First crossing time, overshoot and Appell-Hessenberg type functions

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Abstract: We consider a general insurance risk model with extended flexibility under which claims arrive according to a point process with independent increments, their amounts may have any joint distribution and the premium income is accumulated following any non-decreasing, possibly discontinuous, real valued function. Point processes with independent increments are in general non-stationary, allowing for an arbitrary (possibly discontinuous) claim arrival cumulative intensity function which is appealing for insurance applications. Under these general assumptions, we derive a closed form expression for the joint distribution of the time to ruin and the deficit at ruin, which is remarkable, since as we show, it involves a new interesting class of what we call Appell-Hessenberg type functions. The latter are shown to coincide with the classical Appell polynomials in the Poisson case and to yield a new class of the so called Appell-Hessenberg factorial polynomials in the case of negative binomial claim arrivals. Corollaries of our main result generalize previous ruin formulas e.g., those obtained for the case of stationary Poisson claim arrivals.

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1. Introduction

The problem of first crossing of a boundary by a stochastic process has for long time been of interest in applied probability. Such problems naturally arise in many real-life applications, e.g. in insurance, finance, engineering, operations research, reliability, queuing, chemistry, physics and biology. The literature on first crossing is vast and various probabilistic models with different assumptions on the boundary and the underlying stochastic process have been considered.

In risk theory the first crossing time and the overshoot of the process over the boundary are interpreted as the ruin time and the deficit at ruin of an insurance company. The boundary, most often an increasing linear function, represents the accumulation of premiums over time, whereas the underlying stochastic process is often assumed to be a compound Poisson process modelling the aggregate claim amount. Ruin then occurs if the aggregate claim process exceeds the premium income and the exceedance is the deficit at ruin. Studying the joint distribution of the ruin time and deficit at ruin is important since both ruin time and deficit are directly relevant in measuring and managing the solvency and liquidity risk of an insurance company, applying e.g.,(Tail)-Value-at-Risk type of analysis.

In this paper, we give explicitly this joint distribution in a risk model which generalizes the classical one in several ways. We consider finite rather than infinite time horizon. We relax the assumption of i.i.d claim amounts and consider dependent claim sizes with any joint distribution. In our model, premium income is accumulated following not just a linear function of time but following any non-decreasing function, allowing jumps. Finally and most importantly, we relax the classical assumption of Poisson claim arrivals and assume that claims arrive according to a *point process with independent increments*. Somewhat surprisingly, to the best of our knowledge, this reasonably general class of point processes has not been considered in the literature on first crossing and ruin. As we will demonstrate, it leads to a very elegant risk model which has significant implications and allows to considerably extend the flexibility of modelling claim arrivals. This is because point processes with independent increments are in general non-stationary, allowing for an arbitrary (possibly discontinuous) claim arrival cumulative intensity function. The latter feature is appealing for insurance applications in which the intensity of claim arrivals can vary with time due to, e.g., seasonal effects, environmental and climate changes and other reasons related to economic slowdowns and speedups affecting insurance business. Furthermore, the case of a cumulative intensity function with jumps corresponds to the occurrence with non-zero probability of fixed points in the underlying point process which is also relevant, e.g., in discrete time claim arrival models of ruin. In the latter case a binomial claim arrival process naturally arises if a finite-time ruin problem is considered. In the general case, point processes with independent increments also include both stationary and non-stationary Poisson and negative binomial (NB) point processes as typical special cases of claim arrivals. The latter processes allow for clustering of claims, including arrival of two, three or more claims instantaneously, and/or clusters of arrivals at fixed time instants. Clustering at both fixed and random time instants often occurs in insurance portfolios. Therefore, such point process models have the potential to capture better real claim arrival patterns.

Under these general risk model assumptions, in our main result given by Theorem 2.3, we derive a closed form expression for the joint distribution of the time to ruin and the deficit at ruin. The latter expression is remarkable, since as we show, it involves a new interesting class of functions which are Appell type functions and admit representation as Hessenberg determinants. For this reason we refer to them as Appell-Hessenberg type functions. They generalize the well known classical Appell polynomials introduced by Appell (1880). As has first been shown by Ignatov and Kaishev (2000), classical Appell polynomials naturally arise in ruin probability formulas in relation to the Poisson claim arrivals. Different generalizations of the classical Appell polynomials have been considered in the ruin context by Picard and Lefèvre (1997); Lefèvre and Picard (2014b). It is worth noting however that these generalizations do not yield the classical Appell polynomials as a special case and therefore differ from the Appell-Hessenberg type functions considered here. For brevity in the sequel we will refer to the latter simply as Appell-Hessenberg functions.

We show that in the case of Poisson claim arrivals the Appell-Hessenberg functions coincide with the classical Appell polynomials, whereas when claims arrive according to a negative binomial point process the Appell-Hessenberg functions are expressed in terms of factorial functions. If the negative binomial claim arrival process is stationary, the Appell-Hessenberg functions are shown to yield a new class of polynomials which we call Appell factorial polynomials. Our main result gives the (marginal) distribution of the time to ruin and therefore generalizes the explicit ruin probability expressions, in terms of classical Appell polynomials, obtained by Ignatov and Kaishev (2000, 2004, 2006), Ignatov et al. (2001), and by Lefèvre and Loisel (2009) for the case of stationary Poisson claim arrivals. Furthermore, it applies also to the special case of a stationary mixed Poisson process (see Remark 3.8). This special case has recently been considered by Lefèvre and Picard (2011, 2014a) within the context of processes with the order statistics (OS) property. For more recent ruin-deficit formulas under OS claim arrivals which cover and extend previous results by Lefèvre and Picard (2011), see Dimitrova et al. (2014). It is worth mentioning that asymptotic ruin probability results with respect to the initial capital, under some non-stationary claim arrival processes (e.g. Hawkes and Cox processes with shot-noise intensity) have recently been obtained by Stabile and Torrisi (2010) and Zhu (2013).

The paper is organized as follows. In section 2, we prove our main result given by Theorem 2.3. For the purpose, we formulate and prove Lemmas 2.5, 2.6 and 2.7 (and also Proposition 2.2) which are of interest in their own right, establishing explicit and recurrent representations of the Appell-Hessenberg type functions and sums of them. Corollaries 2.8, 2.9 of Theorem 2.3 give ruin formulas for important special cases. In section 3, we specify the results of section 2 to the special cases of Poisson and negative binomial claim arrivals.

2. A formula for $P(T < x, Y > y)$

We assume that the amounts of consecutive claims to an insurance company are modelled by the random variables $W_1, W_2, \ldots$ with partial sums $Y_1 = W_1, Y_2 = W_1 + W_2, \ldots$ having joint density $f(y_1, \ldots, y_k)$.

We will further assume that claims arrive according to a point process, ξ, defined on $(0, \infty)$, whose consecutive points, i.e., claim arrival times, can be represented by a sequence of random variables $0 < T_1 \leq T_2 \leq \ldots$, independent of $W_1, W_2, \ldots$, with $\lim_{l \to +\infty} T_l = +\infty$ a.s.. Such point processes exclude infinite aggregation of points within finite time domains and are therefore well suited for modelling arrivals of insurance claims and events in other applications. If $\mathcal{B}^+$ is the Borel σ-algebra on $(0, \infty)$ and if $B \in \mathcal{B}^+$, then by $\xi(B)$ we will denote the number of points of ξ in B and in particular, $\xi(0, t]$ will denote the number of points (claim arrivals) on $(0, t]$.

The cumulative premium income of the insurance company up to time t

is modelled by the function $h(t)$ which is assumed a non-negative and non-decreasing real-valued function, defined on $[0, +\infty)$, such that $\lim_{t\to\infty} h(t) = +\infty$. Let us also note that the function $h(t)$ does not need to be necessarily continuous and can therefore model arrivals of lump sum premium amounts. If $h(t)$ is discontinuous, we define $h^{-1}(y) = \inf\{z : h(z) \geq y\}$. The insurance company's surplus process is then expressed as $R_t = h(t) - S_t$, where $S_t = Y_{\xi(0,t]}$ is the aggregate claim amount process, and the instant of ruin, T, is defined as $T := \inf\{t : t > 0, R_t < 0\}$ or $T = \infty$, if $R_t \geq 0$ for all t. Given ruin occurs, i.e. $T < \infty$, the deficit at ruin Y is defined as $Y = -R_T$.

