{66}\right)\\
\mathcal{L}_{23}&=\mathcal{D}_x\left(-\alpha D_{12}-\alpha D_{66}-2\alpha c_1F_{12}-2\alpha c_1F_{66}-\alpha c_1^2H_{12}-\alpha c_1^2H_{66}\right)\\
\mathcal{L}_{31}&=-\alpha\left(A_{44}+c_2(2D_{44}+c_2F_{44})+\alpha^2c_1(F_{22}+c_1H_{22})\right)+\alpha c_1\mathcal{D}_x^2(F_{12}+2F_{66}+c_1H_{12}+2c_1H_{66})\\
\mathcal{L}_{32}&=\alpha\mathcal{D}_x(D_{12}+D_{66}+c_1(2F_{12}+2F_{66}+c_1(H_{12}+H_{66})))\\
\mathcal{L}_{33}&=-A_{44}-c_2(2D_{44}+c_2F_{44})-\alpha^2(D_{22}+c_1(2F_{22}+c_1H_{22}))+\mathcal{D}_x^2(D_{66}+c_1(2F_{66}+c_1H_{66}))
\end{aligned}
\tag{14}
$$

where $\mathcal{D}_x=\frac{\mathrm{d}}{\mathrm{dx}}$ and A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} have already been defined in Eq. (11). Expanding the determinant of the matrix in Eq. (13) the following differential equation is obtained:

$$
\left(a_1\,\mathcal{D}_x^8+a_2\,\mathcal{D}_x^6+a_3\,\mathcal{D}_x^4+a_4\,\mathcal{D}_x^2+a_5\right)\Psi=0 \tag{15}
$$

where

$$
\Psi=W_m,\ \Phi_{y_m},\ \Phi_{x_m} \tag{16}
$$

Using a trial solution e^{λ} in Eq. (15) yields the following auxiliary equation:

$$
a_1\,\lambda^8+a_2\,\lambda^6+a_3\,\lambda^4+a_4\,\lambda^2+a_5\ =0 \tag{17}
$$

Substituting $\mu=\lambda^2$, the 8^{th} order polynomial of Eq. (17) can be reduced to a quartic as:

$$
a_1\,\mu^4+a_2\,\mu^3+a_3\,\mu^2+a_4\,\mu+a_5=0 \tag{18}
$$

the four roots for the quartic equation are given by:

$$
\begin{aligned}
\mu_1 &= -s_1 - \frac{1}{2}\sqrt{-s_5 + s_2 - \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} - \frac{1}{2}\sqrt{s_9} \\
\mu_2 &= -s_1 + \frac{1}{2}\sqrt{-s_5 + s_2 - \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} - \frac{1}{2}\sqrt{s_9} \\
\mu_3 &= -s_1 - \frac{1}{2}\sqrt{-s_5 + s_2 + \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} + \frac{1}{2}\sqrt{s_9} \\
\mu_4 &= -s_1 + \frac{1}{2}\sqrt{-s_5 + s_2 + \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} + \frac{1}{2}\sqrt{s_9}
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
&s_1 = \frac{a_1}{4\,a_2}, \qquad s_2 = -\frac{4\,a_3}{3\,a_1^2} + \frac{a_2^2}{2\,a_1^2}, \qquad s_3 = 2\,a_3 - 9\,a_2\,a_3\,a_4 - 72\,a_2^2\,a_5 + 27\,a_2^2\,a_5 + 27\,a_1\,a_4^2, \\
&s_4 = a_3^2 - 3\,a_2\,a_4 + 12\,a_1\,a_5, \qquad s_5 = \frac{1}{a_1}\left(\frac{s_3 + \sqrt{s_3 - 4\,s_4}}{32}\right)^{\frac{1}{3}}, \qquad s_6 = \sqrt{2^2}^{\,3}\,s_4, \qquad s_7 = \sqrt{32}^{\,3}\,s_5\,a_1, \\
&s_8 = \left(\frac{a_2}{a_1}\right)^3 \frac{4\,a_2\,a_3}{a_1^2} - \frac{8\,a_4}{a_1}, \qquad s_9 = s_5 + \frac{s_2}{2} + \frac{s_6}{3\,s_7\,a_1}
\end{aligned}
\tag{20}
$$

The explicit form of the polynomials coefficients $a_j \;\; (j = 1, 2, 3, 4, 5)$ are given in Appendix B. Some pair or pairs of complex roots may occur when computing $\mu_j \;\; (j = 1, 2, 3, 4)$, but the amplitude of the displacements $W_m(x)$, $\Phi_{x_m}(x)$, $\Phi_{y_m}(x)$ are all real, whilst the associated coefficients can be complex. As complex roots always occur in conjugate pairs, the associated coefficients will also occur as conjugates. The solution of the system of ordinary differential equations in Eq. (13) can thus be written as:

$$
\begin{aligned}
W_m(x) &= A_1\,\mathrm{e}^{+\mu_1 x} + A_2\,\mathrm{e}^{-\mu_1 x} + A_3\,\mathrm{e}^{+\mu_2 x} + A_4\,\mathrm{e}^{-\mu_2 x} \\
&\quad + A_5\,\mathrm{e}^{+\mu_3 x} + A_6\,\mathrm{e}^{-\mu_3 x} + A_7\,\mathrm{e}^{+\mu_4 x} + A_8\,\mathrm{e}^{-\mu_4 x} \\
\Phi_{x_m}(x) &= B_1\,\mathrm{e}^{+\mu_1 x} + B_2\,\mathrm{e}^{-\mu_1 x} + B_3\,\mathrm{e}^{+\mu_2 x} + B_4\,\mathrm{e}^{-\mu_2 x} \\
&\quad + B_5\,\mathrm{e}^{+\mu_3 x} + B_6\,\mathrm{e}^{-\mu_3 x} + B_7\,\mathrm{e}^{+\mu_4 x} + B_8\,\mathrm{e}^{-\mu_4 x} \\
\Phi_{y_m}(x) &= C_1\,\mathrm{e}^{+\mu_1 x} + C_2\,\mathrm{e}^{-\mu_1 x} + C_3\,\mathrm{e}^{+\mu_2 x} + C_4\,\mathrm{e}^{-\mu_2 x} \\
&\quad + C_5\,\mathrm{e}^{+\mu_3 x} + C_6\,\mathrm{e}^{-\mu_3 x} + C_7\,\mathrm{e}^{+\mu_4 x} + C_8\,\mathrm{e}^{-\mu_4 x}
\end{aligned}
\tag{21}
$$

where $A_1 - A_8$, $B_1 - B_8$, $C_1 - C_8$, are three sets of integration constants. The sets of constants are not all independent. Only one set of eight constants are needed to relate each set. Constants $B_1 - B_8$ are chosen to be the independent base. By substituting Eqs. (21) into (13) the following relationships are obtained using symbolic computation:

$$
\begin{aligned}
&A_1 = \delta_1\,B_1, \qquad A_2 = -\delta_1\,B_2, \qquad C_1 = \gamma_1\,B_1, \qquad C_2 = -\gamma_1\,B_2 \\
&A_3 = \delta_2\,B_3, \qquad A_4 = -\delta_2\,B_4, \qquad C_3 = \gamma_2\,B_3, \qquad C_4 = -\gamma_2\,B_4 \\
&A_5 = \delta_3\,B_5, \qquad A_6 = -\delta_3\,B_6, \qquad C_5 = \gamma_3\,B_5, \qquad C_6 = -\gamma_3\,B_6 \\
&A_7 = \delta_4\,B_7, \qquad A_8 = -\delta_4\,B_8, \qquad C_7 = \gamma_4\,B_7, \qquad C_8 = -\gamma_4\,B_8
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\delta_i = &-\Big[- A_{55}\alpha^2 D_{22} - 2A_{55}c_2D_{44} - 2\alpha^2 c_2 D_{22}D_{55} - 4c_2^2 D_{44}D_{55} - \alpha^4 D_{22}D_{66} - 2\alpha^2 c_2 D_{44}D_{66} - 2A_{55}\alpha^2 c_1 F_{22} \\
&- 4\alpha^2 c_1 c_2 D_{55}F_{22} - 2\alpha^4 c_1 D_{66}F_{22} - A_{55}c_2^2 F_{44} - 2c_2^3 D_{55}F_{44} - \alpha^2 c_2^2 D_{66}F_{44} - \alpha^2 c_2^2 D_{22}F_{55} - 2c_2^3 D_{44}F_{55} \\
&- 2\alpha^2 c_1 c_2^2 F_{22}F_{55} - c_2^4 F_{44}F_{55} - 2\alpha^4 c_1 D_{22}F_{66} - 4\alpha^2 c_1 c_2 D_{44}F_{66} - 4\alpha^4 c_1^2 F_{22}F_{66} - 2\alpha^2 c_1 c_2^2 F_{44}F_{66} \\
&- A_{55}\alpha^2 c_1^2 H_{22} - 2\alpha^2 c_1^2 c_2 D_{55}H_{22} - \alpha^4 c_1^2 D_{66}H_{22} - \alpha^2 c_1^2 c_2^2 F_{55}H_{22} - 2\alpha^4 c_1^3 F_{66}H_{22} - \alpha^4 c_1^2 D_{22}H_{66} - 2\alpha^2 c_1^2 c_2 D_{44}H_{66} \\
&- 2\alpha^4 c_1^3 F_{22}H_{66} - \alpha^2 c_1^2 c_2^2 F_{44}H_{66} - \alpha^4 c_1^4 H_{22}H_{66} - A_{44}\Big(A_{55} + 2c_2D_{55} + c_2^2F_{55} + \alpha^2(D_{66} + 2c_1F_{66} + c_1^2H_{66})\Big) \\
&+ A_{44}(D_{11} + c_1(2F_{11} + c_1H_{11}))\mu_i^2 + \Big(A_{55}(D_{66} + c_1(2F_{66} + c_1H_{66})) + 2c_2(D_{11}D_{44} + D_{55}D_{66} + c_1(2D_{44}F_{11} \\
&+ 2D_{55}F_{66} + c_1D_{44}H_{11} + c_1D_{55}H_{66})) + c_2^2(D_{11}F_{44} + D_{66}F_{55} + c_1(2F_{11}F_{44} + 2F_{55}F_{66} + c_1F_{44}H_{11} + c_1F_{55}H_{66})) \\
&- \alpha^2(D_{12}^2 - D_{11}(D_{22} + c_1(2F_{22} + c_1H_{22})) + 2D_{12}(D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66}))) + c_1(4F_{12}(D_{66} + c_1F_{12}) \\
&- D_{22}(2F_{11} + c_1H_{11}) + c_1(8F_{12}F_{66} + 2D_{66}H_{12} - 2F_{11}(2F_{22} + c_1H_{22}) + c_1(-2F_{22}H_{11} + H_{12}(4(F_{12} + F_{66}) + c_1H_{12}) \\
&- c_1H_{11}H_{22} + 4F_{12}H_{66} + 2c_1H_{12}H_{66})))))\Big)\mu_i^2 - (D_{11} + c_1(2F_{11} + c_1H_{11}))(D_{66} + c_1(2F_{66} + c_1H_{66}))\mu_i^4\Big] \\
&/\Big[\alpha^2(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))\mu_i\Big(A_{44} + 2c_2D_{44} + c_2^2F_{44} + \alpha^2c_1(F_{22} + c_1H_{22}) \\
&- c_1(F_{12} + 2F_{66} + c_1H_{12} + 2c_1H_{66})\mu_i^2\Big) - \mu_i\Big(A_{55} + 2c_2D_{55} + c_2^2F_{55} + \alpha^2c_1(F_{12} + 2F_{66} + c_1H_{12} + 2c_1H_{66}) \\
&- c_1(F_{11} + c_1H_{11})\mu_i^2\Big)\Big(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2(D_{22} + 2c_1F_{22} + c_1^2H_{22}) - (D_{66} + 2c_1F_{66} + c_1^2H_{66})\mu_i^2\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_i = &\Big[\alpha\Big(A_{44}A_{55} + 2A_{55}c_2D_{44} + 2A_{44}c_2D_{55} + 4c_2^2D_{44}D_{55} + A_{44}\alpha^2D_{66} + 2\alpha^2c_2D_{44}D_{66} + A_{55}\alpha^2c_1F_{22} + 2\alpha^2c_1c_2D_{55}F_{22} \\
&+ \alpha^4c_1D_{66}F_{22} + A_{55}c_2^2F_{44} + 2c_2^3D_{55}F_{44} + \alpha^2c_2^2D_{66}F_{44} + A_{44}c_2^2F_{55} + 2c_2^3D_{44}F_{55} + \alpha^2c_1c_2^2F_{22}F_{55} + c_2^4F_{44}F_{55} \\
&+ 2A_{44}\alpha^2c_1F_{66} + 4\alpha^2c_1c_2D_{44}F_{66} + 2\alpha^4c_1^2F_{22}F_{66} + 2\alpha^2c_1c_2^2F_{44}F_{66} + A_{55}\alpha^2c_1^2H_{22} + 2\alpha^2c_1^2c_2D_{55}H_{22} + \alpha^4c_1^2D_{66}H_{22} \\
&+ \alpha^2c_1^2c_2^2F_{55}H_{22} + 2\alpha^4c_1^3F_{66}H_{22} + A_{44}\alpha^2c_1^2H_{66} + 2\alpha^2c_1^2c_2D_{44}H_{66} + \alpha^4c_1^3F_{22}H_{66} + \alpha^2c_1^2c_2^2F_{44}H_{66} + \alpha^4c_1^4H_{22}H_{66} \\
&+ \Big(A_{55}(D_{12} + D_{66}) - A_{44}(D_{11} + c_1(2F_{11} + c_1H_{11})) - 2c_2(D_{11}D_{44} - D_{55}(D_{12} + D_{66}) + c_1(2D_{44}F_{11} - D_{55}F_{12} + c_1D_{44}H_{11} \\
&+ c_1D_{55}H_{66})) + c_2^2(-D_{11}F_{44} + (D_{12} + D_{66})F_{55} - c_1(2F_{11}F_{44} - F_{12}F_{55} + c_1F_{44}H_{11} + c_1F_{55}H_{66})) + c_1(A_{55}(F_{12} - c_1H_{66}) \\
&+ \alpha^2(-D_{11}F_{22} + D_{12}(F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66})) + c_1(2F_{12}^2 - D_{11}H_{22} - 2F_{11}(F_{22} + c_1H_{22}) + F_{12}(4F_{66} + 3c_1H_{12} \\
&+ 4c_1H_{66}) + c_1(H_{12}(2F_{66} + c_1H_{12}) - H_{11}(F_{22} + c_1H_{22}) + 2c_1H_{12}H_{66})))))\Big)\mu_i^2 + c_1(D_{11}(F_{12} + 2F_{66}) - D_{12}(F_{11} + c_1H_{11}) \\
&- D_{66}(F_{11} + c_1H_{11}) + c_1(D_{11}(H_{12} + 2H_{66}) + c_1H_{11}(-F_{12} + c_1H_{66}) + F_{11}(2F_{66} + c_1(H_{12} + 3H_{66}))))\mu_i^4\Big)\Big] \\
&/\Big[\mu_i\Big(A_{44}A_{55} - A_{44}\alpha^2D_{12} + A_{55}\alpha^2D_{22} + 2A_{55}c_2D_{44} - 2\alpha^2c_2D_{12}D_{44} + 2A_{44}c_2D_{55} + 2\alpha^2c_2D_{22}D_{55} + 4c_2^2D_{44}D_{55} \\
&- A_{44}\alpha^2D_{66} - 2\alpha^2c_2D_{44}D_{66} - A_{44}\alpha^2c_1F_{12} + \alpha^4c_1D_{22}F_{12} - 2\alpha^2c_1c_2D_{44}F_{12} + 2A_{55}\alpha^2c_1F_{22} - \alpha^4c_1D_{12}F_{22} + 4\alpha^2c_1c_2D_{55}F_{22} \\
&- \alpha^4c_1D_{66}F_{22} + A_{55}c_2^2F_{44} - \alpha^2c_2^2D_{12}F_{44} + 2c_2^3D_{55}F_{44} - \alpha^2c_2^2D_{66}F_{44} - \alpha^2c_1c_2^2F_{12}F_{44} + A_{44}c_2^2F_{55} + \alpha^2c_2^2D_{22}F_{55} \\
&+ 2c_2^3D_{44}F_{55} + 2\alpha^2c_1c_2^2F_{22}F_{55} + c_2^4F_{44}F_{55} + 2\alpha^4c_1D_{22}F_{66} + 2\alpha^4c_1^2F_{22}F_{66} + \alpha^4c_1^2D_{22}H_{12} + \alpha^4c_1^3F_{22}H_{12} + A_{55}\alpha^2c_1^2H_{22} \\
&- \alpha^4c_1^2D_{12}H_{22} + 2\alpha^2c_1^2c_2D_{55}H_{22} - \alpha^4c_1^2D_{66}H_{22} - \alpha^4c_1^3F_{12}H_{22} + \alpha^2c_1^2c_2^2F_{55}H_{22} + A_{44}\alpha^2c_1^2H_{66} + 2\alpha^4c_1^2D_{22}H_{66} \\
&+ 2\alpha^2c_1^2c_2D_{44}H_{66} + 3\alpha^4c_1^3F_{22}H_{66} + \alpha^2c_1^2c_2^2F_{44}H_{66} + \alpha^4c_1^4H_{22}H_{66} - \Big(A_{55}(D_{66} + c_1(2F_{66} + c_1H_{66})) + c_2^2(D_{66}F_{55} \\
&+ c_1(F_{11}F_{44} + 2F_{55}F_{66} + c_1F_{44}H_{11} + c_1F_{55}H_{66})) + 2c_2(c_1D_{44}(F_{11} + c_1H_{11}) + D_{55}(D_{66} + c_1(2F_{66} + c_1H_{66}))) \\
&+ c_1(A_{44}(F_{11} + c_1H_{11}) + \alpha^2(-F_{12}(D_{12} + 2c_1F_{12}) - 2D_{12}F_{66} + D_{22}(F_{11} + c_1H_{11}) + c_1(-4F_{12}F_{66} + F_{11}(2F_{22} + c_1H_{22}) \\
&- D_{12}(H_{12} + 2H_{66}) + c_1(2F_{22}H_{11} - H_{12}(3F_{12} + 2F_{66} + c_1H_{12}) + c_1H_{11}H_{22} - 4F_{12}H_{66} - 2c_1H_{12}H_{66}))))\Big)\mu_i^2 \\
&+ c_1(F_{11} + c_1H_{11})(D_{66} + c_1(2F_{66} + c_1H_{66}))\mu_i^4\Big)\Big]
\end{aligned}
\tag{23}
$$

