ON THE MAGNITUDE OF A FINITE DIMENSIONAL ALGEBRA

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Abstract. There is a general notion of the magnitude of an enriched category, defined subject to hypotheses. In topological and geometric contexts, magnitude is already known to be closely related to classical invariants such as Euler characteristic and dimension. Here we establish its significance in an algebraic context. Specifically, in the representation theory of an associative algebra A, a central role is played by the indecomposable projective A-modules, which form a category enriched in vector spaces. We show that the magnitude of that category is a known homological invariant of the algebra: writing χ_A for the Euler form of A and S for the direct sum of the simple A-modules, it is $\chi_A(S, S)$.

1. Introduction

This paper is part of a large programme to define and investigate cardinality-like invariants of mathematical objects. Given a monoidal category $\mathbf{V}$ together with a notion of the 'size' $|X|$ of each object X of $\mathbf{V}$, there arises automatically a notion of the 'size' or 'magnitude' of each $\mathbf{V}$-category (subject to conditions). Here we apply this general method in the context of associative algebras.

More specifically, for any finite-dimensional algebra A, the category $\mathbf{IP}(A)$ of indecomposable projective A-modules plays a central role (discussed below) in the theory of representations of A. This category is enriched in finite-dimensional vector spaces, and, taking dimension as the base notion of size, we can then consider the magnitude of $\mathbf{IP}(A)$. We show that this is a known homological invariant of the original algebra A.

Little algebra will be assumed on the reader's part; all the necessary background is provided in Section 2.

The general definition of magnitude is as follows [9, §1.3]. Let $\mathbf{V}$ be a monoidal category equipped with a function $|\cdot|$ on its set of objects (taking values in a semiring, say). Let $\mathbf{A}$ be a $\mathbf{V}$-category with finitely many objects. Denote by $Z_{\mathbf{A}} = (Z_{ab})$ the square matrix whose rows and columns are indexed by the objects of $\mathbf{A}$, and with entries

$$Z_{ab} = |\mathbf{A}(a, b)| \tag{1.1}$$

$(a, b \in \mathbf{A})$. If $Z_{\mathbf{A}}$ is invertible, the **magnitude** $|\mathbf{A}|$ of $\mathbf{A}$ is defined to be the sum of all the entries of $Z_{\mathbf{A}}^{-1}$.

Since $Z_{\mathbf{A}}$ need not be invertible, magnitude is not defined for every $\mathbf{A}$. But where magnitude *is* defined, we may harmlessly extend the definition by equivalence, setting $|\mathbf{A}| = |\mathbf{B}|$ whenever $\mathbf{A}$ and $\mathbf{B}$ are equivalent $\mathbf{V}$-categories such that $\mathbf{B}$ has finitely many objects and $Z_{\mathbf{B}}$ is invertible. (There is no problem of consistency, since if $\mathbf{A}$ and $\mathbf{B}$ are equivalent and both $Z_{\mathbf{A}}$ and $Z_{\mathbf{B}}$ are invertible then both $\mathbf{A}$ and $\mathbf{B}$ are skeletal—that is, isomorphic objects are equal—and so $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.)

Unmotivated as this definition may seem, multiple theorems attest that magnitude is the canonical notion of the size of an enriched category. For example, take $\mathbf{V}$ to be the category of finite sets and $|X|$ to be the cardinality of a finite set X. Then we obtain a notion of the magnitude of a finite category. In this context, magnitude is also called Euler characteristic [8], for the following reason. Recall that every small category $\mathbf{A}$ gives rise to a topological space $B\mathbf{A}$, its classifying space or geometric realisation. Proposition 2.11 of [8] states that under finiteness hypotheses,

$$|\mathbf{A}| = \chi(B\mathbf{A}). \tag{1.2}$$

Thus, the Euler characteristic of a category has a similar status to group (co)homology: it is defined combinatorially, but agrees with the topological notion when one passes to the classifying space.