We consider a finite-time interval $[0, x]$ and denote by $P(T > x)$ and $P(T < x, Y > y)$ the probability of non-ruin in $[0, x]$ and the probability that ruin occurs before time x with a deficit, Y, exceeding $y \geq 0$. In what follows, we will give explicit expressions for these and other related probabilities under the assumption that the process of claim arrivals, ξ, belongs to the class of *point processes with independent increments*. Before elaborating on this class, let us introduce some further notation. Denote by $P(\xi(0, z] = i)$ the probability of i points of the process ξ occurring in an interval $(0, z]$, where $P(\xi(0, z] = 0) \neq 0$ for any $z > 0$. Denote also by $\Lambda(B) \equiv E\xi B$, the average number of points in B, i.e., $\Lambda(B)$ is the intensity measure of the process ξ. We will also assume that $\Lambda((0, z])$ is a possibly discontinuous function of $z \in (0, \infty)$ and $\lim_{z\nearrow+\infty} \Lambda((0, z]) = +\infty$. For brevity, we will further denote $\Lambda((0, z]) = \Lambda(z)$ and refer to it as cumulative function of the intensity measure $\Lambda(B)$, i.e. $\Lambda(B) = \int_B d\Lambda((0, z])$, which more concisely will be referred to as the cumulative intensity function. Let us note that the case of jump discontinuous cumulative intensity function $\Lambda(z)$ corresponds to the presence of fixed points in ξ which is also covered within the class of point processes with independent increments.

We will call ξ *a point process with independent increments* if for any $0 < s \leq t$, the number of points, $\xi(0, s]$, in the interval $(0, s]$ and the number of points, $\xi(s, t]$, in the interval $(s, t]$ are independent random variables. As known, a point process with independent increments is in general non-stationary and admits representation as a marked point process. Furthermore, every point process with independent increments can be represented as a sum of a deterministic component, a fixed points component and a compound Poisson point process component (see e.g. Kallenberg (2002), Corollary 12.11 and Karr (1991), Theorem 1.34 and Definition 1.37). In what follows we consider processes without a deterministic component as the latter does not have a relevant interpretation for insurance claim arrivals. Two important members of the class are the Poisson point process and the negative binomial point process. Since these processes are in general non-stationary, they have been widely used in developing point process models in various applications in queuing theory, physics, risk theory, operations research, astronomy etc. Other compound Poisson processes such as the Polya-Aeppli process have also been considered. For further properties of point processes with independent increments we refer to e.g., section 1.5 of Last and Brandt (1995), Chapter 1 of Karr (1991), and Chapter 10 of Daley and Vere-Jones (2008). It should be noted that such processes are also considered within the class of integer-valued Lévy processes.

In order to formulate our main result, we will need to introduce a special class of functions which we call *Appell-Hessenberg (type) functions*.

Definition 2.1. *For a fixed non-negative integer j, let $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z$ be an arbitrary increasing sequence of positive real numbers. Define the function $H_j(z; z_1, \ldots, z_j)$, $z \in (z_j, \infty)$, as*

$$H_j(z; z_1, \ldots, z_j) = (-1)^j \det(\Delta(z_1, \ldots, z_j, z)), \tag{1}$$

where

$$\Delta(z_1, \ldots, z_j, z) = \begin{pmatrix} \frac{P(\xi(0,z_1]=1)}{P(\xi(0,z_1]=0)} & 1 & 0 & 0 & \cdots & 0 \\ \frac{P(\xi(0,z_2]=2)}{P(\xi(0,z_2]=0)} & \frac{P(\xi(0,z_2]=1)}{P(\xi(0,z_2]=0)} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \frac{P(\xi(0,z_j]=j)}{P(\xi(0,z_j]=0)} & \frac{P(\xi(0,z_j]=j-1)}{P(\xi(0,z_j]=0)} & & \cdots & & 1 \\ \frac{P(\xi(0,z]=j)}{P(\xi(0,z]=0)} & \frac{P(\xi(0,z]=j-1)}{P(\xi(0,z]=0)} & & \cdots & & 1 \end{pmatrix}.$$

In Definition 2.1, for $j = 0$, we assume $\Delta(z) \equiv (1)$, $H_0(z) \equiv (-1)^0 \det(\Delta(z)) \equiv 1$.

As will be demonstrated, the functions $H_j(z; z_1, \ldots, z_j)$ of the variable $z \in (z_j, \infty)$, $j = 0, 1, 2, \ldots$, defined by the parameters $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j$, can be viewed as generalizations of the classical Appell polynomials. The latter have been first considered by Appell (1880), (see also Kaz'min (1995) and Vein and Dale (1999)). Since the function $H_j(z; z_1, \ldots, z_j)$ is defined as a determinant of a Hessenberg matrix, we call these functions *Appell-Hessenberg (type) functions*. A matrix whose elements above or below the first subdiagonal are equal to zero (i.e., all elements $a_{ij} = 0$ if $j - i > 1$ or if $i - j > 1$) are called Hessenberg matrixes. For properties of Hessenberg matrixes and their determinants we refer to e.g. Vein and Dale (1999). In what follows, it will some times be convenient to interchangeably use the notation $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z_{j+1}$, with $z_{j+1} \equiv z$, for the sequence $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z$. Let us also note that in Definition 2.1 we have implicitly assumed that the sequence $z_0 < z_1 < z_2 < \ldots < z_j < z_{j+1}$ is such that $\lim_{l \to +\infty} z_l = +\infty$. The following recurrence formula facilitates the numerical evaluation of the Appell-Hessenberg functions $H_j(z; z_1, \ldots, z_j)$.

Proposition 2.2. *For a fixed non-negative integer j, let $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z$ be an arbitrary increasing sequence of positive real numbers. For the Appell-Hessenberg functions, $H_j(z; z_1, \ldots, z_j)$, defined in (1), we have*

$$H_j(z; z_1, \ldots, z_j) = \sum_{i=0}^{j} \frac{P(\xi(0,z] = j - i)}{P(\xi(0,z] = 0)} H_i(0; z_1, \ldots, z_i), j \geq 1, \tag{2}$$

where $H_0(z) \equiv 1$, for $z \geq 0$ and

$$H_i(0; z_1, \ldots, z_i) = -\sum_{k=0}^{i-1} \frac{P(\xi(0,z_i] = i - k)}{P(\xi(0,z_i] = 0)} H_k(0; z_1, \ldots, z_k), i \geq 1.$$

Proof. The proof is similar to the proof given in Ignatov and Kaishev (2000) for the case of classical Appell polynomials (see Lemma 1 therein) and is therefore omitted. □

Next, we state our main result which shows that the joint distribution of the time to ruin and the deficit at ruin in the risk model with claim arrivals following an arbitrary point process with independent increments, ξ, can be expressed in terms of the Appell-Hessenberg functions, $H_j\left(z; z_1, \ldots, z_j\right)$, $j = 0, 1, 2, \ldots$.

Theorem 2.3. *The probability $P(T < x, Y > y)$, $x > 0$, $y \geq 0$, is given by*

$$
\begin{aligned}
P(T < x, Y > y) &= \int_{y}^{+\infty} f\left(y_1\right) dy_1 - \int_{y}^{h(x)+y} P\left(\xi(0, h^{-1}(y_1 - y)] = 0\right) f\left(y_1\right) dy_1 \\
&- P(\xi(0, x] = 0) \int_{h(x)+y}^{+\infty} f\left(y_1\right) dy_1 \\
&+ \sum_{k=2}^{\infty} \int \underset{C_k}{\cdots} \int \left\{ B_{k-2}\left(h^{-1}(y_{k-1}); h^{-1}(y_1), \ldots, h^{-1}(y_{k-2})\right) \right. \\
&\left. - B_{k-1}\left(h^{-1}(y_k - y); h^{-1}(y_1), \ldots, h^{-1}(y_{k-1})\right) \right\} f\left(y_1, \ldots, y_k\right) dy_k \ldots dy_1 \\
&+ \sum_{k=2}^{\infty} \int \underset{D_k}{\cdots} \int \left\{ B_{k-2}\left(h^{-1}(y_{k-1}); h^{-1}(y_1), \ldots, h^{-1}(y_{k-2})\right) \right. \\
&\left. - B_{k-1}\left(x; h^{-1}(y_1), \ldots, h^{-1}(y_{k-1})\right) \right\} f\left(y_1, \ldots, y_k\right) dy_k \ldots dy_1,
\end{aligned} \tag{3}
$$

where $C_k = \{(y_1, \ldots, y_k) : 0 < y_1 < \ldots < y_{k-1} \leq y_{k-1} + y < y_k < h(x) + y\}$, $D_k = \{(y_1, \ldots, y_k) : 0 < y_1 < \ldots < y_{k-1} < h(x) \leq h(x) + y \leq y_k < +\infty\}$ *and*

$$
B_j\left(z; z_1, \ldots, z_j\right) = P(\xi(0, z] = 0) \left[H_0(z) + H_1(z; z_1) + \ldots + H_j(z; z_1, \ldots, z_j)\right], \tag{4}
$$

for $j = 0, 1, 2 \ldots$ *and* $H_j\left(z; z_1, \ldots, z_j\right)$ *are defined as in (1) with* $z_1 = h^{-1}\left(y_1\right), \ldots, z_j = h^{-1}\left(y_j\right)$.