with $i = 1, 2, 3, 4$. The procedure leading to Eqs. (22) and (23) must be undertaken with sufficient care, because if wrong equations are chosen from Eq. (21) to obtain the relationship connecting different sets of constant, numerical instability can occur. When Eqs. (22) are substituted into Eqs. (21) a solution in terms of only 8 constants can be formulated for $W_m(x)$, $\Phi_{x_m}(x)$ and $\Phi_{y_m}(x)$, respectively. Thus

$$\begin{aligned}
W_m(x) &= B_1\,\delta_1\,\mathrm{e}^{+\mu_1 x} - B_2\,\delta_1\,\mathrm{e}^{-\mu_1 x} + B_3\,\delta_2\,\mathrm{e}^{+\mu_2 x} - B_4\,\delta_2\,\mathrm{e}^{-\mu_2 x}\\
&+ B_5\,\delta_3\,\mathrm{e}^{+\mu_3 x} - B_6\,\delta_3\,\mathrm{e}^{-\mu_3 x} + B_7\,\delta_4\,\mathrm{e}^{+\mu_4 x} - B_8\,\delta_4\,\mathrm{e}^{-\mu_4 x}\\
\Phi_{x_m}(x) &= B_1\,\mathrm{e}^{+\mu_1 x} + B_2\,\mathrm{e}^{-\mu_1 x} + B_3\,\mathrm{e}^{+\mu_2 x} + B_4\,\mathrm{e}^{-\mu_2 x}\\
&+ B_5\,\mathrm{e}^{+\mu_3 x} + B_6\,\mathrm{e}^{-\mu_3 x} + B_7\,\mathrm{e}^{+\mu_4 x} + B_8\,\mathrm{e}^{-\mu_4 x}\\
\Phi_{y_m}(x) &= B_1\,\gamma_1\,\mathrm{e}^{+\mu_1 x} - B_2\,\gamma_1\,\mathrm{e}^{-\mu_1 x} + B_3\,\gamma_2\,\mathrm{e}^{+\mu_2 x} - B_4\,\gamma_2\,\mathrm{e}^{-\mu_2 x}\\
&+ B_5\,\gamma_3\,\mathrm{e}^{+\mu_3 x} - B_6\,\gamma_3\,\mathrm{e}^{-\mu_3 x} + B_7\,\gamma_4\,\mathrm{e}^{+\mu_4 x} - B_8\,\gamma_4\,\mathrm{e}^{-\mu_4 x}
\end{aligned} \tag{24}$$

The expressions for forces and moments can also be found in the same way by substituting Eqs. (24) into Eqs. (10) and using symbolic computation. In this way

$$\begin{aligned}
Q_x(x,y) = \Big(&\mathrm{e}^{\mu_1 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_1 x})(A_{55} + A_{55}\,\delta_1\,\mu_1 + 2\,c_2\,(D_{55} + D_{55}\,\delta_1\,\mu_1) + c_2^2\,(F_{55} + \delta_1\,F_{55}\,\mu_1)\\
&+ c_1\,(\alpha\,\gamma_1\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_1 - \mu_1^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_1\,H_{11}\,\mu_1) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_1\,H_{12}\,\mu_1 + 4\,c_1\,\delta_1\,H_{66}\,\mu_1)))+\\
&\mathrm{e}^{\mu_2 x}(B_3 + B_4\,\mathrm{e}^{-2\mu_2 x})(A_{55} + A_{55}\,\delta_2\,\mu_2 + 2\,c_2\,(D_{55} + D_{55}\,\delta_2\,\mu_2) + c_2^2\,(F_{55} + \delta_2\,F_{55}\,\mu_2)\\
&+ c_1\,(\alpha\,\gamma_2\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_2 - \mu_2^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_2\,H_{11}\,\mu_2) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_2\,H_{12}\,\mu_2 + 4\,c_1\,\delta_1\,H_{66}\,\mu_2)))+\\
&\mathrm{e}^{\mu_3 x}(B_5 + B_6\mathrm{e}^{-2\mu_3 x})(A_{55} + A_{55}\,\delta_3\,\mu_3 + 2\,c_2\,(D_{55} + D_{55}\,\delta_3\,\mu_3) + c_2^2\,(F_{55} + \delta_3\,F_{55}\,\mu_3)\\
&+ c_1\,(\alpha\,\gamma_3\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_3 - \mu_3^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_3\,H_{11}\,\mu_3) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_3\,H_{12}\,\mu_3 + 4\,c_1\,\delta_3\,H_{66}\,\mu_3)))+\\
&\mathrm{e}^{\mu_4 x}(B_3 + B_4\,\mathrm{e}^{-2\mu_4 x})(A_{55} + A_{55}\,\delta_4\,\mu_4 + 2\,c_2\,(D_{55} + D_{55}\,\delta_4\,\mu_4) + c_2^2\,(F_{55} + \delta_4\,F_{55}\,\mu_4)\\
&+ c_1\,(\alpha\,\gamma_4\,(F_{12} + 2\,F66 + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_4 - \mu_4^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_4\,H_{11}\,\mu_4) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_4\,H_{12}\,\mu_4 + 4\,c_1\,\delta_4\,H_{66}\,\mu_4)))\Big)\,\sin(\alpha\,y) = \mathcal{Q}_x\,\sin(\alpha\,y)
\end{aligned}$$

$$\begin{aligned}
M_{xx}(x,y) = \Big(&\mathrm{e}^{\mu_1 x}(B_1 + B_2\,\mathrm{e}^{-2\mu_1 x})(\alpha^2\,c_1\,\delta_1\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_1\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_1\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_1\,F_{11}\,\mu_1 + c_1\,\delta_1\,H_{11}\,\mu_1)))+\\
&\mathrm{e}^{\mu_2 x}(B_3 + B_4\,\mathrm{e}^{-2\mu_2 x})(\alpha^2\,c_1\,\delta_2\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_2\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_2\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_2\,F_{11}\,\mu_2 + c_1\,\delta_2\,H_{11}\,\mu_2)))+\\
&\mathrm{e}^{\mu_3 x}(B_5 + B_5\,\mathrm{e}^{-2\mu_3 x})(\alpha^2\,c_1\,\delta_3\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_3\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_3\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_3\,F_{11}\,\mu_3 + c_1\,\delta_3\,H_{11}\,\mu_3)))+\\
&\mathrm{e}^{\mu_4 x}\,(B_7 + B_8\,\mathrm{e}^{-2\mu_4 x})(\alpha^2\,c_1\,\delta_4\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_4\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_4\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_4\,F_{11}\,\mu_4 + c_1\,\delta_4\,H_{11}\,\mu_4)))\Big)\,\sin(\alpha\,y) = \mathcal{M}_{xx}\,\sin(\alpha\,y)
\end{aligned}$$

$$
\begin{aligned}
M_{xy}(x,y) = \Big(& \mathrm{e}^{\mu_1 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_1 x})(\gamma_1\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_1 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
& + 2\,\delta_1\,F_{66}\,\mu_1 + 2\,c_1\,\delta_1\,H_{66}\,\mu_1)))+ \\
& \mathrm{e}^{\mu_2 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_2 x})(\gamma_2\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_2 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
& + 2\,\delta_2\,F_{66}\,\mu_1 + 2\,c_1\,\delta_2\,H_{66}\,\mu_2)))+ \\
& \mathrm{e}^{\mu_3 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_3 x})(\gamma_3\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_3 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
& + 2\,\delta_3\,F_{66}\,\mu_3 + 2\,c_1\,\delta_3\,H_{66}\,\mu_2)))+ \\
& \mathrm{e}^{\mu_4 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_4 x})(\gamma_4\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_4 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
& + 2\,\delta_4\,F_{66}\,\mu_1 + 2\,c_1\,\delta_4\,H_{66}\,\mu_4)))\Big)\,\cos(\alpha\,y) = \mathcal{M}_{xy}\,\cos(\alpha\,y)
\end{aligned}
$$

$$
\begin{aligned}
P_{xx}(x,y) = \Big(& \mathrm{e}^{\mu_1 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_1\,x})\,(\alpha^2\,c_1\,\delta_1\,H_{12} + \alpha\,\gamma_1\,(F_{12} + c_1\,H_{12}) - \mu_1\,(F_{11} + c_1\,H_{11}\,(1 + \delta_1\,\mu_1)))+ \\
& \mathrm{e}^{\mu_2 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_2\,x})\,(\alpha^2\,c_1\,\delta_2\,H_{12} + \alpha\,\gamma_2\,(F_{12} + c_1\,H_{12}) - \mu_2\,(F_{11} + c_1\,H_{11}\,(1 + \delta_2\,\mu_2)))+ \\
& \mathrm{e}^{\mu_3 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_3\,x})\,(\alpha^2\,c_1\,\delta_3\,H_{12} + \alpha\,\gamma_3\,(F_{12} + c_1\,H_{12}) - \mu_3\,(F_{11} + c_1\,H_{11}\,(1 + \delta_3\,\mu_3)))+ \\
& \mathrm{e}^{\mu_4 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_4\,x})\,(\alpha^2\,c_1\,\delta_4\,H_{12} + \alpha\,\gamma_4\,(F_{12} + c_1\,H_{12}) - \mu_4\,(F_{11} + c_1\,H_{11}\,(1 + \delta_4\,\mu_4)))\Big) \\
& \sin(\alpha\,y) = \mathcal{P}_{xx}\,\sin(\alpha\,y)
\end{aligned}
\tag{25}
$$

At this point, zero boundary conditions are generally used to eliminate the constants when using the classical method which establishes the stability equation for a single individual plate. By contrast, the development of the dynamic stiffness matrix entails imposition of general boundary conditions in algebraic form and widens the possibility of the analysis of multi-plate systems. In order to develop the dynamic stiffness matrix, the following boundary conditions are applied next.

$$
\begin{aligned}
x = 0\ : \qquad & W_m = W_{m_1}\,,\ \Phi_{x_m} = \Phi_{x_1}\,,\ \Phi_{y_m} = \Phi_{y_1}\,,\ W_{m,x} = W_{m1,x} \\
x = b\ : \qquad & W_m = W_{m_2}\,,\ \Phi_{x_m} = \Phi_{x_2}\,,\ \Phi_{y_m} = \Phi_{y_2}\,,\ W_{m,x} = W_{m2,x}
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
x = 0\ : \qquad & \mathcal{Q}_x = -\mathcal{Q}_{x_1}\,,\ \mathcal{M}_{xx} = -\mathcal{M}_{xx_1}\,,\ \mathcal{M}_{xy} = -\mathcal{M}_{xy_1}\,,\ \mathcal{P}_{xx} = -\mathcal{P}_{xx_1} \\
x = b\ : \qquad & \mathcal{Q}_x = \quad \mathcal{Q}_{x_2}\,,\ \mathcal{P}_{xx} = \quad \mathcal{P}_{xx_2}\,,\ \mathcal{M}_{xy} = \quad \mathcal{M}_{xy_2}\,,\ \mathcal{P}_{xx} = \quad \mathcal{P}_{xx_2}
\end{aligned}
$$

By substituting Eq. (26) into Eq.(24), the following matrix relations for the displacements are obtained:

$$
\begin{bmatrix} W_1 \\ \Phi_{x_1} \\ \Phi_{y_1} \\ W_{1,x} \\ W_2 \\ \Phi_{x_2} \\ \Phi_{y_2} \\ W_{2,x} \end{bmatrix}
=
\begin{bmatrix}
\delta_1 & -\delta_1 & \delta_2 & -\delta_2 & \delta_3 & -\delta_3 & \delta_4 & -\delta_4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\gamma_1 & -\gamma_1 & \gamma_2 & -\gamma_2 & \gamma_3 & -\gamma_3 & \gamma_4 & -\gamma_4 \\
f_1 & -f_1 & f_2 & -f_2 & f_3 & -f_3 & f_4 & -f_4 \\
\delta_1\,\mathrm{e}^{b\,\mu_{o1}} & -\delta_1\,\mathrm{e}^{-b\,\mu_{o1}} & \delta_2\,\mathrm{e}^{b\,\mu_{o2}} & -\delta_2\,\mathrm{e}^{-b\,\mu_{o2}} & \delta_3\,\mathrm{e}^{b\,\mu_{o3}} & -\delta_3\,\mathrm{e}^{-b\,\mu_{o3}} & \delta_4\,\mathrm{e}^{b\,\mu_{o4}} & -\delta_4\,\mathrm{e}^{-b\,\mu_{o4}} \\
\mathrm{e}^{b\,\mu_{o1}} & -\mathrm{e}^{-b\,\mu_{o1}} & \mathrm{e}^{b\,\mu_{o2}} & -\mathrm{e}^{-b\,\mu_{o2}} & \mathrm{e}^{b\,\mu_{o3}} & -\mathrm{e}^{-b\,\mu_{o3}} & \mathrm{e}^{b\,\mu_{o4}} & -\mathrm{e}^{-b\,\mu_{o4}} \\
\gamma_1\,\mathrm{e}^{b\,\mu_{o1}} & -\gamma_1\,\mathrm{e}^{-b\,\mu_{o1}} & \gamma_2\,\mathrm{e}^{b\,\mu_{o2}} & -\gamma_2\,\mathrm{e}^{-b\,\mu_{o2}} & \gamma_3\,\mathrm{e}^{b\,\mu_{o3}} & -\gamma_3\,\mathrm{e}^{-b\,\mu_{o3}} & \gamma_4\,\mathrm{e}^{b\,\mu_{o4}} & -\gamma_4\,\mathrm{e}^{-b\,\mu_{o4}} \\
f_1\,\mathrm{e}^{b\,\mu_{o1}} & -f_1\,\mathrm{e}^{-b\,\mu_{o1}} & f_2\,\mathrm{e}^{b\,\mu_{o2}} & -f_2\,\mathrm{e}^{-b\,\mu_{o2}} & f_3\,\mathrm{e}^{b\,\mu_{o3}} & -f_3\,\mathrm{e}^{-b\,\mu_{o3}} & f_4\,\mathrm{e}^{b\,\mu_{o4}} & -f_4\,\mathrm{e}^{-b\,\mu_{o4}}
\end{bmatrix}
\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix}
\tag{27}
$$

