Exact Dynamic Stiffness Elements based on One-Dimensional Higher-Order Theories for Free Vibration Analysis of Solid and Thin-Walled Structures

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Abstract

In this paper, an exact dynamic stiffness formulation using one-dimensional (1D) higher-order theories is presented and subsequently used to investigate the free vibration characteristics of solid and thin-walled structures. Higher-order kinematic fields are developed using the Carrera Unified Formulation, which allows for straightforward implementation of any-order theories without the need for ad hoc formulations. Classical beam theories (Euler-Bernoulli and Timoshenko) are also captured from the formulation as degenerate cases. The Principle of Virtual Displacements is used to derive the governing differential equations and the associated natural boundary conditions. An exact dynamic stiffness matrix is then developed by relating the amplitudes of harmonically varying loads to those of the responses. The explicit terms of the dynamic stiffness matrices are also presented. The resulting dynamic stiffness matrix is used with particular reference to the Wittrick-Williams algorithm to carry out the free vibration analysis of solid and thin-walled structures. The accuracy of the theory is confirmed both by published literature and by extensive finite element solutions using the commercial code MSC/NASTRAN®.

Keywords: Dynamic stiffness method; Unified formulation; Free vibration; Higher-order theories; Beams; Thin-walled structures

1 Introduction

Beam models are widely used to analyze the mechanical behavior of slender bodies, such as columns, rotor-blades, aircraft wings, towers, antennae and bridges amongst others. The simplicity of 1D theories and their ease of application coupled with computational efficiency are some of the main reasons why structural analysts prefer them to two-dimensional (2D) and three-dimensional (3D) models.

The classical and best-known beam theories that survived the test of time and still valid to this day, are those by Euler [1] - hereinafter referred to as EBBM - and Timoshenko [2, 3] - hereinafter referred to as TBM. The former does not account for transverse shear deformations and rotatory inertia, whereas the latter assumes a uniform shear distribution along the cross-section of the beam together with the effects of rotatory inertia. These models yield reasonably good results when slender, solid section, homogeneous structures are subjected to flexure. Conversely, the analysis of deep, thin-walled, open section beams may require more sophisticated theories to achieve sufficiently accurate results, see [4].

Over the last century, many refined beam theories have been proposed to overcome the limitation of classical beam modelling. Different approaches have been used to improve the beam models, which include the introduction of shear correction factors, the use of warping functions based on de Saint-Venant's solution, the variational asymptotic solution (VABS), the generalized beam theory (GBT), and others. Some selective references and noteworthy contributions are briefly discussed below, with particular attention to dynamic analysis which is the main focus of this paper.

Early investigators have focused on the use of appropriate shear corrections factors to increase the accuracy of classical 1D formulations, see for examples Timoshenko and Goodier [5], Sokolniko [6], Stephen [7], and Hutchinson [8]. The shear correction factor has generally been used as a static concept which is restrictive. In this respect, Jensen [9] showed how the shear correction factor can vary with the natural frequencies. Furthermore, a review paper by Kaneko [10] and a recent paper by Dong et al. [11] highlighted the difficulty in the definition of a universally accepted formulation for shear correction factors.

Another important class of refinement methods reported in the literature is based on the use of warping functions. The contributions by El Fatmi [12, 13, 14] and Ladevéze et al. [15, 16] are some excellent examples. Rand [17] and Kim and White [18] used more or less the same approach in the free vibration analysis by introducing out-of-plane warping with no in-plane stretching terms.

Asymptotic type expansion in conjunction with variational methods has also been proposed particularly by Berdichevsky et al. [19], in which a commendable review of previous works on beam theory developments is given. Some further valuable contributions are by Volovoi [20], Popescu and Hodges [21], Yu et al. [22], Yu and Hodges [23, 24]. Other related work can be found in the papers published by Kim and Wang [25] and Firouz-Abad et al. [26].

The generalized beam theory (GBT) probably was originated from the work of Schardt [27, 28]. GBT improves classical theories by using piece-wise beam description of thin-walled sections. It has been widely employed and extended in various forms by Silvetre et al. [29, 30, 31] and a dynamic application has been presented by Bebiano et al. [32].

Higher-order theories are generally obtained by using refined displacement fields of the beam cross-sections. Washizu [33] ascertained how the use of an arbitrarily chosen rich displacement fields can lead to closed form exact 3D solutions. Many other higher-order theories have also been introduced to include non-classical effects. A review was compiled by Kapania and Raciti [34, 35] focusing on flexural deformation, vibration analysis, wave propagations, buckling and post-buckling behaviour.

The present work is focused on 1D higher-order theories based on generalized displacement variables to carry out free vibration analysis of solid and thin-walled structures. Refined beam models are developed within the framework of Carrera Unified Formulation (CUF) which is well established in the literature for over a decade [36, 37, 38, 39, 40]. CUF is a hierarchical formulation that considers the order of the model, N, as a free-parameter (i.e. as an input) of the analysis or in other words, refined models are obtained without having the need for any ad hoc formulations. In the present work, beam theories using CUF are obtained on the basis of Taylor-type expansions (TE). EBBM and TBM can be obtained as particular or special cases. The strength of CUF TE 1D models in dealing with arbitrary geometries, thin-walled structures and identifying local effects are well known for both static [41, 42] and free-vibration analysis [43, 44, 45].

In majority of the papers on 1D CUF, the finite element method (FEM) has been used to handle arbitrary geometries and loading conditions. The present work is intended to provide a more powerful approach for CUF TE theories through the application of the dynamic stiffness method (DSM) to carry out the free vibration analysis of solid and thin-walled structures in a much broader context by allowing for the cross-sectional deformation. DSM has been quite extensively developed for beam elements by Banerjee [46, 47, 48, 49, 50], Banerjee et al. [51], and Williams and Wittrick [52]. Plate elements based on DSM were originally formulated by Wittrick [53] and Wittrick and Williams [54].

Recently, DSM has been applied to Mindlin plate assemblies by Boscolo and Banerjee in [55, 56], where a more comprehensive review on the use of DSM can be found.

The DSM is appealing in dynamic analysis because unlike the FEM, it provides exact solution of the equations of motion of a structure once the initial assumptions on the displacements field have been made. This essentially means that, unlike the FEM and other approximate methods, the model accuracy is not unduly compromised when a small number of elements are used in the analysis. For instance, one single structural element can be used in the DSM to compute any number of natural frequencies to any desired accuracy. Of course, the accuracy of the DSM will be as good as the accuracy of the governing differential equations of the structural element in free vibration. In fact, the exact dynamic stiffness (DS) matrix stems from the solution of the governing differential equations.

In this work, CUF is adopted to automatically build any-order beam theory which is the fundamental prerequisite to develop 1D higher-order exact DS elements as the objective. The investigation is carried out in the following steps: (i) first CUF is introduced and higher-order models are formulated, (ii) secondly, the Principle of Virtual Displacements (PVD) is used to derive the differential governing equations and the associated natural boundary conditions for the generic N-order model, (iii) next, by assuming harmonic oscillation, the equilibrium equations and the natural boundary conditions are formulated in the frequency domain by making extensive use of symbolic computation, (iv) the resulting system of ordinary differential equations of second order with constant coefficients is then solved in closed analytical form, (v) subsequently, the frequency dependent DS matrix of the system is derived by relating the amplitudes of the harmonically varying nodal generalised forces to those of the nodal generalized displacements, and (vi) finally, the well-known algorithm of Wittrick and Williams [57] is applied to the resulting DS matrix for free vibration analysis of compact and thin-walled structures.

2 1D unified formulation

2.1 Preliminaries

The adopted rectangular cartesian coordinate system is shown in Fig. 1. Let us introduce the transposed displacement vector,

$$\mathbf{u}(x,y,z;t) = \left\{ \begin{array}{ccc} u_x & u_y & u_z \end{array} \right\}^T \tag{1}$$

The cross-sectional plane of the structure is denoted by Ω, and the beam boundaries over y are $0 \leq y \leq L$. The stress, $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\epsilon}$, components are grouped as follows:

$$\begin{aligned} \boldsymbol{\sigma}_p &= \left\{ \begin{array}{ccc} \sigma_{zz} & \sigma_{xx} & \sigma_{zx} \end{array} \right\}^T, \quad \boldsymbol{\epsilon}_p = \left\{ \begin{array}{ccc} \epsilon_{zz} & \epsilon_{xx} & \epsilon_{zx} \end{array} \right\}^T \\ \boldsymbol{\sigma}_n &= \left\{ \begin{array}{ccc} \sigma_{zy} & \sigma_{xy} & \sigma_{yy} \end{array} \right\}^T, \quad \boldsymbol{\epsilon}_n = \left\{ \begin{array}{ccc} \epsilon_{zy} & \epsilon_{xy} & \epsilon_{yy} \end{array} \right\}^T \end{aligned} \tag{2}$$

In the case of small displacements with respect to a characteristic dimension in the plane of Ω, the strain - displacement relations are

$$\begin{aligned} \boldsymbol{\epsilon}_p &= \mathbf{D}_p \mathbf{u} \\ \boldsymbol{\epsilon}_n &= \mathbf{D}_n \mathbf{u} = (\mathbf{D}_{n\Omega} + \mathbf{D}_{ny}) \mathbf{u} \end{aligned} \tag{3}$$

where $\mathbf{D}_p$ and $\mathbf{D}_n$ are linear differential operators and the subscript "n" stands for terms lying on the cross-section, while "p" stands for terms lying on planes which are orthogonal to Ω.

$$\mathbf{D}_p = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}, \quad \mathbf{D}_{n\Omega} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{ny} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \end{bmatrix} \tag{4}$$

Constitutive laws are now exploited to obtain stress components to give

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}} \boldsymbol{\epsilon} \tag{5}$$

Equation (5) can be split into $\boldsymbol{\epsilon}_p$ and $\boldsymbol{\epsilon}_n$ with the help of Eq. (2) so that

$$\begin{aligned} \boldsymbol{\sigma}_p &= \tilde{\mathbf{C}}_{pp} \boldsymbol{\epsilon}_p + \tilde{\mathbf{C}}_{pn} \boldsymbol{\epsilon}_n \\ \boldsymbol{\sigma}_n &= \tilde{\mathbf{C}}_{np} \boldsymbol{\epsilon}_p + \tilde{\mathbf{C}}_{nn} \boldsymbol{\epsilon}_n \end{aligned} \tag{6}$$

In the case of orthotropic material the matrices $\tilde{\mathbf{C}}_{pp}$, $\tilde{\mathbf{C}}_{nn}$, $\tilde{\mathbf{C}}_{pn}$, and $\tilde{\mathbf{C}}_{np}$ are

$$\tilde{\mathbf{C}}_{pp} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & 0 \\ 0 & 0 & \tilde{C}_{44} \end{bmatrix}, \quad \tilde{\mathbf{C}}_{nn} = \begin{bmatrix} \tilde{C}_{55} & 0 & 0 \\ 0 & \tilde{C}_{66} & \tilde{C}_{36} \\ 0 & \tilde{C}_{36} & \tilde{C}_{33} \end{bmatrix}, \quad \tilde{\mathbf{C}}_{pn} = \tilde{\mathbf{C}}_{np}^T = \begin{bmatrix} 0 & \tilde{C}_{16} & \tilde{C}_{13} \\ 0 & \tilde{C}_{26} & \tilde{C}_{23} \\ \tilde{C}_{45} & 0 & 0 \end{bmatrix} \tag{7}$$

Coefficients $\tilde{C}_{ij}$ depend on the Young's modulus, Poisson's ratio, and fiber orientation angle. For the sake of brevity, the expressions for the coefficients $\tilde{C}_{ij}$ are not reported here, but can be found in standard texts, see for example Tsai [58] and Reddy [59]. In the present paper the governing equations are derived for the generic case of orthotropic material. However, applications to metallic isotropic structures will be shown.

