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Testing Downside Risk Efficiency Under Market Distress

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Abstract

In moments of distress downside risk measures like Lower Partial Moments (LPM) are more appropriate than the standard variance to characterize risk. The goal of this paper is to study how to compare portfolios in these situations. In order to do that we show the close connection between mean-risk efficiency sets and stochastic dominance under distress episodes of the market, and use the latter property to propose a hypothesis test to discriminate between portfolios across risk aversion levels. Our novel family of test statistics for testing stochastic dominance under distress makes allowance for testing orders of dominance higher than zero, for general forms of dependence between portfolios and can be extended to residuals of regression models. These results are illustrated in the empirical application for data from US stocks. We show that mean-variance strategies are stochastically dominated by mean-risk efficient sets in episodes of financial distress.

JEL classification: C1, C2, G1.

Keywords: Comovements, Downside risk, Lower partial moments, Market Distress, Mean-risk models, Mean-variance models, Stochastic dominance

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1 Introduction

It was Markowitz (1952) who formalized the concept of portfolio diversification by showing that investors should choose assets as if they care only about the mean and variance of the returns on an investment portfolio and therefore should penalize equally departures from expected wealth in both sides. Alternatively, Roy (1952) developed the concept of safety-first portfolios where investors' aim consisted on minimizing the likelihood of a dread event, this identified with an outcome in the tail of the distribution of portfolio returns. Roy, as Markowitz, also confined himself to distributions defined by the first two statistical moments. Following this alternative interpretation of risk Markowitz (1959) proposed the semivariance, risk measure that only focused on deviations of the return on the portfolio below a target return determined by the expected return on the investment or the return on the risk-free asset.

Hogan and Warren (1974), Bawa (1975), Arzac and Bawa (1977), and Bawa and Lindenberg (1977) continued on the idea of risk based on dread events introduced by Roy and proposed different risk measures based on penalizing the chance of these events. Thus, building on Roy's (1952) formulation of risk and extending the semivariance of Markowitz (1959) these authors introduced lower partial moments (LPM) of the distribution of returns to describe risk. Bawa ((1975), (1976), (1978)) provided a microeconomic foundation for these risk measures by introducing a family of utility functions consistent with them that described the preferences of downside risk averse investors. These functions take this form:

$$U(R^P; q, \tau) = R^P - k(\tau - R^P)^q I(R^P \leq \tau), \tag{1}$$

where R^P is the return on a portfolio P; τ is the threshold denoting the target return; k a scale parameter, $I(\cdot)$ an indicator function that takes the value one if $R^P \leq \tau$ and zero otherwise, and q the degree of risk aversion of the investor.

Bawa and Lindenberg (1977) and Harlow and Rao (1989) showed that the optimal portfolio choice of downside risk averse investors is the solution of the following equation,

$$\min_{w} LPM_q^P(\tau) = \int_{-\infty}^{\tau} (\tau - x)^q dF(x), \tag{2}$$

where with an abuse of notation x denotes the random variable R^P, with $R^P = \sum_{j=1}^{m} w_j R_j$, m the number of assets, $\mathbf{w} = (w_1, \ldots, w_m)$ is the vector of weights of each asset and $\sum_{j=1}^{m} w_j = 1$. This minimization problem is subject to the following budget constraint $\sum_{j=1}^{m} w_j E[R_j] \leq \mu^*(P)$, with $\mu^*(P)$ some target return level. The distribution of R^P is denoted $F(x)$ and the corresponding curve of all efficient portfolios is called mean-risk efficient set, in contrast to the mean-variance efficient set derived from minimizing the variance.

More recently, Granger (2002) discusses, from an econometrician point of view, lower partial moments risk measures in the spirit of those proposed in Fishburn (1977) as an alternative to processes concerned with describing the dynamics of the conditional variance. Fishburn (1977) in particular, explores the close relationship between mean-risk models derived from these downside risk measures and the concepts of mean-risk dominance and of stochastic dominance. This author shows that the efficient sets obtained from minimizing LPM_q measures are a subset of the different efficient first, second and third stochastic dominance sets. Similar results involving hypothesis tests for stochastic dominance between investment portfolios are found in Post (2003), Post and Versijp (2004) or Linton, Maasoumi, and Whang (2005). Related tests for the hypothesis in different contexts are found in McFadden (1989), Kaur, Rao and Singh (1994), Anderson (1996), Davidson and Duclos (2000) or Barret and Donald (2003).

The concept of stochastic dominance also encompasses under general conditions the mean-variance model. Gotoh and Konno (2000) and Manganelli (2007), among others, discuss the existence of mean-variance portfolio allocations that are dominated in the second order stochastic dominance sense for all risk-averse agents. This can be particularly remarkable in distress episodes of the market where portfolio diversification really matters. In these periods it is common to observe that uncorrelated assets co-move invalidating mean-variance strategies. It is important, therefore, to consider alternative diversification strategies under comovement periods.

The main aims of the paper are to extend the relationship between mean-risk and stochastic dominance efficient sets shown in Fishburn (1977) to distress episodes of the market, and to propose a hypothesis test for stochastic dominance between portfolios under distress. To test for relative optimality of these strategies we introduce consistent test statistics for testing the different forms of stochastic dominance and stochastic dominance under distress for orders of dominance greater or equal than zero. Furthermore, due to a decomposition of the relevant LPM measures introduced in this paper we are able to derive a simple and estimable form of the asymptotic distribution of the different test statistics for each family of hypothesis tests. Also, by a simple transformation of the test statistic our method allows to test the reverse stochastic dominance hypotheses using the same asymptotic critical values and therefore without any extra computational effort. Finally, as in Linton, Maasoumi and Whang (2005) and unlike Barret and Donald (2003), we make allowance for dependence between portfolios when testing for the different hypotheses, and discuss briefly the extension to testing stochastic dominance for residuals of regression models and time series.

In this way our study on stochastic dominance tests complements and extends the pioneering works of Barret and Donald (2003) and Linton, Maasoumi and Whang (2005) in three directions. First, the asymptotic distribution function of our test statistics for testing the relevant hypotheses have a close form easily estimable that allows to approximate the critical value of the tests in small samples without the need of multiplier methods as in Barret and Donald (2003) or subsampling methods as in Linton, Maasoumi and Whang (2005). Second, we use the concept of stochastic dominance in portfolio theory for testing for efficiency among investment portfolios; and finally we extend stochastic dominance tests to stochastic dominance under distress episodes of the market.

The paper is structured as follows. Section 2 introduces the definitions of stochastic dominance and conditional stochastic dominance and its relation with mean-risk efficiency under distress. Section 3 introduces different estimators of the LPM risk measures, derives the relevant hypothesis tests for testing stochastic dominance and conditional stochastic dominance, and the asymptotic theory. In Section 4 we carry out a Monte Carlo simulation experiment to study the size and power of the tests and compare our approximations to those obtained from the p-value transformation advocated in related papers. Section 5 compares the mean-variance and mean-risk efficient portfolios via stochastic and conditional stochastic dominance for real data from US equity market. Finally Section 6 concludes with the main findings of the paper. Proofs and tables are gathered in the appendix.

2 Mean-Risk and Stochastic Dominance Under Comovements

The efficient portfolio frontier in models in which risk is measured by probability weighted dispersions below a target is defined by those portfolios minimizing LPM_q measures under the constraints introduced in (2). Bawa and Lindenberg (1977) and Harlow and Rao (1989) show that these measures are consistent with the maximization of preferences of downside risk averse investors. Those portfolios in the efficient frontier satisfy the following result:

Result 1: (Fishburn (1977), page 118). *Portfolio A dominates Portfolio B in the mean-risk model defined at a τ level if and only if $\mu(A) \geq \mu(B)$ and $LPM_q^A(\tau) \leq LPM_q^B(\tau)$ for $q \geq 0$, with at least one strict inequality.*

The proof of this result is given by observing that

$$E[U(R^i; q, \tau)] = \mu(i) - k\, LPM_q^i(\tau), \tag{3}$$

with $\mu(i)$ denoting the expected values of the random variables R^i, $i = A, B$, and k a scale parameter.

Fishburn (1977) shows the existing close connection between the efficiency of LPM_q portfolios and their stochastic dominance over the rest of possible risky portfolios. Before elaborating on this result we introduce stochastic dominance between portfolios as discussed by this author.

Result 2: (Fishburn (1977), page 118).

- *A first stochastic dominates (FSD) B if and only if* $F^A \neq F^B$ *and* $LPM_0^A(\tau) \leq LPM_0^B(\tau)$ *for all* $\tau \in \Re$.
- *A second stochastic dominates (SSD) B if and only if* $F^A \neq F^B$ *and* $LPM_1^A(\tau) \leq LPM_1^B(\tau)$ *for all* $\tau \in \Re$.
- *A third stochastic dominates (TSD) B if and only if* $F^A \neq F^B$, $\mu(A) \geq \mu(B)$, *and* $LPM_2^A(\tau) \leq LPM_2^B(\tau)$ *for all* $\tau \in \Re$,

with F^A *and* F^B *the distribution functions of two portfolios A and B.*

In particular lemma 1 and theorem 3 in Fishburn (1977) show that if A FSD B then $\mu(A) > \mu(B)$ and $E[v^A(x)] \geq E[v^B(x)]$, for every nondecreasing real valued function $v(x)$ with expected value evaluated at F^A and F^B respectively; and therefore A dominates B in the mean-risk model for LPM_q measures for all $q \geq 0$ and $\tau \in \mathbb{R}$. In the same way if A SSD B then $\mu(A) \geq \mu(B)$ and $E[v^A(x)] \geq E[v^B(x)]$, for every nondecreasing and concave real valued function $v(x)$; and therefore A dominates B in the mean-risk model for LPM_q measures for all $q \geq 1$, except when $\mu(A) = \mu(B)$ and $LPM_q^A(\tau) = LPM_q^B(\tau)$ for all τ. Finally, if A TSD B then $\mu(A) \geq \mu(B)$ and $E[v^A(x)] \geq E[v^B(x)]$, for every nondecreasing and concave real valued function $v(x)$ for which $-\delta v(x)/\delta x$ is concave, $x \in \mathbb{R}$; and therefore A dominates B in the mean-risk model for LPM_q measures for all $q \geq 2$, except when $\mu(A) = \mu(B)$ and $LPM_q^A(\tau) = LPM_q^B(\tau)$ for all τ. Therefore these results show that efficient portfolio sets corresponding to investors minimizing LPM_q measures are a subset of the FSD efficient set for $q \geq 0$; of the SSD efficient set for $q \geq 1$ and of the TSD efficient set for $q \geq 2$; except in the noted cases.

In what follows we extend the results on stochastic dominance shown above to a setting characterized by periods of market distress. This phenomenon is identified in this paper with a state of the market where the return on every risky asset is below a threshold u. This will be measured by $P(R_1 \leq u, \ldots, R_m \leq u)$ and denoted throughout $\lambda(u)$. In this context we define the following risk measure

$$LPM_{q,u}^{P}(\tau) = \int_{-\infty}^{\tau} (\tau - x)^q dF_u(x), \tag{4}$$

where $F_u(x) := P\left(R^P \leq x | R_1 \leq u, \ldots, R_m \leq u\right)$ denotes the distribution function of the returns on portfolio P conditional on being on a comovement regime.

The next proposition shows a very helpful decomposition of the risk measures in (2) and (4) that will enable us to derive the asymptotic distribution of the relevant test statistics and that, in contrast to existing literature, can be easily estimated for any order of q. Specifically, our decomposition improves Anderson (1996) that uses a trapezoidal approximation of the LPM-integrals, and Davidson and Duclos (2000) and Barret and Donald (2003) that integrate directly the empirical processes. Before introducing the different decompositions we need the following three assumptions.

Assumption A.1: The vector of weights characterizing portfolio P satisfies that $0 \leq w_j \leq 1$, for all j, and $\sum_{j=1}^{m} w_j = 1$.

Assumption A.2: The distribution functions $F(\tau)$, $F_u(\tau)$, $LPM_{0,u}^{c}(\tau) := F_u^c(\tau)$ defined by the probability $P\left(R^P \leq \tau | R_1 > u \text{ or } R_2 > u \text{ or} \ldots \text{or } R_m > u\right)$, with the superscript c denoting the complementary conditioning event, and $\lambda(u) = P\left(R_1 \leq u, \ldots, R_m \leq u\right)$ are continuous and differentiable in the $\mathbb{R}$ and $\mathbb{R}^m$ domain respectively.