where

$$
f_i = \delta_i\,\mu_i; \quad \text{with}\, i = 1,2,3,4
$$

Equations (34) and (27) can be written as

$$
\boldsymbol{\delta} = \boldsymbol{A}\,\boldsymbol{C} \tag{28}
$$

By applying the same procedure for forces and moments, i.e. substituting Eq. (26) into Eq.(25) the following matrix relations are obtained:

$$\begin{bmatrix} \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{P}_{xx_1} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \\ \mathcal{M}_{xx_2} \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{Q}_4 \\ \mathcal{T}_1 & -\mathcal{T}_1 & \mathcal{T}_2 & -\mathcal{T}_2 & \mathcal{T}_3 & -\mathcal{T}_3 & \mathcal{T}_4 & -\mathcal{T}_4 \\ -\mathcal{I}_1 & -\mathcal{I}_1 & -\mathcal{I}_2 & -\mathcal{I}_2 & -\mathcal{I}_3 & -\mathcal{I}_3 & -\mathcal{I}_4 & -\mathcal{I}_4 \\ \mathcal{L}_1 & -\mathcal{L}_1 & \mathcal{L}_2 & -\mathcal{L}_2 & \mathcal{L}_3 & -\mathcal{L}_3 & \mathcal{Y}_L & -\mathcal{L}_4 \\ \mathcal{Q}_1\,\mathrm{e}^{b\,\mu_{o1}} & -\mathcal{Q}_1\,\mathrm{e}^{-b\,\mu_{o1}} & \mathcal{Q}_2\,\mathrm{e}^{b\,\mu_{o2}} & -\mathcal{Q}_2\,\mathrm{e}^{-b\,\mu_{o2}} & \mathcal{Q}_3\,\mathrm{e}^{b\,\mu_{o3}} & -\mathcal{Q}_3\,\mathrm{e}^{-b\,\mu_{o3}} & \mathcal{Q}_4\,\mathrm{e}^{b\,\mu_{o4}} & -\mathcal{Q}_4\,\mathrm{e}^{-b\,\mu_{o4}} \\ -\mathcal{T}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{T}_1\mathrm{e}^{-b\,\mu_{o1}} & -\mathcal{T}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{T}_2\mathrm{e}^{-b\,\mu_{o2}} & -\mathcal{T}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{T}_3\mathrm{e}^{-b\,\mu_{o3}} & -\mathcal{T}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{T}_4\mathrm{e}^{-b\,\mu_{o4}} \\ \mathcal{I}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{I}_1\mathrm{e}^{-b\,\mu_{o1}} & \mathcal{I}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{I}_2\mathrm{e}^{-b\,\mu_{o2}} & \mathcal{I}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{I}_3\mathrm{e}^{-b\,\mu_{o3}} & \mathcal{I}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{I}_4\mathrm{e}^{-b\,\mu_{o4}} \\ -\mathcal{L}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{L}_1\mathrm{e}^{-b\,\mu_{o1}} & -\mathcal{L}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{L}_2\mathrm{e}^{-b\,\mu_{o2}} & -\mathcal{L}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{L}_3\mathrm{e}^{-b\,\mu_{o3}} & -\mathcal{L}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{L}_4\mathrm{e}^{-b\,\mu_{o4}} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix} \tag{29}$$

where

$$\begin{aligned} \mathcal{Q}_i &= -A_{55}(1+\delta_i\mu_{oi}) - 2c_2(D_{55}+D_{55}\delta_i\mu_{oi}) - c_2^2(F_{55}+\delta_i F_{55}\mu_{oi}) - c_1(\alpha\gamma_i(F_{12}+2F_{66}+c_1H_{12} \\ &\quad + 2c_1H_{66})\mu_{oi} - \mu_{oi}^2(F_{11}+c_1H_{11}+c_1\delta_iH_{11}\mu_{oi}) + \alpha^2(2F_{66}+2c_1H_{66}+c_1\delta_iH_{12}\mu_{oi}+4c_1\delta_iH_{66}\mu_{oi})) \\[2ex] \mathcal{T}_i &= \alpha^2c_1\delta_i(F_{12}+c_1H_{12}) + \alpha\gamma_i(D_{12}+c_1(2F_{12}+c_1H_{12})) - \mu_{oi}(D_{11}+c_1(2F_{11}+c_1H_{11} \\ &\quad + \delta_iF_{11}\mu_{oi}+c_1\delta_iH_{11}\mu_{oi})) \\[2ex] \mathcal{I}_i &= \gamma_1(D_{66}+c_1(2F_{66}+c_1H_{66}))\mu_{oi} - \alpha(D_{66}+c_1(2F_{66}+c_1H_{66}+2\delta_iF_{66}\mu_{oi}+2c_1\delta_iH_{66}\mu_{oi})) \\[2ex] \mathcal{L}_i &= c_1(\alpha^2c_1\delta_iH_{12}+\alpha\gamma_i(F_{12}+c_1H_{12}) - \mu_{oi}(F_{11}+c_1H_{11}(1+\delta_i\mu_{oi}))) \qquad \text{with} \quad i=1,2,3,4 \end{aligned} \tag{30}$$

Equation (29) can be written as

$$\boldsymbol{F} = \boldsymbol{R}\,\boldsymbol{C} \tag{31}$$

By eliminating the constants vector $\boldsymbol{C}$ form Eqs. (28) and (31) the dynamic stiffness matrix is formulated as follows:

$$\boldsymbol{K} = \boldsymbol{R}\,\boldsymbol{A}^{-1} \tag{32}$$

or more explicitly

$$\boldsymbol{K} = \begin{bmatrix} s_{qq} & s_{qm} & s_{qt} & s_{qh} & f_{qq} & f_{qm} & f_{qt} & f_{qh} \\ & s_{mm} & s_{mt} & s_{mh} & -f_{qm} & f_{mm} & f_{mt} & f_{mh} \\ & & s_{tt} & s_{th} & f_{qt} & -f_{mt} & f_{tt} & f_{th} \\ & & & s_{hh} & -f_{qh} & f_{mh} & -f_{th} & f_{hh} \\ & & Sym & & s_{qq} & -s_{qm} & s_{qt} & -s_{qh} \\ & & & & & s_{mm} & -s_{mt} & s_{mh} \\ & & & & & & s_{tt} & -s_{th} \\ & & & & & & & s_{hh} \end{bmatrix} \tag{33}$$

Finally the dynamic stiffness matrix related to the force and displacement vectors can be written as follows:

$$\begin{bmatrix} \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{P}_{xx_1} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \\ \mathcal{P}_{xx_2} \end{bmatrix} = \begin{bmatrix} s_{qq} & s_{qm} & s_{qt} & s_{qh} & f_{qq} & f_{qm} & f_{qt} & f_{qh} \\ & s_{mm} & s_{mt} & s_{mh} & -f_{qm} & f_{mm} & f_{mt} & f_{mh} \\ & & s_{tt} & s_{th} & f_{qt} & -f_{mt} & f_{tt} & f_{th} \\ & & & s_{hh} & -f_{qh} & f_{mh} & -f_{th} & f_{hh} \\ & & \text{Sym} & & s_{qq} & -s_{qm} & s_{qt} & -s_{qh} \\ & & & & & s_{mm} & -s_{mt} & s_{mh} \\ & & & & & & s_{tt} & -s_{th} \\ & & & & & & & s_{hh} \end{bmatrix} \begin{bmatrix} W_1 \\ \Phi_{x_1} \\ \Phi_{y_1} \\ W_{1,x} \\ W_2 \\ \Phi_{x_2} \\ \Phi_{y_2} \\ W_{2,x} \end{bmatrix} \quad (34)$$

which in compact matrix form:

$$\boldsymbol{F} = \boldsymbol{K}\,\boldsymbol{D} \quad (35)$$

The above dynamic stiffness matrix will now be used in conjunction with the Wittrick-Williams algorithm [13] to analyze composite simple and stiffened plates for their buckling behavior based on HSDT. Explicit expressions for each element of the DS matrix were obtained via symbolic computation, but they are far too extensive and voluminous to report here. The correctness of these expressions was further checked by implementing them in a MATLAB® program and carrying out a wide rage of numerical simulations.

2.4. Assembly procedure, boundary conditions and similarities with FEM

Once the DS matrix of a laminate element has been developed, it can be rotated and/or offset if required and thus can be assembled to form the global DS matrix of the final structure. The assembly procedure is schematically shown in Fig. 2 which is similar to that of FEM. Although like the FEM, a mesh is required in the DSM, it should be noted that unlike the former, the latter is not mesh dependent in the sense that additional elements are required only when there is a change in the geometry of the structure. A single DS laminate element is enough to compute any number of its buckling loads to any desired accuracy, which, of course, is impossible in the FEM. However, for the type of structures under consideration DS plate elements do not have point nodes, but have line nodes instead. Also no change in the geometry along longitudinal direction is admitted. This assumption is in addition to the assumed simple support boundary conditions on two opposite sides. The other two sides of the plate can have any boundary conditions. The application of the boundary conditions of the global dynamic stiffness matrix involves the use of the so-called penalty method. This consists of adding a large stiffness to the appropriate position on the leading diagonal term which corresponds to the degree of freedom of the node that needs to be suppressed. It is thus possible to apply free, simple support and clamped boundary conditions on the structure by penalizing the appropriate degrees of freedom. Note that in accordance with the notation and sign convention used in Fig. 2 for simple support boundary condition V, W and Φ_y have to be penalized whereas for clamped boundary condition U, V, W, Φ_y, Φ_x, W,x have to be penalized. Clearly for the free-edge boundary condition no penalty will be applied. Because of the similarities between DSM and FEM, DS elements can be implemented in FEM codes and thus the accuracy of results can be increased substantially.

2.5. Application of the Wittrick-Williams Algorithm

In order to compute the critical buckling loads of a structure by using the DSM, an efficient way to solve the eigen-problem is to apply the Wittrick and Williams algorithm [13] which has featured in literally hundreds of papers. For the sake of completeness the procedure is briefly summarized as follows. First the global dynamic stiffness matrix of the final structure K^* is computed for an arbitrarily chosen trial critical buckling load λ^*. Next, by applying the usual form of Gauss elimination the global stiffness matrix, is transformed into its upper triangular $K^{*^{\triangle}}$ form. The number of negative terms on the leading diagonal of $K^{*^{\triangle}}$ is now defined as the sign count $s(K^*)$ which forms the fundamental basis of the algorithm. In its simplest form, the algorithm states that j, the number of critical buckling loads (λ) of a structures that lie below an arbitrarily chosen trial buckling load (λ^*) is given by:

$$j = j_0 + s(K^*) \tag{36}$$

where j_0 is the number of critical buckling loads of all single strip elements within the structure which are still lower than the trial buckling load (λ^*) when their opposite sides are fully clamped. It is necessary to account for this clamped-clamped critical buckling loads because exact buckling analysis using DSM allows an infinity number of critical buckling loads to be accounted for when all the nodes of the structures are fully clamped.(i.e. in the overall formulation $\boldsymbol{K}\,\boldsymbol{\delta} = 0$, these critical buckling loads correspond to $\boldsymbol{\delta} = 0$ modes.) Thus j_0 is an integral part of the algorithm and is not a peripheral issue. However, j_0 is usually zero and the dominant term of the algorithm is the sign-count $s(K^*)$ of Eq. (36). One way of avoiding the computation of troublesome j_0 is to split the structure into sufficient number of elements so that the clamped-clamped buckling loads of an individual element in the structure are never exceeded. Once $s(K^*)$ and j_0 of Eq. (36) are known, any suitable method, for example, bi-section technique can be devised to bracket any critical buckling load within any desired accuracy. The mode shapes are routinely computed by using the standard eigen-solution procedure in which the global dynamic stiffness matrix is computed at the critical buckling load and the force vector is set to zero whilst deleting one row of the DS matrix and giving one of the nodal displacement component an arbitrarily chosen value and determining the rest of the displacements in terms of the chosen one.

3. Results ans Discussion

A preliminary validation of the critical buckling load analysis for moderately thick ($a/h = 10$) simply-supported cross-ply square plates uniaxially loaded in the x direction is carried out and the results are shown in Table 1 for different orthotropy ratos. The dimensionless critical buckling load, obtained using HSDT within the framework of the DSM are in excellent agreement when compared with the 3D elasticity solution and the results also lead to the same findings of the classical Lèvy-type closed form solution. Note that for all practical purpose, it is only the first buckling load that matters. Therefore only the first critical loads is presented in this paper. As expected the percentage error, with respect to the 3D elasticity solution, increases when increasing the orthotropic ratio. In Table 2 the dimensionless critical buckling load for the same case study of Table 1 is computed but taking into account the effects of the length-to-thickness and orthotropy ratios and boundary conditions (see Fig. 3). At a fixed length-to-thickness ratio, the dimensionless critical buckling load increases when increasing the orthotropic ratio for all the considered boundary conditions. A similar behavior can be observed when varying the length-to-thickness ratio but by fixing the orthotropic ratio. Understandably, the largest dimensionless critical buckling load is given by the boundary condition S-C-S-C and the lowest by S-F-S-F. In Table 3 the results are given for composite plates that are uniaxially loaded in the y direction, instead of the x direction for different values of length-to-thickness and orthotropy ratios. The dimensionless critical buckling load is generally lower for all the boundary conditions but for for the case with one or two sides free, namely, S-S-S-F and S-F-S-F, it decreases significantly. The biaxial compression effect is examined and the results in Table 4 show as expected, a notable reduction in the critical buckling load, with respect to the uniaxial load considered along x and y axes, respectively. In Table 5, results for moderately thick plate (a/h=10) with the effect of the in-plane ratio (L/b) are presented for different values of orthotropy ratio. As can be seen from this table, the critical buckling load decreases when increasing the in-plane ratio, independently for all boundary conditions.

3.1. Buckling analysis of cross-ply composite stiffened plates

A particular feature of the DSM is that it allows the buckling analysis of stiffened plates in an exact sense. Two different stiffened composite plate configurations shown in Fig. 5 and Fig. 6 respectively, are analyzed and the results are discussed in this section. The geometrical parameters of the two stiffened composite plates are given below:

1. **First stiffened composite plate configuration** (Figure 5)
   $b = 1\,\text{m}$; $L = 1\,\text{m}$; $b_1 = 0.20\,\text{m}$; $b_2 = 0.5\,\text{m}$; $b_3 = 0.10\,\text{m}$; $t_1 = 0.10\,\text{m}$; $t_2 = 0.20\,\text{m}$;
2. **Second stiffened composite plate configuration** (Figure 6)
   $b = 1\,\text{m}$; $L = 1\,\text{m}$; $b_1 = 0.15\,\text{m}$; $b_2 = 0.5\,\text{m}$; $b_3 = 0.20\,\text{m}$; $t_1 = 0.10\,\text{m}$; $t_2 = 0.20\,\text{m}$;

In addition to the above stiffened panels, a simple uniform panel of thickness 0.002 m and with the same of the dimensions has also been analyzed for comparative purposes. In Fig. 7 the dimensionless critical buckling load for different boundary conditions are shown against the length-to-thickness ratio for the two stiffened composite plates alongside the results of the simple panel of uniform thickness. The panels were loaded in the x direction. As can be seen from the figure, the introduction of the stiffeners increases the buckling loads very considerably, particularly for thin composite plates. On the contrary, when thick composite plates are analyzed the increase is not so prominent. Clearly, the results are dependent on the applied boundary conditions. Indeed, in the case of S-S-S-S boundary condition the use of stiffeners increases the critical buckling load for both of the stiffened plate configurations. By contrast, using a S-C-S-C boundary condition does not affect the critical buckling load so significantly. It is interesting to note that the relative advantages of using the first or the second stiffened composite plate configurations depend on the applied boundary conditions, although the differences in the dimensionless buckling loads between the two configurations are sometimes rather small. In particular, when applying S-S-S-S, S-S-S-C and S-C-S-C boundary conditions, the first stiffened composite plate configuration leads to higher buckling load. Whilst, applying S-S-S-F, S-F-S-F and S-C-S-F the highest buckling loads are reached with the second stiffened panel configuration. Finally in Figs. 8-10 the representative mode shapes corresponding to simple composite plates, first and second configurations of stiffened composite plates are respectively shown. When presenting modes, square symmetric cross-ply plates made of five layers and uniaxially loaded along the y direction are considered. It is now possible to provide an overview of how the mode shapes change when changing the geometrical characteristics of the composite plate assemblies. It should be noted that the mode shape of a S-S-S-S simple composite plate of uniform thickness related to its critical buckling load is made up of two half-waves in the x direction and one in the y direction. On the contrary, in either of the two stiffened composite plates the mode shape is characterized by only one half-wave in both direction x and y. Other changes in the mode shapes related to different boundary conditions can be observed in the same figures. The results are particularly useful when controlling the mode shapes which generally have significant impact on response.