Within the framework of the CUF, the displacement field $\mathbf{u}(x, y, z; t)$ can be expressed as

$$\mathbf{u}(x, y, z; t) = F_\tau(x, z)\mathbf{u}_\tau(y; t), \qquad \tau = 1, 2, ..., M \tag{8}$$

where F_τ are the functions of the coordinates x and z on the cross-section. $\mathbf{u}_\tau$ is the vector of the *generalized* displacements, M stands for the number of terms used in the expansion, and the repeated subscript, τ, indicates summation. The choice of F_τ determines the class of the 1D CUF model that is required and subsequently to be adopted. TE (Taylor expansion) 1D CUF models - described by Eq. (8) - consists of a Maclaurin series that uses the 2D polynomials $x^i\ z^j$ as base, where i and j are positive integers. Table 1 shows M and F_τ as functions of the expansion order, N. For instance, the displacement field of the second-order ($N = 2$) TE model can be expressed as

$$\begin{aligned}
u_x &= u_{x_1} + x\ u_{x_2} + z\ u_{x_3} + x^2\ u_{x_4} + xz\ u_{x_5} + z^2\ u_{x_6} \\
u_y &= u_{y_1} + x\ u_{y_2} + z\ u_{y_3} + x^2\ u_{y_4} + xz\ u_{y_5} + z^2\ u_{y_6} \\
u_z &= u_{z_1} + x\ u_{z_2} + z\ u_{z_3} + x^2\ u_{z_4} + xz\ u_{z_5} + z^2\ u_{z_6}
\end{aligned} \tag{9}$$

The order N of the expansion is set as an input option of the analysis; the integer N is arbitrary and defines the order the beam theory. The Timoshenko beam model (TBM) can be realised by using a suitable F_τ expansion. Two conditions have to be imposed: (1) a first-order ($N = 1$) approximation kinematic field:

$$\begin{aligned}
u_x &= u_{x_1} + x\ u_{x_2} + z\ u_{x_3} \\
u_y &= u_{y_1} + x\ u_{y_2} + z\ u_{y_3} \\
u_z &= u_{z_1} + x\ u_{z_2} + z\ u_{z_3}
\end{aligned} \tag{10}$$

(2) the displacement components u_x and u_z have to be constant above the cross-section:

$$u_{x_2} = u_{z_2} = u_{x_3} = u_{z_3} = 0 \tag{11}$$

By contrast, the Euler-Bernoulli beam model (EBBM) can be obtained through the penalization of ϵ_{xy} and ϵ_{zy}. This condition can be imposed by using a penalty value χ in the constitutive equations

to give

$$\begin{aligned}\sigma_{xy} &= \chi\tilde{C}_{55}\epsilon_{xy} + \chi\tilde{C}_{45}\epsilon_{zy} \\ \sigma_{zy} &= \chi\tilde{C}_{45}\epsilon_{xy} + \chi\tilde{C}_{55}\epsilon_{xy}\end{aligned} \tag{12}$$

Classical theories and first-order models ($N = 1$) require the necessary assumption of reduced material stiffness coefficients to correct Poisson's locking (see [39]). In this paper, Poisson's locking is corrected according to the method outlined by Carrera et al. [40].

2.2 Governing equations of the N-order TE model

The principle of virtual displacements is used to derive the equations of motion.

$$\delta L_{\text{int}} = \int_V (\delta\boldsymbol{\epsilon}_p^T\boldsymbol{\sigma}_p + \delta\boldsymbol{\epsilon}_n^T\boldsymbol{\sigma}_n)\,\mathrm{d}V = -\delta L_{\text{ine}} \tag{13}$$

where L_{int} stands for the strain energy and δL_{ine} is the work done by the inertial loadings. δ stands for the usual virtual variation operator. The virtual variation of the strain energy is rewritten using Eq.s (3), (6) and (8). After integrations by part, Eq. (13) becomes

$$\delta L_{\text{int}} = \int_L \delta\mathbf{u}_\tau^T\mathbf{K}^{\tau s}\mathbf{u}_s\,\mathrm{d}y + \left[\delta\mathbf{u}_\tau^T\boldsymbol{\Pi}^{\tau s}\mathbf{u}_s\right]_{y=0}^{y=L} \tag{14}$$

where $\mathbf{K}^{\tau s}$ is the differential linear stiffness matrix and $\boldsymbol{\Pi}^{\tau s}$ is the matrix of the natural boundary conditions in the form of 3×3 fundamental nuclei. The components of $\mathbf{K}^{\tau s}$ are provided as follows and they are referred to as $K_{(ij)}^{\tau s}$, where i is the row number ($i = 1, 2, 3$) and j denotes the column

number ($j = 1, 2, 3$)

$$
\begin{aligned}
K^{\tau s}_{(11)} &= E^{22}_{\tau,x s,x} + E^{44}_{\tau,z s,z} + \big(E^{26}_{\tau,x s} - E^{26}_{\tau s,x}\big)\frac{\partial}{\partial y} - E^{66}_{\tau s}\frac{\partial^2}{\partial y^2} \\
K^{\tau s}_{(12)} &= E^{26}_{\tau,x s,x} + E^{45}_{\tau,z s,z} + \big(E^{23}_{\tau,x s} - E^{66}_{\tau s,x}\big)\frac{\partial}{\partial y} - E^{36}_{\tau s}\frac{\partial^2}{\partial y^2} \\
K^{\tau s}_{(13)} &= E^{12}_{\tau,x s,z} + E^{44}_{\tau,z s,x} + \big(E^{45}_{\tau,z s} - E^{16}_{\tau s,z}\big)\frac{\partial}{\partial y} \\
K^{\tau s}_{(21)} &= E^{26}_{\tau,x s,x} + E^{45}_{\tau,z s,z} + \big(E^{66}_{\tau,x s} - E^{23}_{\tau s,x}\big)\frac{\partial}{\partial y} - E^{36}_{\tau s}\frac{\partial^2}{\partial y^2} \\
K^{\tau s}_{(22)} &= E^{66}_{\tau,x s,x} + E^{55}_{\tau,z s,z} + \big(E^{36}_{\tau,x s} - E^{36}_{\tau s,x}\big)\frac{\partial}{\partial y} - E^{33}_{\tau s}\frac{\partial^2}{\partial y^2} \\
K^{\tau s}_{(23)} &= E^{16}_{\tau,x s,z} + E^{45}_{\tau,z s,x} + \big(E^{55}_{\tau,z s} - E^{13}_{\tau s,z}\big)\frac{\partial}{\partial y} \\
K^{\tau s}_{(31)} &= E^{44}_{\tau,x s,z} + E^{12}_{\tau,z s,x} + \big(E^{16}_{\tau,z s} - E^{45}_{\tau s,z}\big)\frac{\partial}{\partial y} \\
K^{\tau s}_{(32)} &= E^{45}_{\tau,x s,z} + E^{16}_{\tau,z s,x} + \big(E^{13}_{\tau,z s} - E^{55}_{\tau s,z}\big)\frac{\partial}{\partial y} \\
K^{\tau s}_{(33)} &= E^{44}_{\tau,x s,x} + E^{11}_{\tau,z s,z} + \big(E^{45}_{\tau,x s} - E^{45}_{\tau s,x}\big)\frac{\partial}{\partial y} - E^{55}_{\tau s}\frac{\partial^2}{\partial y^2}
\end{aligned}
\tag{15}
$$

The generic term $E^{\alpha\beta}_{\tau,\theta s,\zeta}$ above is a cross-sectional moment parameter

$$
E^{\alpha\beta}_{\tau,\theta s,\zeta} = \int_\Omega \tilde{C}_{\alpha\beta} F_{\tau,\theta} F_{s,\zeta}\,\mathrm{d}\Omega \tag{16}
$$

The suffix after the comma in Eq. (15) denotes the derivatives. As far as the boundary conditions are concerned, the components of $\mathbf{\Pi}^{\tau s}$ are

$$
\begin{aligned}
&\Pi^{\tau s}_{(11)} = E^{26}_{\tau s,x} + E^{66}_{\tau s}\frac{\partial}{\partial y}, \quad \Pi^{\tau s}_{(12)} = E^{66}_{\tau s,x} + E^{36}_{\tau s}\frac{\partial}{\partial y}, \quad \Pi^{\tau s}_{(13)} = E^{16}_{\tau s} \\
&\Pi^{\tau s}_{(21)} = E^{23}_{\tau s,x} + E^{36}_{\tau s}\frac{\partial}{\partial y}, \quad \Pi^{\tau s}_{(22)} = E^{36}_{\tau s,x} + E^{33}_{\tau s}\frac{\partial}{\partial y}, \quad \Pi^{\tau s}_{(23)} = E^{13}_{\tau s,z} \\
&\Pi^{\tau s}_{(31)} = E^{45}_{\tau s}, \qquad\qquad \Pi^{\tau s}_{(32)} = E^{55}_{\tau s,z}, \qquad\qquad \Pi^{\tau s}_{(33)} = E^{45}_{\tau s,x} + E^{55}_{\tau s}\frac{\partial}{\partial y}
\end{aligned}
\tag{17}
$$

The virtual variation of the inertial work is given by

$$
\delta L_{\mathrm{ine}} = \int_L \delta\mathbf{u}_\tau \int_\Omega \rho F_\tau F_s\,\mathrm{d}\Omega\;\ddot{\mathbf{u}}_s\,\mathrm{d}y = \int_L \delta\mathbf{u}_\tau \mathbf{M}^{\tau s}\ddot{\mathbf{u}}_s\,\mathrm{d}y \tag{18}
$$