Assumption A.3: Let q define the intensity of risk aversion in utility function (1). Then $E[(R^P)^q] < \infty$ for R^P the return on portfolio P.

Assumption A.1 ensures that investors can only take long positions in the assets comprising the portfolio and implies that $LPM_{0,u}(u) = 1$. This assumption is very standard in the literature, see for instance Post (2003). Assumption A.2 and A.3 guarantee the existence of the different LPM measures determined by q.

Proposition 1: *Assume A.1-A.3 hold, and let $LPM_q^P(\cdot)$ and $LPM_{q,u}^P(\cdot)$ for $q \geq 0$ be the downside risk measures defined in* (2) *and* (4) *respectively. Then*

$$LPM_q^P(\tau) = E[(\tau - R^P)^q | R^P \leq \tau] LPM_0^P(\tau), \tag{5}$$

and

$$LPM_{q,u}^P(\tau) = E[(\tau - R^P)^q | R^P \leq \tau, R_1 \leq u, \ldots, R_m \leq u] LPM_{0,u}^P(\tau). \tag{6}$$

Proposition 1 can be used to derive a decomposition of the unconditional downside risk measure for any order q. For example, for $q = 0$ the conditional probability theorem implies that $LPM_0^P(\tau)$ can be decomposed as

$$LPM_0^P(\tau) = \lambda(u)LPM_{0,u}^P(\tau) + (1 - \lambda(u))LPM_{0,u}^{c\,P}(\tau). \tag{7}$$

The following corollary extends this decomposition to higher orders of q.

Corollary 1: *Let LPM_q^P for $q \geq 0$ be the downside risk measure defined in* (2)*. Then*

$$LPM_q^P(\tau) = \lambda(u)\gamma_{q,u}(\tau)LPM_{q,u}^P(\tau) + (1 - \lambda(u))\gamma_{q,u}^c(\tau)LPM_{q,u}^{c\,P}(\tau), \tag{8}$$

with $\gamma_{q,u}(\tau) = \frac{E[(\tau - R^P)^q | R^P \leq \tau]}{E[(\tau - R^P)^q | R^P \leq \tau, R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u]}$ and
$\gamma_{q,u}^c(\tau) = \frac{E[(\tau - R^P)^q | R^P \leq \tau]}{E[(\tau - R^P)^q | R^P \leq \tau, R_1 > u \text{ or } R_2 > u \text{ or} \ldots \text{or } R_m > u]}$.

Furthermore, under comovements $LPM_q^P(\tau) = \gamma_{q,u}(\tau)LPM_{q,u}^P(\tau)$.

This decomposition allow us to disentangle the risk exposure of the portfolio due to the probability $\lambda(u)$ of market distress, from the risk exposure produced by the allocation of weights in each market regime. In particular there can be two portfolios A and B consisting of different assets and such that $LPM_q^A(\tau) \leq LPM_q^B(\tau)$ for every $\tau \in \mathbb{R}$, but not under comovements. In this scenario there can be other asset allocations more efficient to diversify risk. This is explored in the remaining of the section. Following result 2 we define first the concept of stochastic dominance conditional on comovements.

Definition 1:

- *A first conditional stochastic dominates (FCSD) B if and only if $F_u^A \neq F_u^B$ and $LPM_{0,u}^A(\tau) \leq LPM_{0,u}^B(\tau)$ for all $\tau \leq u$.*
- *A second conditional stochastic dominates (SCSD) B if and only if $F_u^A \neq F_u^B$ and $LPM_{1,u}^A(\tau) \leq LPM_{1,u}^B(\tau)$ for all $\tau \leq u$.*
- *A third conditional stochastic dominates (TCSD) B if and only if $F_u^A \neq F_u^B$, $\mu_u(A) \geq \mu_u(B)$, and $LPM_{2,u}^A(\tau) \leq LPM_{2,u}^B(\tau)$ for all $\tau \leq u$,*

with F_u^A and F_u^B the relevant conditional distribution functions introduced before, and $\mu_u(A) := E[R^A | R_1 \leq u, \ldots, R_m \leq u]$ and $\mu_u(B) := E[R^B | R_1 \leq u, \ldots, R_m \leq u]$ the corresponding conditional expected values.

Using lemma 1 in Fishburn (1977) we obtain that if A FCSD B then $\mu_u(A) > \mu_u(B)$ and $E_u[v^A(x)] \geq E_u[v^B(x)]$ for every nondecreasing real valued function $v(x)$ with expected value evaluated at F_u^A and F_u^B, respectively. If A SCSD B then $\mu_u(A) \geq \mu_u(B)$ and $E_u[v^A(x)] \geq E_u[v^B(x)]$ for every nondecreasing and concave real valued function $v(x)$; and finally, if A TCSD B then $\mu_u(A) \geq \mu_u(B)$ and $E_u[v^A(x)] \geq E_u[v^B(x)]$, for every nondecreasing and concave real valued function $v(x)$ for which $-\delta v(x)/\delta x$ is concave, with $x \in \mathbb{R}$. In the particular case $v = U$, with $U(\cdot)$ defined in (1), the definition above allows us to extend naturally the relationship between mean-risk and stochastic dominance efficient frontiers to a conditional environment characterized by the occurrence of market distress.

Theorem 1:

- *If A FCSD B then A dominates B in the mean-risk model defined by $LPM_{q,u}$ measures for all $q \geq 0$.*
- *If A SCSD B then A dominates B in the mean-risk model defined by $LPM_{q,u}$ measures for all $q \geq 1$, except when $\mu_u(A) = \mu_u(B)$ and $LPM_{q,u}^A(\tau) = LPM_{q,u}^B(\tau)$ for all $\tau \leq u$.*
- *If A TCSD B then A dominates B in the mean-risk model defined by $LPM_{q,u}$ measures for all $q \geq 2$, except when $\mu_u(A) = \mu_u(B)$ and $LPM_{q,u}^A(\tau) = LPM_{q,u}^B(\tau)$ for all $\tau \leq u$.*

This result entails different optimal portfolio choices contingent on the state of the market. In order to make the conditions for stochastic dominance in Fishburn (1977) and in theorem 1 above statistically testable we will develop in the next section hypothesis tests for unconditional stochastic dominance and stochastic dominance under distress of different orders.

3 Estimation and Inference

Suppose we have n independent and identically distributed vectors of observations obtained from m different random variables $R_1, \ldots, R_m$. Then, natural estimators of $LPM_0(\tau)$ and $LPM_{0,u}(\tau)$, for τ nonstochastic are

$$\widehat{LPM}_0(\tau) := \frac{1}{n}\sum_{i=1}^{n} I(R_i^P \leq \tau), \tag{9}$$

and

$$\widehat{LPM}_{0,u}(\tau) := \frac{1}{n_u}\sum_{i=1}^{n} I(R_i^P \leq \tau | R_{1,i} \leq u, R_{2,i} \leq u, \ldots, R_{m,i} \leq u), \tag{10}$$

with n_u the number of vectors satisfying $R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u$. The multivariate version of these empirical estimators is employed to estimate $\lambda(u)$. Thus,

$$\widehat{\lambda}(u) := \frac{1}{n}\sum_{i=1}^{n} I(R_{1,i} \leq u, R_{2,i} \leq u, \ldots, R_{m,i} \leq u). \tag{11}$$

The different expected values necessary to compute LPM_q measures of higher orders are estimated by their corresponding empirical counterparts

$$\widehat{E}[(\tau - R^P)^q | R^P \leq \tau] := \frac{1}{n_p}\sum_{i=1}^{n} (\tau - R_i^P)^q I(R_i^P \leq \tau), \tag{12}$$

and

$$\widehat{E}[(\tau - R^P)^q | R^P \leq \tau, R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u] :=$$

$$= \frac{1}{n'_p}\sum_{i=1}^{n} (\tau - R_i^P)^q I(R_i^P \leq \tau, R_{1,i} \leq u, \ldots, R_{m,i} \leq u), \tag{13}$$

with n_p the number of observations in the sample satisfying $R^P \leq \tau$ and n'_p the number of observations satisfying $R^P \leq \tau$ and $R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u$.

By the strong law of large numbers in the univariate and multivariate setting and by Slutsky theorem these estimators and linear functions of them necessary to estimate LPM_q and $LPM_{q,u}$ are strongly consistent estimators of the population parameters for $n'_p \to \infty$. Note that this implies n_p, $n_u \to \infty$ since $n'_p \leq n_p$, $n'_p \leq n_u$.

These estimators allow to construct consistent tests for the hypotheses involving different types of stochastic dominance and for any order q. Since we are interested in a portfolio investment environment we will concentrate on first, second and third orders of stochastic dominance, although our results can be extended to any q.

3.1 A Hypothesis Test for Stochastic Dominance

This is an open problem widely investigated in economics and finance in general; and in particular, in the income distribution literature and more recently in portfolio theory, see McFadden (1989), Larsen and Resnick (1993), Kaur, Rao and Singh (1994), Anderson (1996), Davidson and Duclos (2000), Barret and Donald (2003) or recently Linton, Maasoumi and Whang (2005) and Davidson and Duclos (2006). Our approach for testing stochastic dominance differs from these influential papers in three aspects: first, due to the decompositions of the LPM measures in proposition 1 we can test for any order of stochastic dominance by using simple modifications of the test statistics. Further, the critical values of the asymptotic distribution of the tests can be approximated by uniformly consistent estimation procedures. Second, the different tests for stochastic dominance make allowance for dependence between portfolios;

and third, we extend these tests to scenarios of market distress, characterized by values of the vector of random variables comprising the portfolio below a given threshold u.

Our test statistic is of Kolmogorov-Smirnov type and shares the spirit of the test statistic proposed in McFadden (1989), Anderson (1996), Davidson and Duclos (2000) or more recently in Barret and Donald (2003). Since the utility function (1) is increasing for $q = 0$ and nondecreasing and concave for $q > 0$ the results in Fishburn (1977) apply, and we can focus on the hypothesis test

$$\begin{cases} H_{0,\gamma} : LPM_{\gamma}^{A}(\tau) \leq LPM_{\gamma}^{B}(\tau), & \text{for all } \tau \in \mathbb{R}, \\ H_{1,\gamma} : LPM_{\gamma}^{A}(\tau) > LPM_{\gamma}^{B}(\tau), & \text{for some } \tau \in \mathbb{R}, \end{cases} \tag{14}$$

rather than on the strict inequality for testing first ($\gamma = 0$), second ($\gamma = 1$)[1] and third ($\gamma = 2$) stochastic dominance between two portfolios A and B. Alternatively, and following the notation in Linton, Maasoumi and Whang (2005) we define $D_{\gamma}(\tau) := LPM_{\gamma}^{A}(\tau) - LPM_{\gamma}^{B}(\tau)$ and write the hypothesis test above as

$$\begin{cases} H_{0,\gamma} : D_{\gamma}(\tau) \leq 0, & \text{for all } \tau \in \mathbb{R}, \\ H_{1,\gamma} : D_{\gamma}(\tau) > 0, & \text{for some } \tau \in \mathbb{R}. \end{cases} \tag{15}$$

Under $H_{0,0}$ A dominates B in the mean-risk sense for risk-neutral and risk-averse investors, under $H_{0,1}$ A dominates B for risk-averse investors except when $\mu(A) = \mu(B)$, and under $H_{0,2}$ and $\mu(A) \geq \mu(B)$ A dominates B for risk-averse investors with increasing absolute risk aversion levels. Other testing methods for this hypothesis reverse the roles of the hypotheses and have the alternative hypothesis as corresponding to strong stochastic dominance. These methods are formulated using a slightly different definition of stochastic dominance that involves strict inequality (strong stochastic dominance) in (14), and are usually based on the minimum distance rather than on the maximum, see for example Kaur, Rao and Singh (1994).

The asymptotic theory for LPM risk measures determined by τ fixed is given in the following proposition. In contrast to most of the existing literature this result is possible for general orders of q due to the decompositions discussed in proposition 1.