The structure of formula (3) is reasonably straightforward: the first three summands represent the probability of ruin at the first claim with deficit at least y; the two sums correspond to the probability of survival after the first $k-1$ claims and ruin at the k-th claim with deficit at least y for $k = 2, 3, \ldots$ (cf. the proof). Practical applications (with numerical illustrations) of $P(T < x, Y > y)$ for the special case of stationary Poisson and Erlang claim arrivals in various risk analysis problems, e.g. insurance solvency, systems reliability, flood risk management, inventory management, are extensively discussed in Dimitrova et al. (2015).

Remark 2.4. *For the efficient numerical evaluation of (3), it is essential to: 1) appropriately truncate the infinite summations; 2) compute the underlying multiple integrals; 3) efficiently compute the integrand functions $B_j\left(z; z_1, \ldots, z_j\right)$. The latter can be done using recurrence formula (2) to compute each of the* *functions* $H_l(z; z_1, \ldots, z_j)$*,* $l = 0, 1, \ldots, j$ *in (4). Methods for solving 1) and 2) developed in Dimitrova et al. (2016) for the special case of stationary Poisson claim arrivals could be generalized to the case of claim arrivals following a point process with independent increments and in particular to the case of negative binomial claim arrivals. Details of how this could be done are outside the scope of the present paper and will be considered separately.*

In order to prove Theorem 2.3 and some related corollaries, we will need the following lemmas.

Lemma 2.5. *For the real sequence* $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z$*, and* $H_j(z; z_1, \ldots, z_j)$ *defined as in (1), we have*

$$P(\xi(0,z] = 0)\, H_j(z; z_1, \ldots, z_j) = \sum_{(g_0,\ldots,g_j)\in E(0,j)} \left(\prod_{l=0}^{j-1} P(\xi(z_l, z_{l+1}] = g_l) \right) P(\xi(z_j, z] = g_j), \tag{5}$$

for $j = 0, 1, 2, \ldots$*, where it is assumed that* $\prod_{l=0}^{-1}(\cdot) = 1$*,* $E(0,m)$ *is the set of* $(m+1)$*-tuples of non-negative integers such that*

$$E(0,m) = \{(g_0, \ldots, g_m) : g_0 \le 0, g_0 + g_1 \le 1, \ldots, g_0 + \ldots + g_{m-1} \le m-1, g_0 + \ldots + g_m = m\}, \tag{6}$$

where m *is a non-negative integer.*

Proof. Expressing $H_j(z; z_1, \ldots, z_j)$ from (1), we have

$$P(\xi(0,z] = 0)\, H_j(z; z_1, \ldots, z_j) = (-1)^j \det\left(\tilde{\Delta}(z_1, \ldots, z_j, z) \right), \tag{7}$$

where $\tilde{\Delta}(z_1, \ldots, z_j, z)$ is equal to $\Delta(z_1, \ldots, z_j, z)$, introduced in Definition 2.1, with its last row multiplied by $P(\xi(0,z] = 0)$.

First, we verify that Lemma 2.5 holds in the cases $j = 0$ and $j = 1$. When $j = 0$, and $0 \equiv z_0 < z$, for the left-hand side of (5), we have

$$P(\xi(0,z] = 0)\, H_0(z) \equiv P(\xi(0,z] = 0),$$

and for the right-hand side, we have

$$\sum_{g_0 \in E(0,0)} P(\xi(0,z] = g_0) = P(\xi(0,z] = 0)$$

and therefore, Lemma 2.5 holds. When $j = 1$ and $0 \equiv z_0 < z_1 < z$, for the left-hand side of (5), we have

$$P(\xi(0,z] = 0)\, H_1(z; z_1) = (-1) \det \begin{pmatrix} \frac{P(\xi(0,z_1]=1)}{P(\xi(0,z_1]=0)} & 1 \\ P(\xi(0,z] = 1) & P(\xi(0,z] = 0) \end{pmatrix}$$

$$= P\left(\xi(0,z]=1\right) \times 1 - \frac{P\left(\xi(0,z_1]=1\right) P\left(\xi(0,z]=0\right)}{P\left(\xi(0,z_1]=0\right)}$$

$$= P\big(\left(\xi(0,z_1]=0\right) \cap \left(\xi(z_1,z]=1\right) \cup \left(\xi(0,z_1]=1\right) \cap \left(\xi(z_1,z]=0\right) \big)$$

$$- \frac{P\left(\xi(0,z_1]=1\right) P\left(\left(\xi(0,z_1]=0\right) \cap \left(\xi(z_1,z]=0\right)\right)}{P\left(\xi(0,z_1]=0\right)}$$

$$= P\left(\xi(0,z_1]=0\right) P\left(\xi(z_1,z]=1\right) + P\left(\xi(0,z_1]=1\right) P\left(\xi(z_1,z]=0\right)$$

$$- \frac{P\left(\xi(0,z_1]=1\right) P\left(\xi(0,z_1]=0\right) P\left(\xi(z_1,z]=0\right)}{P\left(\xi(0,z_1]=0\right)},$$

where in the last equality we have used the fact that ξ is a process with independent increments and also that some of the events are mutually exclusive. Finally, for the left-hand side of (5), after some trivial algebra, we obtain

$$P(\xi(0,z]=0) H_1\left(z; z_1\right) = P\left(\xi(0,z_1]=0\right) P\left(\xi(z_1,z]=1\right),$$

where it can be directly verified that the right-hand side coincides with

$$\sum_{(g_0,g_1)\in E(0,1)} P\left(\xi(0,z_1]=g_0\right) P\left(\xi(z_1,z]=g_1\right)$$

and therefore, equality (5) is again valid.

We will continue the proof by induction. We showed that Lemma 2.5 holds for $j=0$ and $j=1$. Assume it is true for all indexes up to $j-1$. Lemma 2.5 will be proved if we show that (5) is true also for the index j. Expanding the determinant on the right-hand side of equality (7) with respect to the first column of $\tilde{\Delta}\left(z_1,\ldots,z_j,z\right)$ and after some matrix algebra we obtain

$$\begin{aligned} P\left(\xi(0,z]=0\right) H_j\left(z;z_1,\ldots,z_j\right) &= (-1)^j \det\left(\tilde{\Delta}\left(z_1,\ldots,z_j,z\right)\right) \\ = -\frac{P\left(\xi(0,z_1]=1\right)}{P\left(\xi(0,z_1]=0\right)} P(\xi(0,z)=0) H_{j-1}\left(z;z_2,\ldots,z_j\right) &- \frac{P\left(\xi(0,z_2]=2\right)}{P\left(\xi(0,z_2]=0\right)} P(\xi(0,z)=0) \\ \times H_{j-2}\left(z;z_3,\ldots,z_j\right) - \ldots - \frac{P\left(\xi(0,z_j]=j\right)}{P\left(\xi(0,z_j]=0\right)} P(\xi(0,z)=0) H_0(z) &+ P\left(\xi(0,z]=j\right). \end{aligned} \tag{8}$$

Let us denote by

$$G_{j-n}\left(z_{n+1},\ldots,z_j,z\right) = \bigcup_{(g_1,\ldots,g_{j-n})\in E(1,j-n)} \left(\bigcap_{l=n}^{j-1} \left(\xi(z_{l+1},z_{l+2}]=g_{l-n+1}\right) \right), \tag{9}$$

for $n=0,\ldots,j-1$, where on the right-hand side we assume that $z_{j+1}\equiv z$, $E(l,m)$, $1\le l\le m\le j$, is the set of $(m-l+1)$-tuples of non-negative integers such that

$$E(l,m) = \{(g_l,\dots,g_m) : g_l \le 1, g_l+g_{l+1} \le 2,\dots, g_l+\dots+g_{m-1} \le m-l, g_l+\dots+g_m = m-l+1\}\,, \tag{10}$$

and where for $n=j$, $G_0(z) \equiv \Omega$.