The explicit form of the governing equations is

$$
\begin{aligned}
\delta u_{x\tau}: \quad & -E^{66}_{\tau s}u_{xs,yy} + \big(E^{26}_{\tau,xs} - E^{26}_{\tau s,x}\big)u_{xs,y} + \big(E^{22}_{\tau,xs,x} + E^{44}_{\tau,zs,z}\big)u_{xs} \\
& -E^{36}_{\tau s}u_{ys,yy} + \big(E^{23}_{\tau,xs} - E^{66}_{\tau s,x}\big)u_{ys,y} + \big(E^{26}_{\tau,xs,x} + E^{45}_{\tau,zs,z}\big)u_{ys} \\
& +\big(E^{45}_{\tau,zs} - E^{16}_{\tau s,z}\big)u_{zs,y} + \big(E^{44}_{\tau,zs,x} + E^{12}_{\tau,xs,z}\big)u_{zs} = -E^{\rho}_{\tau s}\ddot{u}_{xs} \\
\delta u_{y\tau}: \quad & -E^{36}_{\tau s}u_{xs,yy} + \big(E^{66}_{\tau,xs} - E^{23}_{\tau s,x}\big)u_{xs,y} + \big(E^{26}_{\tau,xs,x} + E^{45}_{\tau,zs,z}\big)u_{xs} \\
& -E^{33}_{\tau s}u_{ys,yy} + \big(E^{36}_{\tau,xs} - E^{36}_{\tau s,x}\big)u_{ys,y} + \big(E^{66}_{\tau,xs,x} + E^{55}_{\tau,zs,z}\big)u_{ys} \\
& +\big(E^{55}_{\tau,zs} - E^{13}_{\tau s,z}\big)u_{zs,y} + \big(E^{16}_{\tau,xs,z} + E^{45}_{\tau,zs,x}\big)u_{zs} = -E^{\rho}_{\tau s}\ddot{u}_{ys} \\
\delta u_{z\tau}: \quad & \big(E^{16}_{\tau,zs} - E^{45}_{\tau s,z}\big)u_{xs,y} + \big(E^{44}_{\tau,xs,z} + E^{12}_{\tau,zs,x}\big)u_{xs} \\
& +\big(E^{13}_{\tau,zs} - E^{55}_{\tau s,z}\big)u_{ys,y} + \big(E^{45}_{\tau,xs,z} + E^{16}_{\tau,zs,x}\big)u_{ys} - E^{55}_{\tau s}u_{zs,yy} \\
& +\big(E^{45}_{\tau,xs} - E^{45}_{\tau s,x}\big)u_{zs,y} + \big(E^{44}_{\tau,xs,x} + E^{11}_{\tau,zs,z}\big)u_{zs} = -E^{\rho}_{\tau s}\ddot{u}_{zs}
\end{aligned}
\tag{19}
$$

where

$$
E^{\rho}_{\tau s} = \int_{\Omega} \rho F_{\tau} F_{s} \, \mathrm{d}\Omega \tag{20}
$$

Double over dots stand as second derivative with respect to time (t). Letting $\mathbf{P}_{\tau} = \left\{ \begin{array}{ccc} P_{x\tau} & P_{y\tau} & P_{z\tau} \end{array} \right\}^{T}$ to be the vector of the generalized forces, the natural boundary conditions are

$$
\begin{aligned}
\delta u_{x\tau}: \quad & P_{xs} = E^{66}_{\tau s}u_{xs,y} + E^{26}_{\tau s,x}u_{xs} + E^{36}_{\tau s}u_{ys,y} + E^{66}_{\tau s,x}u_{ys} + E^{16}_{\tau s,z}u_{zs} \\
\delta u_{y\tau}: \quad & P_{ys} = E^{36}_{\tau s}u_{xs,y} + E^{23}_{\tau s,x}u_{xs} + E^{33}_{\tau s}u_{ys,y} + E^{36}_{\tau s,x}u_{ys} + E^{13}_{\tau s,z}u_{zs} \\
\delta u_{z\tau}: \quad & P_{zs} = E^{45}_{\tau s,z}u_{xs} + E^{55}_{\tau s,z}u_{ys} + E^{55}_{\tau s}u_{zs,y} + E^{45}_{\tau s,x}u_{zs}
\end{aligned}
\tag{21}
$$

For a fixed approximation order N, Eq.s (19) and (21) have to be expanded using the indices τ and s in order to obtain the governing differential equations and the natural boundary conditions of the

desired model. The equations for classical TBM and EBBM could also be obtained from Eq.s (19) and (21). For instance, the governing equations of TBM can be found from Eq. (19) by expanding the terms u_{x1}, u_{y1}, u_{z1} (i.e. the translational displacements of the beam axis) and u_{y2}, u_{y3} (i.e. the rotations of the cross-section about the z- and x-axis). Similarily EBBM could also be captured from TBM formulation by choosing a proper combination of F_τ in order to write the rotations unknowns (i.e. u_{y2}, u_{y3}) as functions of the cross-sectional displacements u_{x1} and u_{z1}. However, in the present paper, EBBM is straightforwardly obtained from TBM by penalizing the shear stiffness as discussed in Section 2.1.

In the case of harmonic motion, the solution of Eq.s (19) is sought in the form

$$\mathbf{u}_s(y;t) = \mathbf{U}_s(y)\,\mathrm{e}^{\mathrm{i}\omega t} \tag{22}$$

where $\mathbf{U}_s(y)$ is the amplitude function of the motion, ω is an arbitrary circular or angular frequency, and i is $\sqrt{-1}$. Equations (22) allows the formulation of the equilibrium equations and the natural boundary conditions in the frequency domain. Substituting Eq. (22) into Eq.s (19), a set of three coupled ordinary differential equations is obtained which can be written in a matrix form as follows:

$$\delta\mathbf{U}_\tau: \quad \mathbf{L}^{\tau s}\,\tilde{\mathbf{U}}_s = 0 \tag{23}$$

where

$$\tilde{\mathbf{U}}_s = \left\{ \begin{array}{ccccccccc} U_{xs} & U_{xs,y} & U_{xs,yy} & U_{ys} & U_{ys,y} & U_{ys,yy} & U_{zs} & U_{zs,y} & U_{zs,yy} \end{array} \right\}^T \tag{24}$$

and $\mathbf{L}^{\tau s}$ is the 3×9 fundamental nucleus which contains the coefficients of the ordinary differential equations. The components of matrix $\mathbf{L}^{\tau s}$ are provided below and they are referred to as $L^{\tau s}_{(ij)}$, where

i is the row number ($i = 1, 2, 3$) and j is the column number ($j = 1, 2, ..., 9$)

$$
\begin{array}{lll}
L^{\tau s}_{(11)} = -\omega^2 E^{\rho}_{\tau s} + E^{22}_{\tau,xs,x} + E^{44}_{\tau,zs,z}, & L^{\tau s}_{(12)} = E^{26}_{\tau,xs} - E^{26}_{\tau s,x}, & L^{\tau s}_{(13)} = -E^{66}_{\tau s} \\
L^{\tau s}_{(14)} = E^{26}_{\tau,xs,x} + E^{45}_{\tau,zs,z}, & L^{\tau s}_{(15)} = E^{23}_{\tau s,x} - E^{66}_{\tau s,x}, & L^{\tau s}_{(16)} = -E^{36}_{\tau s} \\
L^{\tau s}_{(17)} = E^{12}_{\tau,xs,z} + E^{44}_{\tau,zs,x}, & L^{\tau s}_{(18)} = E^{45}_{\tau,zs} - E^{16}_{\tau s,z}, & L^{\tau s}_{(19)} = 0 \\
L^{\tau s}_{(21)} = E^{26}_{\tau,xs,x} + E^{45}_{\tau,zs,z}, & L^{\tau s}_{(22)} = E^{66}_{\tau,xs} - E^{23}_{\tau s,x}, & L^{\tau s}_{(23)} = -E^{36}_{\tau s} \\
L^{\tau s}_{(24)} = -\omega^2 E^{\rho}_{\tau s} + E^{66}_{\tau,xs,x} + E^{55}_{\tau,zs,z}, & L^{\tau s}_{(25)} = E^{36}_{\tau,xs} - E^{36}_{\tau s,x}, & L^{\tau s}_{(26)} = -E^{33}_{\tau s} \\
L^{\tau s}_{(27)} = E^{16}_{\tau,xs,z} + E^{45}_{\tau,zs,x}, & L^{\tau s}_{(28)} = E^{55}_{\tau,zs} - E^{13}_{\tau s,z}, & L^{\tau s}_{(29)} = 0 \\
L^{\tau s}_{(31)} = E^{44}_{\tau,xs,z} + E^{12}_{\tau,zs,x}, & L^{\tau s}_{(32)} = E^{16}_{\tau,zs,x} - E^{45}_{\tau s,z}, & L^{\tau s}_{(33)} = 0 \\
L^{\tau s}_{(34)} = E^{45}_{\tau,xs,z} + E^{16}_{\tau,zs,x}, & L^{\tau s}_{(35)} = E^{13}_{\tau,zs} - E^{55}_{\tau s,z}, & L^{\tau s}_{(36)} = 0 \\
L^{\tau s}_{(37)} = -\omega^2 E^{\rho}_{\tau s} + E^{44}_{\tau,xs,x} + E^{11}_{\tau,zs,z}, & L^{\tau s}_{(38)} = E^{45}_{\tau,xs} - E^{45}_{\tau s,x}, & L^{\tau s}_{(39)} = -E^{55}_{\tau s}
\end{array}
\tag{25}
$$

For a given expansion order, N, the equilibrium equations can be obtained in the form of Eq. (26) as given below by expanding $\mathbf{L}^{\tau s}$ for $\tau = 1, 2, ..., (N+1)(N+2)/2$ and $s = 1, 2, ..., (N+1)(N+2)/2$ as shown in Fig. 2. It reads:

$$\mathbf{L}\,\tilde{\mathbf{U}} = 0 \tag{26}$$

In a similar way, the boundary conditions of Eq.s (21) can be written in a matrix form as

$$\delta\mathbf{U}_\tau : \quad \mathbf{P}_\tau = \mathbf{B}^{\tau s}\,\hat{\mathbf{U}}_s \tag{27}$$

where

$$\hat{\mathbf{U}}_s = \left\{ \begin{array}{cccccc} U_{xs} & U_{xs,y} & U_{ys} & U_{ys,y} & U_{zs} & U_{zs,y} \end{array} \right\}^T \tag{28}$$

and $\mathbf{B}^{\tau s}$ is the 3×6 fundamental nucleus which contains the coefficients of the natural boundary

conditions

$$\mathbf{B}^{\tau s} = \begin{bmatrix} E^{26}_{\tau s,x} & E^{66}_{\tau s} & E^{66}_{\tau s,x} & E^{36}_{\tau s} & E^{16}_{\tau s,z} & 0 \\ E^{23}_{\tau s,x} & E^{36}_{\tau s} & E^{36}_{\tau s,x} & E^{33}_{\tau s} & E^{13}_{\tau s,z} & 0 \\ E^{45}_{\tau s,z} & 0 & E^{55}_{\tau s,z} & 0 & E^{45}_{\tau s,x} & E^{55}_{\tau s} \end{bmatrix} \tag{29}$$