Proposition 2: *Suppose we have n independent and identically distributed observations from a random variable R, and let $\widehat{LPM}_{\gamma}(\tau)$ be a $\sqrt{n}-$consistent estimator of $LPM_{\gamma}(\tau)$, and*

[1]Hereafter q denotes the order of investor's risk aversion and γ the order of stochastic dominance.

assume A.1-A.3 hold. Then

$$\sqrt{n}\left(\widehat{LPM}_{\gamma}(\tau)-LPM_{\gamma}(\tau)\right)\xrightarrow{d}N\left(0,E[(\tau-R)^{2\gamma}|R\leq\tau]F(\tau)-E[(\tau-R)^{\gamma}|R\leq\tau]^{2}F^{2}(\tau)\right),\tag{16}$$

for all fixed τ in the real line, and $\gamma \geq 0$.

Before introducing the asymptotic theory relevant to the composite hypothesis test we need the following notation and two further assumptions. Let A and B denote two portfolios with returns characterized by two random variables R^A and R^B respectively. Denote $F^{A,B}(\tau,\tau) := P(R^A \leq \tau, R^B \leq \tau)$, $k^i_\gamma(\tau) = E[(\tau - R^i)^\gamma | R^i \leq \tau]$ with $i = A, B$, and $k^{A,B}_\gamma(\tau,\tau) = E[(\tau - R^A)^\gamma(\tau - R^B)^\gamma | R^A \leq \tau, R^B \leq \tau]$, and $\Sigma(\tau,\tau)$ is the asymptotic covariance function of the vector $(\widehat{LPM}^A_\gamma(\tau) - LPM^A_\gamma(\tau), \widehat{LPM}^B_\gamma(\tau) - LPM^B_\gamma(\tau))$.

Assumption A.4: $\inf\limits_{\tau,\tau^*\in\mathbb{R}} det(\Sigma(\tau,\tau^*)) > 0$.

Assumption A.5: The empirical counterparts of $k^A_\gamma(\tau)$, $k^B_\gamma(\tau)$ and $k^{A,B}_\gamma(\tau,\tau^*)$ introduced above converge uniformly to $k^A_\gamma(\tau)$, $k^B_\gamma(\tau)$ and $k^{A,B}_\gamma(\tau,\tau^*)$, respectively, over $\tau,\tau^* \in \mathbb{R}$.

Assumption A.4 ensures that result (16) can be extended to describe the asymptotic bivariate distribution of $\widehat{LPM}^A_\gamma(\tau)$ and $\widehat{LPM}^B_\gamma(\tau)$ for all fixed $\tau \in \mathbb{R}$. Assumption A.5 and Glivenko-Cantelli theorem ensure the uniform convergence of the different estimators to the parameters of interest. Now, the Cramer-Wold device guarantees that the limit distribution of the difference between the random variables also converges to a normal distribution. Then

$$\sqrt{n}\left(\widehat{D}_\gamma(\tau) - D_\gamma(\tau)\right) \xrightarrow{d} N\left(0, V_\gamma(\tau)\right),\tag{17}$$

with

$$\begin{aligned}V_\gamma(\tau) = \left(k^A_{2\gamma}(\tau)F^A(\tau) - (k^A_\gamma(\tau)F^A(\tau))^2\right) + \left(k^B_{2\gamma}(\tau)F^B(\tau) - (k^B_\gamma(\tau)F^B(\tau))^2\right) - \\ 2\left(k^{A,B}_\gamma(\tau,\tau)F^{A,B}(\tau,\tau) - k^A_\gamma(\tau)F^A(\tau)k^B_\gamma(\tau)F^B(\tau)\right).\end{aligned}$$

Furthermore, this result can be extended to the associated continuous random process indexed by $\tau \in \mathbb{R}$.

Theorem 2: *Under assumptions A.1-A.5,*

$$\sqrt{n}\sup_{\tau\in\mathbb{R}}(\widehat{D}_\gamma(\tau) - D_\gamma(\tau)) \xrightarrow{d} \sup_{\tau\in\mathbb{R}} G_\gamma(\tau), \tag{18}$$

with $G_\gamma(\tau)$ a Gaussian process with zero mean and covariance function given by

$$E[G_\gamma(\tau_s)G_\gamma(\tau_t)] = \tag{19}$$

$$\begin{aligned}
&\big(k_{2\gamma}^A(\tau_s\wedge\tau_t)F^A(\tau_s\wedge\tau_t) - k_\gamma^A(\tau_s)F^A(\tau_s)k_\gamma^A(\tau_t)F^A(\tau_t)\big) + \\
&\big(k_{2\gamma}^B(\tau_s\wedge\tau_t)F^B(\tau_s\wedge\tau_t) - k_\gamma^B(\tau_s)F^B(\tau_s)k_\gamma^B(\tau_t)F^B(\tau_t)\big) - \\
&\big(k_\gamma^{A,B}(\tau_s,\tau_t)F^{A,B}(\tau_s,\tau_t) - k_\gamma^A(\tau_s)F^A(\tau_s)k_\gamma^B(\tau_t)F^B(\tau_t)\big) - \\
&\big(k_\gamma^{A,B}(\tau_t,\tau_s)F^{A,B}(\tau_t,\tau_s) - k_\gamma^A(\tau_t)F^A(\tau_t)k_\gamma^B(\tau_s)F^B(\tau_s)\big),
\end{aligned}$$

for all $\tau_s, \tau_t \in \mathbb{R}$, and $0 \le \gamma \le q$.

Remark: For the multivariate version of (19) defined by a finite grid of points in the real line $-\infty < \tau_1 < \tau_2 < \ldots < \tau_t < \infty$ we observe that $\Sigma(\tau_s,\tau_t) := E[G_\gamma(\tau_s)G_\gamma(\tau_t)]$ for $s, t = 1, \ldots$

Our family of test statistics is defined by $T_{n,\gamma} := \sqrt{n}\sup_{\tau\in\mathbb{R}}\widehat{D}_\gamma(\tau)$. The null hypothesis is the equality of functions $F^A(\tau) = F^B(\tau)$ for every $\tau\in\mathbb{R}$. Under A.1-A.5, and $H_{0,\gamma}$,

$$T_{n,\gamma} \xrightarrow{d} \sup_{\tau\in\mathbb{R}} G_\gamma(\tau). \tag{20}$$

Further, the asymptotic critical values of these tests indexed by γ are given by

$$c_\gamma(1-\alpha) := \inf_{x\in\mathbb{R}}\{x \,|\, P\left(\sup_{\tau\in\mathbb{R}} G_\gamma(\tau) \le x\right) \ge 1-\alpha\}, \tag{21}$$

with α denoting the significance level.

Barret and Donald (2003) and particularly Linton, Maasoumi and Whang (2005) discuss the problem of assuming equality of functions for the null hypothesis version of the test. These authors argue that the convergence of test statistics of Kolmogorv-Smirnov and Cramér-von Mises type is not uniform over the probabilities under the null hypothesis. More recently, Linton, Song and Whang (2008) show that discontinuity of convergence arises precisely between the interior points of the null hypothesis and the boundary points of the null hypothesis. In order to solve this these authors propose bootstrap procedures to obtain stochastic dominance tests with asymptotic coverage exactly equal to the nominal level of the test over the boundary of points and therefore valid over the whole null hypothesis. We will not discuss this technical issue further in the paper and will derive asymptotic critical values with correct coverage under the least favorable case (equality of distributions) that in our testing framework coincides with the boundary of the null hypothesis.

Proposition 3: *Given Assumptions A.1-A.5 and the test statistic $T_{n,\gamma}$, then:*

(i) *Under $H_{0,\gamma}$,*

$$\lim_{n\to\infty} P(\textit{reject } H_{0,\gamma}) = \lim_{n\to\infty} P(T_{n,\gamma} > c_\gamma(1-\alpha)) \leq \alpha, \tag{22}$$

with equality when $F^A(\tau) = F^B(\tau)$ for every $\tau \in \mathbb{R}$.

(ii) *If $H_{0,\gamma}$ is false,*

$$\lim_{n\to\infty} P(\text{reject } H_{0,\gamma}) = \lim_{n\to\infty} P(T_{n,\gamma} > c_\gamma(1-\alpha)) = 1. \tag{23}$$

Next we determine the power of the test against a sequence of contiguous alternatives converging to the boundary $D_\gamma(\tau) = 0$ for all τ, at a rate $n^{-1/2}$. We define the sequence of local alternatives $F^A(\tau) = F^B(\tau) + \frac{\delta(\tau)}{\sqrt{n}}$, that implies $D_\gamma(\tau) = \frac{\delta(\tau)}{\sqrt{n}}$ for each $\tau \in \mathbb{R}$, and with $\delta(\tau)$ such that $\sup\limits_{\tau\in\mathbb{R}} \delta(\tau) > 0$.

Proposition 4: *Under $H_{1,\gamma} : D_\gamma(\tau) = \frac{\delta(\tau)}{\sqrt{n}}$ with $\sup\limits_{\tau\in\mathbb{R}} \delta(\tau) > 0$, we have*

$$\lim_{n\to\infty} P\left(\text{reject } H_{0,\gamma}\right) = \lim_{n\to\infty} P\left(T_{n,\gamma} > c_\gamma(1-\alpha)\right) \geq \lim_{n\to\infty} P\left(\sup_{\tau\in\mathbb{R}} G_\gamma(\tau) > c_\gamma(1-\alpha) - \sup_{\tau\in\mathbb{R}} \delta(\tau)\right). \tag{24}$$

Then, the power of the test against local alternatives is nontrivial since

$$\lim_{n\to\infty} P\left(\sup_{\tau\in\mathbb{R}} G_\gamma(\tau) > c_\gamma(1-\alpha) - \sup_{\tau\in\mathbb{R}} \delta(\tau)\right) > \alpha.$$

In practice, the asymptotic critical value of the different tests depends on the marginal and joint distribution functions evaluated at the different points of a finite grid of random points $\tau_1 < \tau_2 < \ldots < \tau_t$, and on the corresponding conditional expected values. This, as acknowledged by other authors as well, implies that $c_\gamma(1-\alpha)$ is not distribution-free and cannot be universally tabulated. This value, if F^A, F^B and $F^{A,B}$ are known, can be approximated by Monte-Carlo simulation of the asymptotic distribution function of the supremum of the

Gaussian process G_γ. The choice of the number of Monte-Carlo iterations and the partition of the grid is up to the econometrician, making the accuracy of this approximation as fine as the econometrician desires.

The interest of these tests is, however, when the nuisance parameters of the asymptotic distribution are not known. In this case there are two alternatives explored in the literature, namely, the p-value transformation in the spirit of Hansen (1996) or multiplier method, see Van der Vaart and Wellner (1996) or Barret and Donald (2003); and resampling methods, bootstrap as in Barret and Donald (2003) or subsampling as in Linton, Maasoumi and Whang (2005). Alternatively, we propose here to exploit the parametric form of the asymptotic distribution of the functional of G_γ, and approximate the critical values of the true sampling distribution of the test with the critical value of the asymptotic distribution with covariance function estimated by the $\sqrt{n}-$nonparametric consistent estimators introduced above. This methodology to approximate the critical value is not new. Koul and Ossiander (1994) and Koul and Ling (2006), for example, propose it in a context of goodness of fit tests for the error distribution of autoregressive and heteroscedastic time series models. The choice of this method has two main advantages over the other two standard simulation techniques. These are now discussed.

In contrast to the multiplier method our asymptotic distribution makes allowance for dependence between the random variables A and B and therefore covers a higher spectrum of possibilities. Also, our method can be implemented very easily to higher orders of stochastic dominance. It is not clear that this is the case, in practice, for the multiplier method since the sequence of normal random variables have to multiply complicated functionals of the empirical processes defining $T_{n,\gamma}$.

Bootstrap resampling techniques offer a good alternative to approximate the finite-sample distribution of the test under the null hypothesis. For power studies, however, bootstrap versions of the hypothesis tests are not consistent when the null hypothesis of stochastic dominance is not known, that is, one does not know whether inequality $LPM_q^A \leq LPM_q^B$ or $LPM_q^B \leq LPM_q^A$ holds and thereby whether the bootstrap for the test statistic $T_{n,\gamma}$ approximates the null or the alternative distribution. To solve this Linton, Maasoumi and Whang (2005) propose the use of subsampling tests that are consistent against $H_{1,\gamma}$, see the monograph of Politis, Romano and Wolf (1999) for the consistency of subsampling tests. This alternative relies heavily on the choice of an optimal subsample size and can be difficult to implement in practice.