4. Concluding Remarks

An exact dynamic stiffness theory for composite plate elements using higher order shear deformation theory is developed for the first time using the principle of minimum potential energy and symbolic algebra to carry out buckling analysis in an exact sense. The theory is implemented in a computer program to carry out buckling analysis of complex composite structures modelled as plate assemblies. The proposed theory is a significant refinement over other dynamic stiffness theories using classical plate theory and/or first order shear deformation theory. The developed DSM model is particularly useful when analyzing thick composite plates with moderate to high orthotropic ratios for which the FEM may become unreliable. A detailed parametric study has been carried out by varying significant plate parameters and boundary conditions. The results have been critically examined and the theory has been assessed using existing theories. Two different stepped composite plate configurations have been analyzed for their stability behavior. Based on the computed results the following conclusions can be drawn:

- The exact HSDT plate element has been shown to be extremely accurate in terms of results and computational efficiency when carrying out buckling analysis of composite plate assemblies.
- The exact HSDT plate element provides a significant refinement over to the FSDT element particularly when thick plate with a high orthotropy ratio are analyzed.
- The boundary conditions affect conspicuously the buckling modes.
- Stiffeners, if properly introduced increase the buckling load considerably.
- The buckling load of stiffened composite plates change prominently with respect to the simple composite plate depending on the stiffeners position and on the applied boundary conditions.

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Tables

Table 1: Dimensionless uniaxial buckling load (along x direction) $\mathrm{N}_{cr} = \bar{\mathrm{N}}_{cr}\,\frac{b^2}{E_2\,h^3}$, for simply supported cross-ply square plates with $b/h = 10$, $E_1/E_2 =$ open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Stacking Sequence	Models	E_1/E_2					
$[0°/90°/90°/0°]$		3		20		40	
	3D-Elasticity [45]	5.304	$\Delta^{\dagger}_{\mathrm{3D}}\%$	15.019	$\Delta_{\mathrm{3D}}\%$	22.881	$\Delta_{\mathrm{3D}}\%$
Classical Lèvy's solution	HSDT	5.393	(1.68)	15.298	(1.86)	23.340	(2.01)
	FSDT	5.399	(1.79)	15.351	(2.21)	23.453	(2.50)
	CLPT	5.754	(8.48)	19.712	(31.2)	36.160	(58.0)
DSM	HSDT	5.393	(1.68)	15.298	(1.86)	23.340	(2.01)

† $\Delta_{\mathrm{3D}}\% = \frac{\hat{\omega}-\hat{\omega}_{\mathrm{3D}}}{\hat{\omega}_{\mathrm{3D}}} \times 100$.

Table 2: Dimensionless uniaxial buckling load (along x direction) $\mathrm{N}_{cr} = \bar{\mathrm{N}}_{cr} \frac{b^2}{E_2 h^3}$, for simply supported cross-ply square plates, stacking sequence [0°/90°/90°/0°] and E_1/E_2 = open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Uniaxial Compression (UA-C)							
		b/h					
E_1/E_2	BCs	2	5	10	25	50	100
3	S-S-S-S	1.9557	4.5458	5.3933	5.6928	5.7384	5.7499
	S-S-S-F	1.5912	3.3640	4.3496	5.4623	6.2621	6.8168
	S-S-S-C	1.9821	5.4309	7.1169	7.8176	7.9304	7.9592
	S-F-S-F	1.5839	3.2200	4.1087	5.1745	6.0137	6.7261
	S-C-S-F	1.5954	3.3805	4.3651	5.4686	6.2624	6.8599
	S-C-S-C	2.0555	6.8502	10.273	12.012	12.314	12.392
10	S-S-S-S	2.2810	7.1554	9.9406	11.209	11.420	11.473
	S-S-S-F	2.0267	5.1516	6.9540	8.6366	9.7745	10.536
	S-S-S-C	2.3426	8.3895	14.133	17.865	18.587	18.778
	S-F-S-F	2.0040	4.5982	6.0871	7.9721	9.8185	11.836
	S-C-S-F	2.0746	7.1554	7.0981	9.0614	10.724	12.305
	S-C-S-C	2.5134	9.9930	20.515	30.068	32.273	32.878
20	S-S-S-S	2.5689	9.4219	15.298	18.825	19.482	19.654
	S-S-S-F	2.3144	6.7586	9.8649	12.398	14.130	15.513
	S-S-S-C	2.6643	10.376	20.977	31.075	33.495	34.166
	S-F-S-F	2.2874	5.8285	8.1129	10.893	13.785	17.359
	S-C-S-F	2.4425	7.0390	10.416	13.613	16.334	19.282
	S-C-S-C	2.9945	11.723	28.670	52.336	59.682	61.870
40	S-S-S-S	3.0749	11.997	23.340	33.131	35.347	35.953
	S-S-S-F	2.6827	8.7789	14.723	19.374	22.292	24.982
	S-S-S-C	3.1136	12.301	29.414	54.096	62.190	64.645
	S-F-S-F	2.6558	7.5209	11.386	15.744	20.251	26.398
	S-C-S-F	2.9731	9.4057	15.987	21.955	26.517	31.952
	S-C-S-C	3.5992	13.357	37.045	87.668	110.86	118.82

Table 3: Dimensionless uniaxial buckling load (along y direction) $\mathrm{N}_{cr} = \bar{\mathrm{N}}_{cr} \frac{b^2}{E_2 h^3}$, for simply supported cross-ply square plates, stacking sequence [0°/90°/90°/0°] and E_1/E_2 = open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Uniaxial Compression (UA-C)							
		b/h					
E_1/E_2	BCs	2	5	10	25	50	100
3	S-S-S-S	1.7611	4.5458	5.3933	5.6928	5.7384	5.7499
	S-S-S-F	0.9288	1.4363	1.5677	1.6136	1.6219	1.6247
	S-S-S-C	1.8052	5.0418	7.0967	8.0978	8.2713	8.3164
	S-F-S-F	0.6575	0.9484	1.0136	1.0338	1.0369	1.0377
	S-C-S-F	1.0287	1.7630	2.0138	2.1123	2.1309	2.1371
	S-C-S-C	1.8552	5.3906	8.0926	9.6793	9.9819	10.062
10	S-S-S-S	1.8421	6.0365	9.2387	10.877	11.161	11.234
	S-S-S-F	1.0769	1.9529	2.2269	2.3238	2.3402	2.3452
	S-S-S-C	1.9124	6.4505	10.676	13.497	14.069	14.222
	S-F-S-F	0.8416	1.4919	1.6799	1.7416	1.7508	1.7531
	S-C-S-F	1.2723	2.6410	3.2331	3.4805	3.5247	3.5380
	S-C-S-C	1.9998	6.9366	12.551	17.665	18.940	19.301
20	S-S-S-S	1.9042	7.2304	12.728	16.249	16.923	17.100
	S-S-S-F	1.1977	2.5793	3.1263	3.3311	3.3647	3.3741
	S-S-S-C	2.0019	7.6623	14.591	20.721	22.214	22.634
	S-F-S-F	0.9876	2.1487	2.5906	2.7494	2.7737	2.7798
	S-C-S-F	1.4691	3.6342	4.8200	5.3662	5.4629	5.4902
	S-C-S-C	2.1313	8.2008	16.897	27.735	31.291	32.391
40	S-S-S-S	1.9744	8.5588	18.034	26.395	28.285	28.801
	S-S-S-F	1.3172	3.5410	4.7817	5.3152	5.4037	5.4272
	S-S-S-C	2.1210	8.9972	20.099	33.690	37.996	39.317
	S-F-S-F	1.1294	3.1544	4.2674	4.7367	4.8124	4.8317
	S-C-S-F	1.5871	5.0774	7.6226	9.0324	9.2956	9.3669
	S-C-S-C	2.3293	9.5963	22.614	44.455	54.516	58.136

Table 4: Dimensionless biaxial buckling load $\text{N}_{cr} = \bar{\text{N}}_{cr} \frac{b^2}{E_2\, h^3}$, for simply supported cross-ply square plates, stacking sequence $[0°/90°/90°/0°]$ and $E_1/E_2 =$ open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Biaxial Compression (BA-C)							
		b/h					
E_1/E_2	BCs	2	5	10	25	50	100
3	S-S-S-S	1.1023	2.2729	2.6967	2.8464	2.8692	2.8750
	S-S-S-F	0.7380	1.1497	1.2769	1.3521	1.3852	1.4067
	S-S-S-C	1.1933	2.9361	3.8771	4.2793	4.3446	4.3614
	S-F-S-F	0.6667	0.9694	1.0398	1.0645	1.0700	1.0724
	S-C-S-F	0.7982	1.3461	1.5636	1.7234	1.8048	1.8605
	S-C-S-C	1.3088	3.8427	5.7863	6.8039	6.9824	7.0288
10	S-S-S-S	1.2983	3.5777	4.9703	5.6045	5.7098	5.7368
	S-S-S-F	0.8614	1.5689	1.8270	1.9946	2.0841	2.1516
	S-S-S-C	1.3940	4.4697	7.6189	9.7570	10.177	10.289
	S-F-S-F	0.8433	3.5777	1.6939	1.7622	1.7749	1.7794
	S-C-S-F	0.9938	2.0208	2.5026	2.9029	3.1589	3.3695
	S-C-S-C	1.5359	5.4652	9.8391	13.602	14.490	14.737
20	S-S-S-S	1.4291	4.7109	7.6492	9.4124	9.7410	9.8271
	S-S-S-F	0.9639	2.0760	2.5673	2.8809	3.0531	3.1949
	S-S-S-C	1.5490	5.5364	11.204	18.325	16.920	18.717
	S-F-S-F	0.9876	2.1483	2.5978	2.7668	2.7959	2.8048
	S-C-S-F	1.1486	2.7954	3.7130	4.4580	4.9542	5.4076
	S-C-S-C	1.7409	6.6190	15.867	29.452	33.783	35.082
40	S-S-S-S	1.5757	2.8554	11.670	16.566	17.673	17.976
	S-S-S-F	1.0707	15.582	3.9151	4.5807	4.9204	5.2192
	S-S-S-C	1.7588	6.7056	15.582	29.298	33.968	35.400
	S-F-S-F	1.1290	4.7504	4.2561	4.7504	4.8338	4.8566
	S-C-S-F	1.3103	3.9332	5.8466	7.3603	8.3119	9.2536
	S-C-S-C	2.0693	7.7486	20.486	49.045	62.622	67.338

Table 5: Dimensionless uniaxial buckling load (along x direction) $\mathrm{N}_{cr} = \bar{\mathrm{N}}_{cr} \frac{b^2}{E_2 h^3}$, for simply supported cross-ply square plates, stacking sequence $[0°/90°/90°/0°]$ and E_1/E_2 = open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$ and $b/h = 10$.

Uniaxial Compression (UA-C)							
		BCs					
E_1/E_2	L/b	S-S-S-S	S-S-S-F	S-S-S-C	S-F-S-F	S-C-S-F	S-C-S-C
3	0.5	18.183	12.767	19.233	12.776	12.794	21.724
	1	5.3933	4.3495	7.1169	4.1087	4.3651	10.273
	2	2.8193	0.9613	4.8157	1.4112	1.6563	8.1889
	3	2.4587	0.4922	4.4819	0.8937	1.2931	7.8872
10	0.5	28.622	18.999	29.988	19.095	19.232	28.622
	1	9.9406	6.9540	14.133	6.0871	7.0981	20.515
	2	6.9366	1.3132	11.576	2.0381	3.3398	18.337
	3	6.5724	0.7331	11.259	1.4060	2.9245	18.074
20	0.5	37.688	25.547	38.880	25.650	25.796	41.504
	1	15.298	9.8649	20.977	8.1129	10.416	28.670
	2	11.650	1.7286	17.929	2.6008	5.4332	26.143
	3	11.270	0.9827	17.609	1.8724	4.9325	25.888
40	0.5	47.531	34.730	47.734	34.765	34.798	48.892
	1	23.340	14.723	29.414	11.386	15.987	37.045
	2	18.414	2.4441	25.291	3.4376	9.0464	33.705
	3	17.982	1.3694	24.928	2.5440	8.3704	33.416

Figures

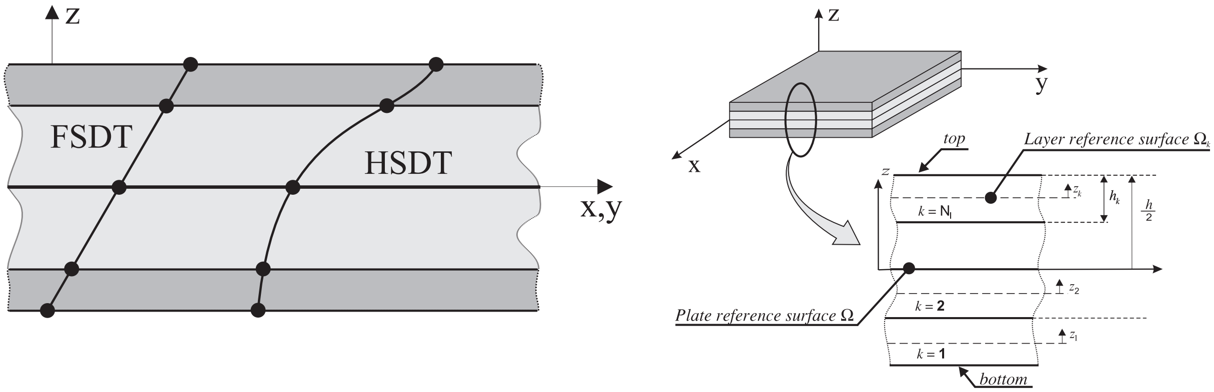


Figure 1: Kinematic descriptions of FSDT and HSDT for a multilayered plates.

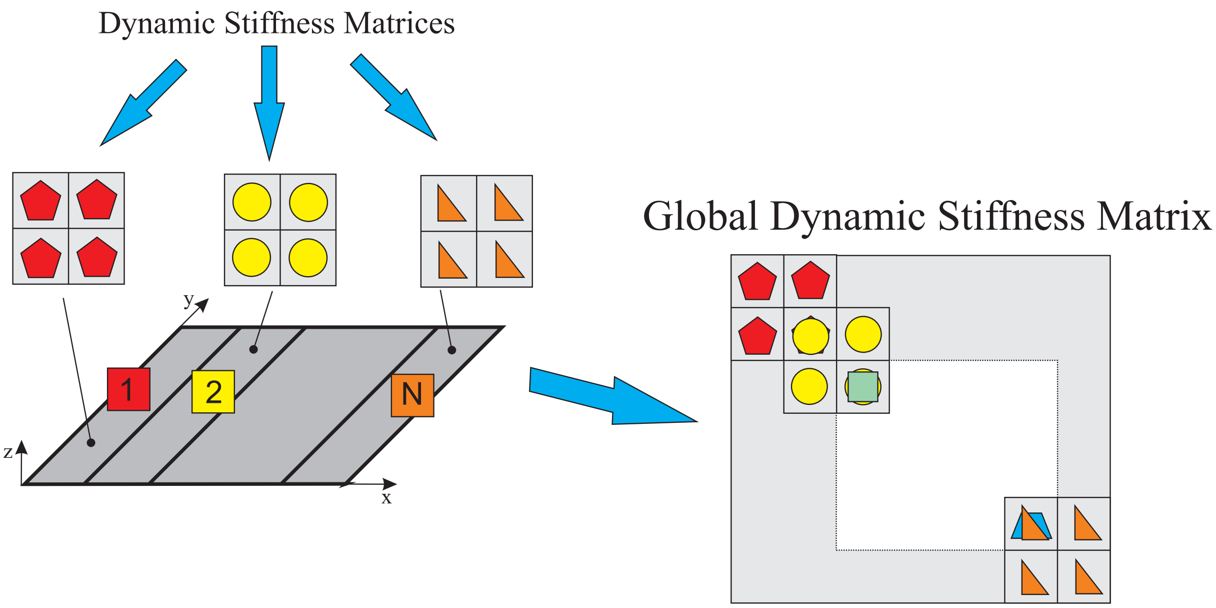


Figure 2: Direct assembly of dynamic stiffness elements.