For a given expansion order, N, the natural boundary conditions can be obtained in the form of Eq. (30) by expanding $\mathbf{B}^{\tau s}$ in the same way as $\mathbf{L}^{\tau s}$ to finally give

$$\mathbf{P} = \mathbf{B}\,\hat{\mathbf{U}} \tag{30}$$

3 Solution of the differential equations

The procedure to solve a system of ordinary differential equations of second order with constant coefficients is shown in APPENDIX A once the matrices $\tilde{\mathbf{S}}$ (Eq. (A.3)) and $\mathbf{S}$ (Eq. (A.7)) are formulated. As explained in APPENDIX A, a change of variables to reduce the second order system to a first order system is sought in the following form:

$$\begin{aligned} \mathbf{Z} = \left\{ \begin{array}{cccc} Z_1 & Z_2 & \dots & Z_{\mathcal{N}} \end{array} \right\}^T = \hat{\mathbf{U}} = \\ \left\{ \begin{array}{cccccccccccc} U_{x1} & U_{x1,y} & U_{y1} & U_{y1,y} & U_{z1} & U_{z1,y} & \dots & U_{xn} & U_{xn,y} & U_{yn} & U_{yn,y} & U_{zn} & U_{zn,y} \end{array} \right\}^T \end{aligned} \tag{31}$$

where $\hat{\mathbf{U}}$ is the expansion of $\hat{\mathbf{U}}_s$ for a given theory order, $n = 3 \times M$ is the number of the degrees of freedom for the given N-order beam theory, and $\mathcal{N} = 2 \times n$ is the dimension of the unknown vector as well as the number of differential equations. The main problem now is to find an algorithm to transform the expanded $\mathbf{L}$ matrix of Eq. (26) into the matrix $\tilde{\mathbf{S}}$. In fact, by looking at Eq. (A.2), it could be seen that there are only second derivatives on the left hand side (LHS) of the differential equations whereas by looking at Eq.s (19), it is clear that for each equations more than one second derivative appears. In order to obtain the matrix $\tilde{\mathbf{S}}$ from the $\mathbf{L}$ matrix, decoupling between the second derivatives can be done line by line so that only one second derivative remains on each line. Moreover, for each line, the coefficient of the second order derivative which is left has to be set as -1 by means of a factorization. By performing the above procedure on the $\mathbf{L}$ matrix and by removing the columns which contain the coefficient (-1) of the second order derivative, the matrix of the coefficients of the differential equations is formulated in the form of the matrix $\tilde{\mathbf{S}}$ as it appears in APPENDIX A. The procedure to transform the matrix $\mathbf{L}$ into the matrix $\tilde{\mathbf{S}}$ consists of performing a number of Gauss eliminations. This procedure is discussed in APPENDIX B. Subsequently, $\mathbf{S}$ can be

obtained from $\tilde{\mathbf{S}}$ by adding rows with 0's and 1's in order to account for the change of variables (see APPENDIX A, Eq.s (A.3) and (A.7)). Once the matrix $\tilde{\mathbf{S}}$ (Eq. (A.3)) is obtained, and subsequently transformed into $\mathbf{S}$ (Eq. (A.7)), by following the procedure in APPENDIX A, the solution can be written in matrix form as follows:

$$\begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{21} & \dots & \delta_{\mathcal{N}1} \\ \delta_{12} & \delta_{22} & \dots & \delta_{\mathcal{N}2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1\mathcal{N}} & \delta_{2\mathcal{N}} & \dots & \delta_{\mathcal{N}\mathcal{N}} \end{bmatrix} \begin{bmatrix} C_1 \mathrm{e}^{\lambda_1 y} \\ C_2 \mathrm{e}^{\lambda_2 y} \\ \vdots \\ C_{\mathcal{N}} \mathrm{e}^{\lambda_{\mathcal{N}} y} \end{bmatrix} \tag{32}$$

where λ_i is the i-th eigenvalue of the $\mathbf{S}$ matrix, δ_{ij} is the j-th element of the i-th eigenvector of the $\mathbf{S}$ matrix and C_i are the integration constants which need to be determined by using the boundary conditions. The above equation can be written in matrix form as:

$$\mathbf{Z} = \boldsymbol{\delta} \mathbf{C} \mathrm{e}^{\boldsymbol{\lambda} y} \tag{33}$$

It should be noted that the vector $\mathbf{Z}$ does not only contain the displacements but also their first derivatives which will come at hand when computing the boundary conditions. If only the displacements are needed, by recalling Eq. (31), only the lines $1, 3, 5, \dots, \mathcal{N}-1$ should be taken into account, giving a solution in the following form:

$$\begin{aligned} U_{x1}(y) &= C_1 \delta_{11} \mathrm{e}^{\lambda_1 y} + C_2 \delta_{21} \mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}} \delta_{\mathcal{N}1} \mathrm{e}^{\lambda_{\mathcal{N}} y} \\ U_{y1}(y) &= C_1 \delta_{13} \mathrm{e}^{\lambda_1 y} + C_2 \delta_{23} \mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}} \delta_{\mathcal{N}3} \mathrm{e}^{\lambda_{\mathcal{N}} y} \\ U_{z1}(y) &= C_1 \delta_{15} \mathrm{e}^{\lambda_1 y} + C_2 \delta_{25} \mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}} \delta_{\mathcal{N}5} \mathrm{e}^{\lambda_{\mathcal{N}} y} \\ &\vdots \\ U_{zn}(y) &= C_1 \delta_{1(\mathcal{N}-1)} \mathrm{e}^{\lambda_1 y} + C_2 \delta_{2(\mathcal{N}-1)} \mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}} \delta_{\mathcal{N}(\mathcal{N}-1)} \mathrm{e}^{\lambda_{\mathcal{N}} y} \end{aligned} \tag{34}$$

Once the displacements and their first derivatives are known, the boundary conditions can be easily obtained by remembering that $\hat{\mathbf{U}}$ is equal to $\mathbf{Z}$ (Eq. (31)) and by substituting the solution of Eq. (33) into the boundary conditions (Eq. (30)) to give

$$\mathbf{P} = \mathbf{B} \boldsymbol{\delta} \mathbf{C} \mathrm{e}^{\boldsymbol{\lambda} y} = \boldsymbol{\Lambda} \mathbf{C} e^{\boldsymbol{\lambda} y} \tag{35}$$

The boundary conditions can be written in explicit form as follows:

$$\begin{aligned}
P_{x1}(y) &= C_1\Lambda_{11}\mathrm{e}^{\lambda_1 y} + C_2\Lambda_{12}\mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}}\Lambda_{1\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} y} \\
P_{y1}(y) &= C_1\Lambda_{21}\mathrm{e}^{\lambda_1 y} + C_2\Lambda_{22}\mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}}\Lambda_{2\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} y} \\
P_{z1}(y) &= C_1\Lambda_{31}\mathrm{e}^{\lambda_1 y} + C_2\Lambda_{32}\mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}}\Lambda_{3\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} y} \\
&\vdots \\
P_{zn}(y) &= C_1\Lambda_{\mathcal{N}1}\mathrm{e}^{\lambda_1 y} + C_2\Lambda_{\mathcal{N}2}\mathrm{e}^{\lambda_2 y} + \ldots + C_{\mathcal{N}}\Lambda_{\mathcal{N}\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} y}
\end{aligned} \tag{36}$$

Although resorting to the $\mathbf{L}$ matrix seems extremely convoluted and complicated it is in fact the simplest and most effective way to solve the problem. The matrix $\mathbf{L}^{\tau s}$ is simply a different way to write the differential equations but the greatest advantage is that it allows for automatic formulation of the differential equations of any order beam theories in a systematic way. In sharp contrast to the structural problems sowed in the literature, where by using a Navier-type solution the system becomes algebraic, here by using $\mathbf{L}$ the differential equations can be written automatically, thus allowing the solution for any-order theory possible with relative ease.

4 The Dynamic Stiffness formulation

4.1 Dynamic Stiffness matrix

Once the boundary conditions (BCs) and displacements are expressed in terms of the $\mathcal{N}$ integration constants, the classical method to solve the problem would be to set $\mathcal{N}$ displacements and/or forces in order to eliminate the constants using the boundary conditions. These would give rise to the following scenarios (i) free boundaries: forces equal to zero at $y = 0$ and $y = L$; (ii) clamped boundaries: displacements equal zero at $y = 0$ and $y = L$; (iii) simply supported: a combination of displacements and forces equal to zero at $y = 0$ and $y = L$. A limitation to the classical method is that it can only be applied to study simple structures such as an individual structural element. By contrast, the solution obtained thus far, can be used to obtain the DS matrix of an element which can be assembled to obtain the closed form exact result for complex structures.
The procedure to obtain the DS matrix for a structural problem can be summarised as follows:

(i) Seek a closed form analytical solution of the governing differential equations of motion of the structural element in free vibration.

(ii) Apply a number of general boundary conditions equal to twice the number of integration

constants in algebraic form which are usually the nodal displacements and forces.

(iii) Eliminate the integration constants by relating the harmonically varying amplitudes of the generalized nodal forces to the corresponding generalized displacements which generates the frequency dependent DS matrix.

The closed form solution has already been found in the previous section and now the generic boundary conditions for generalized displacements and forces need to be applied (see Fig. 3) to develop the DS matrix.

Starting from the displacements, the boundary conditions can be written as

$$\begin{aligned} &\text{At} \quad y = 0 : \\ &\qquad U_{x1}(0) = -\overline{U}1_{x1} \\ &\qquad U_{y1}(0) = -\overline{U}1_{y1} \\ &\qquad U_{z1}(0) = -\overline{U}1_{z1} \\ &\qquad \vdots \\ &\qquad U_{zn}(0) = -\overline{U}1_{zn} \end{aligned} \tag{37}$$

$$\begin{aligned} &\text{At} \quad y = L : \\ &\qquad U_{x1}(L) = \overline{U}2_{x1} \\ &\qquad U_{y1}(L) = \overline{U}2_{y1} \\ &\qquad U_{z1}(L) = \overline{U}2_{z1} \\ &\qquad \vdots \\ &\qquad U_{zn}(L) = \overline{U}2_{zn} \end{aligned} \tag{38}$$

By evaluating Eq.s (34) in 0 and L and applying the boundary conditions of Eq.s (37) and (38), the following matrix relation for the nodal displacements is obtained:

$$\begin{Bmatrix} \overline{U}1_{x1} \\ \overline{U}1_{y1} \\ \overline{U}1_{z1} \\ \vdots \\ \overline{U}1_{zn} \\ \overline{U}2_{x1} \\ \overline{U}2_{y1} \\ \overline{U}2_{z1} \\ \vdots \\ \overline{U}2_{zn} \end{Bmatrix} = \begin{bmatrix} -\delta_{11} & -\delta_{21} & \dots & -\delta_{\mathcal{N}1} \\ -\delta_{13} & -\delta_{23} & \dots & -\delta_{\mathcal{N}3} \\ -\delta_{15} & -\delta_{25} & \dots & -\delta_{\mathcal{N}5} \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_{1(n-1)} & -\delta_{2(n-1)} & \dots & -\delta_{\mathcal{N}(n-1)} \\ \delta_{11}\mathrm{e}^{\lambda_1 L} & \delta_{21}\mathrm{e}^{\lambda_2 L} & \dots & \delta_{\mathcal{N}1}\mathrm{e}^{\lambda_{\mathcal{N}} L} \\ \delta_{13}\mathrm{e}^{\lambda_1 L} & \delta_{23}\mathrm{e}^{\lambda_2 L} & \dots & \delta_{\mathcal{N}3}\mathrm{e}^{\lambda_{\mathcal{N}} L} \\ \delta_{15}\mathrm{e}^{\lambda_1 L} & \delta_{25}\mathrm{e}^{\lambda_2 L} & \dots & \delta_{\mathcal{N}5\mathrm{e}^{\lambda_{\mathcal{N}} L}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1(n-1)}\mathrm{e}^{\lambda_1 L} & \delta_{2(n-1)}\mathrm{e}^{\lambda_2 L} & \dots & \delta_{\mathcal{N}(n-1)}\mathrm{e}^{\lambda_{\mathcal{N}} L} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \\ C_{n+1} \\ C_{n+2} \\ C_{n+3} \\ \vdots \\ C_{\mathcal{N}} \end{Bmatrix} \tag{39}$$

where $n = 3 \times M$ is the number of degrees of freedom (DOFs) per node. The above equation can be written in a more compact form as

$$\overline{\mathbf{U}} = \mathbf{AC} \tag{40}$$

Similarly, boundary conditions for generalized nodal forces are as follows:

$$\begin{aligned} \text{At} \quad & y = 0 : \\ & P_{x1}(0) = -\overline{P}1_{x1} \\ & P_{y1}(0) = -\overline{P}1_{y1} \\ & P_{z1}(0) = -\overline{P}1_{z1} \\ & \vdots \\ & P_{zn}(0) = -\overline{P}1_{zn} \end{aligned} \tag{41}$$

$$\begin{aligned} \text{At} \quad & y = L : \\ & P_{x1}(L) = \overline{P}2_{x1} \\ & P_{y1}(L) = \overline{P}2_{y1} \\ & P_{z1}(L) = \overline{P}2_{z1} \\ & \vdots \\ & P_{zn}(L) = \overline{P}2_{zn} \end{aligned} \tag{42}$$