Like in the bootstrap and multiplier method the critical value obtained from estimating the asymptotic distribution is data dependent. This is so because each draw from the data generating process produces a different set of estimates of the nuisance parameters, and therefore, a proper Monte-Carlo exercise for studying the properties of the test should generate different critical values for each sample, all of them converging uniformly at $\sqrt{n}-$rate to $c_\gamma(1-\alpha)$. Note instead that the parametric nature of our approximation and the certainty that we are using the correct asymptotic distribution under $H_{0,\gamma}$ allows us to use universally, given the sample size, the critical value obtained from one single iteration in the Monte-Carlo study. This fact improves considerably, in computational terms, the efficiency of the tests with very little sacrifice in terms of accuracy of size and power. This can be observed in the Monte-Carlo exercises reported in Section 4. Before, we formalize this choice of critical value.

Proposition 5: *Assume A.1-A.5 hold, and let* $\mathbf{x}_n^{(j)} := (x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)})'$, $j = 1, \ldots,$ *be a collection of random samples of dimension* $n \times 2$ *drawn from a bivariate distribution* $F^{A,B}(\tau,\tau)$*. Let* $T_{n,\gamma}^{(j)}$ *be the test statistic associated to each sample, and* $c_\gamma^{(j)}(1-\alpha)$ *the critical values obtained from the corresponding estimated functional of* G_γ*. Then*

(i) *Under* $H_{0,\gamma}$,

$$\lim_{n\to\infty} P(\textit{reject } H_{0,\gamma}) = \lim_{n\to\infty} P(T_{n,\gamma}^{(j)} > c_\gamma^{(1)}(1-\alpha)) \leq \alpha, \tag{25}$$

almost surely for every random sample $\mathbf{x}_n^{(j)}$*, and with equality when* $F^A(\tau) = F^B(\tau)$ *for every* $\tau \in \mathbb{R}$.

(ii) *If* $H_{0,\gamma}$ *is false,*

$$\lim_{n\to\infty} P(\textit{reject } H_{0,\gamma}) = \lim_{n\to\infty} P(T_{n,\gamma}^{(j)} > c_\gamma^{(1)}(1-\alpha)) = 1, \tag{26}$$

almost surely for every random sample $\mathbf{x}_n^{(j)}$.

Finally, it is worth mentioning that the spherical symmetry of the asymptotic distribution of the different test statistics $T_{n,\gamma}$ under $H_{0,\gamma}$ allows us to carry out the reverse hypothesis test $H_{0,\gamma}^* : LPM_q^B \leq LPM_q^A$ without the need of extra calculations. The asymptotic critical value of this test is also $c_\gamma(1-\alpha)$, and the relevant test statistic $T_{n,\gamma}^*$ can be computed from $T_{n,\gamma}$ by exploiting that $T_{n,\gamma}^* = -\sqrt{n} \inf_{\tau\in\mathbb{R}} \widehat{D}_\gamma$. In practice then we need to compute this value along with $T_{n,\gamma}$ to extract meaningful conclusions about the reverse test in case $H_{0,\gamma}$ is rejected.

3.2 Stochastic Dominance Hypothesis Tests Under Distress

The results above can be easily extended to testing stochastic dominance under distress and with it the mean-risk dominance set in episodes of market turmoil. For ease of exposition we will assume that both portfolios have the same number of observations n_u below the threshold u, and same number of assets m. More formally, denote $n_u^A := \sum_{i=1}^{n} I(R_{1,i}^A \leq u, \ldots, R_{m,i}^A \leq u)$, $n_u^B := \sum_{i=1}^{n} I(R_{1,i}^B \leq u, \ldots, R_{m,i}^B \leq u)$ with R_j^A and R_j^B, $j = 1, \ldots, m$ the assets comprising portfolio A and B respectively.

Assumption A.6: $n_u := n_u^A = n_u^B$.

REMARK: This assumption can be relaxed and use instead two threshold values u^A and u^B satisfying $n_{u^A}^A = n_{u^B}^B$.

The relevant hypothesis test in this environment of comovements is

$$\begin{cases} H_{0,\gamma,u} : D_{\gamma,u}(\tau) \leq 0, & \text{for all } \tau \in \mathbb{R}, \\ H_{1,\gamma,u} : D_{\gamma,u}(\tau) > 0, & \text{for some } \tau \in \mathbb{R}, \end{cases} \tag{27}$$

where $D_{\gamma,u}(\tau) = LPM_{\gamma,u}^A(\tau) - LPM_{\gamma,u}^B(\tau)$.

The asymptotic theory follows from the previous results for the unconditional stochastic dominance tests. Assumptions A.2-A.4 are sufficient to guarantee that the covariance function of the corresponding conditional functional process is well defined. Assumption A.5 guarantees the uniform convergence of the corresponding conditional moments that we introduce now. Let $F_u^{A,B}(\tau_s, \tau_t) := P\{R^A \leq \tau_s, R^B \leq \tau_t | R_1 \leq u, \ldots, R_m \leq u\}$, $k_{\gamma,u}^i(\tau) := E[(\tau - R^i)^\gamma | R^i \leq \tau, R_1 \leq u, \ldots, R_m \leq u]$ with $i = A, B$, and $k_{\gamma,u}^{A,B}(\tau, \tau) := E[(\tau - R^A)^\gamma (\tau - R^B)^\gamma | R^A \leq \tau, R^B \leq \tau, R_1 \leq u, \ldots, R_m \leq u]$.

Theorem 3: *Under A.1-A.6,*

$$\sqrt{n_u} \sup_{\tau \in (-\infty, u]} (\widehat{D}_{\gamma,u}(\tau) - D_{\gamma,u}(\tau)) \xrightarrow{d} \sup_{\tau \in (-\infty, u]} G_{\gamma,u}(\tau), \tag{28}$$

with $G_{\gamma,u}(\tau)$ a Gaussian process with zero mean and covariance function given by

$$E[G_{\gamma,u}(\tau_s) G_{\gamma,u}(\tau_t)] = \tag{29}$$

$$\begin{aligned} &\left(k_{2\gamma,u}^A(\tau_s \wedge \tau_t) F_u^A(\tau_s \wedge \tau_t) - k_{\gamma,u}^A(\tau_s) F_u^A(\tau_s) k_{\gamma,u}^A(\tau_t) F_u^A(\tau_t)\right) + \\ &\left(k_{2\gamma,u}^B(\tau_s \wedge \tau_t) F_u^B(\tau_s \wedge \tau_t) - k_{\gamma,u}^B(\tau_s) F_u^B(\tau_s) k_{\gamma,u}^B(\tau_t) F_u^B(\tau_t)\right) - \end{aligned}$$

$$\left(k^{A,B}_{\gamma,u}(\tau_s,\tau_t)F^{A,B}_u(\tau_s,\tau_t) - k^{A}_{\gamma,u}(\tau_s)F^{A}_u(\tau_s)k^{B}_{\gamma,u}(\tau_t)F^{B}_u(\tau_t)\right) -$$
$$\left(k^{A,B}_{\gamma,u}(\tau_t,\tau_s)F^{A,B}_u(\tau_t,\tau_s) - k^{A}_{\gamma,u}(\tau_t)F^{A}_u(\tau_t)k^{B}_{\gamma,u}(\tau_s)F^{B}_u(\tau_s)\right),$$

for all $\tau_s, \tau_t \leq u$, *and* $0 \leq \gamma \leq q$.

The processes $G_{0,u}(\tau)$ and $G_0(\tau)$ are identical in distribution. For higher orders of γ this is not the case since the asymptotic distribution depends on the conditional versions of the different expected values entering the covariance function and on the choice of threshold parameter.

The family of test statistics for testing stochastic dominance under distress are $T_{n_u,\gamma} := \sqrt{n_u} \sup\limits_{\tau\in(-\infty,u]} \widehat{D}_{\gamma,u}(\tau)$, that under A.1-A.6, and $H_{0,\gamma,u}$, with $u \in \mathbb{R}$, satisfy

$$T_{n_u,\gamma} \xrightarrow{d} \sup_{\tau\in(-\infty,u]} G_{\gamma,u}(\tau), \tag{30}$$

with $G_{\gamma,u}(\tau)$ a Gaussian process with zero mean and covariance function given in expression (29). Further, the asymptotic critical values of these tests are given by

$$c_{\gamma,u}(1-\alpha) := \inf_{x\in\mathbb{R}}\{x \,\big|\, P\left(\sup_{\tau\in(-\infty,u]} G_{\gamma,u}(\tau) \leq x\right) \geq 1-\alpha\}, \tag{31}$$

with α denoting the significance level. In contrast to the unconditional case, this critical value cannot be tabulated even for $\gamma = 0$ due to the dependence of the supremum functional process on u. Simulation procedures as a p-value transformation or bootstrap can be proposed to approximate the critical value of the test. Note that in this conditional context it is convenient to make allowance for mutual tail dependence between the prospects even if A and B are unconditionally uncorrelated. This dependence makes the p-value transformation inadequate for testing stochastic dominance under comovements. On the other hand, the inconsistency of bootstrap tests under the alternative hypothesis remains in this context. Alternatively, we propose to estimate the asymptotic covariance function (29) from the data and approximate the critical value of the test by Monte-Carlo simulation of the restricted supremum of the estimated gaussian process. The validity of this method and the consistency of the test can be shown applying proposition 5 to an environment of comovement periods.

The next subsection discusses the extensions of these tests to residuals of linear regression models and time series models.

3.3 Stochastic Dominance Hypothesis Tests for Residual Processes

In many situations of practical interest the realizations of the random variables under study are serially dependent or depend on other observed covariates. To account for these different forms of dependence in the tests introduced above the researcher can proceed in two ways. One possibility is to develop hypothesis tests for stochastic dominance robust to the presence of serial dependence. In this case the asymptotic distributions in theorems 2 and 3 need to incorporate the presence of serial correlation and heteroscedasticity in the data, implying more convoluted covariance structures of the respective asymptotic distributions. Appropriate heteroscedastic and autocorrelation consistent (HAC) estimators of the conditional expected values and distribution functions need to be used instead. The sequences under study also need to satisfy some mixing conditions. Alternatively, one can apply filters to the data in order to transform the observations from each random variable into *iid* observations and use the tests above. This methodology is based on the residuals of regression and time series models and is explored as follows.

Let $Z_t^T = \{(1, R_{t-j}^A, R_{t-j}^B, X_{t+1-j}),\ j = 1, \dots\}$ be a vector of regressors, where X_t denotes a vector of random variables different from R_t^A and R_t^B. The relevant regression equation is

$$R_t^i = Z_t^T \beta^i + a_t^i, \tag{32}$$

with β^i the parameter vector and $a_t^i = h_t^i \varepsilon_t^i$, the innovation variables corresponding to each regression equation. These sequences consist of a volatility process h_t^i and an error sequence ε_t^i that satisfies $E[\varepsilon^i|Z] = 0$ for $i = A, B$. Consider the family of test statistics $T_{n,\gamma}$ of the unconditional tests above and let $\widehat{T}_{n,\gamma}$ be the family of test statistics computed from the residual sequences $\widehat{\varepsilon}_t^i := \frac{R_t^i - Z_t^T \widehat{\beta}^i}{\widehat{h}_t^i}$ for $i = A, B$, where $\widehat{\beta}^i$ is the vector of parameter estimates and $\widehat{h}_t^i$ the estimated volatility process. In what follows we show that theorems 2 and 3 still hold for these alternative tests based on the residual sequences and for $0 \leq \gamma \leq 2$.

Assumption A.7: (i) $\big\{(R_t^i, Z_t) : t = 1, \dots, n\big\}$ is a strictly stationary and ergodic sequence for $i = A, B$. (ii) The conditional distribution of ε_t^i given the vector Z_t has bounded density with respect to Lebesgue measure almost sure $(a.s.)$ for $i = A, B$, and $t \geq 1$. (iii) $\sqrt{n}(\widehat{\beta}^i - \beta^i) = O_p(1)$ and $\sqrt{n}(\widehat{h}_t^i - h_t^i) = O_p(1)$.