Assume that Lemma 2.5 is true for any index $n = 1,\dots,j-1$, i.e. apply (5) noting that the right-hand side is equal to $P\left((\xi(0,z_{n+1}] = 0) \cap G_{j-n}\left(z_{n+1},\dots,z_j,z\right)\right)$, evaluated at $n=0$. Then, for the n-th term, $n=1,\dots,j$, on the right-hand side of (8), we have

$$\begin{aligned}
&-\frac{P\left(\xi(0,z_n] = n\right)}{P\left(\xi(0,z_n] = 0\right)} P(\xi(0,z) = 0) H_{j-n}\left(z; z_{n+1},\dots,z_j\right)\\
=&\frac{P\left(\xi(0,z_n] = n\right)}{P\left(\xi(0,z_n] = 0\right)} P\left((\xi(z_0,z_{n+1}] = g_0) \cap \left(G_{j-n}\left(z_{n+1},\dots,z_j,z\right)\right)\right)\\
=&\frac{P\left(\xi(0,z_n] = n\right)}{P\left(\xi(0,z_n] = 0\right)} P\left(\xi(0,z_n] = 0\right) P\left(\xi(z_n,z_{n+1}] = 0\right) P\left(G_{j-n}\left(z_{n+1},\dots,z_j,z\right)\right)\\
&= P\left(\xi(0,z_n] = n\right) P\left(\xi(z_n,z_{n+1}] = 0\right) P\left(G_{j-n}\left(z_{n+1},\dots,z_j,z\right)\right) = P(G(n)),
\end{aligned} \tag{11}$$

where we assume that $z_{j+1} \equiv z$ and where in the last equality in (11)

$$G(n) = (\xi(0,z_n] = n) \cap (\xi(z_n,z_{n+1}] = 0) \cap G_{j-n}\left(z_{n+1},\dots,z_j,z\right), \tag{12}$$

$n=1,\dots,j$. When $n=0$, definition (12) shall be interpreted as

$$\begin{aligned}
G(0) &= (\xi(0,z_0] = 0) \cap (\xi(z_0,z_1] = 0) \cap G_j\left(z_1,\dots,z_j,z\right)\\
&= (\xi(0,z_1] = 0) \cap G_j\left(z_1,\dots,z_j,z\right)
\end{aligned} \tag{13}$$

where we have assumed that $(\xi(0,z_0] = 0)$ is the sure event, i.e. $(\xi(0,z_0] = 0) \equiv \Omega$. We also recall that the probability of the event on the right-hand side of the last equality in (13) is equal to the expression on the right-hand side of (5).

Now, applying (11), one can then rewrite (8) as

$$P(\xi(0,z] = 0) H_j\left(z; z_1,\dots,z_j\right) = P\left(\xi(0,z] = j\right) - \Big(P\left(G(1)\right) + \dots + P\left(G(j)\right)\Big). \tag{14}$$

Let us note that for the events $G(n)$, $n=1,\dots,j$, appearing on the right hand side of (14), the following statements are true

$$G(n) \subset \Big(\xi(0,z] = j\Big), n = 0,\dots,j, \tag{15}$$

$$G(n) \cap G(m) = \emptyset, \text{ if } 1 \le n \ne m \le j, \tag{16}$$

where $\emptyset$ is the impossible event, and

$$(\xi(0,z] = j) \setminus \left(\bigcup_{l=1}^{j} G(l)\right) \equiv G(0). \tag{17}$$

Verification of (15), (16) and (17) is technically involved and will be omitted. Now, in view of (16), (17) and (13), one can rewrite (14) as

$$
\begin{aligned}
&P\left(\xi(0,z]=0\right)H_j\left(z;z_1,\ldots,z_j\right)=P\left(\xi(0,z]=j\right)-\Big(P\left(G(1)\cup\ldots\cup G(j)\right)\Big)\\
=&P\Big(\left(\xi(0,z]=j\right)\setminus\left(G(1)\cup\ldots\cup G(j)\right)\Big)=P(G(0))=P\Big(\left(\xi(0,z_1]=0\right)\cap G_j\left(z_1,\ldots,z_j,z\right)\Big)\\
=&\sum_{(g_0,\ldots,g_j)\in E(0,j)}\left(\prod_{l=0}^{j-1}P\left(\xi(z_l,z_{l+1}]=g_l\right)\right)P\left(\xi(z_j,z)=g_j\right),
\end{aligned}
$$

where in the last equality, $E(0,j)$ is defined in (6) for $m=j$ and we have used definitions (13), and (9) with $n=0$, the properties of the probability measure P, and the fact that ξ is a process with independent increments. This completes the proof of Lemma 2.5. □

Lemma 2.6. *For the real sequence, $0\equiv z_0<z_1<z_2<\ldots<z_k<z$, we have*

$$
B_k\left(z;z_1,\ldots,z_k\right)=\sum_{(g_0,\ldots,g_k)\in I(0,k)}\left(\prod_{l=0}^{k-1}P\left(\xi(z_l,z_{l+1}]=g_l\right)\right)P\left(\xi(z_k,z)=g_k\right),\tag{18}
$$

where $B_k\left(z;z_1,\ldots,z_k\right)$ is defined as in (4) with $j=k$, $\prod_{l=0}^{-1}(\cdot)=1$, and where $I(0,k)$ is the set of $(k+1)$-tuples of non-negative integers such that

$$
I(0,m)=\left\{(g_0,\ldots,g_m):g_0\le 0,g_0+g_1\le 1,\ldots,g_0+\ldots+g_{m-1}\le m-1,g_0+\ldots+g_m\le m\right\}.
$$

Proof. Applying an appropriate mapping which relates each element $(g_0,\ldots,g_j)\in\bigcup_{j=0}^{k}E(0,j)$, where $E(0,j)$ is defined in (6), to a set of elements from $I(0,k)$, it can be shown that, for the right-hand side of (18), we have

$$
\begin{aligned}
&\sum_{(g_0,\ldots,g_k)\in I(0,k)}\left(\prod_{l=0}^{k-1}P\left(\xi(z_l,z_{l+1}]=g_l\right)\right)P\left(\xi(z_k,z)=g_k\right)\\
&\qquad=\sum_{j=0}^{k}\sum_{(g_0,\ldots,g_j)\in E(0,j)}\left(\prod_{l=0}^{j-1}P\left(\xi(z_l,z_{l+1}]=g_l\right)\right)P\left(\xi(z_j,z)=g_j\right).
\end{aligned}\tag{19}
$$

The assertion of Lemma 2.6 follows, applying Lemma 2.5 to the second sum on the right-hand side of (19) which gives

$$
\begin{aligned}
\sum_{j=0}^{k}\sum_{(g_0,\ldots,g_j)\in E(0,j)}\left(\prod_{l=0}^{j-1}P\left(\xi(z_l,z_{l+1}]=g_l\right)\right)P\left(\xi(z_j,z]=g_j\right)&=\sum_{j=0}^{k}P\left(\xi(0,z)=0\right)H_j\left(z;z_1,\ldots,z_j\right)\\
&\equiv B_k\left(z;z_1,\ldots,z_k\right).
\end{aligned}
$$

□

Lemma 2.7. *Let $0 < T_1 \le T_2 \le \ldots \le T_k \le T_{k+1} \le \ldots$ be the consecutive points of a point process with independent increments, ξ, and let $0 \equiv z_0 < z_1 < z_2 < \ldots < z_k < z$ be a sequence of positive real numbers. For a fixed k, we have*

$$P\left(T_1 > z_1, \ldots, T_k > z_k, T_{k+1} > z\right) = B_k\left(z; z_1, \ldots, z_k\right),$$

where $B_k\left(z; z_1, \ldots, z_k\right)$ is defined as in (4) with $j = k$.

Proof. Let us consider the intersection of events

$$\left(T_1 > z_1\right) \cap \ldots \cap \left(T_k > z_k\right) \cap \left(T_{k+1} > z\right).$$

For $k = 0$ the event $(T_1 > z)$ coincides with the event $(\xi(0, z] = 0)$, i.e. $(T_1 > z) = (\xi(0, z] = 0)$, since if the realization $T_1(\omega)$ of T_1 of the process ξ has occurred within the interval (z, ∞), then the interval $(0, z]$ remains empty, since the sequence $T_1(\omega) \le T_2(\omega) \le \ldots$ is non-decreasing.