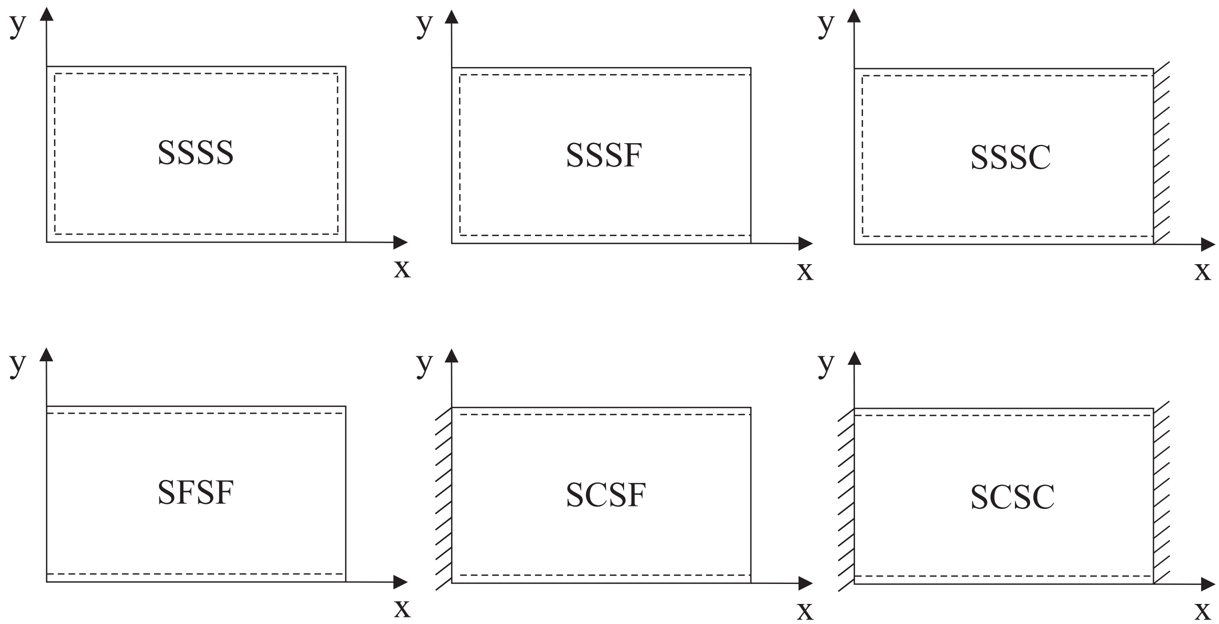


Figure 3: Boundary conditions.

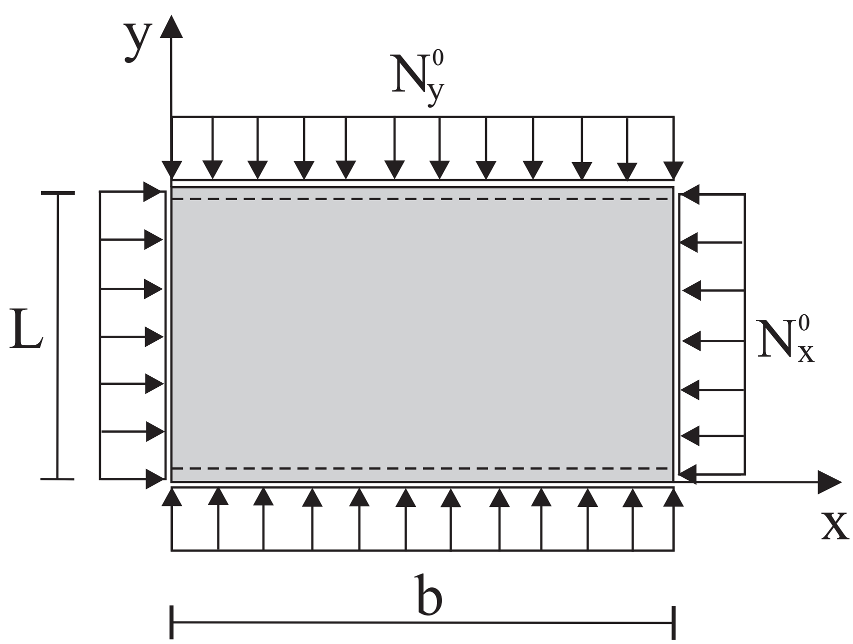


Figure 4: Laminated composite plate under in-plane loadings.

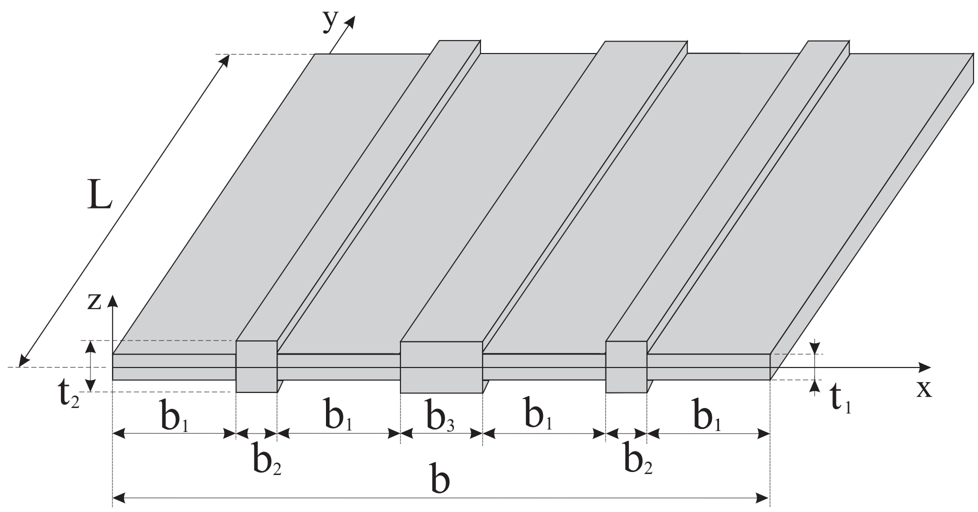


Figure 5: 1st stiffened composite plate configuration.

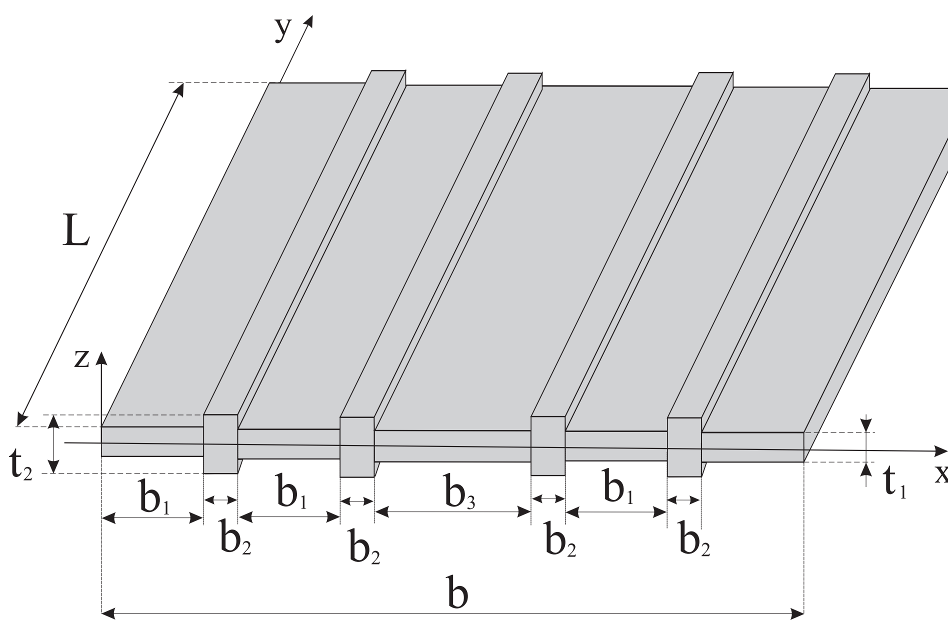
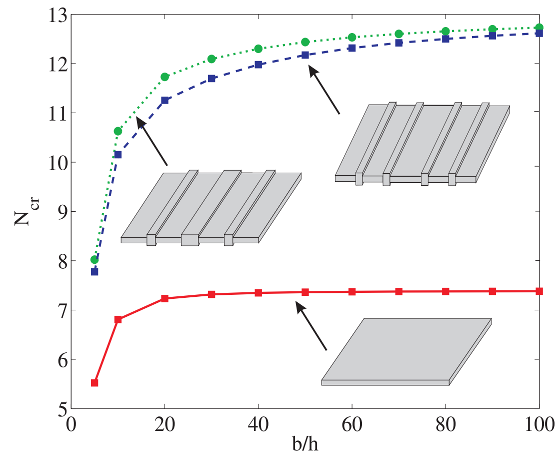
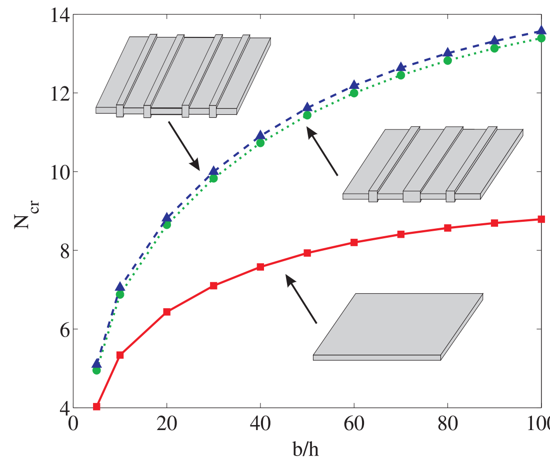


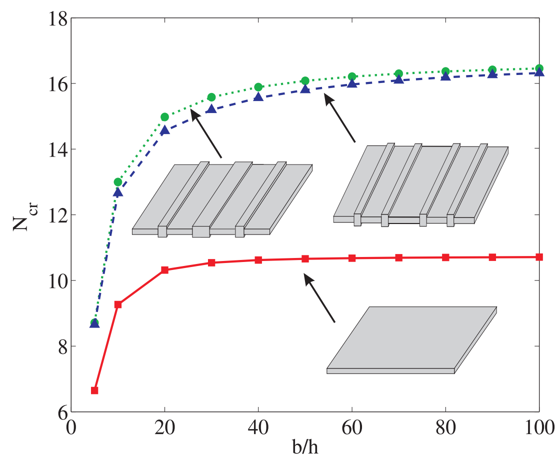
Figure 6: 2nd stiffened composite plate configuration.



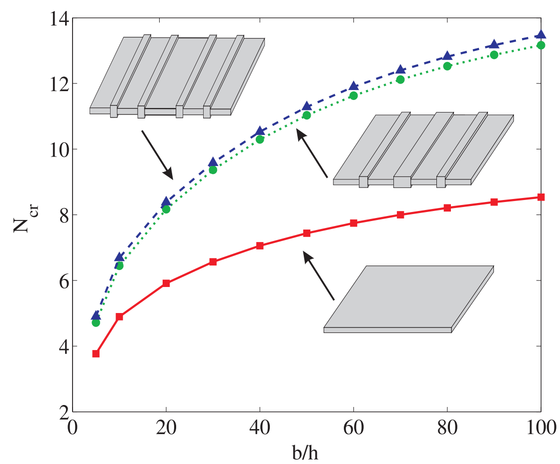
(a) S-S-S-S



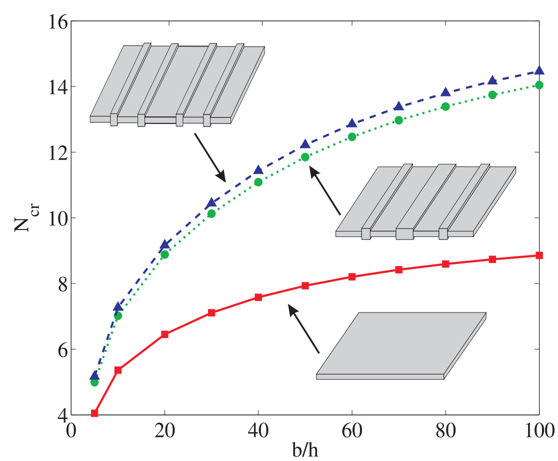
(b) S-S-S-F



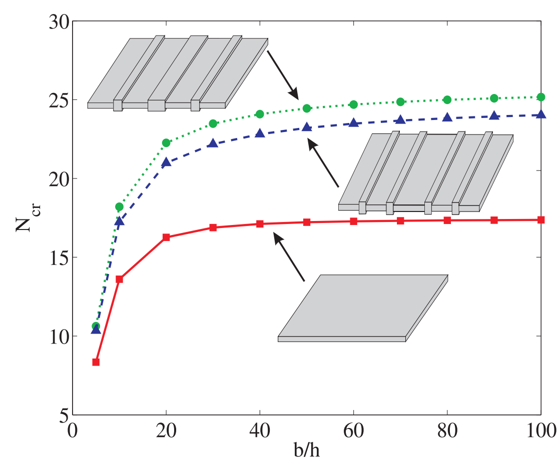
(c) S-S-S-C



(d) S-F-S-F

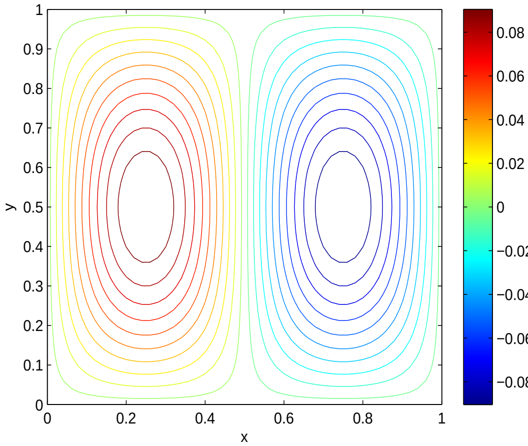


(e) S-C-S-F

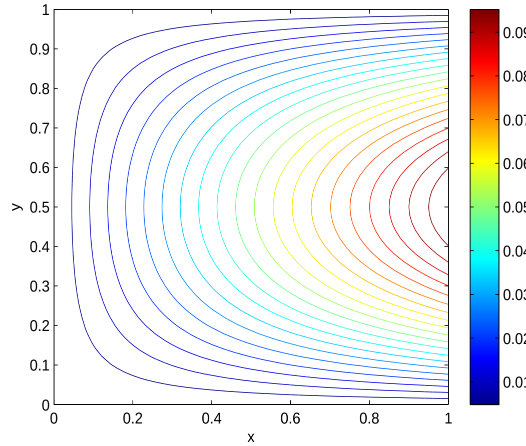


(f) S-C-S-C

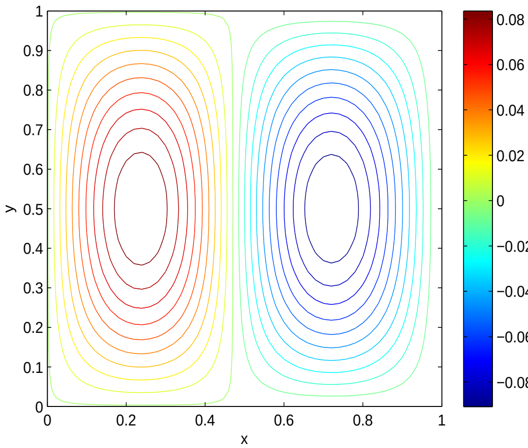
Figure 7: Dimensionless uniaxial buckling loads $N_{cr} = \bar{N}_{cr} \frac{b^2}{E_2\, h^3}$ along the x direction varying the length-to-thickness ratio for simple and stiffened cross-ply plates, stacking sequence $[0°/90°/0°/90°/0°]$, step ratio $t_2/t_1 = 2$ and $E_1/E_2 = 5$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.