Corollary 2: *Suppose assumptions A.1-A.5, and A.7, are satisfied. Then, under* $H_{0,\gamma}$ *for*

$0 \leq \gamma \leq 2$,

$$\widehat{T}_{n,\gamma} \xrightarrow{d} \sup_{\tau \in \mathbb{R}} G_{\gamma}(\tau), \tag{33}$$

with $G_{\gamma}(\tau)$ *the Gaussian process introduced in* (19).

This corollary can be also formulated for stochastic dominance under distress using residual processes, and without the need of imposing more assumptions. This result is omitted for sake of space.

The battery of tests proposed in this section extends in three ways the existing methods for testing stochastic dominance. First, by deriving a testing framework for general degrees of stochastic dominance that makes allowance for different forms of dependence between portfolios without relying on bootstrap and subsampling techniques; second, by introducing alternative tests for the hypothesis of stochastic dominance under distress episodes of the market, and third by showing the applicability of these techniques to residuals from regression and time series models. The implications of these techniques in optimal portfolio theory are of much interest. A simple application for financial data is described in Section 5. Next section illustrates via simulation experiments the findings of this section.

4 Mote-Carlo Simulation Experiments

In this section we consider a small Monte Carlo experiment to gauge the extent to which the preceding asymptotic arguments hold in finite samples. We are interested, in particular, in comparing the approximation of the critical values given by our asymptotic theory and the approximation offered by the multiplier method discussed in Barret and Donald (2003). The critical values of both methods are conditional on a given sample. In our method this is due to the estimation of the nuisance parameters in the covariance function of the Gaussian process, and in the p-value transformation or multiplier method due to the generation of random versions of the relevant test statistic. For comparison purposes the multiplier method implemented in this section differs slightly from Barret and Donald procedure. In our case we multiply the raw observations of the bivariate data generating process by two independent vectors of standard normal random variables and use these simulated observations to compute the modified versions of the different test statistics. This is plausible due to the linear form of our test statistics and the continuous mapping theorem.

We study these approximations for stochastic dominance tests of first and second order; and also, for the corresponding tests of stochastic dominance under distress. In the second block of simulations we carry out a small study of the power of the tests against local alternatives. In this case we only focus on our method to derive the critical values in order to study the power of the tests.

Tables 1 and 2 report empirical sizes under both methods for $H_{0,\gamma}$ for $\gamma = 0, 1$ and when the correlation parameter between the random variables is $\rho = 0, 0.4$, and 0.8. The significance levels studied are 10%, 5% and 1% and the data generating processes are bivariate Student-t distributions with $\nu = 5$ and $\nu = 10$ degrees of freedom. We choose these distributions as plausible candidates to describe the unconditional generating process for pairs of financial returns, or more usually, to describe the sequence of innovations of the standard processes encountered in the modeling of financial time series, see Bollerslev (1987). These distributions belong to the elliptical family of distribution functions and are therefore completely characterized by the first two statistical moments and the correlation function. Nevertheless, unlike the gaussian distribution these processes are capable of generating asymptotic tail dependence as ρ increases. The impact of this phenomenon in the size and power of the tests can be observed in the different simulations reported.

[INSERT TABLES 1 AND 2 ABOUT HERE]

The results for the empirical size for stochastic dominance under comovements are reported in tables 3 and 4. Note that in order to have a simulation exercise comparable to the unconditional case we need to have conditional samples of $n_u = 50, 100$ and 500 observations. This can achieved for the independent case, for a threshold $u = 0$ and for such data generating processes, by generating random samples of $n = 200, 400$ and 2000 observations. For values of ρ greater than zero, the asymptotic tail dependence present in the data, generates subsamples in the conditioning region with more than $n_u = 50, 100$ and 500 observations and yield in turn better approximations of size and power.

[INSERT TABLES 3 and 4 ABOUT HERE]

The study of the power of the tests against local alternatives is designed as follows. The family of alternative hypotheses is defined by a random variable $R^A = X - \frac{c}{\sqrt{n}}$, where X, as R^B, follows an standardized mean-zero Student-t distribution with ν degrees of freedom, and such that $Cov(X, R^B) = \rho$. The distribution function of R^A is given by $F^A(\tau) := F^B(\tau + \frac{c}{\sqrt{n}})$, that by a Taylor expansion satisfies $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$ with $f^B(\tau)$ the density function of the centered Student-t distribution.[2] The distribution of R^A can be written,

[2]The supremum of $f^B(\tau)$ is achieved at $\tau = 0$ and takes the value 0.380 for $\nu = 5$ and 0.389 for $\nu = 10$.

therefore, as $F^A(\tau) = F^B(\tau) + \frac{\delta(\tau)}{\sqrt{n}}$ with $\delta(\tau) = cf^B(\tau)$ and such that $\sup\limits_{\tau \in \mathbb{R}} \delta(\tau) > 0$. For our examples we consider $c = 0.5, 1$ and 5.

We study the power for the three dependence structures considered before. Tables 5 and 6 report the results for the unconditional tests and tables 7 and 8 the results corresponding to the conditional tests representing financial distress. The data generating processes are Student-t distributions with $\nu = 5$ and 10.

[INSERT TABLES 5 - 8 ABOUT HERE]

Some remarks on the simulations:

1. Our family of test statistics shows an adequate finite sample performance in terms of size and power when $n > 50$.
2. The approximation of the different critical values by the asymptotic theory that we postulate in the paper is in general more accurate than under the p-value transformation. This is particularly remarkable under the presence of dependence between portfolios A and B, where the p-value method fails completely to report accurate approximations of the asymptotic critical values.
3. The choice of the grid used for the Monte-Carlo simulation only plays an important role for small sample sizes ($n = 50$). In these cases the econometrician must fine tune the lower and upper limit of the grid to avoid simulated covariance matrices that are not well defined. Unfortunately, the constraints imposed on the grid, and therefore on the process, distort considerably the approximations of the size, and one should opt in these few cases for the p-value transformation.
4. The power of the tests increases as the correlation between the random variables is higher.
5. The conclusions from the simulations for stochastic dominance under distress are very similar and are omitted for sake of space. It is remarkable the substantial increase in power in these cases compared to their unconditional counterparts with same sample sizes.

In the next section we implement these tests for evaluating efficient investment portfolios and compare them in normal and crises episodes of the market.

5 An Empirical Study of Mean-risk Efficiency

We study a portfolio of risky and heavily traded stocks in the US economy that cover very different and important sectors: Microsoft (MSFT), General Electric (GE), Bank of America Corporation (BAC) and Verizon Communications (VZ). The data set we propose to use spans the period 02/01/2000 - 30/12/2007 and are obtained from Yahoo Finance website. In contrast to studies using financial indexes each asset in this case is not a diversified instrument per se and can be dramatically affected by negative and positive idiosyncratic shocks. The marginal unconditional distribution functions exhibit rather heavy tails and can invalidate, in turn, approximations of the distribution of the portfolio given by normal distributions, and that thereby support mean-variance efficient sets consisting of aggregation of uncorrelated assets.

We concentrate on two portfolio candidates, w_o denoting the mean-risk efficient portfolio derived from minimizing a LPM_0 measure for $\tau = 0$, and characterized by the following weights: $w_o := [0.05 \quad 0.85 \quad 0.05 \quad 0.05]$; and $w_{mv} := [0.20 \quad 0.15 \quad 0.30 \quad 0.35]$ obtained from minimizing the corresponding unconditional variance. The left panel in figure 5.1 shows the unconditional distribution function of returns from each strategy. A simple visual inspection of the plot indicates the rejection of both $H_{0,0}$ and the reverse hypothesis. The relevant hypothesis test for first stochastic dominance confirms the findings of no dominance of either portfolio. Both hypotheses are rejected at 5%. In particular the simulated critical values are 1.029, 1.158 and 1.364 at 10%, 5% and 1%, respectively. The test statistics are 2.710 and 2.377.

The test for second stochastic dominance shows a different picture. In this case the critical values are 1.594, 1.982 and 2.813 at 10%, 5% and 1%, respectively, with test statistics 7.305 for $H_{0,1}$ and -2.480 for $H^*_{0,1}$. There is sufficient evidence to reject the null and accept the reverse hypothesis. This test implies that risk-averse investors prefer the mean-variance strategy to the mean-risk efficient portfolio. This order of convergence is sufficient to infer the dominance of the mean-variance strategy over the other for higher orders of stochastic dominance.

The efficiency analysis between portfolios is repeated now under comovements defined by a threshold $u = 0$. In this case the efficient portfolios under each strategy are $w_{o,o} := [0.05 \quad 0.05 \quad 0.05 \quad 0.85]$ and $w_{mv,o} := [0.05 \quad 0.05 \quad 0.50 \quad 0.40]$. The critical values of the test $H_{0,0,0} : LPM_{0,0}^{w_{o,o}} \leq LPM_{0,0}^{w_{mv,o}}$ are 0.990, 1.095 and 1.368, and the relevant test statistics 0.306 and 1.733. Therefore whereas we find no evidence to reject the null hypothesis $H_{0,0,0}$ we do to reject the reverse hypothesis. We conclude that the mean-variance strategy is dominated under comovement episodes of the market by the mean-risk frontier for risk-neutral and risk-averse investors. The right panel of figure 5.1 supports these findings.

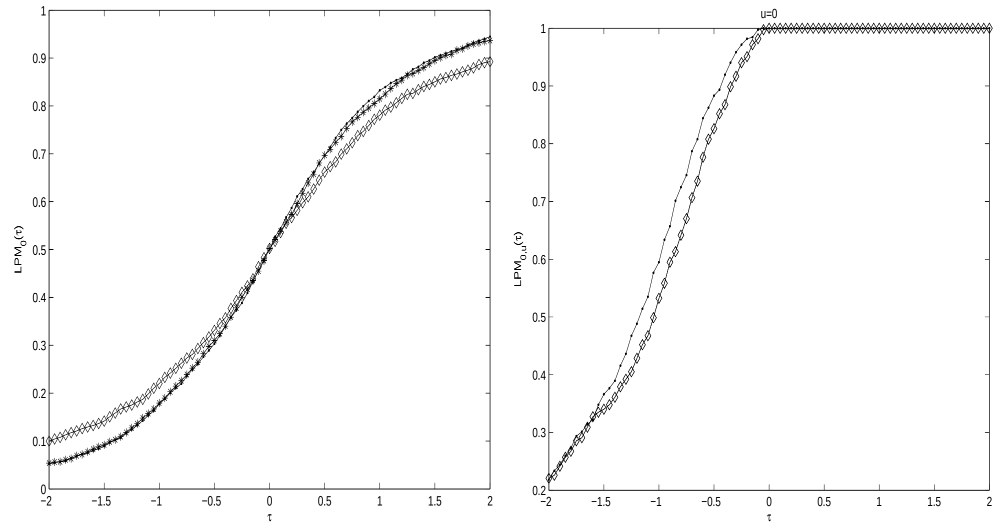

Figure 5.1. *Nonparametric estimates of the conditional risk measures LPM_0 (left panel) and $LPM_{0,0}$ (right panel). ($\Diamond-$) describes Portfolio w_o and $w_{o,o}$, and ($\cdot-$) Portfolio w_{mv} and $w_{mv,o}$. The assets comprising the portfolio are $(GE, MSFT, VZ, BAC)$ and the relevant period 02/01/2000 - 30/12/2007.*

To confirm our findings we also carry out this experiment using two more methodologies. For the first alternative, we entertain the abnormal returns of each portfolio obtained from removing the dependence from the market portfolio, proxied in this example by the Dow-Jones Industrial Average Stock Index over the same period. We find, however, no statistical significance at 5% of the systematic risk (β) parameter. Therefore, the results on stochastic dominance obtained before do not vary now. The second experiment contemplates the residual sequence of each time series after filtering for the presence of serial dependence in the data. In particular, we have estimated each optimal portfolio independently using an ARMA(1,1)-GARCH(1,1) process and a pure GARCH(1,1) process. Whereas the ARMA components are not statistical significant at 5%, the parameters of the volatility model are highly significant. The process for the downside risk portfolio is

$$R_t^{w_o} = h_{o,t}\varepsilon_t^{w_0}, \; with \; h_{o,t}^2 = \underset{(0.006)}{0.033} + \underset{(0.010)}{0.124} R_{t-1}^2 + \underset{(0.010)}{0.876} h_{o,t-1}^2,$$

with $\varepsilon_t^{w_0}$ the corresponding error term, and where standard errors of the estimates are in brackets. For the mean-variance efficient portfolio,

$$R_t^{w_{mv}} = h_{mv,t}\varepsilon_t^{w_{mv}}, \; with \; h_{mv,t}^2 = \underset{(0.002)}{0.006} + \underset{(0.007)}{0.060} R_{mv,t-1}^2 + \underset{(0.007)}{0.938} h_{mv,t-1}^2,$$

with $\varepsilon_t^{w_{mv}}$ the error term.