Let us now sequentially transform the intersections

$$(T_1 > z)\,; \Big((T_1 > z_1) \cap (T_2 > z) \Big); \ldots; \Big((T_1 > z_1) \cap \ldots \cap (T_k > z_k) \cap (T_{k+1} > z) \Big).$$

We have

$$(T_1 > z) = (\xi(0, z] = 0) \equiv \bigcup_{g_0 \in I(0,0)} \left(\xi(0, z] = g_0\right),$$

since $I(0, 0) = \{g_0 = 0\}$, and

$$\begin{aligned}
(T_1 > z_1) \cap (T_2 > z) &= (\xi(0, z_1] = 0) \cap (\xi(z_1, z] = 0) \cup (\xi(0, z_1] = 0) \cap (\xi(z_1, z] = 1) \\
&= (\xi(0, z_1] = 0) \cap \Big((\xi(z_1, z] = 0) \cup (\xi(z_1, z] = 1) \Big) \\
&= \left(\bigcup_{g_0 \in I(0,0)} (\xi(0, z_1] = g_0) \right) \cap \left(\bigcup_{(g_0, g_1) \in J(1)} \left((\xi(z_0, z_1] = g_0) \cap (\xi(z_1, z] = g_1) \right) \right) \\
&= \bigcup_{(g_0, g_1) \in I(0,1)} \left((\xi(z_0, z_1] = g_0) \cap (\xi(z_1, z] = g_1) \right),
\end{aligned}$$

where $J(k) = \{(g_0, g_1, \ldots, g_k) : g_0 + g_1 + \ldots + g_k \le k\}$. Similarly, we also have

$$\begin{aligned}
&(T_1 > z_1) \cap (T_2 > z_2) \cap (T_3 > z) \\
&= \left(\bigcup_{(g_0, g_1) \in I(0,1)} \left(\bigcap_{l=0}^{1} (\xi(z_l, z_{l+1}] = g_l) \right) \right) \cap \left(\bigcup_{(g_0, g_1, g_2) \in J(2)} \left(\bigcap_{l=0}^{1} (\xi(z_l, z_{l+1}] = g_l) \cap (\xi(z_2, z] = g_2) \right) \right) \\
&= \bigcup_{(g_0, g_1, g_2) \in I(0,2)} \left(\bigcap_{l=0}^{1} (\xi(z_l, z_{l+1}] = g_l) \cap (\xi(z_2, z] = g_2) \right).
\end{aligned}$$

By straightforward induction, one can write

$$\begin{aligned}
&(T_1 > z_1) \cap (T_2 > z_2) \cap \ldots \cap (T_k > z_k) \cap (T_{k+1} > z) \\
&\qquad = \bigcup_{(g_0,\ldots,g_k) \in I(0,k)} \left(\bigcap_{l=0}^{k-1} (\xi(z_l, z_{l+1}] = g_l) \cap (\xi(z_k, z] = g_k) \right).
\end{aligned} \tag{20}$$

Therefore (20) gives

$$\begin{aligned}
P(T_1 > z_1, \ldots, T_k > z_k, T_{k+1} > z) &= P\left(\bigcup_{(g_0,\ldots,g_k) \in I(0,k)} \left(\bigcap_{l=0}^{k-1} (\xi(z_l, z_{l+1}] = g_l) \cap (\xi(z_k, z] = g_k) \right) \right) \\
&= \sum_{(g_0,\ldots,g_k) \in I(0,k)} \left(\prod_{l=0}^{k-1} P(\xi(z_l, z_{l+1}] = g_l) \right) P(\xi(z_k, z] = g_k).
\end{aligned} \tag{21}$$

The assertion of Lemma 2.7 now follows applying Lemma 2.6 to the sum on the right-hand side of (21). □

We are now in position to prove Theorem 2.3.
Proof of Theorem 2.3: It is not difficult to see that

$$\begin{aligned}
P(T < x, Y > y) &= P\left(\left(T_1 < h^{-1}(Y_1 - y)\right) \cap (T_1 < x) \right) \\
&\quad + \sum_{k=2}^{\infty} P\left(\left(\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}(Y_l)\right) \right) \cap \left(T_k < h^{-1}(Y_k - y)\right) \cap (T_k < x) \right),
\end{aligned} \tag{22}$$

where $\left(T_1 < h^{-1}(Y_1 - y)\right) \cap (T_1 < x)$, is the event of ruin at the first claim with deficit at least y, and $\left(\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}(Y_l)\right) \right) \cap \left(T_k < h^{-1}(Y_k - y)\right) \cap (T_k < x)$ is the event of survival after the first $k-1$ claims have arrived and ruin at the k-th claim with deficit at least y.

Let us now transform the probabilities in (22). By means of conditional probabilities, and after appropriate transformation of the domain of integration, it is easy to show that

$$\begin{aligned}
&P\left(\left(T_1 < h^{-1}(Y_1 - y)\right) \cap (T_1 < x) \right) \\
&= \int_y^{+\infty} f(y_1)\, dy_1 - \int_y^{h(x)+y} P\left(\xi(0, h^{-1}(y_1 - y)] = 0\right) f(y_1)\, dy_1 \\
&\quad - P(\xi(0, x] = 0) \int_{h(x)+y}^{+\infty} f(y_1)\, dy_1.
\end{aligned} \tag{23}$$

It is straightforward to show that the remaining probabilities in (22), can be expressed as

$$P\left(\left(\bigcap_{l=1}^{k-1}\left(T_l > h^{-1}\left(Y_l\right)\right)\right) \cap \left(T_k < h^{-1}\left(Y_k - y\right)\right) \cap \left(T_k < x\right)\right) = \int_{C_k}\!\!\dots\!\int \left\{P\left[\bigcap_{l=1}^{k-1}\left(T_l > h^{-1}\left(y_l\right)\right)\right]\right.$$
$$\left. -P\left[\left(\bigcap_{l=1}^{k-1}\left(T_l > h^{-1}\left(y_l\right)\right)\right) \cap \left(T_k > h^{-1}\left(y_k - y\right)\right)\right]\right\} f\left(y_1, \dots, y_k\right) dy_1 \dots dy_k$$
$$+\int_{D_k}\!\!\dots\!\int \left\{P\left[\bigcap_{l=1}^{k-1}\left(T_l > h^{-1}\left(y_l\right)\right)\right]\right.$$
$$\left. -P\left[\left(\bigcap_{l=1}^{k-1}\left(T_l > h^{-1}\left(y_l\right)\right)\right) \cap \left(T_k > x\right)\right]\right\} f\left(y_1, \dots, y_k\right) dy_1 \dots dy_k, \tag{24}$$

where we have set $C_k = \{(y_1, \dots, y_k) : 0 \leq y_1 \leq \dots \leq y_{k-1} \leq y_{k-1} + y \leq y_k \leq h(x) + y\}$ and $D_k = \{(y_1, \dots, y_k) : 0 \leq y_1 \leq \dots \leq y_{k-1} \leq h(x) \leq h(x) + y \leq h(x) < y_k < +\infty\}$.

From (22),(23) and (24), applying Lemma 2.7 to the probabilities on the right-hand side of (24) we obtain the asserted formula (3). □

The following two corollaries of Theorem 2.3 give explicitly formulas for the joint distribution of the ruin time and deficit and for the finite and infinite time probability of ruin, under a claim arrival process with independent increments, which generalize previous results of Ignatov and Kaishev (2000, 2004) obtained for the Poisson case.

Corollary 2.8. *In the case of discrete claim amounts $W_1, W_2, \dots$ with joint probability mass function $P_{w_1,\dots,w_k} = P\left(W_1 = w_1, \dots, W_k = w_k\right)$, $k = 1, 2, \dots$, we have*

$$P(T < x, Y > y) = 1 - \sum_{w_1=1}^{m-1} P_{w_1} - \sum_{w_1=m}^{l} P_{w_1} \times P\left(\xi(0, h^{-1}(w_1 - y)) = 0\right)$$
$$-\left(1 - \sum_{w_1=1}^{l-1} P_{w_1}\right) \times P\left(\xi(0, x) = 0\right)$$
$$+\sum_{k=2}^{l} \sum_{(w_1,\dots,w_k) \in \tilde{C}_k} P_{w_1,\dots,w_k} \left\{B_{k-2}\left(h^{-1}\left(w_1 + \dots + w_{k-1}\right); h^{-1}\left(w_1\right), \dots, h^{-1}\left(w_1 + \dots + w_{k-2}\right)\right)\right.$$
$$\left. -B_{k-1}\left(h^{-1}\left(w_1 + \dots + w_k - y\right); h^{-1}\left(w_1\right), \dots, h^{-1}\left(w_1 + \dots + w_{k-1}\right)\right)\right\}$$
$$+\sum_{k=2}^{n+1} \sum_{(w_1,\dots,w_k) \in \tilde{D}_k} P_{w_1,\dots,w_k} \left\{B_{k-2}\left(h^{-1}\left(w_1 + \dots + w_{k-1}\right); h^{-1}\left(w_1\right), \dots, h^{-1}\left(w_1 + \dots + w_{k-2}\right)\right)\right.$$
$$\left. -B_{k-1}\left(x; h^{-1}\left(w_1\right), \dots, h^{-1}\left(w_1 + \dots + w_{k-1}\right)\right)\right\},$$

where $m = [y] + 1$, $n = [h(x)]$, $l = [h(x) + y]$, *with* $[\cdot]$ *denoting the integer part,* $\tilde{C}_k = \{(w_1, \ldots, w_k) : 1 \leq w_i, i = 1, \ldots, k, y < w_k, w_1 + \ldots + w_k < h(x) + y\}$, $\tilde{D}_k = \{(w_1, \ldots, w_k) : 1 \leq w_i, i = 1, \ldots, k, w_1 + \ldots + w_k \leq h(x) \leq h(x) + y \leq w_1 + \ldots + w_k < +\infty\}$, *and* $B_j\left(z; h^{-1}(w_1), \ldots, h^{-1}(w_1 + \ldots + w_j)\right)$
$= P(\xi(0, z) = 0)\left[H_0(z) + H_1\left(z; h^{-1}(w_1)\right) + \ldots + H_j\left(z; h^{-1}(w_1), \ldots, h^{-1}(w_1 + \ldots + w_j)\right)\right]$.