By evaluating Eq.s (36) in 0 and L and applying the BCs of Eq.s (41) and (42), the following matrix relation for the nodal forces is obtained:

$$\left\{\begin{array}{c} \overline{P}1_{x1} \\ \overline{P}1_{y1} \\ \overline{P}1_{z1} \\ \vdots \\ \overline{P}1_{n1} \\ \overline{P}2_{x1} \\ \overline{P}2_{y1} \\ \overline{P}2_{z1} \\ \vdots \\ \overline{P}2_{n1} \end{array}\right\} = \left[\begin{array}{cccc} -\Lambda_{11} & -\Lambda_{12} & \dots & -\Lambda_{1\mathcal{N}} \\ -\Lambda_{21} & -\Lambda_{22} & \dots & -\Lambda_{2\mathcal{N}} \\ -\Lambda_{31} & -\Lambda_{32} & \dots & -\Lambda_{3\mathcal{N}} \\ \vdots & \vdots & \ddots & \vdots \\ -\Lambda_{n1} & -\Lambda_{n2} & \dots & -\Lambda_{n\mathcal{N}} \\ \Lambda_{11}\mathrm{e}^{\lambda_1 L} & \Lambda_{12}\mathrm{e}^{\lambda_2 L} & \dots & \Lambda_{1\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} L} \\ \Lambda_{21}\mathrm{e}^{\lambda_1 L} & \Lambda_{22}\mathrm{e}^{\lambda_2 L} & \dots & \Lambda_{2\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} L} \\ \Lambda_{31}\mathrm{e}^{\lambda_1 L} & \Lambda_{32}\mathrm{e}^{\lambda_2 L} & \dots & \Lambda_{3\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} L} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{n1}\mathrm{e}^{\lambda_1 L} & \Lambda_{n2}\mathrm{e}^{\lambda_2 L} & \dots & \Lambda_{n\mathcal{N}}\mathrm{e}^{\lambda_{\mathcal{N}} L} \end{array}\right] \left\{\begin{array}{c} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \\ C_{n+1} \\ C_{n+2} \\ C_{n+3} \\ \vdots \\ C_{\mathcal{N}} \end{array}\right\} \tag{43}$$

The above equation can be written in a more compact form as

$$\overline{\mathbf{P}} = \mathbf{R}\mathbf{C} \tag{44}$$

The constants vector $\mathbf{C}$ from Eq.s (40) and (44) can now be eliminated to give the DS matrix of one beam element as follows:

$$\overline{\mathbf{P}} = \boldsymbol{\mathcal{K}}\overline{\mathbf{U}} \tag{45}$$

where

$$\boldsymbol{\mathcal{K}} = \mathbf{R}\mathbf{A}^{-1} \tag{46}$$

is the required DS matrix. It should be noted that the DS matrix consists of both the inertia and stiffness properties of the structure element unlike the FEM for which they are separately identified.

4.2 Assembly of the DS elements

The DS matrix given by Eq. (46) is the basic building block to compute the exact natural frequencies of a higher-order beam. The DSM has also many of the general features of the FEM. In particular, it is possible to assemble elemental DS matrices to form the overall DS matrix of any complex structures consisting of beam elements (see Fig. 4). The global DS matrix can be written as

$$\overline{\mathbf{P}}_{\mathrm{G}} = \boldsymbol{\mathcal{K}}_{\mathrm{G}}\overline{\mathbf{U}}_{\mathrm{G}} \tag{47}$$

where $\boldsymbol{\mathcal{K}}_G$ is the square global DS matrix of the final structure. For the sake of simplicity, the subscript "G" is omitted hereafter.

4.3 Boundary conditions

The boundary conditions can be applied by using the well-known penalty method (often used in FEM) or by simply removing rows and columns of the stiffness matrix corresponding to the degrees of freedom which are zeroes. Due to the presence of higher-order degrees of freedom at each interface, a multitude of boundary condition can be applied at the required nodes. Although there are multiple possibilities, the implemented constrain types and the associated degrees of freedom that are penalized are as follow:

- Free end (F): no penalty;
- Clamped end (C): penalty applied to U_{x1}, U_{y1}, U_{z1}, U_{x2}, U_{y2}, U_{z2}, ..., U_{zn};
- Simply supported (SS): penalty applied to U_{x1}, U_{z1}, U_{x2}, U_{z2}, ..., U_{zn} at each end.

Note that it is possible to add a spring or a concentrated (lumped) mass at a node if required when using the DSM.

4.4 The Wittrick-Williams algorithm

For free vibration analysis of structures, FEM generally leads to a linear eigenvalue problem. By contrast, the DSM leads to a transcendental (non-linear) eigenvalue problem for which the Wittrick - Williams algorithm [60] is recognisably the best available solution technique at present. The basic working principle of the algorithm can be briefly summarised in the following steps:

(i) A trial frequency ω^* is chosen to compute the dynamic stiffness matrix $\boldsymbol{\mathcal{K}}^*$ of the final structure;

(ii) $\boldsymbol{\mathcal{K}}^*$ is reduced to its upper triangular form by the usual form of Gauss elimination to obtain ${\boldsymbol{\mathcal{K}}^*}^{\triangle}$ and the number of negative terms on the leading diagonal of ${\boldsymbol{\mathcal{K}}^*}^{\triangle}$ is counted; this is known as the sign count $s(\boldsymbol{\mathcal{K}}^*)$ of the algorithm;

(iii) The number, j, of natural frequencies (ω) of the structure which lie below the trial frequency (ω^*) is given by:

$$j = j_0 + s(\boldsymbol{\mathcal{K}}^*) \tag{48}$$

where j_0 is the number of natural frequencies of all individual elements with clamped-clamped (CC) boundary conditions on their opposite sides which still lie below the trial frequency ω^*.

Note that j_0 is required because the DSM allows for an infinite number of natural frequencies to be accounted for when all the nodes of the structure are fully clamped so that one or more individual elements of the structure can still vibrate on their own between the nodes. j_0 corresponds to $\overline{\mathbf{U}} = 0$ modes of Eq. (47) when $\overline{\mathbf{P}} = 0$. Assuming that j_0 is known, and $s(\boldsymbol{\mathcal{K}}^*)$ can be obtained by counting the number of negative terms in $\boldsymbol{\mathcal{K}}^{*^{\triangle}}$, a suitable procedure can be devised, for example the bi-section method, to bracket any natural frequency between an upper and lower bound of the trial frequency ω^* to any desired accuracy. The computation of j_0 can be cumbersome and may require additional analysis to compute the CC frequencies of the single elements within the structure. The problem can be overcome by splitting the element into many smaller elements for which the CC frequencies will be exceptionally high and hence j_0 will be zero within all practical range of frequency interest.

4.5 Mode shapes computation

Once the natural frequency has been computed and the related global DS matrix evaluated, the corresponding nodal generalized displacements can be obtained by solving the associated homogeneous system of Eq. (47). By utilizing the nodal generalized displacements $\overline{\mathbf{U}}$, the integration constants $\mathbf{C}$ of the element can be computed with the help of Eq. (40). In this way, using Eq. (34), the unknown generalized displacements can be computed as a function of y. Finally, by using Eq.s (1) and (22), the complete displacement field can be generated as a function of x, y, z and the time t (if an animated plot is needed). Clearly, the plot of the required mode and required element can be visualised on a fictitious 3D mesh. By following this procedure it is possible to compute the exact mode shapes using just one element which is impossible in FEM.

5 Numerical Results

The accuracy and computational efficiency of the present exact, higher-order DS elements are demonstrate by carrying out the free vibration analysis of both solid and thin-walled structures and the results are presented in this section. First, free vibration of beams with rectangular cross-section are addressed so as to make an easy and straightforward comparisons with classical beam theories. Exact DSM solutions are also compared with approximate results based on higher-order TE models built using FEM. Next, a thin-walled cylindrical cross-section is considered. Free vibration analysis is carried out for different BCs. Finally, a thin-walled beam with a semi-circular cross-section is analyzed to highlight the usefulness of the present method, particularly when capturing

the flexural-torsion coupling effects. The present DSM-CUF models are also compared with reference solutions from the literature together with the results obtained from the finite element commercial code MSC/NASTRAN®.

5.1 Solid Structures

A beam with a solid rectangular cross-section such as the one shown in Fig. 5 is considered first. For illustrative purposes, it is assumed that the beam has a square cross-section ($a = b$), with $b = 0.2$ m. The material data are: Young modulus, $E = 75$ GPa, Poisson ratio, $\nu = 0.33$, material density, $\rho = 2700$ Kg m^{-3}.

First, the predictable accuracy of the DSM when applied in the context of classical beam theories (EBBM,TBM) is established. Table 2 shows the first ten flexural frequencies in non-dimensional form ($\omega^* = \frac{\omega L^2}{b}\sqrt{\frac{\rho}{E}}$) for a simply-supported (SS) beam with a solid square cross-section and a L/b ratio equal to 10. The results are compared with 1D FEM solutions obtained using the commercial code MSC/NASTRAN®for which 1D FEM models were constructed by using 2-node CBAR elements. The generic one-dimensional MSC/NASTRAN®model is addressed here as "NAS1D$_{\text{el}}$" where the subscript stands for the number of beam elements. Figure 6 shows the convergence rate of the MSC/NASTRAN®models. Based on these results the following remarks can be made:

- FEM models require much finer meshes to achieve acceptable accuracy, particularly when higher frequencies are required.
- The DSM being distinct from the FEM, provides exact natural frequencies since it is mesh independent.