The results in this case are more supportive of the stochastic dominance of downside risk strategies for the complete domain of the random variables. The test statistics for the unconditional case are $\widehat{T}_n = 0.866$ and $\widehat{T}_n^* = 1.577$, and the critical value at 5% is 1.163. Hence we do not reject the hypothesis of stochastic dominance of the downside risk portfolio. Finally, the results from the hypothesis test under market distress confirm these findings. In this case the relevant test statistics are $\widehat{T}_{n,0} = 0.038$ and $\widehat{T}_{n,0}^* = 5.045$, and the critical value at 5% is 1.267.

6 Conclusions

The number of articles in the financial literature postulating alternatives to the variance to measure risk has been steadily increasing during the last thirty years. One of the main reasons for this is the belief that financial markets are more interconnected and therefore more likely to enjoy or collapse together. This phenomenon is particularly intense under distress episodes of the markets. In these scenarios and under very general conditions mean-variance strategies can fail to account for these stronger links surging between markets. Natural measures to account properly for these comovements are lower partial moments of the distribution of the portfolio returns. These measures have been studied in portfolio theory and asset pricing since long ago but not for gauging risk under financial distress. We propose in this paper to refine these measures to account explicitly for the presence of comovements in periods of distress and more importantly, to be able to construct efficient portfolios for downside risk averse investors independently of their specific level of risk aversion.

In order to compare the efficiency of two portfolios using lower partial moments we propose a set of statistical tests that allow dependence between prospects and whose critical values can be obtained without resampling methods. These tests can be easily extended to testing stochastic dominance under financial distress. A portfolio that stochastically dominates another portfolio in an scenario of financial distress is a portfolio that is in the mean-risk efficient frontier, and therefore it should be the preferred choice by investors. Further, mean-variance strategies designed to be efficient unconditionally can be dominated in market distress by these alternative portfolios derived from downside risk measures.

These findings and the methodologies derived in this paper can be of much interest for researchers and practitioners interested in the optimal portfolio choices of downside risk averse investors. In particular for those investors where the level of risk aversion cannot be modeled by a simple threshold level given, for example, by the return on the risk-free asset or by a zero return, but is within an interval of possible threshold values. In these cases tests of stochastic dominance and of stochastic dominance under distress can be employed as valid techniques to discriminate among portfolios. These tests can be also of much interest in portfolio theory for economies consisting of downside risk averse heterogeneous agents with risk aversion levels described by different thresholds across investors.

Extensions of our tests for stochastic dominance and mean-risk efficiency to more than two risky prospects are straightforward by using the formulations for the relevant joint tests as in Barret and Donald (2003) and Linton, Maasoumi and Whang (2005), and modifying accordingly the asymptotic theory presented above.

Mathematical appendix

Proof of proposition 1: The proof of the first result in this proposition is trivial. For the second equality denote $F_u(\tau) := P(R^P \leq \tau \big| R_1 \leq u, \ldots, R_m \leq u)$, and note that

$$\int_{-\infty}^{\tau} dF_u(x) = \int_{-\infty}^{\tau} \frac{dF_u(x)}{F_u(\tau)} F_u(\tau),$$

with $\frac{F_u(x)}{F_u(\tau)} = P\{R^P \leq x | R^P \leq \tau, R_1 \leq u, \ldots, R_m \leq u\}$. Also, by an abuse of notation we have that

$$LPM_{0,u}^P(\tau) = \int_{-\infty}^{\tau} dF_u(x) = F_u(\tau).$$

Therefore

$$LPM_{q,u}^P(\tau) = \int_{-\infty}^{\tau} (\tau - x)^q \frac{dF_u(x)}{F_u(\tau)} LPM_{0,u}^P(\tau),$$

that yields result (6).

Proof of theorem 1: It follows from Definition 1 that if $A \; FCSD \; B$ then $LPM_{0,u}^A(\tau) \leq LPM_{0,u}^B(\tau)$ for all $\tau \leq u$ and $\mu_u(A) > \mu_u(B)$. Further, this definition also implies that $FCSD$ implies $SCSD$ and so on; therefore $LPM_{q,u}^A(\tau) \leq LPM_{q,u}^B(\tau)$ for all $\tau \leq u$ and $q \geq 0$. Now, given that

$$E_u[U(R^i; q, \tau)] = \int_{-\infty}^{u} x dF_u^i(x) - k \int_{-\infty}^{\tau} (\tau - x)^q dF_u^i(x) = \mu_u(i) - k \, LPM_{q,u}^i(\tau), \tag{34}$$

with $i = A, B$, it follows that $E_u[U(R^A; q, \tau)] \geq E_u[U(R^B; q, \tau)]$ for all $\tau \leq u$ and $q \geq 0$. The proof for higher orders of conditional stochastic dominance is analogous.

Proof of proposition 2: Suppose we have n independent and identically distributed vectors of observations from a random variable R, and let $\widehat{LPM}_\gamma(\tau)$ be the estimator of $LPM_\gamma(\tau)$ introduced in (9). This estimator can be written as:

$$\widehat{LPM}_\gamma(\tau) = \frac{1}{n} \sum_{i=1}^{n} (\tau - x_i)^\gamma I(x_i \leq \tau).$$

By the law of iterated expectations

$$E[\widehat{LPM}_\gamma(\tau)] = E[(\tau - X)^\gamma | X \leq \tau] E[I(x_i \leq \tau)] = E[(\tau - X)^\gamma | X \leq \tau] F(\tau),$$

with $F(\tau)$ the distribution function of the random variable R. Note that $LPM_0(\tau) := F(\tau)$ and therefore by (2) we obtain the unbiasedness of the estimator.

The proof of the variance is similar but more tedious. By definition we know that

$$V(\widehat{LPM}_\gamma(\tau)) = E[\widehat{LPM}^2_\gamma(\tau)] - E^2[(\tau - X)^\gamma | X \leq \tau] F^2(\tau).$$

By the serial independence between the observations and the law of iterated expectations we can express the first term on the right as

$$E[\widehat{LPM}^2_\gamma(\tau)] =$$
$$E\left[\frac{1}{n}\sum_{i=1}^{n}(\tau - x_i)^{2\gamma}\big|X \leq \tau\right] F(\tau) + E\left[\frac{n(n-1)}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\tau - x_i)^\gamma(\tau - x_j)^\gamma\big|x_i \leq \tau, x_j \leq \tau\right] F^2(\tau).$$

After some algebra we obtain

$$E[\widehat{LPM}^2_\gamma(\tau)] =$$
$$\frac{1}{n}\Big(E\left[(\tau - X)^{2\gamma}|X \leq \tau\right] F(\tau) - E\left[(\tau - X)^\gamma|X \leq \tau\right]^2 F^2(\tau)\Big) + E\left[(\tau - X)^\gamma|X \leq \tau\right]^2 F^2(\tau).$$

It follows then that

$$V[\widehat{LPM}^2_\gamma(\tau)] = \frac{1}{n}\Big(E\left[(\tau - X)^{2\gamma}|X \leq \tau\right] F(\tau) - E\left[(\tau - X)^\gamma|X \leq \tau\right]^2 F^2(\tau)\Big),$$

implying that

$$\sqrt{n}\frac{\widehat{LPM}_\gamma(\tau) - LPM_\gamma(\tau)}{\sqrt{\Big(E\left[(\tau - X)^{2\gamma}|X \leq \tau\right] F(\tau) - E\left[(\tau - X)^\gamma|X \leq \tau\right]^2 F^2(\tau)\Big)}} \xrightarrow{d} N(0,1). \tag{35}$$

Proof of theorem 2: The proof of this result consists of different steps. First, we need to derive the multivariate version of (17). After this we show the tightness of the process, and finally, by using the continuous mapping theorem we derive the asymptotic distribution of the supremum. Thus, suppose we have a partition of the real line given by $-\infty < \tau_1 < \tau_2 < \ldots < \tau_t < \infty$, and n serially independent and identically distributed observations from two random variables R^A and R^B. Let $\widehat{D}_\gamma(\tau)$ be the consistent estimator of $D_\gamma(\tau)$ introduced above. Then, under A.1 -A.5,

$$\sqrt{n}\left(\widehat{D}_\gamma(\tau_1) - D_\gamma(\tau_1), \ldots, \widehat{D}_\gamma(\tau_t) - D_\gamma(\tau_t)\right) \xrightarrow{d} \left(G_\gamma(\tau_1), \ldots, G_\gamma(\tau_t)\right), \tag{36}$$

with the vector on the right following a multivariate normal distribution with mean zero and covariance matrix given by

$$\begin{aligned} E[G_\gamma(\tau_s)G_\gamma(\tau_t)] = {} & \big(k^A_{2\gamma}(\tau_s \wedge \tau_t)F^A(\tau_s \wedge \tau_t) - k^A_\gamma(\tau_s)F^A(\tau_s)k^A_\gamma(\tau_t)F^A(\tau_t)\big) + \\ & \big(k^B_{2\gamma}(\tau_s \wedge \tau_t)F^B(\tau_s \wedge \tau_t) - k^B_\gamma(\tau_s)F^B(\tau_s)k^B_\gamma(\tau_t)F^B(\tau_t)\big) - \\ & \big(k^{A,B}_\gamma(\tau_s, \tau_t)F^{A,B}(\tau_s, \tau_t) - k^A_\gamma(\tau_s)F^A(\tau_s)k^B_\gamma(\tau_t)F^B(\tau_t)\big) - \\ & \big(k^{A,B}_\gamma(\tau_t, \tau_s)F^{A,B}(\tau_t, \tau_s) - k^A_\gamma(\tau_t)F^A(\tau_t)k^B_\gamma(\tau_s)F^B(\tau_s)\big), \end{aligned}$$

for all $\tau_s, \tau_t \in \mathbb{R}$.

The proof of this result follows the same steps as in the previous proof. We will only show the proof for $Cov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{A}(\tau_t)\right)$ and $Cov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{B}(\tau_t)\right)$. The other two terms follow the same algebra. Thus

$$E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{A}(\tau_t)] = E\left[\tfrac{1}{n}\sum_{i=1}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - x_i)^{\gamma}\big|X \leq \tau_s \wedge \tau_t\right] F^{A}(\tau_s \wedge \tau_t) +$$
$$E\left[\tfrac{n(n-1)}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - x_j)^{\gamma}\big|x_i \leq \tau_s, x_j \leq \tau_t\right] F^{A}(\tau_s)F^{A}(\tau_t).$$

By the serial independence between the observations the former expression reads as

$$E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{A}(\tau_t)] =$$
$$\tfrac{1}{n}\left(E\left[(\tau_s - X)^{\gamma}(\tau_t - X)^{\gamma}|X \leq \tau_s \wedge \tau_t\right] F^{A}(\tau_s \wedge \tau_t) - E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - X)^{\gamma}|X \leq \tau_t\right] F^{A}(\tau_s)F^{A}(\tau_t)\right) +$$
$$E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - X)^{\gamma}|X \leq \tau_t\right] F^{A}(\tau_s)F^{A}(\tau_t).$$

It follows then that

$$\lim_{n\to\infty} nCov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{A}(\tau_t)\right) = E\left[(\tau_s - X)^{\gamma}(\tau_t - X)^{\gamma}|X \leq \tau_s \wedge \tau_t\right] F^{A}(\tau_s \wedge \tau_t) -$$
$$E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - X)^{\gamma}|X \leq \tau_t\right] F^{A}(\tau_s)F^{A}(\tau_t).$$

For the covariance term denoting cross dependence the procedure is similar. Let $\{y_j\}_{j=1}^{n}$ denote the sequence of observations from B. Now,

$$E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{B}(\tau_t)] = E\left[\tfrac{1}{n}\sum_{i=1}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - y_i)^{\gamma}\big|X \leq \tau_s, Y \leq \tau_t\right] F^{A,B}(\tau_s, \tau_t) +$$
$$E\left[\tfrac{n(n-1)}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - y_j)^{\gamma}\big|x_i \leq \tau_s, y_j \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t).$$

By the serial independence between the observations, and the cross independence between x_i and y_j for $i \neq j$ the former expression reads as

$$E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{B}(\tau_t)] = \tfrac{1}{n}E\left[(\tau_s - X)^{\gamma}(\tau_t - X)^{\gamma}|X \leq \tau_s, Y \leq \tau_t\right] F^{A,B}(\tau_s, \tau_t) -$$
$$\tfrac{1}{n}E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - Y)^{\gamma}|Y \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t) +$$
$$E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - Y)^{\gamma}|Y \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t).$$

It follows then that

$$\lim_{n\to\infty} nCov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{B}(\tau_t)\right) = E\left[(\tau_s - X)^{\gamma}(\tau_t - Y)^{\gamma}|X \leq \tau_s, Y \leq \tau_t\right] F^{A,B}(\tau_s, \tau_t) -$$
$$E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - Y)^{\gamma}|Y \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t).$$

Now, we can extend this result to the sequence of empirical processes in (36). Since the class of functions we are interested in belongs to the Donsker class, see Van der Vaart (1998, chapter 19), this process converges in distribution in the Skorohod space $D[-\infty, \infty]$, equipped with the uniform norm, to a Gaussian process $G_{\gamma}(\tau)$ with zero mean and the above covariance function. Finally, by the continuous mapping theorem we obtain the weak convergence of the supremum of the process stated in theorem 2.