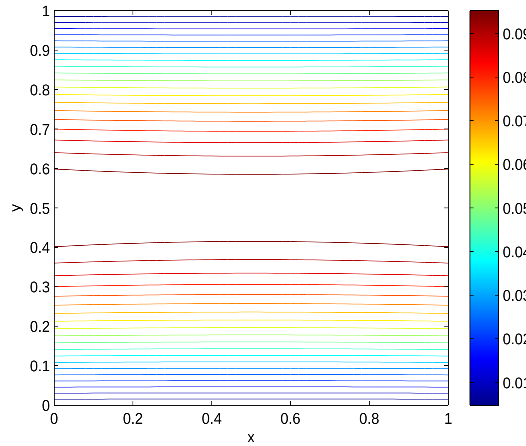
(a) $N_{cr} = 17.025$, S-S-S-S



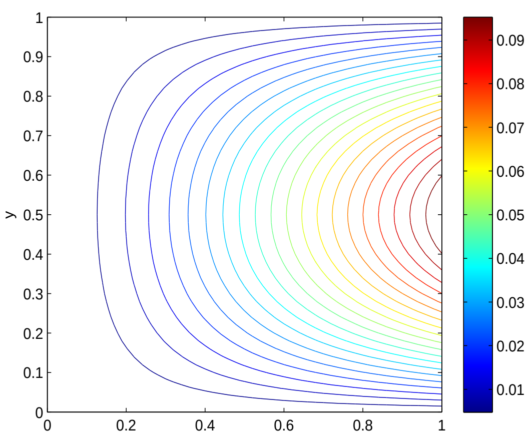
(b) $N_{cr} = 4.8539$, S-S-S-F



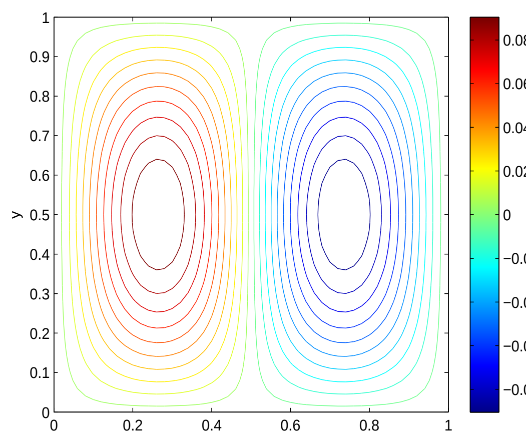
(c) $N_{cr} = 19.086$, S-S-S-C



(d) $N_{cr} = 4.3424$, S-F-S-F

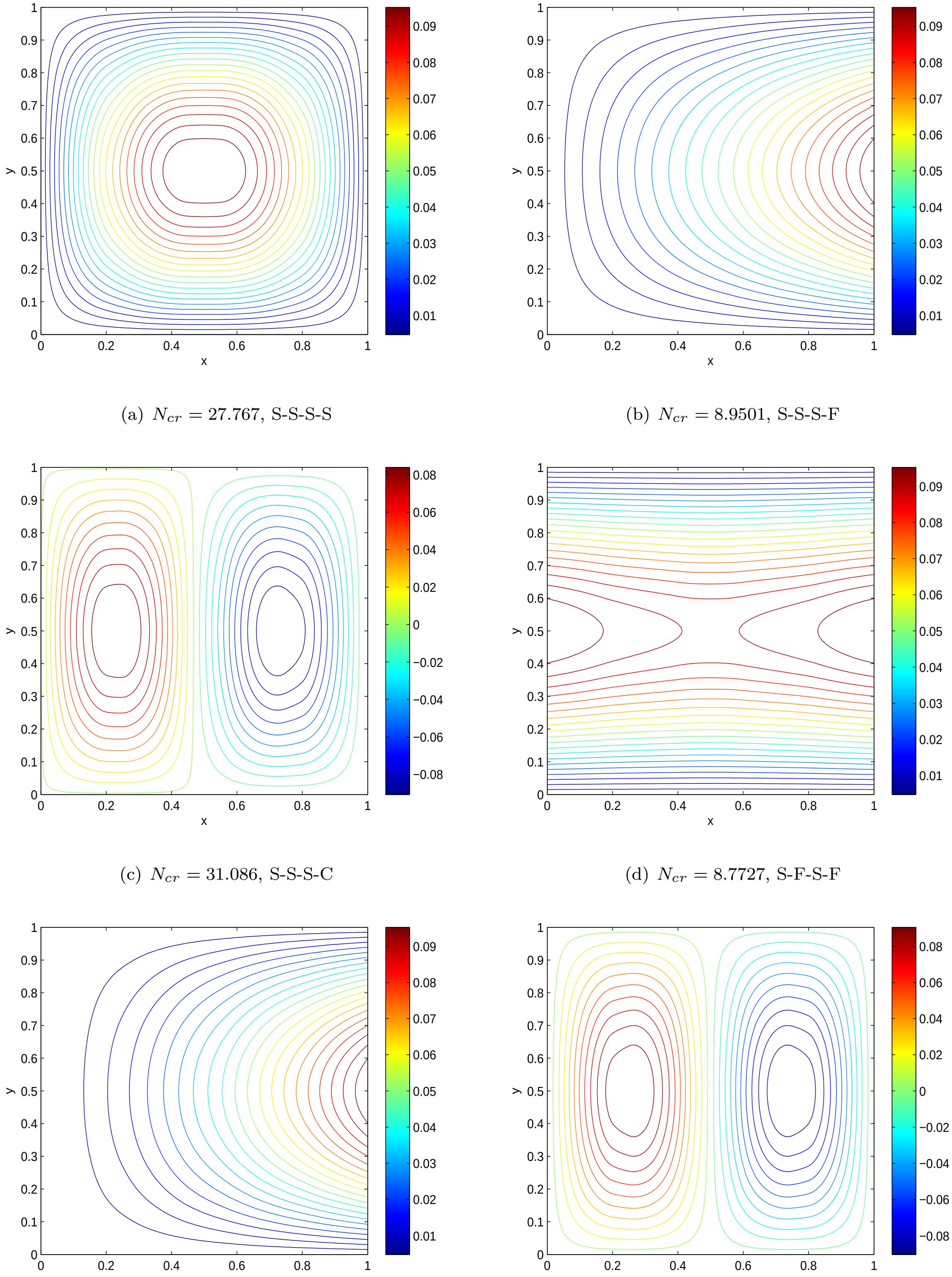


(e) $N_{cr} = 6.7430$, S-C-S-F



(f) $N_{cr} = 21.640$, S-C-S-C

Figure 8: First buckling modes of simple cross-ply plates, lamination scheme [0°/90°/0°/90°/0°] under uniaxial compression along the y direction, length-to-thickness ratio b/h=10 and orthotropic ratio E_1/E_2=25.



(a) $N_{cr} = 27.767$, S-S-S-S

(b) $N_{cr} = 8.9501$, S-S-S-F

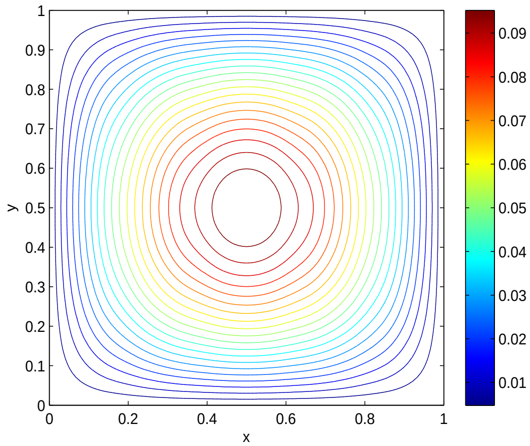
(c) $N_{cr} = 31.086$, S-S-S-C

(d) $N_{cr} = 8.7727$, S-F-S-F

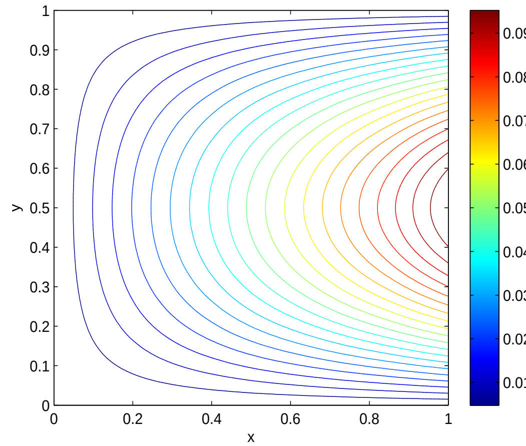
(e) $N_{cr} = 10.769$, S-C-S-F

(f) $N_{cr} = 33.907$, S-C-S-C

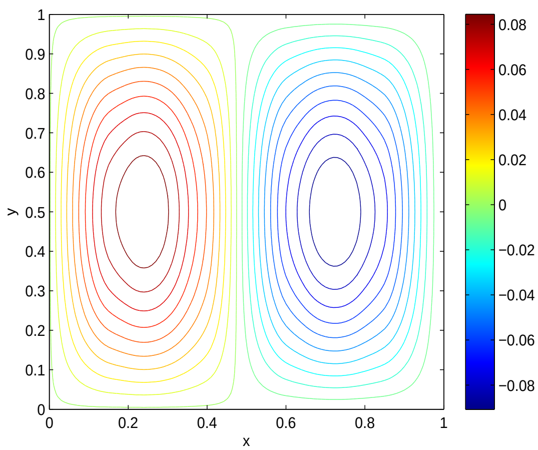
Figure 9: First buckling modes for the first stiffened cross-ply plate configurations Fig. 5, lamination scheme [0°/90°/0°/90°/0°] under uniaxial compression along the y direction, length-to-thickness ratio b/h=10 and orthotropic ratio E_1/E_2=25.



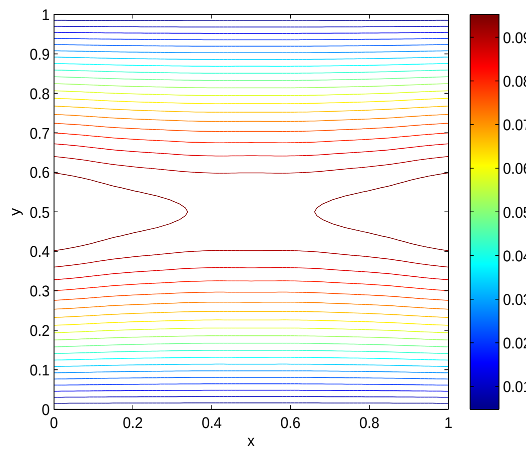
(a) $N_{cr} = 26.320$, S-S-S-S



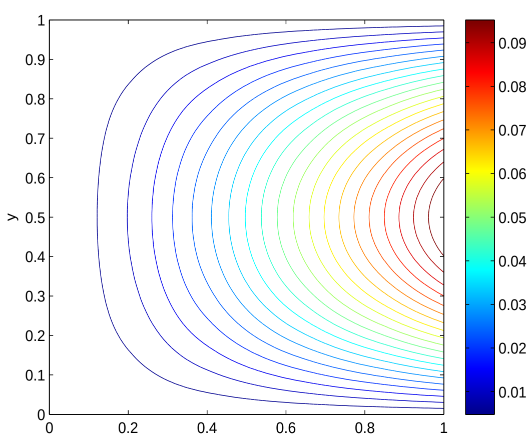
(b) $N_{cr} = 9.3288$, S-S-S-F



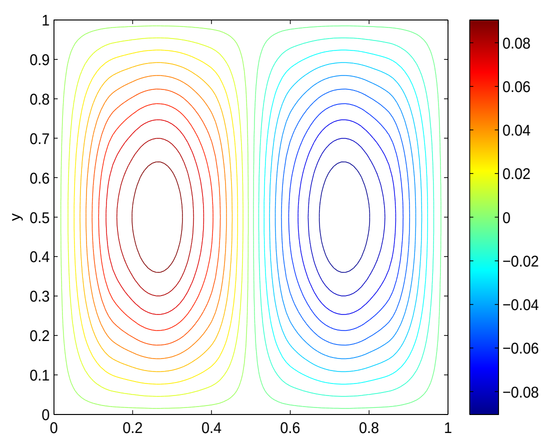
(c) $N_{cr} = 29.371$, S-S-S-C



(d) $N_{cr} = 8.9209$, S-F-S-F



(e) $N_{cr} = 11.373$, S-C-S-F



(f) $N_{cr} = 32.012$, S-C-S-C

Figure 10: First buckling modes for the second stiffened cross-ply plate configurations Fig. 6, lamination scheme under [0°/90°/0°/90°/0°] uniaxial compression along the y direction, length-to-thickness ratio b/h=10 and orthotropic ratio E_1/E_2=25.

Appendix A. Laminate Geometric and Constitutive Equations

The geometric relation for a lamina in the local or lamina reference system can be written as:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x & 0 & 0 \\ 0 & \mathcal{D}_y & 0 \\ \mathcal{D}_y & \mathcal{D}_x & 0 \\ 0 & \mathcal{D}_z & \mathcal{D}_y \\ \mathcal{D}_z & 0 & \mathcal{D}_x \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{A.1}$$

and in terms of the functional degrees of freedom:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x & 0 & \left(c_1\, z^3\right) \mathcal{D}_{xx} & \left(z + c_1\, z^3\right) \mathcal{D}_x & 0 \\ 0 & \mathcal{D}_y & \left(c_1\, z^3\right) \mathcal{D}_{yy} & 0 & \left(z + c_1\, z^3\right) \mathcal{D}_y \\ \mathcal{D}_y & \mathcal{D}_x & \left(c_1\, z^3\, \mathcal{D}_{xy} + c_1\, z^3\, \mathcal{D}_{yx}\right) & \left(z + c_1\, z^3\right) \mathcal{D}_y & \left(z + c_1\, z^3\right) \mathcal{D}_x \\ 0 & 0 & \left(1 + 3\, c_1\, z^2\right) \mathcal{D}_y & 0 & \left(1 + 3\, c_1\, z^2\right) \\ 0 & 0 & \left(1 + 3\, c_1\, z^2\right) \mathcal{D}_x & \left(1 + 3\, c_1\, z^2\right) & 0 \end{bmatrix} \begin{bmatrix} u^0 \\ v^0 \\ w^0 \\ \phi_x \\ \phi_y \end{bmatrix} \tag{A.2}$$

where $\mathcal{D}_x$ and $\mathcal{D}_y$ are the derivatives in x and y respectively and $c_1 = -\frac{4}{3\,h^2}$. The constitutive equations in the lamina reference system can be written, in terms of reduced stiffness coefficients, as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 & 0 & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & 0 & 0 & 0 \\ 0 & 0 & \tilde{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \tilde{C}_{44} & 0 \\ 0 & 0 & 0 & 0 & \tilde{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} \tag{A.3}$$

where the $\tilde{C}_{ij}$ are expressed in terms of stiffness coefficients C_{ij}, as:

$$\begin{aligned} &\tilde{C}_{11} = C_{11} - \frac{C_{13}^2}{C_{33}} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \qquad \tilde{C}_{12} = C_{12} - \frac{C_{13}\,C_{23}}{C_{33}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \qquad \tilde{C}_{22} = C_{22} - \frac{C_{23}^2}{C_{33}} = \frac{E_2}{1-\nu_{12}\nu_{21}} \\ &\tilde{C}_{44} = C_{44} = G_{23}, \qquad \tilde{C}_{55} = C_{55} = G_{13} \qquad \tilde{C}_{66} = C_{66} = G_{12} \end{aligned} \tag{A.4}$$