Corollary 2.9. *In the case of an infinite time horizon, i.e. when* $x = \infty$, *formula (3) simplifies to*

$$P(T < \infty, Y > y) = \int_y^{+\infty} f(y_1)\, dy_1 - \int_y^{+\infty} P\left(\xi(0, h^{-1}(y_1 - y)) = 0\right) f(y_1)\, dy_1$$
$$+ \sum_{k=2}^{\infty} \int \underset{C_k}{\ldots} \int \left\{ B_{k-2}\left(h^{-1}(y_{k-1}); h^{-1}(y_1), \ldots, h^{-1}(y_{k-2})\right) \right.$$
$$\left. - B_{k-1}\left(h^{-1}(y_k - y); h^{-1}(y_1), \ldots, h^{-1}(y_{k-1})\right) \right\} f(y_1, \ldots, y_k)\, dy_k \ldots dy_1,$$

where $C_k = \{(y_1, \ldots, y_k) : 0 < y_1 < \ldots < y_{k-1} \leq y_{k-1} + y < y_k < +\infty\}$, *and* B_k *is defined as in Theorem 2.3.*

Let us note that further Corollaries which generalize previous ruin probability formulas of Ignatov and Kaishev (2000, 2004, 2006) obtained for the Poisson case can be easily obtained by directly substituting in (3), $y = 0$ and $y = 0, x = \infty$.

3. $P(T < x, Y > y)$ for some special cases of the claim arrival process ξ

In this section we provide some further corollaries of our main result given by Theorem 2.3, which cover important special cases of claim arrival point processes with independent increments, namely, the (non-)stationary Poisson and negative binomial cases. To the best of our knowledge such models have not been extensively considered in the ruin probability literature (with the exception of the stationary Poisson claim arrival case).

3.1. Non-stationary Poisson claim arrivals

Let $G(z)$ be the cumulative function of the measure $\Lambda(\cdot)$, i.e. $G(z) = \Lambda((0, z])$, such that $\lim_{z\to\infty} G(z) = \infty$. If the process of claim arrivals, ξ, is a Poisson point process with cumulative intensity function $G(z) = E\xi((0, z])$ then it is not difficult to see that the definition of the Appell-Hessenberg functions, given in (1) specifies to

$$\Phi_j\left(z; z_1, \ldots, z_j\right) = (-1)^j \det \begin{pmatrix} \frac{G(z_1)^1}{1!} & 1 & 0 & 0 & \ldots & 0 \\ \frac{G(z_2)^2}{2!} & \frac{G(z_2)^1}{1!} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & & \ddots & \\ \frac{G(z_j)^j}{j!} & \frac{G(z_j)^{j-1}}{(j-1)!} & & & \ldots & 1 \\ \frac{G(z)^j}{j!} & \frac{G(z)^{j-1}}{(j-1)!} & & & \ldots & 1 \end{pmatrix}, \quad (25)$$

where $j = 0, 1, 2, \ldots$ and $\Phi_0(z) \equiv 1$. The latter can be efficiently evaluated using the following corollary of Proposition 2.2.

Corollary 3.1. *For the Appell-Hessenberg functions, $\Phi_j\left(z; z_1, \ldots, z_j\right)$, defined in (25), we have*

$$\Phi_j\left(z; z_1, \ldots, z_j\right) = \sum_{i=0}^{j} \frac{G(z)^{j-i}}{(j-i)!} \Phi_i\left(0; z_1, \ldots, z_i\right), j \geq 0,$$

where $\Phi_0(z) \equiv 1$,

$$\Phi_i\left(0; z_1, \ldots, z_i\right) = -\sum_{k=0}^{i-1} \frac{G\left(z_i\right)^{i-k}}{(i-k)!} \Phi_k\left(0; z_1, \ldots, z_k\right), i \geq 1,$$

with $\Phi_0(0) \equiv 1$.

From Lemma 2.7, for the Poisson case, we have

Lemma 3.2. *Let $0 < T_1 \leq T_2 \leq \ldots \leq T_k \leq T_{k+1} \leq \ldots$ be the consecutive points of a Poisson point process, ξ, and let $0 \equiv z_0 < z_1 < z_2 < \ldots < z_k < z$ be a sequence of positive real numbers. For a fixed k, we have*

$$P\left(T_1 > z_1, \ldots, T_k > z_k, T_{k+1} > z\right) = \beta_k\left(z; z_1, \ldots, z_k\right),$$

where

$$\beta_k\left(z; z_1, \ldots, z_k\right) = e^{-G(z)}\left[\Phi_0(z) + \Phi_1\left(z; z_1\right) + \ldots + \Phi_k\left(z; z_1, \ldots, z_k\right)\right]$$

is the particular Poisson case version of the function $B_k\left(z; z_1, \ldots, z_k\right)$, given in Theorem 2.3.

For an arbitrary $G(z)$, the Appell-Hessenberg type function $\Phi_j\left(z; z_1, \ldots, z_j\right)$ coincides with a classical Appell polynomial, as established by the following.

Corollary 3.3. *For a non-stationary Poisson point process, ξ, with cumulative intensity $G(z)$, the Appell-Hessenberg type functions $\Phi_j\left(z; z_1, \ldots, z_j\right)$, $j = 0, 1, 2, \ldots$, defined in (25), coincide with the classical Appell polynomials $A_j\left(G(z); G\left(z_1\right), \ldots, G\left(z_j\right)\right) \equiv A_j(G(z))$ of degree j with a coefficient in front of $G(z)^j$ equal to $1/j!$, i.e.,*

$$\Phi_j\left(z; z_1, \ldots, z_j\right) \equiv A_j\left(G(z); G\left(z_1\right), \ldots, G\left(z_j\right)\right)$$

where

$$\begin{aligned} A_0(G(z)) &= 1, \\ A_j'(G(z)) &= A_{j-1}(G(z)), \text{ and} \\ A_j\left(G\left(z_j\right)\right) &= 0, \end{aligned} \tag{26}$$

$j = 1, 2, \ldots,$ *with* $0 \le z_1 \le \ldots \le z_j,\ z_j \in \mathbb{R}$.

The following corollary is a direct consequence of Corollary 3.3 and Lemma 3.2.

Corollary 3.4. *With the notation of Lemma 3.2, we have*

$$P\left(T_1 > z_1, \ldots, T_k > z_k, T_{k+1} > z\right) = b_k\left(z; z_1, \ldots, z_k\right),$$

where

$$b_k\left(z; z_1, \ldots, z_k\right) = e^{-G(z)}\left[A_0(G(z)) + A_1\left(G(z); G\left(z_1\right)\right) + \ldots + A_k\left(G(z); G\left(z_1\right), \ldots, G\left(z_k\right)\right)\right], \tag{27}$$

and $A_0(G(z))$, $A_1\left(G(z); G\left(z_1\right)\right)$, $\ldots$, $A_k\left(G(z); G\left(z_1\right), \ldots, G\left(z_k\right)\right)$ *are the classical Appell polynomials defined as in (26) with* $j = k$, *i.e. evaluated at* $G(z)$ *and defined by the sequence* $G\left(z_1\right), \ldots, G\left(z_k\right)$.

Lemma 2.5 in the non-stationary Poisson case can be reformulated as follows.

Lemma 3.5. *We have*

$$e^{-G(z)}\Phi_j\left(z; z_1, \ldots, z_j\right) = e^{-G(z_1)}\sum_{(g_1,\ldots,g_j)\in E(1,j)} e^{-(G(z_2)-G(z_1))}\frac{\left(G\left(z_2\right) - G\left(z_1\right)\right)^{g_1}}{g_1!} \times \ldots$$

$$\times\, e^{-(G(z_j)-G(z_{j-1}))}\frac{\left(G\left(z_j\right) - G\left(z_{j-1}\right)\right)^{g_{j-1}}}{g_{j-1}!} e^{-(G(z)-G(z_j))}\frac{\left(G(z) - G\left(z_j\right)\right)^{g_j}}{g_j!},$$

where $E(1, j)$ *is defined as in (10), with* $l = 1$ *and* $m = j$.