Proof of proposition 3: It is similar to proof of proposition 1 in Barret and Donald (2003). The proof of (i) involves characterizing the distribution of the test statistic and then using the covariance structure in theorem 2 to prove an inequality between suprema of Gaussian random variables.

Proof of proposition 4: The power of the asymptotic test in theorem 2 is defined under $H_{1,\gamma}$ by $P(T_{n,\gamma} > c_\gamma(1-\alpha))$. Substracting in both sides of the probability expression we obtain

$$P(T_{n,\gamma} > c_\gamma(1-\alpha)) = P\left(\sqrt{n}\sup_{\tau\in\mathbb{R}}\widehat{D}_\gamma(\tau) - \sup_{\tau\in\mathbb{R}}\delta(\tau) > c_\gamma(1-\alpha) - \sup_{\tau\in\mathbb{R}}\delta(\tau)\right) \geq$$

$$\geq P\left(\sqrt{n}\sup_{\tau\in\mathbb{R}}\left(\widehat{D}_\gamma(\tau) - \frac{\delta(\tau)}{\sqrt{n}}\right) > c_\gamma(1-\alpha) - \sup_{\tau\in\mathbb{R}}\delta(\tau)\right),$$

and

$$\lim_{n\to\infty} P\left(\sqrt{n}\sup_{\tau\in\mathbb{R}}\left(\widehat{D}_\gamma(\tau) - \frac{\delta(\tau)}{\sqrt{n}}\right) > c_\gamma(1-\alpha) - \sup_{\tau\in\mathbb{R}}\delta(\tau)\right) > \alpha, \tag{37}$$

since $\sqrt{n}\sup_{\tau\in\mathbb{R}}\left(\widehat{D}_\gamma(\tau) - \frac{\delta(\tau)}{\sqrt{n}}\right)$ converges to $\sup_{\tau\in\mathbb{R}} G_\gamma(\tau)$, as does $T_{n,\gamma}$ under H_0. Now, by definition of the process $\sup_{\tau\in\mathbb{R}}\delta(\tau)$, the quantile of the asymptotic distribution in (37) is to the left of the asymptotic critical value $c_\gamma(1-\alpha)$ and implies therefore a rejection probability greater than α.

Proof of proposition 5: Let $\mathbf{x}_n^{(j)} := (x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)})'$, $j = 1, \ldots$, be a collection of random samples of dimension $n \times 2$ drawn from a bivariate distribution $F^{A,B}(\tau,\tau)$. Define $T_{n,\gamma}^{(j)}$ as the corresponding family of test statistics associated to $\mathbf{x}_n^{(j)}$. Under $H_{0,\gamma}$, proposition 3 shows that this test statistic is $O_P(1)$ of the functional of the gaussian process $\sup_\tau G_\gamma(\tau)$. Mathematically,

$$\lim_{n\to\infty} P\left(T_{n,\gamma}^{(j)} > c_\gamma(1-\alpha)\right) \leq \alpha,$$

with $c_\gamma(1-\alpha)$ the critical value at an α significance level of the asymptotic distribution. Further, each sample indexed by j defines a gaussian process $\sup_\tau \widehat{G}_\gamma(\tau)$ determined by $\sqrt{n}-$consistent estimates of the nuisance parameters in the covariance function (19). Glivenko-Cantelli and

Slutsky theorems plus assumption A.5 ensure that this convergence is uniform and almost sure. Now, the uniform continuity of the gaussian processes implies

$$\sup_{\tau}\left|\widehat{G}_{\gamma}^{(j)}(\tau) - G_{\gamma}(\tau)\right| \xrightarrow{a.s.} 0, \quad \text{for all } j = 1, \ldots \tag{38}$$

where $a.s.$ stands for almost surely, and denotes convergence with probability one. Now, using the properties of the supremum functional we obtain

$$\left|\sup_{\tau}\widehat{G}_{\gamma}^{(j)}(\tau) - \sup_{\tau}G_{\gamma}(\tau)\right| \xrightarrow{a.s.} 0, \quad \text{for all } j = 1, \ldots \tag{39}$$

Note that the uniform convergence in (38) is a sufficient condition to show (39).

Each functional of the collection of $\widehat{G}_{\gamma}^{(j)}(\tau)$ processes defines a data dependent critical value $c_{\gamma}^{(j)}(1-\alpha)$ satisfying

$$P\left(\sup_{\tau}\widehat{G}_{\gamma}^{(j)} > c_{\gamma}^{(j)}(1-\alpha)\right) = \alpha, \quad \text{for all } j = 1, \ldots \tag{40}$$

The uniform convergence in (39) and the fact that the distribution function of $\sup_{\tau} G_{\gamma}(\tau)$ is strictly increasing in τ implies that

$$c_{\gamma}^{(j)}(1-\alpha) \xrightarrow{a.s.} c_{\gamma}(1-\alpha), \quad \text{for all } j = 1, \ldots \tag{41}$$

Note that this result is sufficient for our purpose but it also holds uniformly in $\alpha \in (0,1)$.

Consider now a sample $\mathbf{x}_n^{(1)}$ and retain the associated critical value $c_{\gamma}^{(1)}(1-\alpha)$. By using basic algebra in (39) it is simple to show that

$$\left|\sup_{\tau}\widehat{G}_{\gamma}^{(j)}(\tau) - \sup_{\tau}\widehat{G}_{\gamma}^{(1)}(\tau)\right| \xrightarrow{a.s.} 0, \tag{42}$$

and therefore, using the same arguments as before, we obtain that

$$c_{\gamma}^{(j)}(1-\alpha) - c_{\gamma}^{(1)}(1-\alpha) \xrightarrow{a.s.} 0, \quad \text{for all } j = 1, \ldots \tag{43}$$

Furthermore, it can be shown that this property also holds uniformly in α, that is,

$$\sup_{\alpha \in (0,1)}\left|c_{\gamma}^{(j)}(1-\alpha) - c_{\gamma}^{(1)}(1-\alpha)\right| \xrightarrow{a.s.} 0, \quad \text{for all } j = 1, \ldots$$

From this convergence results we obtain the desired result since

$$\lim_{n\to\infty} P\left(T_{n,\gamma}^{(j)} > c_\gamma^{(1)}(1-\alpha)\right) = \lim_{n\to\infty} P\left(T_{n,\gamma}^{(j)} > c_\gamma^{(j)}(1-\alpha) - \left(c_\gamma^{(j)}(1-\alpha) - c_\gamma^{(1)}(1-\alpha)\right)\right) \leq \alpha. \tag{44}$$

The proof for the consistency of the test under $H_{\gamma,1}$ follows from observing that whereas the test statistic $T_{n,\gamma}$ diverges to infinity, the collection of critical values $c_\gamma^{(j)}(1-\alpha)$ is simulated from the respective estimated gaussian processes under the null hypothesis. The convergence in (41) and (43) hold, but now

$$\lim_{n\to\infty} P\left(T_{n,\gamma}^{(j)} > c_\gamma^{(1)}(1-\alpha)\right) = \lim_{n\to\infty} P\left(T_{n,\gamma}^{(j)} > c_\gamma^{(j)}(1-\alpha) - \left(c_\gamma^{(j)}(1-\alpha) - c_\gamma^{(1)}(1-\alpha)\right)\right) = 1. \tag{45}$$

Proof of theorem 3: Note that $n_u = \lambda(u) + o_P(n)$ implying that $\sqrt{n_u} = \sqrt{\lambda(u)}\sqrt{n} + o_P(\sqrt{n})$. The rest of the proof is then analogous to the proof of theorem 2 but replacing the relevant unconditional distribution functions by their conditional counterparts, with the conditioning event defined by a threshold u.

Proof of corollary 2: Let ε_t^i be the error sequence of a possibly heteroscedastic time series defined in (32), and $\widehat{\varepsilon}_t^i$ be the corresponding residual sequence. The relevant test statistics are $T_{n,\gamma}$ and $\widehat{T}_{n,\gamma}$ respectively. The latter test statistic can be expressed as

$$\widehat{T}_{n,\gamma} := \sqrt{n}\sup_\tau \left(\left(\widehat{LPM}_\gamma^{\widehat{A}}(\tau) - \widehat{LPM}_\gamma^{A}(\tau) \right) - \left(\widehat{LPM}_\gamma^{\widehat{B}}(\tau) - \widehat{LPM}_\gamma^{B}(\tau) \right) + \widehat{D}_\gamma(\tau) \right), \tag{46}$$

with $\widehat{LPM}_\gamma^{\widehat{i}}$ denoting the downside risk measure computed from the estimated residuals of the regression models for $i = A, B$, and $\gamma = 0, 1, 2$.

Now, it is sufficient to show that $\sqrt{n}\left(\widehat{LPM}_\gamma^{\widehat{i}}(\tau) - \widehat{LPM}_\gamma^{i}(\tau)\right) \xrightarrow{p} 0$ for all $\tau \in \mathbb{R}$ and $i = A, B$, to obtain the desired result. Without loss of generality and to ease notation we will denote the error and residual variables without using the index i. Then, the difference above can be written as

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(\tau - \widehat{\varepsilon}_t)^\gamma I(\widehat{\varepsilon}_t \leq x) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n}(\tau - \varepsilon_t)^\gamma I(\varepsilon_t \leq x), \tag{47}$$

for both portfolios A and B, and with $\gamma = 0, 1, 2$.

This expression is upper bounded by the product: $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[(\tau-\widehat{\varepsilon}_t)^{\gamma}-(\tau-\varepsilon_t)^{\gamma}\right]\left[I(\widehat{\varepsilon}_t\leq x)-I(\varepsilon_t\leq x)\right]$. Now, using Newton's formula we obtain the following inequality:

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[(\tau-\widehat{\varepsilon}_t)^{\gamma}-(\tau-\varepsilon_t)^{\gamma}\right]\left[I(\widehat{\varepsilon}_t\leq x)-I(\varepsilon_t\leq x)\right]\leq \tag{48}$$

$$\leq\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[(\tau-\widehat{\varepsilon}_t)^{\gamma}-(\tau-\varepsilon_t)^{\gamma}\right]^2+\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[I(\widehat{\varepsilon}_t\leq x)-I(\varepsilon_t\leq x)\right]^2. \tag{49}$$

Operating with the first right term and using assumption A.7. we observe that it is of order $o_p(1)$. To derive the convergence of the second term we note that $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[I(\widehat{\varepsilon}_t\leq x)-I(\varepsilon_t\leq x)\right]^2=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|I(\widehat{\varepsilon}_t\leq x)-I(\varepsilon_t\leq x)\right|$. Reordering the terms inside the sum operator, this expression can be decomposed as

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|I(\widehat{\varepsilon}_t\leq x)-I(\varepsilon_t\leq x)\right|=\sqrt{\left(1-\frac{n_o}{n}\right)}\frac{1}{\sqrt{n-n_o}}\sum_{t=1}^{n-n_0}\left[I(\widehat{\varepsilon}_t\leq x)-I(\varepsilon_t\leq x)\right] \tag{50}$$

$$+\sqrt{\frac{n_o}{n}}\frac{1}{\sqrt{n_o}}\sum_{t=n-n_0+1}^{n}\left[I(\varepsilon_t\leq x)-I(\widehat{\varepsilon}_t\leq x)\right], \tag{51}$$

with n_o indicating the number of observations where the difference of indicators inside the absolute value operator is negative. Now, using Koul and Ling (2006, theorem 4.1 and lemma 4.1) we note that both terms (50) and (51) converge to zero in probability, and therefore the proof of corollary 2 follows.