As far as higher-order beam theories are concerned, Tables 3 to 5 show results using both DSM and FEM solutions based on TE models. Approximate higher-order TE FEM results were obtained using the recent works by Carrera et al. [40, 41, 43, 42], which showed that TE models are able to deal with 3D-like solutions. Higher-order TE finite elements with 2 (B2), 3 (B3) and 4 (B4) nodes were used in the FEM solutions, or in other words, linear, quadratic and cubic approximations along the y-axis were adopted. Table 3 shows the first non-dimensional natural frequency of a SS square beam with a L/b ratio equal to 100. Column 1 shows the number of finite elements used in the analysis, whereas the second column quotes the element type. Columns 3 and 4 show the results by classical beam models (EBBM, TBM) alongside the results in Column 5 where the complete linear expansion model ($N = 1$) is considered. Column 6 shows the natural frequencies computed by using

the second-order ($N = 2$) TE model. For the same structure, Table 4 shows the second, the third and the fourth natural flexural frequencies for up-to-the-third TE models. Table 5 shows the first four flexural natural frequencies for a SS square beam with $L/b = 10$. Classical beam theories, linear ($N = 1$), quadratic ($N = 2$), cubic ($N = 3$) and fourth-order ($N = 4$) TE models are considered. It is clearly shown that, as far as FEM solutions of CUF higher-order models are concerned, the number of beam elements that are necessary to obtain accurate results - provided by the DSM - increases as natural frequencies as well as beam theory order increase. Figure 7 gives an estimation of the errors incurred when using FEM as opposed to DSM. The figure shows that FEM gives errors ranging from 0.005 % (finer mesh) to 19.7 % (coarse mesh).

Figure 8 shows the first three flexural modes of the beam with SS boundary conditions obtained from the DSM analysis when using a $N = 4$ TE model. It should be emphasized that DSM results are mesh independent and the mesh used in Fig. 8 is merely a plotting grid for convenience.

One of the most important features of the DSM is that it provides exact solutions for any kind of boundary conditions. Moreover, TE higher-order theories are able to take into account several non-classical effects such as warping, in-plane deformations, shear effects and flexural-torsion couplings. In Table 6, the first two flexural modes and the first two torsional modes for a clamped-free (CF) short ($L/b = 10$) square beam are shown. The exact solutions for classical, linear and higher-order beam theories are also shown and they were computed using the DSM. The results are compared with 3D FEM models using MSC/NASTRAN®. The generic three-dimensional FEM solution is herein referred to as "NAS3D$_{\text{el}}$" where the subscript "el" stands for the number of elements along one cross-sectional coordinate. In the results shown in Table 6, 3D FEM models are built using 8-node solid elements with an aspect-ratio equal to 10 were used. Figure 9 shows some representative modal shapes for the seventh-order TE model of the CF beam. Some comments are relevant:

- According to 3D MSC/NASTRAN®, the present lower-order DSM-TE models are able to characterize the flexural behaviour of solid cross-section beams.
- A fourth-order ($N = 4$) model is necessary to correctly detect torsional frequencies.

Finally, analyses were carried out for different values of the thickness for a rectangular cross-section beam with length-to-side ratio, L/a, equal to 10 and $a = 0.2$. The effect of the aspect ratio, a/b, on results is shown in Fig. 10, where the variation of the natural frequencies of the first flexural mode on plane xy and the first torsional mode is plotted for different values of a/b.

5.2 Thin-walled Structures

Free vibration analyses of a thin-walled cylindrical beam were carried out next. The cross-section of the beam is shown in Fig. 11. The outer diameter, d, was set to 2 m, whereas the thickness, t, was 0.02 m. The length-to-diameter ratio, L/d, was taken to be equal to 10. The cylinder was made of the same material as in the previous examples.

Table 7 shows the natural frequencies of the beam for different BCs, namely, free-free (FF), clamped-free (CF), clamped-clamped (CC), and simply-supported (SS) BCs. The natural frequencies, in Hz, of the first and second flexural, shell-like and torsional modes are shown. The solutions for classical beam theories and up-to-fifth-order TE models were obtained using the DSM. The results are compared to 2D FEM MSC/NASTRAN®solutions, which are referred to as NAS2D_{32}. The subscript "32" stands for the number of shell elements along the circumference. In the analysis, 4-node shell elements with an aspect-ratio equal to 10 were used. The authors have been highly selective when presenting the modes. Figure 12 shows the important modes of the cylinder for CC boundary condition. The following comments arise:

- Only the flexural modes are provided by the classical beam theories.
- Torsional modes are correctly detected by the linear TE ($N = 1$) model.
- 1D higher-order model are necessary to detect shell-like modes as evident from the 2D FEM solutions provided by MSC/NASTRAN®.

In Fig. 13 the positions of the first two flexural frequencies in the eigenvalue vector are plotted for different models of the free-free cylinder. According to Fig. 13, the following remarks can be made:

- In the case of classical theories and the lower-order TE models, the first two flexural frequencies hold the first two positions of the vector.
- New vibration modes appear as the model is refined. In particular, shell-like modes were found in between the flexural ones.

The ability of 1D CUF models in dealing with 2D shell-like analyses is widely documented in previous works, such as [40, 45, 61, 43]. In this paper, the attention is particularly focused on the advantages related to using the DSM formulation. As it has been said, the DSM provides the exact solution of the differential equations of the motion once the structural model has been formulated and importantly it has all the essential features of the FEM. The following analysis is carried out

to confirm the strength and elegance of the DSM when applied to refined 1D CUF models. A thin-walled beam with the semi-circular cross-section shown in Fig. 14 is analyzed. The geometrical dimensions are chosen from the literature [62, 63, 64, 57, 65, 66] so that a comparison of the results is possible. The radius, r, was assumed to be equal to 2.45×10^{-2} m, the thickness, t, was equal to 4×10^{-3} m, the length, L, of the beam was set to 0.82 m. The beam was made of aluminum with the Young modulus, E, equal to 68.9 GPa, the Poisson ratio, ν, equal to 0.3, and the density, ρ, equal to 2700 $\mathrm{Kg\,m^{-3}}$. Both clamped-free (CF) and simply supported (SS) boundary conditions were considered. Table 8 shows the first three coupled and uncoupled natural frequencies of the beam for cantilever boundary conditions and the results from the present theory are compared with those from the literature. The superiority of the present DSM-CUF models is clearly evident. Table 9 shows the first three coupled and uncoupled natural frequencies for a SS beam and also shows comparisons with the literature results. Figure 15 shows both the uncoupled and coupled modal shapes for the SS semi-circular beam. The following statements are worthy of careful study:

- The present models give good accuracy in the evaluation of the uncoupled frequencies for thin-walled beams.
- The present DS formulation is preferable to FEM in solving 1D CUF models, especially when higher-order natural frequencies.
- The increase in the order of the theory provides greater accuracy on the evaluation of the coupled frequencies. The present sixth-order ($N = 6$) DSM-TE model is able to compute the first two coupled frequencies for the SS beam, whereas only the first coupled frequency is detected if CF boundary conditions are considered.

6 Conclusions

A higher-order exact DS matrix has been developed using the CUF, which allows for the formulation of any-order beam theories by setting the expansion order as an input of the analysis. The resulting DS matrix is applied using the Wittrick-Williams algorithm to compute the natural frequencies and mode shapes of some solid and thin-walled structures. The results agree with those obtained using MSC/NASTRAN®FEM models and with those from the literature. The investigation provides optimism for future studies on the dynamic analysis of composite structures.

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APPENDIX A Solution of a system of second order differential equations

A system of differential equations of the second order in x can be written as

$$\frac{d^2\mathbf{y}(x)}{dx^2} = \ddot{\mathbf{y}}(x) = f\left(\mathbf{y}(x), \dot{\mathbf{y}}(x)\right) \tag{A.1}$$

where $\mathbf{y}(x) = [y_1, y_2, ..., y_n]^T$ are the n unknown functions. This can be written in matrix form as

$$\ddot{\mathbf{y}}(x) = \tilde{\mathbf{S}} \left\{ y_1 \ \dot{y}_1 \ y_2 \ \dot{y}_2 \ \dots \ y_n \ \dot{y}_n \right\}^T \tag{A.2}$$

where $\tilde{\mathbf{S}}$ is the matrix of coefficient whose dimension is $n \times 2n$ and can be written as:

$$\tilde{\mathbf{S}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & \dots & S_{1(2n-1)} & S_{1(2n)} \\ S_{21} & S_{22} & S_{23} & S_{24} & \dots & S_{2(2n-1)} & S_{2(2n)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{n1} & S_{n2} & S_{n3} & S_{n4} & \dots & S_{n(2n-1)} & S_{n(2n)} \end{bmatrix} \tag{A.3}$$

With a simple change of variables, the system of second order differential equations can be transformed into a system of first order differential equations. The change of variables is

$$\begin{aligned} & Z_1(x) = y_1(x) \ , \ Z_2(x) = \dot{y}_1(x) \\ & Z_3(x) = y_2(x) \ , \ Z_4(x) = \dot{y}_2(x) \\ & \vdots \\ & Z_{(2n-1)}(x) = y_n(x) \ , \ Z_{(2n)}(x) = \dot{y}_n(x) \end{aligned} \tag{A.4}$$

By doing this, a number of first order differential equations, such as $\dot{Z}_1 = Z_2$, $\dot{Z}_3 = Z_4$ and $\dot{Z}_{n-1} = Z_n$, will be added to the system of Eq. (A.1) - and consequently to Eq. (A.2) - which becomes a first order differential system. If the differential system is linear and the coefficients are constant, the set

of equations can be re-written in a matrix form as

$$\dot{\mathbf{Z}}(x) = \mathbf{S}\mathbf{Z}(x) \tag{A.5}$$

where the unknown functions are now:

$$\mathbf{Z} = \{Z_1\ Z_2\ Z_3\ Z_4\ \ldots\ Z_{2n-1}\ Z_{2n}\}^T = \{y_1\ \dot{y}_1\ y_2\ \dot{y}_2\ \ldots\ y_n\ \dot{y}_n\}^T \tag{A.6}$$

and the new matrix of coefficients $\mathbf{S}$, whose dimension now is $2n \times 2n$ can be written as:

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ S_{11} & S_{12} & S_{13} & S_{14} & \ldots & S_{1(2n-1)} & S_{1(2n)} \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ S_{21} & S_{22} & S_{23} & S_{24} & \ldots & S_{2(2n-1)} & S_{2(2n)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ S_{n1} & S_{n2} & S_{n3} & S_{n4} & \ldots & S_{n(2n-1)} & S_{n(2n)} \end{bmatrix} \tag{A.7}$$

The solution of first order differential equations of Eq. (A.5) can be written as

$$Z_i = \sum_{j=1}^{2n} C_j \delta_{ji} e^{\lambda_j x} \tag{A.8}$$

where C_j are the constant of integration, λ_j is the j-th eigenvalue of the matrix $\mathbf{S}$ and δ_{ji} is i-th value in the j-th eigenvector of the matrix $\mathbf{S}$. For the sake of simplicity, the solution for Z_1, i.e. y_1 (see Eq. (A.4)) is given in explicit form

$$y_1(x) = C_1\delta_{11}e^{\lambda_1 x} + C_2\delta_{21}e^{\lambda_2 x} + \ldots + C_{2n}\delta_{(2n)1}e^{\lambda_{2n} x} \tag{A.9}$$

if the eigenvectors are written as a matrix $\boldsymbol{\delta}$ in the following form:

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_{11} & \delta_{21} & \ldots & \delta_{(2n)1} \\ \delta_{12} & \delta_{22} & \ldots & \delta_{(2n)2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1(2n)} & \delta_{2(2n)} & \ldots & \delta_{(2n)(2n)} \end{bmatrix} \tag{A.10}$$

where, for δ_{ji}, j is the eigenvector number and i is the position in the eigenvector, and the eigenvalues with the constants are written in the following form:

$$\mathbf{Ce^{\lambda x}} = \left\{ C_1 e^{\lambda_1 x} \; C_2 e^{\lambda_2 x} \; \ldots \; C_{2n} e^{\lambda_{2n} x} \right\}^T \tag{A.11}$$

then the solution of Eq. (A.8) can be written in a more compact matrix form as

$$\mathbf{Z} = \boldsymbol{\delta} \mathbf{Ce^{\lambda x}} \tag{A.12}$$

APPENDIX B Forward and backward Gauss elimination

In this section, the procedure to transform the matrix $\mathbf{L}$ (Eq. (25)) into $\tilde{\mathbf{S}}$ (Eq. (A.3)) is described in details. In matrix $\mathbf{L}$, the coefficients of the second derivatives are located in the columns which are multiple of 3. In order to decouple the equations, the first row should have -1 in the third column and zero below it, the second row should have -1 in the sixth column and zeros above and below that and so on. This matrix has been called $\hat{\mathbf{L}}$.