It can directly be seen from Lemma 2.6 that for the non-stationary Poisson case we have

Lemma 3.6. *For* $b_k\left(z; z_1, \ldots, z_k\right)$, *defined as in (27) we have*

$$b_k\left(z; z_1, \ldots, z_k\right) = e^{-G(z_1)}\sum_{(g_1,\ldots,g_k)\in I(1,k)} e^{-(G(z_2)-G(z_1))}\frac{\left(G\left(z_2\right) - G\left(z_1\right)\right)^{g_1}}{g_1!} \times \ldots$$

$$\times\, e^{-(G(z_k)-G(z_{k-1}))}\frac{\left(G\left(z_k\right) - G\left(z_{k-1}\right)\right)^{g_{k-1}}}{g_{k-1}!} \times e^{-(G(z)-G(z_k))}\frac{\left(G(z) - G\left(z_k\right)\right)^{g_k}}{g_k!},$$

where $I(1, k) = \{(g_1, \ldots, g_k) : g_1 \le 1, g_1 + g_2 \le 2, \ldots, g_1 + \ldots + g_k \le k\}$.

From Theorem 2.3, Lemma 3.2 and Corollary 3.4, we have

Corollary 3.7. *In the case of Poisson claim arrivals with cumulative intensity function* $G(z)$*, the probability* $P(T<x,Y>y)$*,* $x>0$*,* $y\geq 0$*, is given by*

$$
\begin{aligned}
&P(T<x,Y>y)=\int_{y}^{+\infty} f\left(y_1\right) dy_1-\int_{y}^{h(x)+y} e^{-G\left(h^{-1}\left(y_1-y\right)\right)} f\left(y_1\right) dy_1 \\
&-e^{-G(x)} \int_{h(x)+y}^{+\infty} f\left(y_1\right) dy_1 \\
&+\sum_{k=2}^{\infty} \int_{C_k}\cdots\int \left\{b_{k-2}\left(h^{-1}\left(y_{k-1}\right); h^{-1}\left(y_1\right), \ldots, h^{-1}\left(y_{k-2}\right)\right)\right. \\
&\quad\left.-b_{k-1}\left(h^{-1}\left(y_k-y\right); h^{-1}\left(y_1\right), \ldots, h^{-1}\left(y_{k-1}\right)\right)\right\} f\left(y_1, \ldots, y_k\right) dy_k \ldots dy_1 \\
&+\sum_{k=2}^{\infty} \int_{D_k}\cdots\int \left\{b_{k-2}\left(h^{-1}\left(y_{k-1}\right); h^{-1}\left(y_1\right), \ldots, h^{-1}\left(y_{k-2}\right)\right)\right. \\
&\quad\left.-b_{k-1}\left(x; h^{-1}\left(y_1\right), \ldots, h^{-1}\left(y_{k-1}\right)\right)\right\} f\left(y_1, \ldots, y_k\right) dy_k \ldots dy_1,
\end{aligned} \tag{28}
$$

where C_k*, and* D_k*, are defined as in Theorem 2.3 and*

$$
\begin{aligned}
&b_j\left(z; h^{-1}\left(y_1\right), \ldots, h^{-1}\left(y_j\right)\right) \\
&=e^{-G(z)}\left[A_0(z)+A_1\left(G(z); G(h^{-1}(y_1))\right)+\ldots+A_j\left(G(z); G(h^{-1}(y_1)), \ldots, G(h^{-1}(y_j))\right)\right].
\end{aligned} \tag{29}
$$

for $j=0,1,2\ldots$ *are the functions defined as in (27) (see Corollary 3.4).*

Consider the case of claims arriving according to a stationary Poisson point process, ξ, with intensity 1, i.e. with cumulative intensity $G(z)=z$. All results from Section 3.1 directly hold for the stationary Poisson case replacing $G(z)$ with z.

Remark 3.8. *In the case when claim arrivals follow a stationary Poisson process with intensity* λ*, formula (28) for the probability* $P(T<x,Y>y)$ *holds with* $G(z)=\lambda z$ *in (29), i.e. with*

$$
\begin{aligned}
&b_j\left(z; h^{-1}(y_1), \ldots, h^{-1}(y_j)\right) \\
&=e^{-\lambda z}\left[A_0(z)+A_1\left(\lambda z; \lambda h^{-1}(y_1)\right)+\ldots+A_j\left(\lambda z; \lambda h^{-1}(y_1), \ldots, \lambda h^{-1}(y_j)\right)\right].
\end{aligned}
$$

Furthermore, if λ *is a positive random variable, say* Λ*, formula (28) holds true with*

$$
\begin{aligned}
&b_j\left(z; h^{-1}(y_1), \ldots, h^{-1}(y_j)\right) \\
&=\mathbb{E}\Big[e^{-\Lambda z}\left[A_0(z)+A_1\left(\Lambda z; \Lambda h^{-1}(y_1)\right)+\ldots+A_j\left(\Lambda z; \Lambda h^{-1}(y_1), \ldots, \Lambda h^{-1}(y_j)\right)\right]\Big].
\end{aligned}
$$

Let us note that this case corresponds to the mixed Poisson process of claim arrivals considered also by Lefèvre and Picard (2011) as an order statistics point process, i.e., formula (28) with $P\left(T<x,Y>0\right)$ *covers formula (4.1) therein.*

3.2. Negative binomial claim arrivals

Let the claim arrival process, ξ, be a negative binomial point process. Let $G(z)$ be the cumulative function of its intensity measure $\Lambda(\cdot)$, i.e. $G(z) = \Lambda((0, z])$, such that $\lim_{z\to\infty} G(z) = \infty$. In other words, we assume that the random variable $\xi((0, z])$ has a negative binomial distribution with parameters q and $r(z) = \frac{p}{q}G(z)$, $(p = 1 - q)$, i.e.

$$P(\xi(0, z] = k) = \begin{pmatrix} -r(z) \\ k \end{pmatrix} p^{r(z)}(-q)^k, k = 1, 2, \ldots.$$

Clearly, the process of claim arrivals, ξ, is a non-stationary negative binomial point process with independent increments. Then, for the process ξ the definition of the Appell-Hessenberg functions given in (1) specifies as

$$\Psi_j\left(z; z_1, \ldots, z_j\right)$$
$$= (-1)^j q^j \det \begin{pmatrix} \begin{pmatrix} r(z_1) \\ 1 \end{pmatrix} & 1 & 0 & 0 & \ldots & 0 & 0 \\ \begin{pmatrix} r(z_2)+1 \\ 2 \end{pmatrix} & \begin{pmatrix} r(z_2) \\ 1 \end{pmatrix} & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \\ \begin{pmatrix} r(z_j)+j-1 \\ j \end{pmatrix} & \begin{pmatrix} r(z_j)+j-2 \\ j-1 \end{pmatrix} & & & \ldots & \begin{pmatrix} r(z_j) \\ 1 \end{pmatrix} & 1 \\ \begin{pmatrix} r(z)+j-1) \\ j \end{pmatrix} & \begin{pmatrix} r(z)+j-2 \\ j-1 \end{pmatrix} & & & \ldots & \begin{pmatrix} r(z) \\ 1 \end{pmatrix} & 1 \end{pmatrix}, \tag{30}$$

where, $j = 0, 1, 2, \ldots$ and $\Psi_0(z) \equiv 1$. As seen from (30), $\Psi_j\left(z; z_1, \ldots, z_j\right)$ are expressed in terms of factorial functions. The following corollary is a direct consequence of Proposition 2.2.

Corollary 3.9. *For the Appell-Hessenberg functions, $\Psi_j\left(z; z_1, \ldots, z_j\right)$, defined in (30), we have*

$$\Psi_j\left(z; z_1, \ldots, z_j\right) = \sum_{i=0}^{j} \begin{pmatrix} r(z)+j-i-1 \\ j-i \end{pmatrix} \Psi_i\left(0; z_1, \ldots, z_i\right), j \geq 0,$$

where $\Psi_0(z) \equiv 1$,

$$\Psi_i\left(0; z_1, \ldots, z_i\right) = -\sum_{k=0}^{i-1} \begin{pmatrix} r(z_i)+i-k-1 \\ i-k \end{pmatrix} \Psi_k\left(0; z_1, \ldots, z_k\right), i \geq 1,$$

with $\Psi_0(0) \equiv 1$.