where E_1 is the elastic modulus in the fibre direction, E_2 the elastic modulus in perpendicular to the fibre, ν_{12} and $\nu_{21} = \nu_{12}\,E_2/E_1$ the Poisson's ratios, $G_{12} = G_{13}$ and G_{23} the shear modulus of each single orthotropic lamina. If the lamina is placed at an angle θ in the laminate or global reference system, the equation need to be transformed as follows:

$$\begin{aligned} \overline{C}_{11} =& \tilde{C}_{11}\mathcal{C}^4 + 2(\tilde{C}_{12} + 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{22}\mathcal{S}^4 \\ \overline{C}_{12} =& (\tilde{C}_{11} + \tilde{C}_{22} - 4\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{12}(\mathcal{S}^4 + \mathcal{C}^4) \\ \overline{C}_{16} =& (\tilde{C}_{11} - \tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}\mathcal{C}^3 + (\tilde{C}_{12} - \tilde{C}_{22} + 2\tilde{C}_{66})\mathcal{S}^3 \\ \overline{C}_{22} =& \tilde{C}_{11}\mathcal{S}^4 + 2(\tilde{C}_{12} + 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{22}\mathcal{C}^4 \\ \overline{C}_{26} =& (\tilde{C}_{11} - \tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}^3\mathcal{C} + (\tilde{C}_{12} - \tilde{C}_{22} + 2\tilde{C}_{66})\mathcal{S}\mathcal{C}^3 \\ \overline{C}_{66} =& (\tilde{C}_{11} + \tilde{C}_{22} - 2\tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{66}(\mathcal{S}^4 + \mathcal{C}^4) \\ \overline{C}_{44} =& \tilde{C}_{44}\mathcal{C}^2 + \tilde{C}_{55}\mathcal{S}^2 \\ \overline{C}_{55} =& \tilde{C}_{44}\mathcal{S}^2 + \tilde{C}_{55}\mathcal{C}^2 \\ \overline{C}_{45} =& (\tilde{C}_{55} - \tilde{C}_{44})\mathcal{C}\mathcal{S} \end{aligned} \tag{A.5}$$

where $\mathcal{C} = \cos(\theta)$ and $\mathcal{S} = \sin(\theta)$. This leads to the constitutive equation for the k-th lamina in the laminate or global reference system:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} & 0 & 0 \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} & 0 & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{A.6}$$

that in compact form can be written for each k-th lamina as:

$$\boldsymbol{\sigma}^k = \overline{\boldsymbol{C}}^k \boldsymbol{\varepsilon}^k \tag{A.7}$$

Appendix B. Polynomial Coefficients

The polynomial coefficients are following defined:

$$a_1 = c_1^2\,(F_{11}^2 - D_{11}\,H_{11})(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))$$

$$a_2 = -c_1^2\,(A_{44} + \alpha^2 D_{22})\,F_{11}^2 + 2\alpha^2 c_1^2 D_{12} F_{11} F_{12} - \alpha^2 c_1^2 D_{11} F_{12}^2 + 2\alpha^2 c_1^3 F_{11} F_{12}^2 - 2\alpha^2 c_1^3 F_{11}^2 F_{22} + 4\alpha^2 c_1^2 D_{12}$$
$$F_{11} F_{66} - 4\alpha^2 c_1^2 D_{11} F_{12} F_{66} - 4\alpha^2 c_1^2 D_{11} F_{66}^2 - 8\alpha^2 c_1^3 F_{11} F_{66}^2 + A_{44} c_1^2 D_{11} H_{11} - \alpha^2 c_1^2 D_{12}^2 H_{11} + \alpha^2 c_1^2 D_{11} D_{22} H_{11}$$
$$- 2\alpha^2 c_1^2 D_{12} D_{66} H_{11} - 2\alpha^2 c_1^3 D_{12} F_{12} H_{11} - 4\alpha^2 c_1^3 D_{66} F_{12} H_{11} - \alpha^2 c_1^4 F_{12}^2 H_{11} + 2\alpha^2 c_1^3 D_{11} F_{22} H_{11} - 4\alpha^2 c_1^4 F_{12} F_{66}$$
$$H_{11} - 4\alpha^2 c_1^4 F_{66}^2 H_{11} + 2\alpha^2 c_1^2 D_{11} D_{66} H_{12} + 2\alpha^2 c_1^3 D_{12} F_{11} H_{12} + 4\alpha^2 c_1^3 D_{66} F_{11} H_{12} - 2\alpha^2 c_1^3 D_{11} F_{12} H_{12} + 2\alpha^2 c_1^4 F_{11}$$
$$F_{12} H_{12} + 4\alpha^2 c_1^4 F_{11} F_{66} H_{12} - \alpha^2 c_1^4 D_{11} H_{12}^2 - \alpha^2 c_1^4 F_{11}^2 H_{22} + \alpha^2 c_1^4 D_{11} H_{11} H_{22} + 4\alpha^2 c_1^2 D_{11} D_{66} H_{66} + 4\alpha^2 c_1^3 D_{12} F_{11}$$
$$H_{66} + 8\alpha^2 c_1^3 D_{66} F_{11} H_{66} - 4\alpha^2 c_1^3 D_{11} F_{12} H_{66} + 2\alpha^2 c_1^4 D_{12} H_{11} H_{66} + 4\alpha^2 c_1^4 D_{66} H_{11} H_{66} - 2\alpha^2 c_1^4 D_{11} H_{12} H_{66} + A_{55} D_{11}$$
$$(D_{66} + c_1(2F_{66} + c_1 H_{66})) + c_2^2\big(D_{11} D_{66} F_{55} + 2c_1 D_{11} F_{55} F_{66} + c_1^2\big(-F_{11}^2 F_{44} + D_{11} F_{44} H_{11} + D_{11} F_{55} H_{66}\big)\big) + 2c_2$$
$$\big(-c_1^2 D_{44} F_{11}^2 + D_{11}\big(c_1^2 D_{44} H_{11} + D_{55}\big(D_{66} + 2c_1 F_{66} + c_1^2 H_{66}\big)\big)\big) + (D_{11} + c_1(2F_{11} + c_1 H_{11}))(D_{66} + c_1(2F_{66} + c_1 H_{66}))\lambda N_{x0}$$

$$a_3 = 2\alpha^2 c_2 D_{12}^2 D_{55} - 2\alpha^2 c_2 D_{11} D_{22} D_{55} - 4c_2^2 D_{11} D_{44} D_{55} - 2\alpha^2 c_2 D_{11} D_{44} D_{66} + 4\alpha^2 c_2 D_{12} D_{55} D_{66} - 4\alpha^2 c_1 c_2 D_{12} D_{44} F_{11}$$
$$- 8\alpha^2 c_1 c_2 D_{44} D_{66} F_{11} + 4\alpha^2 c_1 c_2 D_{11} D_{44} F_{12} + 4\alpha^2 c_1 c_2 D_{12} D_{55} F_{12} + 8\alpha^2 c_1 c_2 D_{55} D_{66} F_{12} + 2\alpha^4 c_1^2 D_{22} F_{11} F_{12} + 4\alpha^2$$
$$c_1^2 c_2 D_{44} F_{11} F_{12} - 2\alpha^4 c_1^2 D_{12} F_{12}^2 + 2\alpha^2 c_1^2 c_2 D_{55} F_{12}^2 - 4\alpha^4 c_1^3 F_{12}^3 - 4\alpha^2 c_1 c_2 D_{11} D_{55} F_{22} - 2\alpha^4 c_1^2 D_{12} F_{11} F_{22} - 2$$
$$\alpha^4 c_1^2 D_{66} F_{11} F_{22} + 2\alpha^4 c_1^2 D_{11} F_{12} F_{22} + 4\alpha^4 c_1^3 F_{11} F_{12} F_{22} - 2c_2^3 D_{11} D_{55} F_{44} - \alpha^2 c_2^2 D_{11} D_{66} F_{44} - 2\alpha^2 c_1 c_2^2 D_{12} F_{11}$$
$$F_{44} - 4\alpha^2 c_1 c_2^2 D_{66} F_{11} F_{44} + 2\alpha^2 c_1 c_2^2 D_{11} F_{12} F_{44} + 2\alpha^2 c_1^2 c_2^2 F_{11} F_{12} F_{44} + \alpha^2 c_2^2 D_{12}^2 F_{55} - \alpha^2 c_2^2 D_{11} D_{22} F_{55} - 2c_2^3$$
$$D_{11} D_{44} F_{55} + 2\alpha^2 c_2^2 D_{12} D_{66} F_{55} + 2\alpha^2 c_1 c_2^2 D_{12} F_{12} F_{55} + 4\alpha^2 c_1 c_2^2 D_{66} F_{12} F_{55} + \alpha^2 c_1^2 c_2^2 F_{12}^2 F_{55} - 2\alpha^2 c_1 c_2^2 D_{11}$$
$$F_{22} F_{55} - c_2^4 D_{11} F_{44} F_{55} + 4\alpha^2 c_1 c_2 D_{11} D_{44} F_{66} + 4\alpha^4 c_1^2 D_{22} F_{11} F_{66} + 8\alpha^2 c_1^2 c_2 D_{44} F_{11} F_{66} - 8\alpha^4 c_1^2 D_{12} F_{12} F_{66} + 8\alpha^2 c_1^2$$
$$c_2 D_{55} F_{12} F_{66} - 16\alpha^4 c_1^3 F_{12}^2 F_{66} + 4\alpha^4 c_1^2 D_{11} F_{22} F_{66} + 12\alpha^4 c_1^3 F_{11} F_{22} F_{66} + 2\alpha^2 c_1 c_2^2 D_{11} F_{44} F_{66} + 4\alpha^2 c_1^2 c_2^2 F_{11} F_{44}$$
$$F_{66} + 4\alpha^2 c_1^2 c_2^2 F_{12} F_{55} F_{66} - 8\alpha^4 c_1^2 D_{12} F_{66}^2 + 8\alpha^2 c_1^2 c_2 D_{55} F_{66}^2 - 16\alpha^4 c_1^3 F_{12} F_{66}^2 + 4\alpha^2 c_1^2 c_2^2 F_{55} F_{66}^2 - 4\alpha^2 c_1^2$$
$$c_2 D_{12} D_{44} H_{11} - \alpha^4 c_1^2 D_{22} D_{66} H_{11} - 8\alpha^2 c_1^2 c_2 D_{44} D_{66} H_{11} + 2\alpha^4 c_1^3 D_{22} F_{12} H_{11} - 2\alpha^4 c_1^3 D_{12} F_{22} H_{11} - 4\alpha^4 c_1^3 D_{66} F_{22}$$
$$H_{11} + 2\alpha^4 c_1^4 F_{12} F_{22} H_{11} - 2\alpha^2 c_1^2 c_2^2 D_{12} F_{44} H_{11} - 4\alpha^2 c_1^2 c_2^2 D_{66} F_{44} H_{11} + 2\alpha^4 c_1^3 D_{22} F_{66} H_{11} + 4\alpha^4 c_1^4 F_{22} F_{66} H_{11} + 2$$
$$\alpha^4 c_1^2 D_{12}^2 H_{12} - 2\alpha^4 c_1^2 D_{11} D_{22} H_{12} + 4\alpha^4 c_1^2 D_{12} D_{66} H_{12} - 2\alpha^4 c_1^3 D_{22} F_{11} H_{12} + 4\alpha^4 c_1^3 D_{12} F_{12} H_{12} + 8\alpha^4 c_1^3 D_{66} F_{12}$$
$$H_{12} - 2\alpha^4 c_1^4 F_{12}^2 H_{12} - 2\alpha^4 c_1^3 D_{11} F_{22} H_{12} - 2\alpha^4 c_1^4 F_{11} F_{22} H_{12} - 8\alpha^4 c_1^4 F_{12} F_{66} H_{12} - 8\alpha^4 c_1^4 F_{66}^2 H_{12} + 2\alpha^4 c_1^4 D_{12}$$
$$H_{12}^2 + 4\alpha^4 c_1^4 D_{66} H_{12}^2 - 2\alpha^2 c_1^2 c_2 D_{11} D_{55} H_{22} - \alpha^4 c_1^2 D_{11} D_{66} H_{22} - 2\alpha^4 c_1^3 D_{12} F_{11} H_{22} - 4\alpha^4 c_1^3 D_{66} F_{11} H_{22} + 2\alpha^4 c_1^3$$
$$D_{11} F_{12} H_{22} + 2\alpha^4 c_1^4 F_{11} F_{12} H_{22} - \alpha^2 c_1^2 c_2^2 D_{11} F_{55} H_{22} + 2\alpha^4 c_1^3 D_{11} F_{66} H_{22} + 4\alpha^4 c_1^4 F_{11} F_{66} H_{22} - 2\alpha^4 c_1^4 D_{12} H_{11} H_{22}$$
$$- 4\alpha^4 c_1^4 D_{66} H_{11} H_{22} + 4\alpha^4 c_1^2 D_{12}^2 H_{66} - 4\alpha^4 c_1^2 D_{11} D_{22} H_{66} - 2\alpha^2 c_1^2 c_2 D_{11} D_{44} H_{66} - 4\alpha^2 c_1^2 c_2 D_{12} D_{55} H_{66} + 8\alpha^4 c_1^2 D_{12}$$
$$D_{66} H_{66} - 8\alpha^2 c_1^2 c_2 D_{55} D_{66} H_{66} - 4\alpha^4 c_1^3 D_{22} F_{11} H_{66} + 8\alpha^4 c_1^3 D_{12} F_{12} H_{66} + 16\alpha^4 c_1^3 D_{66} F_{12} H_{66} - 4\alpha^4 c_1^3 D_{11} F_{22} H_{66}$$
$$- 2\alpha^4 c_1^4 F_{11} F_{22} H_{66} - \alpha^2 c_1^2 c_2^2 D_{11} F_{44} H_{66} - 2\alpha^2 c_1^2 c_2^2 D_{12} F_{55} H_{66} - 4\alpha^2 c_1^2 c_2^2 D_{66} F_{55} H_{66} - \alpha^4 c_1^4 D_{22} H_{11} H_{66} + 4\alpha^4 c_1^4$$
$$D_{12} H_{12} H_{66} + 8\alpha^4 c_1^4 D_{66} H_{12} H_{66} - \alpha^4 c_1^4 D_{11} H_{22} H_{66} + \alpha^2 D_{12}^2 \lambda N_{x0} - \alpha^2 D_{11} D_{22} \lambda N_{x0} - 2c_2 D_{11} D_{44} \lambda N_{x0} + 2\alpha^2 D_{12} D_{66}$$
$$\lambda N_{x0} - 2c_2 D_{55} D_{66} \lambda N_{x0} - 2\alpha^2 c_1 D_{22} F_{11} \lambda N_{x0} - 4c_1 c_2 D_{44} F_{11} \lambda N_{x0} + 4\alpha^2 c_1 D_{12} F_{12} \lambda N_{x0} + 4\alpha^2 c_1 D_{66} F_{12} \lambda$$
$$N_{x0} + 4\alpha^2 c_1^2 F_{12}^2 \lambda N_{x0} - 2\alpha^2 c_1 D_{11} F_{22} \lambda N_{x0} - 4\alpha^2 c_1^2 F_{11} F_{22} \lambda N_{x0} - c_2^2 D_{11} F_{44} \lambda N_{x0} - 2c_1 c_2^2 F_{11} F_{44} \lambda$$
$$N_{x0} - c_2^2 D_{66} F_{55} \lambda N_{x0} + 4\alpha^2 c_1 D_{12} F_{66} \lambda N_{x0} - 4c_1 c_2 D_{55} F_{66} \lambda N_{x0} + 8\alpha^2 c_1^2 F_{12} F_{66} \lambda N_{x0} - 2c_1 c_2^2 F_{55} F_{66} \lambda$$
$$N_{x0} - \alpha^2 c_1^2 D_{22} H_{11} \lambda N_{x0} - 2c_1^2 c_2 D_{44} H_{11} \lambda N_{x0} - 2\alpha^2 c_1^3 F_{22} H_{11} \lambda N_{x0} - c_1^2 c_2^2 F_{44} H_{11} \lambda N_{x0} + 2\alpha^2 c_1^2 D_{12} H_{12} \lambda$$
$$N_{x0} + 2\alpha^2 c_1^2 D_{66} H_{12} \lambda N_{x0} + 4\alpha^2 c_1^3 F_{12} H_{12} \lambda N_{x0} + 4\alpha^2 c_1^3 F_{66} H_{12} \lambda N_{x0} + \alpha^2 c_1^4 H_{12}^2 \lambda N_{x0} - \alpha^2 c_1^2 D_{11} H_{22} \lambda$$
$$N_{x0} - 2\alpha^2 c_1^3 F_{11} H_{22} \lambda N_{x0} - \alpha^2 c_1^4 H_{11} H_{22} \lambda N_{x0} + 2\alpha^2 c_1^2 D_{12} H_{66} \lambda N_{x0} - 2c_1^2 c_2 D_{55} H_{66} \lambda N_{x0} + 4\alpha^2 c_1^3 F_{12} H_{66}$$
$$\lambda N_{x0} - c_1^2 c_2^2 F_{55} H_{66} \lambda N_{x0} + 2\alpha^2 c_1^4 H_{12} H_{66} \lambda N_{x0} - A_{44}\Big(A_{55} D_{11} + 2c_2 D_{11} D_{55} + c_2^2 D_{11} F_{55} + \alpha^2(2c_1(F_{11}(D_{12} + 2D_{66}$$