Let us examine a 3 by 9 $\mathbf{L}$ matrix which is fully populated. The algorithm can easily be extended to a matrix of N by $N \times 3$ dimension. The matrix $\hat{\mathbf{L}}$ and subsequently the matrix $\tilde{\mathbf{S}}$ (see Eq. (A.3)) can be obtained by following four steps.

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & l_{19} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} & l_{27} & l_{28} & l_{29} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} & l_{36} & l_{37} & l_{38} & l_{39} \end{bmatrix} \tag{B.1}$$

(i) Forward Gauss elimination. Gauss elimination is carried out on entries below l_{13}, l_{26}. This is achieved by the following algorithm for the third column

$$\begin{aligned} l_{2i} &= l_{2i} - \frac{l_{23}}{l_{13}} l_{1i} \quad \text{for} \quad i = 1, \ldots, 9 \\ l_{3i} &= l_{3i} - \frac{l_{33}}{l_{13}} l_{1i} \quad \text{for} \quad i = 1, \ldots, 9 \end{aligned} \tag{B.2}$$

and for the sixth column[1]

$$l_{3i} = l_{3i} - \frac{l_{36}}{l_{26}} l_{2i} \quad \text{for} \quad i = 1, \ldots, 9 \tag{B.3}$$

[1]this algorithm can be generalised for any matrix dimension in a couple of lines

note that the name of the new element has not been changed for sake of simplicity.

The results would be a new $\mathbf{L}$ matrix in the following form

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & l_{19} \\ l_{21} & l_{22} & 0 & l_{24} & l_{25} & l_{26} & l_{27} & l_{28} & l_{29} \\ l_{31} & l_{32} & 0 & l_{34} & l_{35} & 0 & l_{37} & l_{38} & l_{39} \end{bmatrix} \tag{B.4}$$

(ii) Backward Gauss Elimination. As before but starting from the third row, ninth column and eliminating everything that is above that element in order to obtain the following new $\mathbf{L}$ matrix

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & 0 & l_{17} & l_{18} & 0 \\ l_{21} & l_{22} & 0 & l_{24} & l_{25} & l_{26} & l_{27} & l_{28} & 0 \\ l_{31} & l_{32} & 0 & l_{34} & l_{35} & 0 & l_{37} & l_{38} & l_{39} \end{bmatrix} \tag{B.5}$$

(iii) Factorisation. It is required to have -1 on the coefficient corresponding to the second derivative so to imply that if that coefficient were to be moved on the other side of the differential equation, its value would be 1. In order to do that the first row is divided by $-l_{13}$, the second by $-l_{26}$and the third by $-l_{39}$. in this way, the matrix $\hat{\mathbf{L}}$ can be obtained and it has the following form

$$\hat{\mathbf{L}} = \begin{bmatrix} l_{11} & l_{12} & -1 & l_{14} & l_{15} & 0 & l_{17} & l_{18} & 0 \\ l_{21} & l_{22} & 0 & l_{24} & l_{25} & -1 & l_{27} & l_{28} & 0 \\ l_{31} & l_{32} & 0 & l_{34} & l_{35} & 0 & l_{37} & l_{38} & -1 \end{bmatrix} \tag{B.6}$$

(iii) Eliminate the columns. By eliminating the columns corresponding to the position 3 and it multiples, is equal to move the term containing the second derivatives on the other side of the equations and give the matrix of coefficients associated to the second order differential equation. This matrix has been called $\tilde{\mathbf{S}}$ (see Eq. (A.3)) and following the notation in Eq. (B.6) it can be written as

$$\tilde{\mathbf{S}} = \begin{bmatrix} l_{11} & l_{12} & l_{14} & l_{15} & l_{17} & l_{18} \\ l_{21} & l_{22} & l_{24} & l_{25} & l_{27} & l_{28} \\ l_{31} & l_{32} & l_{34} & l_{35} & l_{37} & l_{38} \end{bmatrix} \tag{B.7}$$

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Tables

N	M	F_τ
0	1	$F_1 = 1$
1	3	$F_2 = x\ F_3 = z$
2	6	$F_4 = x^2\ F_5 = xz\ F_6 = z^2$
3	10	$F_7 = x^3\ F_8 = x^2z\ F_9 = xz^2\ F_{10} = z^3$
$\vdots$	$\vdots$	$\vdots$
N	$\frac{(N+1)(N+2)}{2}$	$F_{(N^2+N+2)/2} = x^N\ F_{(N^2+N+4)/2} = x^{N-1}\ \ldots\ F_{N(N+3)/2} = xz^{N-1}\ F_{(N+1)(N+2)/2} = z^N$

Table 1: MacLaurin's polynomials

Model	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9	Mode 10
$NAS1D_5$	2.811	10.745	21.997	32.700	-*	-	-	-	-	-
$NAS1D_7$	2.812	10.800	22.644	36.232	48.904	57.929	-	-	-	-
$NAS1D_{10}$	2.812	10.823	22.881	37.409	52.736	67.328	79.929	93.782	95.629	-
$NAS1D_{20}$	2.812	10.837	23.020	38.063	54.763	72.175	89.616	106.623	122.873	138.149
$NAS1D_{50}$	2.813	10.841	23.055	38.225	55.256	73.331	91.907	110.643	129.308	147.850
$NAS1D_{100}$	2.813	10.841	23.060	38.246	55.323	73.491	92.225	111.199	130.223	149.195
$NAS1D_{200}$	2.813	10.842	23.062	38.254	55.340	73.532	92.296	111.337	130.447	149.529
TBM - DSM	2.807	10.779	22.849	37.858	54.856	73.192	92.334	112.049	132.111	152.388
EBBM - DSM	2.838	11.213	24.742	42.847	64.869	90.330	117.859	147.586	178.779	211.040

* not provided by the model

Table 2: First 10 non-dimensional flexural frequencies $\omega^* = \frac{\omega L^2}{b}\sqrt{\frac{\rho}{E}}$ for the SS square beam, $L/b = 10$

No. Elem.	Elem. Type	EBBM	TBM	$N = 1$	$N = 2$
		FEM			
10	B2	2.885	2.885	2.885	2.890
20	B2	2.859	2.856	2.856	2.859
40	B2	2.852	2.852	2.852	2.852
10	B3	2.849	2.849	2.849	2.849
20	B3	2.849	2.849	2.849	2.849
		DSM			
		2.849	2.849	2.849	2.849

Table 3: First non-dimensional flexural frequency $\omega^* = \frac{\omega L^2}{b}\sqrt{\frac{\rho}{E}}$ for the SS square beam, $L/b = 100$

No. Elem.	Elem. Type	EBBM	TBM	$N=1$	$N=2$	$N=3$
		II Flexural Mode				
		FEM				
10	B2	11.979	11.974	11.974	12.062	12.062
20	B2	11.535	11.531	11.531	11.552	11.552
40	B2	11.430	11.426	11.426	11.430	11.430
10	B3	11.397	11.392	11.392	11.392	11.392
20	B3	11.395	11.390	11.390	11.390	11.390
40	B3	11.395	11.390	11.390	11.390	11.390
		DSM				
		11.395	11.390	11.390	11.390	11.390
		III Flexural Mode				
		FEM				
10	B2	28.748	28.714	28.714	29.215	29.210
20	B2	26.359	26.330	26.332	26.442	26.437
40	B2	25.810	25.786	25.786	25.812	25.803
10	B3	25.672	25.648	25.648	25.650	25.645
20	B3	25.636	25.610	25.610	25.610	25.607
40	B3	25.631	25.607	25.607	25.607	25.603
10	B4	25.634	25.607	25.607	25.607	25.605
20	B4	25.631	25.607	25.607	25.607	25.603
40	B4	25.631	25.607	25.607	25.607	25.603
		DSM				
		25.631	25.607	25.607	25.607	25.603
		IV Flexural Mode				
		FEM				
10	B2	56.157	56.026	56.026	57.855	57.831
20	B2	47.891	47.800	47.805	48.161	48.146
40	B2	46.122	46.041	46.041	46.127	46.112
10	B3	45.776	45.695	45.695	45.709	45.695
20	B3	45.569	45.490	45.490	45.493	45.481
40	B3	45.557	45.478	45.478	45.478	45.466
10	B4	45.557	45.478	45.478	45.481	45.468
20	B4	45.555	45.476	45.476	45.478	45.466
40	B4	45.555	45.476	45.476	45.478	45.464
		DSM				
		45.555	45.476	45.476	45.478	45.464

Table 4: Second to fourth non-dimensional flexural frequencies $\omega^* = \frac{\omega L^2}{b}\sqrt{\frac{\rho}{E}}$ for the SS square beam, $L/b = 100$