TABLES

$\nu = 5$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.080	0.046	0.000	0.170	0.100	0.036
	p-value	0.100	0.064	0.012	0.078	0.044	0.016
$n = 100$	Gp-value	0.104	0.044	0.004	0.158	0.022	0.016
	p-value	0.114	0.048	0.010	0.132	0.070	0.022
$n = 500$	Gp-value	0.132	0.066	0.016	0.138	0.056	0.010
	p-value	0.152	0.104	0.018	0.124	0.082	0.022
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.126	0.044	0.012	0.128	0.080	0.030
	p-value	0.046	0.008	0.000	0.036	0.022	0.002
$n = 100$	Gp-value	0.108	0.048	0.012	0.148	0.086	0.022
	p-value	0.032	0.018	0.000	0.084	0.026	0.008
$n = 500$	Gp-value	0.112	0.044	0.014	0.104	0.066	0.008
	p-value	0.112	0.058	0.0008	0.066	0.024	0.000
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.164	0.088	0.010	0.164	0.106	0.056
	p-value	0.010	0.002	0.000	0.000	0.000	0.000
$n = 100$	Gp-value	0.092	0.052	0.006	0.158	0.072	0.022
	p-value	0.022	0.0004	0.000	0.012	0.002	0.000
$n = 500$	Gp-value	0.120	0.068	0.012	0.104	0.050	0.014
	p-value	0.020	0.002	0.000	0.000	0.000	0.000

TABLE 1. Empirical size for $H_{0,\gamma}$, $\gamma = 0, 1$ for a standardized bivariate Student-t with $\nu = 5$ degrees of freedom and correlation parameter ρ. Gp : asymptotic p-value, p : Multiplier method p-value. n sample size. $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the asymptotic Gaussian process with covariance function $\widehat{\Sigma}$.

$\nu = 10$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.027	0.024	0.008	0.208	0.136	0.050
	p-value	0.110	0.074	0.114	0.072	0.052	0.014
$n = 100$	Gp-value	0.108	0.044	0.018	0.104	0.056	0.008
	p-value	0.108	0.048	0.0006	0.134	0.074	0.014
$n = 500$	Gp-value	0.086	0.026	0.006	0.118	0.058	0.010
	p-value	0.150	0.114	0.026	0.134	0.082	0.010
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.140	0.080	0.010	0.118	0.064	0.018
	p-value	0.044	0.024	0.000	0.046	0.020	0.000
$n = 100$	Gp-value	0.090	0.040	0.018	0.128	0.068	0.010
	p-value	0.034	0.016	0.000	0.076	0.024	0.006
$n = 500$	Gp-value	0.092	0.044	0.014	0.122	0.052	0.008
	p-value	0.122	0.054	0.006	0.060	0.020	0.002
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.182	0.092	0.010	0.152	0.104	0.034
	p-value	0.018	0.000	0.000	0.002	0.000	0.000
$n = 100$	Gp-value	0.086	0.050	0.012	0.118	0.066	0.022
	p-value	0.024	0.004	0.000	0.004	0.000	0.000
$n = 500$	Gp-value	0.128	0.066	0.028	0.140	0.064	0.012
	p-value	0.006	0.002	0.000	0.002	0.000	0.000

TABLE 2. Empirical size for $H_{0,\gamma}$, $\gamma = 0, 1$ for a standardized bivariate Student-t with $\nu = 10$ degrees of freedom and correlation parameter ρ. *Gp* : asymptotic p-value, *p* : Multiplier method p-value. n sample size. $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the asymptotic Gaussian process with covariance function $\widehat{\Sigma}$.

$\nu = 5$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.048	0.024	0.000	0.324	0.234	0.128
$(n_u \approx 50)$	p-value	0.120	0.050	0.014	0.092	0.048	0.014
$n = 400$	Gp-value	0.106	0.036	0.002	0.102	0.048	0.006
$(n_u \approx 100)$	p-value	0.086	0.045	0.008	0.050	0.022	0.002
$n = 2000$	Gp-value	0.074	0.042	0.010	0.116	0.068	0.020
$(n_u \approx 500)$	p-value	0.094	0.056	0.008	0.050	0.022	0.002
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.118	0.066	0.010	0.136	0.052	0.018
$(n_u \approx 50)$	p-value	0.142	0.074	0.022	0.088	0.040	0.006
$n = 400$	Gp-value	0.162	0.056	0.024	0.170	0.098	0.032
$(n_u \approx 100)$	p-value	0.078	0.040	0.004	0.040	0.010	0.004
$n = 2000$	Gp-value	0.088	0.026	0.0004	0.066	0.020	0.000
$(n_u \approx 500)$	p-value	0.114	0.050	0.010	0.072	0.026	0.002
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.104	0.042	0.012	0.156	0.086	0.024
$(n_u \approx 50)$	p-value	0.074	0.032	0.006	0.024	0.010	0.002
$n = 400$	Gp-value	0.096	0.048	0.008	0.128	0.066	0.018
$(n_u \approx 100)$	p-value	0.070	0.038	0.006	0.014	0.002	0.000
$n = 2000$	Gp-value	0.104	0.034	0.014	0.092	0.054	0.016
$(n_u \approx 500)$	p-value	0.176	0.094	0.008	0.038	0.012	0.000

TABLE 3. Empirical size for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$, for a standardized bivariate Student-t with $\nu = 5$ degrees of freedom and correlation parameter ρ. Gp : asymptotic p-value, p : Multiplier method p-value. n is length of original sample (n_u observations available for the tests). $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the asymptotic Gaussian process with covariance function $\widehat{\Sigma}$.

$\nu = 10$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.106	0.036	0.008	0.058	0.020	0.008
($n_u \approx 50$)	p-value	0.106	0.054	0.006	0.068	0.026	0.014
$n = 400$	Gp-value	0.098	0.046	0.001	0.098	0.052	0.006
($n_u \approx 100$)	p-value	0.082	0.054	0.016	0.044	0.018	0.002
$n = 2000$	Gp-value	0.090	0.034	0.006	0.074	0.028	0.000
($n_u \approx 500$)	p-value	0.086	0.038	0.012	0.052	0.016	0.002
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.124	0.052	0.004	0.118	0.066	0.020
($n_u \approx 50$)	p-value	0.144	0.074	0.012	0.072	0.026	0.004
$n = 400$	Gp-value	0.108	0.062	0.014	0.120	0.060	0.014
($n_u \approx 100$)	p-value	0.100	0.034	0.004	0.032	0.018	0.002
$n = 2000$	Gp-value	0.104	0.046	0.008	0.074	0.044	0.010
($n_u \approx 100$)	p-value	0.116	0.060	0.014	0.048	0.018	0.002
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.102	0.050	0.008	0.103	0.046	0.012
($n_u \approx 50$)	p-value	0.080	0.036	0.006	0.014	0.002	0.000
$n = 400$	Gp-value	0.134	0.068	0.022	0.108	0.056	0.082
($n_u \approx 100$)	p-value	0.068	0.032	0.002	0.004	0.000	0.000
$n = 2000$	Gp-value	0.118	0.074	0.024	0.076	0.038	0.010
($n_u \approx 500$)	p-value	0.156	0.068	0.018	0.030	0.004	0.000

TABLE 4. Empirical size for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$, for a standardized bivariate Student-t with $\nu = 10$ degrees of freedom and correlation parameter ρ. Gp : asymptotic p-value, p : Multiplier method p-value. n is length of original sample (n_u observations available for the tests). $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the asymptotic Gaussian process with covariance function $\widehat{\Sigma}$.

$\nu = 5,\ \alpha = 0.05$	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.102	0.172	0.980	0.162	0.230	0.912
$n = 100$	0.136	0.300	1.000	0.178	0.286	0.988
$n = 500$	0.336	0.768	1.000	0.240	0.522	1.000
$\rho = 0.4$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.110	0.196	0.996	0.134	0.216	0.978
$n = 100$	0.136	0.332	1.000	0.182	0.344	1.000
$n = 500$	0.408	0.890	1.000	0.290	0.692	0.618
$\rho = 0.8$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.226	0.448	1.000	0.264	0.464	1.000
$n = 100$	0.232	0.554	1.000	0.284	0.570	1.000
$n = 500$	0.714	0.998	1.000	0.578	0.986	1.000

TABLE 5. Empirical power for $H_{0,\gamma}$, $\gamma = 0, 1$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 5$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n sample size. $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the asymptotic Gaussian process with covariance function $\widehat{\Sigma}$.

$\nu = 10,\ \alpha = 0.05$	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.056	0.106	0.926	0.198	0.268	0.988
$n = 100$	0.134	0.276	0.798	0.108	0.218	0.980
$n = 500$	0.070	0.150	0.952	0.074	0.168	0.984
$\rho = 0.4$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.166	0.276	0.990	0.114	0.192	0.974
$n = 100$	0.132	0.290	1.000	0.164	0.308	0.996
$n = 500$	0.324	0.792	1.000	0.274	0.662	1.000
$\rho = 0.8$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.212	0.406	1.000	0.226	0.430	1.000
$n = 100$	0.202	0.480	1.000	0.244	0.528	1.000
$n = 500$	0.580	0.990	1.000	0.570	0.986	1.000

TABLE 6. Empirical power for $H_{0,\gamma}$, $\gamma = 0, 1$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 10$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n sample size. $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the asymptotic Gaussian process with covariance function $\widehat{\Sigma}$.

$\nu = 5$, $\alpha = 0.05$	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$ / $c =$	0.5	1	5	0.5	1	5
$n = 200$	0.100	0.282	1.000	0.386	0.548	0.996
$n = 400$	0.044	0.052	0.230	0.070	0.040	0.070
$n = 2000$	0.826	1.000	1.000	0.492	0.870	1.000
$\rho = 0.4$ / $c =$	0.5	1	5	0.5	1	5
$n = 200$	0.246	0.556	1.000	0.178	0.360	1.000
$n = 400$	0.302	0.722	1.000	0.218	0.472	1.000
$n = 2000$	0.864	1.000	1.000	0.526	0.950	1.000
$\rho = 0.8$ / $c =$	0.5	1	5	0.5	1	5
$n = 200$	0.328	0.752	1.000	0.246	0.536	1.000
$n = 400$	0.470	0.954	1.000	0.284	0.704	1.000
$n = 2000$	0.984	1.000	1.000	0.852	1.000	1.000

TABLE 7. Empirical power for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 5$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n is length of original sample ($n_u \approx n/4$ observations available for the tests). $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the relevant asymptotic Gaussian process.

$\nu = 10$, $\alpha = 0.05$	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$ / $c =$	0.5	1	5	0.5	1	5
$n = 200$	0.144	0.350	1.000	0.046	0.124	0.968
$n = 400$	0.200	0.606	1.000	0.152	0.338	1.000
$n = 2000$	0.684	1.000	1.000	0.318	0.834	1.000
$\rho = 0.4$ / $c =$	0.5	1	5	0.5	1	5
$n = 200$	0.150	0.372	1.000	0.164	0.294	1.000
$n = 400$	0.236	0.610	1.000	0.142	0.378	1.000
$n = 2000$	0.766	1.000	1.000	0.532	0.944	1.000
$\rho = 0.8$ / $c =$	0.5	1	5	0.5	1	5
$n = 200$	0.296	0.714	1.000	0.264	0.566	1.000
$n = 400$	0.406	0.920	1.000	0.256	0.712	1.000
$n = 2000$	0.944	1.000	1.000	0.808	1.000	1.000

TABLE 8. Empirical power for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 10$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n is length of original sample ($n_u \approx n/4$ observations available for the tests). $B = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the relevant asymptotic Gaussian process.

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