In the case when $r(z) = z$, i.e. when ξ is a stationary NB process, from (30) we obtain that

$$\mathcal{A}_j\left(z; z_1, \ldots, z_j\right) = \frac{\Psi_j\left(z; z_1, \ldots, z_j\right)}{q^j} \tag{31}$$

are polynomials which we can view as a certain generalization of the classical Appell polynomials. In view of (30), the polynomials $\mathcal{A}_j(z; z_1, \ldots, z_j)$, $j = 0, 1, 2, \ldots$ can be referred to as Appell factorial polynomials, noting that

$$\binom{z+j-1}{j} = \Big((z+j-1)(z+j-2)\ldots(z+0)\Big)/j!. \qquad (32)$$

The Appell factorial polynomials, $\mathcal{A}_j(z; z_1, \ldots, z_j)$, $j = 0, 1, 2, \ldots$, can be recurrently computed by the following recurrence expression which follows from Corollary 3.9.

Corollary 3.10. *For the Appell-Hessenberg functions, $\mathcal{A}_j(z; z_1, \ldots, z_j)$, defined in (31), we have*

$$\mathcal{A}_j(z; z_1, \ldots, z_j) = \sum_{i=0}^{j} \binom{z+j-i-1}{j-i} \mathcal{A}_i(0; z_1, \ldots, z_i), j \geq 0,$$

where $\mathcal{A}_0(z) \equiv 1$,

$$\mathcal{A}_i(0; z_1, \ldots, z_i) = -\sum_{k=0}^{i-1} \binom{z_i+i-k-1}{i-k} \mathcal{A}_k(0; z_1, \ldots, z_k), i \geq 1,$$

with $\mathcal{A}_0(0) \equiv 1$.

Let us recall that the classical Appell polynomials admit the Hessenberg determinant representation (25) with $G(z)$ replaced by z. So, comparing (25) and (30), with (32) in mind, one can see that $\mathcal{A}_j(z; z_1, \ldots, z_j)$ can formally be obtained from $A_j(z; z_1, \ldots, z_j)$, $j = 0, 1, 2, \ldots$, by replacing multiplication in (25) with factorial multiplication, as in (32).

In the general case when $r(z) \neq z$ we have that

$$\mathcal{A}_j(r(z); r(z_1), \ldots, r(z_j)) = \frac{\Psi_j(z; z_1, \ldots, z_j)}{q^j},$$

i.e. the Appell-Hessenberg type functions, $\Psi_j(z; z_1, \ldots, z_j)$, $j = 0, 1, 2, \ldots$, are expressed through the Appell factorial polynomials, $\mathcal{A}_j(r(z); r(z_1), \ldots, r(z_j))$.

Now from Lemmas 2.5, 2.6 and 2.7, for the case when ξ is non-stationary negative binomial point process, we have the corollaries

Corollary 3.11. *For the consecutive points $T_1, T_2, \ldots,$ of a negative binomial point process, ξ, we have*

$$P(T_1 > z_1, \ldots, T_k > z_k, T_{k+1} > z) = \gamma_k(z; z_1, \ldots, z_k),$$

where $0 \equiv z_0 < z_1 < z_2 < \ldots < z_k < z$ is a sequence of positive real numbers,

$$\begin{aligned}\gamma_k(z; z_1, \ldots, z_k) &= p^{r(z)}\left[\Psi_0(z) + \Psi_1(z; z_1) + \ldots + \Psi_{k-1}(z; z_1, \ldots, z_k)\right] \\ &= p^{r(z)}\left[\mathcal{A}_0(r(z)) + q\mathcal{A}_1(r(z); r(z_1)) + \ldots + q^{k-1}\mathcal{A}_k(r(z); r(z_1), \ldots, r(z_k))\right]\end{aligned}$$

is the negative binomial special case of the function $B_k(z; z_1, \ldots, z_k)$ *given in Theorem 2.3,* $\Psi_k(z; z_1, \ldots, z_k)$, $k = 0, 1, 2, \ldots$, *are the Appell-Hessenberg functions defined in (30), and* $\mathcal{A}_k(r(z); r(z_1), \ldots, r(z_k))$ *are the Appell factorial polynomials evaluated at* $r(z)$ *and defined as in (31) by the sequence* $r(z_1), \ldots, r(z_k)$.

Corollary 3.12. *For the Appell-Hessenberg functions defined in (30), we have*

$$\begin{aligned}&p^{r(z)}\Psi_j(z; z_1, \ldots, z_j)\\&= q^j \sum_{E(1,j)} \binom{r(z_2) - r(z_1) + g_1 - 1}{g_1} \times \binom{r(z_3) - r(z_2) + g_2 - 1}{g_2} \times \ldots \times \binom{r(z) - r(z_j) + g_j - 1}{g_j}.\end{aligned}$$

Corollary 3.13. *We have*

$$\begin{aligned}&\gamma_k(z; z_1, \ldots, z_k) = p^{r(z)} \sum_{I(1,k)} q^{g_1 + \ldots + g_k} \binom{r(z_2) - r(z_1) + g_1 - 1}{g_1} \times\\&\binom{r(z_3) - r(z_2) + g_2 - 1}{g_2} \times \ldots \times \binom{r(z_k) - r(z_{k-1}) + g_{k-1} - 1}{g_{k-1}} \times \binom{r(z) - r(z_k) + g_k - 1}{g_k},\end{aligned}$$

where $I(1, k)$ *is defined as in Lemma 3.6.*

From Theorem 2.3 and Corollary 3.11 and Corollary 3.12, we have

Corollary 3.14. *In the case when claim arrivals follow a negative binomial point process with intensity function* $G(z)$, *(i.e. NB parameters* q *and* $r(z) = \frac{p}{q}G(z)$*), the probability* $P(T < x, Y > y)$, $x > 0$, $y \geq 0$, *is given by*

$$\begin{aligned}P(T < x, Y > y) &= \int_y^{+\infty} f(y_1)\, dy_1 - \int_y^{h(x)+y} p^{r\left(h^{-1}(y_1 - y)\right)} f(y_1)\, dy_1\\&-p^{r(x)} \int_{h(x)+y}^{+\infty} f(y_1)\, dy_1\\&+\sum_{k=2}^{\infty} \int \cdots \int_{C_k} \left\{\gamma_{k-2}\left(h^{-1}(y_{k-1}); h^{-1}(y_1), \ldots, h^{-1}(y_{k-2})\right)\right.\\&\quad\left.-\gamma_{k-1}\left(h^{-1}(y_k - y); h^{-1}(y_1), \ldots, h^{-1}(y_{k-1})\right)\right\} f(y_1, \ldots, y_k)\, dy_k \ldots dy_1\\&+\sum_{k=2}^{\infty} \int \cdots \int_{D_k} \left\{\gamma_{k-2}\left(h^{-1}(y_{k-1}); h^{-1}(y_1), \ldots, h^{-1}(y_{k-2})\right)\right.\\&\quad\left.-\gamma_{k-1}\left(x; h^{-1}(y_1), \ldots, h^{-1}(y_{k-1})\right)\right\} f(y_1, \ldots, y_k)\, dy_k \ldots dy_1,\end{aligned} \tag{33}$$

where C_k *and* D_k *are defined as in Theorem 2.3, and*

$$\begin{aligned}&\gamma_j\left(z; h^{-1}(y_1), \ldots, h^{-1}(y_j)\right)\\&= p^{r(z)}\left[\mathcal{A}_0(r(z)) + q\mathcal{A}_1\left(r(z); r(h^{-1}(y_1))\right) + \ldots + q^j \mathcal{A}_j\left(r(z); r(h^{-1}(y_1)), \ldots, r(h^{-1}(y_j))\right)\right],\end{aligned}$$

for $j = 0, 1, 2\ldots$, *are the functions defined as in Corollary 3.11.*

Remark 3.15. *Let us note that the NB process with independent increments considered here is in general different from the NB process considered by Lefèvre and Picard (2011) in the context of OS risk processes. While both NB processes are non-stationary and the distribution of the number of points in a fixed time interval is negative binomial, the NB-process considered by Lefèvre and Picard (2011), is a linear birth process with immigration which is Markovian and is not a process with independent increments. Another difference is that the latter NB-process does not allow for the instantaneous arrival of clusters of points which is possible under the definition adopted here. In conclusion, the two definitions are underpinned by different stochastic constructions, e.g. a NB process with independent increments can be constructed by taking a compound Poisson process with logarithmically distributed summands, whereas the NB process with the OS property cannot (see Kozubowski and Podgórski 2009), leading to their different properties and in particular different clustering of the points.*

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