No. Elem.	Elem. Type	EBBM	TBM	$N=1$	$N=2$	$N=3$	$N=4$
			I Flexural Mode				
			FEM				
10	B2	2.873	2.842	2.842	2.847	2.843	2.843
20	B2	2.846	2.816	2.816	2.818	2.813	2.813
40	B2	2.840	2.809	2.809	2.810	2.806	2.806
10	B3	2.838	2.807	2.807	2.808	2.803	2.803
20	B3	2.838	2.807	2.807	2.808	2.803	2.803
			DSM				
		2.838	2.807	2.807	2.808	2.803	2.803
			II Flexural Mode				
			FEM				
10	B2	11.775	11.292	11.292	11.378	11.304	11.304
20	B2	11.350	10.904	10.904	10.931	10.864	10.863
40	B2	11.247	10.810	10.810	10.823	10.758	10.757
10	B3	11.216	10.782	10.782	10.791	10.726	10.725
20	B3	11.213	10.779	10.779	10.788	10.723	10.722
40	B3	11.213	10.779	10.779	10.787	10.723	10.722
10	B4	11.213	10.779	10.779	10.787	10.723	10.722
			DSM				
		11.213	10.779	10.779	10.787	10.723	10.722
			III Flexural Mode				
			FEM				
10	B2	27.587	25.209	25.209	25.611	25.266	25.260
20	B2	25.409	23.409	23.409	23.526	23.245	23.241
40	B2	24.905	22.988	22.988	23.042	22.775	22.771
10	B3	24.777	22.881	22.881	22.916	22.653	22.649
20	B3	24.743	22.852	22.852	22.886	22.623	22.619
40	B3	24.740	22.850	22.850	22.884	22.621	22.617
10	B4	24.740	22.850	22.850	22.884	22.621	22.617
20	B4	24.740	22.849	22.849	22.884	22.621	22.617
40	B4	24.740	22.849	22.849	22.884	22.621	22.617
			DSM				
		24.742	22.849	22.849	22.884	22.621	22.617
			IV Flexural Mode				
			FEM				
10	B2	51.823	44.543	44.543	45.676	44.680	44.647
20	B2	44.865	39.400	39.400	39.707	38.995	38.975
40	B2	43.339	38.236	38.237	38.371	37.713	37.697
10	B3	43.038	38.006	38.005	38.097	37.448	37.432
20	B3	42.860	37.868	37.868	37.950	37.309	37.292
40	B3	42.848	37.859	37.859	37.940	37.300	37.283
10	B4	42.849	37.860	37.860	37.941	37.301	37.284
			DSM				
		42.853	37.858	37.858	37.939	37.298	37.282

Table 5: First to fourth non-dimensional flexural frequencies $\omega^* = \frac{\omega L^2}{b}\sqrt{\frac{\rho}{E}}$ for the SS square beam, $L/b = 10$

Model	I Flexural	II Flexural	I Torsional	II Torsional
$NAS3D_{24}$	1.016	6.088	8.852	26.516
$NAS3D_{12}$	1.021	6.117	8.822	26.318
$N = 7$	1.012	6.067	8.863	26.588
$N = 6$	1.013	6.068	8.864	26.590
$N = 5$	1.013	6.069	8.868	26.603
$N = 4$	1.013	6.070	8.871	26.619
$N = 3$	1.014	6.075	9.631	28.893
$N = 2$	1.015	6.107	9.631	28.893
$N = 1$	1.008	6.069	9.631	28.893
TBM	1.008	6.069	-*	-
EBBM	1.013	6.276	-	-

*: not provided by the model

Table 6: Non-dimensional natural periods $\omega^* = \frac{\omega L^2}{b}\sqrt{\frac{\rho}{E}}$ for the CF square beam, $L/b = 10$

BCs	Model	I Flexural	II Flexural	I Shell-like	II Shell-like	I Torsional	II Torsional
FF	$NAS2D_{32}$	$30.829^{(7)}$	$76.806^{(31)}$	$14.129^{(1)}$	$14.171^{(3)}$	$80.415^{(39)}$	$160.810^{(93)}$
	$N = 5$	$30.932^{(7)}$	$77.041^{(23)}$	$17.709^{(1)}$	$17.777^{(3)}$	$80.788^{(27)}$	$161.576^{(51)}$
	$N = 4$	$30.932^{(7)}$	$77.043^{(17)}$	$22.987^{(1)}$	$23.053^{(3)}$	$80.789^{(19)}$	$161.577^{(33)}$
	$N = 3$	$30.935^{(3)}$	$77.090^{(9)}$	$22.987^{(1)}$	$34.700^{(5)}$	$80.789^{(11)}$	$161.576^{(19)}$
	$N = 2$	$31.345^{(1)}$	$80.336^{(3)}$	-*	-	$80.788^{(5)}$	$161.576^{(9)}$
	$N = 1$	$31.338^{(1)}$	$80.274^{(3)}$	-	-	$80.789^{(5)}$	$161.576^{(9)}$
	TBM	$31.338^{(1)}$	$80.275^{(3)}$	-	-	-	-
	EBBM	$31.892^{(1)}$	$85.030^{(3)}$	-	-	-	-
CF	$NAS2D_{32}$	$5.059^{(1)}$	$29.001^{(7)}$	$14.235^{(3)}$	$17.435^{(5)}$	$40.209^{(13)}$	$120.620^{(55)}$
	$N = 5$	$5.076^{(1)}$	$29.088^{(7)}$	$17.805^{(3)}$	$20.580^{(5)}$	$40.394^{(11)}$	$121.181^{(35)}$
	$N = 4$	$5.077^{(1)}$	$29.090^{(7)}$	$23.069^{(3)}$	$25.239^{(5)}$	$40.393^{(11)}$	$121.181^{(27)}$
	$N = 3$	$5.079^{(1)}$	$29.104^{(5)}$	$26.882^{(3)}$	$49.252^{(8)}$	$40.393^{(7)}$	$121.181^{(17)}$
	$N = 2$	$5.138^{(1)}$	$30.388^{(3)}$	-	-	$40.394^{(5)}$	$121.181^{(9)}$
	$N = 1$	$5.108^{(1)}$	$30.237^{(3)}$	-	-	$40.393^{(5)}$	$121.182^{(9)}$
	TBM	$5.108^{(1)}$	$30.237^{(3)}$	-	-	-	-
	EBBM	$5.147^{(1)}$	$31.724^{(3)}$	-	-	-	-
CC	$NAS2D_{32}$	$28.498^{(3)}$	$68.960^{(17)}$	$17.396^{(1)}$	$30.225^{(5)}$	$80.415^{(29)}$	$160.810^{(79)}$
	$N = 5$	$28.576^{(3)}$	$69.110^{(13)}$	$20.484^{(1)}$	$32.222^{(5)}$	$80.786^{(21)}$	$161.573^{(41)}$
	$N = 4$	$28.579^{(3)}$	$69.116^{(9)}$	$25.158^{(1)}$	$35.357^{(5)}$	$80.787^{(13)}$	$161.573^{(27)}$
	$N = 3$	$28.605^{(1)}$	$69.199^{(5)}$	$38.690^{(3)}$	$70.333^{(7)}$	$80.787^{(9)}$	$161.572^{(17)}$
	$N = 2$	$30.595^{(1)}$	$77.051^{(3)}$	-	-	$80.787^{(5)}$	$161.574^{(9)}$
	$N = 1$	$30.302^{(1)}$	$76.443^{(3)}$	-	-	$80.786^{(5)}$	$161.574^{(9)}$
	TBM	$30.302^{(1)}$	$76.443^{(3)}$	-	-	-	-
	EBBM	$32.601^{(1)}$	$88.087^{(3)}$	-	-	-	-
SS	$NAS2D_{32}$	$13.978^{(1)}$	$51.366^{(15)}$	$14.913^{(3)}$	$22.917^{(5)}$	$80.415^{(29)}$	$160.810^{(81)}$
	$N = 5$	$14.022^{(1)}$	$51.503^{(9)}$	$18.405^{(3)}$	$25.460^{(5)}$	$80.786^{(21)}$	$161.573^{(43)}$
	$N = 4$	$14.022^{(1)}$	$51.505^{(9)}$	$23.493^{(3)}$	$29.304^{(5)}$	$80.787^{(15)}$	$161.574^{(27)}$
	$N = 3$	$14.022^{(1)}$	$51.520^{(5)}$	$34.935^{(3)}$	$61.300^{(7)}$	$80.787^{(9)}$	$161.572^{(17)}$
	$N = 2$	$14.185^{(1)}$	$53.584^{(3)}$	-	-	$80.787^{(5)}$	$161.574^{(9)}$
	$N = 1$	$14.182^{(1)}$	$53.542^{(3)}$	-	-	$80.787^{(5)}$	$161.574^{(9)}$
	TBM	$14.182^{(1)}$	$53.542^{(3)}$	-	-	-	-
	EBBM	$14.402^{(1)}$	$56.605^{(3)}$	-	-	-	-

*: not provided by the model

Table 7: Natural frequencies (Hz) of the thin-walled cylinder for different boundary conditions. In brackets, the position of the frequencies in the eigenvalue vector is reported

	References					CUF - FEM [45]			CUF - DSM			
	Jun [62]	Jun [63]	Bercing [64]	Wittrick [57]	Friberg [65]	EBBM	$N=2$	$N=4$	EBBM	$N=2$	$N=4$	$N=6$
					Uncoupled frequencies							
1	31.80	-*	-	-	31.81	31.86	31.98	31.00	31.95	32.02	31.95	31.93
2	199.31	-	-	-	199.30	199.57	199.07	193.28	199.97	199.34	198.57	198.51
3	558.09	-	-	-	558.10	557.77	551.50	532.99	558.86	552.27	548.86	548.80
					Coupled frequencies							
1	63.79	63.50	63.51	60.21	63.79	72.84	72.81	67.33	73.00	72.90	68.63	64.41
2	137.68	137.38	137.39	128.3	137.7	453.71	445.18	357.81	454.65	445.73	349.40	276.83
3	278.35	275.81	275.82	257.90	278.40	1257.94	1202.23	593.76	1260.56	1065.97	592.78	481.96

*: not provided

Table 8: First three coupled and uncoupled natural frequencies (Hz) for a CF thin-walled beam with a semi-circular cross-section

	References			CUF - FEM [45]			CUF - DSM			
	Jun [62]	Jun [63]	Borbón [66]	EBBM	$N=2$	$N=4$	EBBM	$N=2$	$N=4$	$N=6$
				Uncoupled frequencies						
1	89.27	-*	89.23	89.32	89.15	86.28	89.66	89.48	89.44	89.44
2	365.81	-	364.31	354.85	352.16	353.12	358.18	355.40	354.71	354.61
3	803.50	-	-	785.65	773.21	761.81	804.20	790.40	786.93	786.07
				Coupled frequencies						
1	150.44	149.66	149.74	204.25	202.90	180.99	204.67	203.31	176.50	150.22
2	320.32	317.25	317.78	811.64	791.22	478.63	813.31	792.83	483.99	317.30
3	357.11	364.02	356.44	1806.57	1711.86	784.25	1810.32	1715.34	796.64	603.70

*: not provided

Table 9: First three coupled and uncoupled natural frequencies (Hz) for a SS thin-walled beam with a semi-circular cross-section

List of Figure Captions

- (Fig. 12) First flexural (a), second flexural (b), first shell-like (c), second shell-like (d), first torsional (e) and second torsional (f) modes for a CC thin-walled cylinder, $N = 5$
  - (a)
  - (b)
  - (c)
  - (d)
  - (e)
  - (f)
- (Fig. 13) Position in the eigenvalue vector of flexural and shell-like modal frequencies versus the beam model, FF thin-walled cylinder
- (Fig. 14) Semi-circular cross-section
- (Fig. 15) Uncoupled (a, b, c) and coupled (d, e, f) modal shapes for a SS semi-circular beam, $N = 6$
  - (a)
  - (b)
  - (c)
  - (d)
  - (e)
  - (f)