$$- c_1(F_{12} + 2F_{66})) + c_1(D_{12} + 2D_{66})H_{11}) + D_{11}(D_{66} + c_1(-2(F_{12} + F_{66}) + c_1 H_{66}))) + (D_{11} + c_1(2F_{11} + c_1 H_{11}))\lambda N_{x0}\Big)$$
$$+ A_{55}\Big(- c_2 D_{11}(2D_{44} + c_2 F_{44}) + \alpha^2\Big(D_{12}^2 - D_{11}(D_{22} + c_1(2F_{22} + c_1 H_{22})) + 2D_{12}(D_{66} + c_1(F_{12} - c_1 H_{66}))$$
$$+ c_1\left(c_1(F_{12} + 2F_{66})^2 + 4D_{66}(F_{12} - c_1 H_{66})\right)\Big) - (D_{66} + c_1(2F_{66} + c_1 H_{66}))\lambda N_{x0}\Big) - \alpha^2(D_{11} + c_1(2F_{11} + c_1 H_{11}))$$
$$(D_{66} + c_1(2F_{66} + c_1 H_{66}))\lambda N_{y0}$$

$$a_4 = 4A_{55}\alpha^2 c_2 D_{12} D_{44} - 2\alpha^4 c_2 D_{12}^2 D_{44} + 2\alpha^4 c_2 D_{11} D_{22} D_{44} + 8\alpha^2 c_2^2 D_{12} D_{44} D_{55} + A_{55}\alpha^4 D_{22} D_{66} + 8A_{55}\alpha^2 c_2 D_{44} D_{66}$$
$$- 4\alpha^4 c_2 D_{12} D_{44} D_{66} + 2\alpha^4 c_2 D_{22} D_{55} D_{66} + 16\alpha^2 c_2^2 D_{44} D_{55} D_{66} + 4\alpha^4 c_1 c_2 D_{22} D_{44} F_{11} - 2A_{55}\alpha^4 c_1 D_{22} F_{12} - 4\alpha^4 c_1 c_2 D_{12}$$
$$D_{44} F_{12} - 4\alpha^4 c_1 c_2 D_{22} D_{55} F_{12} - 8\alpha^4 c_1 c_2 D_{44} D_{66} F_{12} - \alpha^6 c_1^2 D_{22} F_{12}^2 - 2\alpha^4 c_1^2 c_2 D_{44} F_{12}^2 + 2A_{55}\alpha^4 c_1 D_{12} F_{22} + 4\alpha^4 c_1$$
$$c_2 D_{12} D_{55} F_{22} + 4A_{55}\alpha^4 c_1 D_{66} F_{22} + 8\alpha^4 c_1 c_2 D_{55} D_{66} F_{22} - 2A_{55}\alpha^4 c_1^2 F_{12} F_{22} + 2\alpha^6 c_1^2 D_{12} F_{12} F_{22} - 4\alpha^4 c_1^2 c_2 D_{55} F_{12}$$
$$F_{22} + 2\alpha^6 c_1^3 F_{12}^2 F_{22} - \alpha^6 c_1^2 D_{11} F_{22}^2 - 2\alpha^6 c_1^3 F_{11} F_{22}^2 + 2A_{55}\alpha^2 c_2^2 D_{12} F_{44} - \alpha^4 c_2^2 D_{12}^2 F_{44} + \alpha^4 c_2^2 D_{11} D_{22} F_{44} + 4\alpha^2$$
$$c_2^3 D_{12} D_{55} F_{44} + 4A_{55}\alpha^2 c_2^2 D_{66} F_{44} - 2\alpha^4 c_2^2 D_{12} D_{66} F_{44} + 8\alpha^2 c_2^3 D_{55} D_{66} F_{44} + 2\alpha^4 c_1 c_2^2 D_{22} F_{11} F_{44} - 2\alpha^4 c_1 c_2^2 D_{12}$$
$$F_{12} F_{44} - 4\alpha^4 c_1 c_2^2 D_{66} F_{12} F_{44} - \alpha^4 c_1^2 c_2^2 F_{12}^2 F_{44} + 4\alpha^2 c_2^3 D_{12} D_{44} F_{55} + \alpha^4 c_2^2 D_{22} D_{66} F_{55} + 8\alpha^2 c_2^3 D_{44} D_{66} F_{55} - 2\alpha^4$$
$$c_1 c_2^2 D_{22} F_{12} F_{55} + 2\alpha^4 c_1 c_2^2 D_{12} F_{22} F_{55} + 4\alpha^4 c_1 c_2^2 D_{66} F_{22} F_{55} - 2\alpha^4 c_1^2 c_2^2 F_{12} F_{22} F_{55} + 2\alpha^2 c_2^4 D_{12} F_{44} F_{55} + 4\alpha^2$$
$$c_2^4 D_{66} F_{44} F_{55} - 2A_{55}\alpha^4 c_1 D_{22} F_{66} - 4\alpha^4 c_1 c_2 D_{22} D_{55} F_{66} - 4\alpha^6 c_1^2 D_{22} F_{12} F_{66} - 8\alpha^4 c_1^2 c_2 D_{44} F_{12} F_{66} - 4A_{55}\alpha^4 c_1^2 F_{22}$$
$$F_{66} + 4\alpha^6 c_1^2 D_{12} F_{22} F_{66} - 8\alpha^4 c_1^2 c_2 D_{55} F_{22} F_{66} - 4\alpha^4 c_1^2 c_2^2 F_{12} F_{44} F_{66} - 2\alpha^4 c_1 c_2^2 D_{22} F_{55} F_{66} - 4\alpha^4 c_1^2 c_2^2 F_{22} F_{55}$$
$$F_{66} - 4\alpha^6 c_1^2 D_{22} F_{66}^2 - 8\alpha^4 c_1^2 c_2 D_{44} F_{66}^2 - 8\alpha^6 c_1^3 F_{22} F_{66}^2 - 4\alpha^4 c_1^2 c_2^2 F_{44} F_{66}^2 + 2\alpha^4 c_1^2 c_2 D_{22} D_{44} H_{11} - \alpha^6 c_1^4$$
$$F_{22}^2 H_{11} + \alpha^4 c_1^2 c_2^2 D_{22} F_{44} H_{11} + 2\alpha^6 c_1^2 D_{22} D_{66} H_{12} - 2\alpha^6 c_1^3 D_{22} F_{12} H_{12} + 2\alpha^6 c_1^3 D_{12} F_{22} H_{12} + 4\alpha^6 c_1^3 D_{66} F_{22} H_{12}$$
$$+ 2\alpha^6 c_1^4 F_{12} F_{22} H_{12} + 4\alpha^6 c_1^4 F_{22} F_{66} H_{12} - \alpha^6 c_1^4 D_{22} H_{12}^2 + 2A_{55}\alpha^4 c_1^2 D_{12} H_{22} - \alpha^6 c_1^2 D_{12}^2 H_{22} + \alpha^6 c_1^2 D_{11} D_{22} H_{22}$$
$$+ 4\alpha^4 c_1^2 c_2 D_{12} D_{55} H_{22} + 4A_{55}\alpha^4 c_1^2 D_{66} H_{22} - 2\alpha^6 c_1^2 D_{12} D_{66} H_{22} + 8\alpha^4 c_1^2 c_2 D_{55} D_{66} H_{22} + 2\alpha^6 c_1^3 D_{22} F_{11} H_{22} - 2\alpha^6 c_1^3$$
$$D_{12} F_{12} H_{22} - 4\alpha^6 c_1^3 D_{66} F_{12} H_{22} - \alpha^6 c_1^4 F_{12}^2 H_{22} + 2\alpha^4 c_1^2 c_2^2 D_{12} F_{55} H_{22} + 4\alpha^4 c_1^2 c_2^2 D_{66} F_{55} H_{22} - 4\alpha^6 c_1^4 F_{12} F_{66} H_{22}$$
$$- 4\alpha^6 c_1^4 F_{66}^2 H_{22} + \alpha^6 c_1^4 D_{22} H_{11} H_{22} + A_{55}\alpha^4 c_1^2 D_{22} H_{66} + 4\alpha^4 c_1^2 c_2 D_{12} D_{44} H_{66} + 2\alpha^4 c_1^2 c_2 D_{22} D_{55} H_{66} + 4\alpha^6 c_1^2 D_{22}$$
$$D_{66} H_{66} + 8\alpha^4 c_1^2 c_2 D_{44} D_{66} H_{66} - 4\alpha^6 c_1^3 D_{22} F_{12} H_{66} + 4\alpha^6 c_1^3 D_{12} F_{22} H_{66} + 8\alpha^6 c_1^3 D_{66} F_{22} H_{66} + 2\alpha^4 c_1^2 c_2^2 D_{12} F_{44} H_{66}$$
$$+ 4\alpha^4 c_1^2 c_2^2 D_{66} F_{44} H_{66} + \alpha^4 c_1^2 c_2^2 D_{22} F_{55} H_{66} - 2\alpha^6 c_1^4 D_{22} H_{12} H_{66} + 2\alpha^6 c_1^4 D_{12} H_{22} H_{66} + 4\alpha^6 c_1^4 D_{66} H_{22} H_{66} + A_{55}\alpha^2$$
$$D_{22}\lambda N_{x0} + 2A_{55} c_2 D_{44}\lambda N_{x0} + 2\alpha^2 c_2 D_{22} D_{55}\lambda N_{x0} + 4c_2^2 D_{44} D_{55}\lambda N_{x0} + \alpha^4 D_{22} D_{66}\lambda N_{x0} + 2\alpha^2 c_2 D_{44} D_{66}\lambda N_{x0}$$
$$+ 2A_{55}\alpha^2 c_1 F_{22}\lambda N_{x0} + 4\alpha^2 c_1 c_2 D_{55} F_{22}\lambda N_{x0} + 2\alpha^4 c_1 D_{66} F_{22}\lambda N_{x0} + A_{55} c_2^2 F_{44}\lambda N_{x0} + 2c_2^3 D_{55} F_{44}\lambda N_{x0} + \alpha^2$$
$$c_2^2 D_{66} F_{44}\lambda N_{x0} + \alpha^2 c_2^2 D_{22} F_{55}\lambda N_{x0} + 2c_2^3 D_{44} F_{55}\lambda N_{x0} + 2\alpha^2 c_1 c_2^2 F_{22} F_{55}\lambda N_{x0} + c_2^4 F_{44} F_{55}\lambda N_{x0} + 2\alpha^4 c_1$$
$$D_{22} F_{66}\lambda N_{x0} + 4\alpha^2 c_1 c_2 D_{44} F_{66}\lambda N_{x0} + 4\alpha^4 c_1^2 F_{22} F_{66}\lambda N_{x0} + 2\alpha^2 c_1 c_2^2 F_{44} F_{66}\lambda N_{x0} + A_{55}\alpha^2 c_1^2 H_{22}\lambda N_{x0}$$
$$+ 2\alpha^2 c_1^2 c_2 D_{55} H_{22}\lambda N_{x0} + \alpha^4 c_1^2 D_{66} H_{22}\lambda N_{x0} + \alpha^2 c_1^2 c_2^2 F_{55} H_{22}\lambda N_{x0} + 2\alpha^4 c_1^3 F_{66} H_{22}\lambda N_{x0} + \alpha^4 c_1^2 D_{22}$$
$$H_{66}\lambda N_{x0} + 2\alpha^2 c_1^2 c_2 D_{44} H_{66}\lambda N_{x0} + 2\alpha^4 c_1^3 F_{22} H_{66}\lambda N_{x0} + \alpha^2 c_1^2 c_2^2 F_{44} H_{66}\lambda N_{x0} + \alpha^4 c_1^4 H_{22} H_{66}\lambda N_{x0} + \alpha^2$$
$$\Big(A_{55}(D_{66} + c_1(2F_{66} + c_1 H_{66})) + 2c_2(D_{11} D_{44} + D_{55} D_{66} + c_1(2D_{44} F_{11} + 2D_{55} F_{66} + c_1 D_{44} H_{11} + c_1 D_{55} H_{66}))$$
$$+ c_2^2(D_{11} F_{44} + D_{66} F_{55} + c_1(2F_{11} F_{44} + 2F_{55} F_{66} + c_1 F_{44} H_{11} + c_1 F_{55} H_{66})) - \alpha^2\Big(D_{12}^2 - D_{11}(D_{22} + c_1(2F_{22}$$
$$+ c_1 H_{22})) + 2D_{12}(D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66}))) + c_1(4F_{12}(D_{66} + c_1 F_{12}) - D_{22}(2F_{11} + c_1 H_{11})$$
$$+ c_1(8F_{12} F_{66} + 2D_{66} H_{12} - 2F_{11}(2F_{22} + c_1 H_{22}) + c_1(-2F_{22} H_{11} + H_{12}(4(F_{12} + F_{66}) + c_1 H_{12}) - c_1 H_{11} H_{22}$$
$$+ 4F_{12} H_{66} + 2c_1 H_{12} H_{66}))))\Big)\Big)\lambda N_{y0} + A_{44}\Big(2A_{55}\alpha^2(D_{12} + 2D_{66}) + \alpha^4\Big(- D_{12}^2 + D_{11} D_{22} + c_1\Big(2D_{22} F_{11} - 4D_{66}$$
$$F_{12} - c_1(F_{12} + 2F_{66})^2 + c_1 D_{22} H_{11} + 4c_1 D_{66} H_{66}\Big) - 2D_{12}(D_{66} + c_1(F_{12} - c_1 H_{66}))\Big) + A_{55}\lambda N_{x0} + c_2(2D_{55} + c_2$$
$$F_{55})\lambda N_{x0} + \alpha^2\Big(4c_2 D_{55}(D_{12} + 2D_{66}) + 2c_2^2(D_{12} + 2D_{66})F_{55} + (D_{66} + c_1(2F_{66} + c_1 H_{66}))\lambda N_{x0} + (D_{11} + c_1$$
$$(2F_{11} + c_1 H_{11}))\lambda N_{y0}\Big)\Big)$$

$$a_5 = -\alpha^2\left(A_{55} + 2c_2 D_{55} + c_2^2 F_{55} + \alpha^2\left(D_{66} + 2c_1 F_{66} + c_1^2 H_{66}\right)\right)\Big(\alpha^2\Big(A_{44} D_{22} + 2c_2 D_{22} D_{44} + c_2^2 D_{22} F_{44} + \alpha^2 c_1^2$$
$$\left(-F_{22}^2 + D_{22} H_{22}\right)\Big) + \left(A_{44} + c_2(2D_{44} + c_2 F_{44}) + \alpha^2\left(D_{22} + 2c_1 F_{22} + c_1^2 H_{22}\right)\right)\lambda N_{y0}\Big) \tag{B.1}$$