For another example, let $\mathbf{V}$ be the ordered set $([0,\infty],\geq)$ with addition as the monoidal structure, so that metric spaces can be viewed as $\mathbf{V}$-categories [6]. For $x \in [0,\infty]$, put $|x| = e^{-x}$. (The virtue of this choice is that $|x \otimes y| = |x|\,|y|$.) Then we obtain a notion of the magnitude of a finite metric space. This extends naturally to a large class of compact metric spaces [9, 11, 12]. The magnitude of a compact subset of $\mathbb{R}^n$ is always well-defined, and is closely related to classical quantities of geometric measure. For example, a theorem of Meckes [12, Corollary 7.4] shows that Minkowski dimension can be recovered from magnitude, and conjectures of Leinster and Willerton [10] state that magnitude also determines invariants such as volume and surface area.

Here we study the case where $\mathbf{V}$ is the category of finite-dimensional vector spaces and $|X| = \dim X$. We then obtain a notion of the magnitude of a linear (that is, $\mathbf{V}$-enriched) category. Our main theorem is this:

Theorem 1.1. *Let A be an algebra of finite dimension and finite global dimension over an algebraically closed field. Write $\mathbf{IP}(A)$ for the linear category of indecomposable projective A-modules, $(S_i)_{i\in I}$ for representatives of the isomorphism classes of simple A-modules, and $S = \bigoplus_{i\in I} S_i$. Then*

$$|\mathbf{IP}(A)| = \sum_{n=0}^{\infty} (-1)^n \dim \operatorname{Ext}^n_A(S,S). \tag{1.3}$$

We now explain the context of this result; background can be found in the next section.

Any associative algebra A gives rise to several linear categories, including the category of all A-modules and the one-object category corresponding to A itself (which trivially has magnitude $1/\dim A$). But it also gives rise to the category $\mathbf{IP}(A)$ of indecomposable projective A-modules, whose main significance is that its representation theory is the same as that of A:

$$\begin{array}{ccc} A\text{-}\mathbf{Mod} & \simeq & [\mathbf{IP}(A)^{\mathrm{op}}, \mathbf{Vect}] \\ M & \mapsto & \operatorname{Hom}_A(-,M) \end{array} \tag{1.4}$$

where the right-hand side is the category of contravariant linear functors from $\mathbf{IP}(A)$ to vector spaces. In other words, $\mathbf{IP}(A)^{\text{op}}$ and the one-object linear category A are Morita equivalent.

The Krull–Schmidt theorem states that every finitely generated A-module can be expressed as a direct sum of indecomposable modules, in an essentially unique way. It implies that the A-module A is a direct sum of indecomposable *projective* modules, and that, moreover, every indecomposable projective appears at least once in this sum. Thus, the indecomposable projectives are the 'atoms' of A, in the sense of being its constituent parts.

This explains the equivalence (1.4). The absolute colimits in linear categories are the finite direct sums and idempotent splittings (that is, direct summands). Every finitely generated projective module is a direct sum of indecomposable projectives, so the category of finitely generated projectives is the Cauchy completion of $\mathbf{IP}(A)$. On the other hand, every finitely generated projective is a direct summand of a direct sum of copies of the A-module A, so the category of finitely generated projectives is also the Cauchy completion of the one-object category A^{op}. Hence $\mathbf{IP}(A)$ and A^{op} have the same Cauchy completion, and are therefore Morita equivalent.

The simple modules, too, can be thought of as 'atomic' in a different sense. A simple module need not be indecomposable projective, nor vice versa. However, the two conditions are closely related: as recounted in Section 2, there is a canonical bijection between the isomorphism classes of simple modules and the isomorphism classes of indecomposable projectives.

The condition that A has finite global dimension guarantees that the sum in (1.3) has only finitely many nonzero terms. The condition that A has finite dimension guarantees that the linear category $\mathbf{IP}(A)$ is equivalent to one with finitely many objects and finite-dimensional hom-spaces, as we shall see. This is a necessary condition in order for the magnitude of $\mathbf{IP}(A)$ to be defined. It is not a sufficient condition, but part of the statement of Theorem 1.1 is that $|\mathbf{IP}(A)|$ *is* defined.

Theorem 1.1 was first noted by Catharina Stroppel under the additional hypothesis that A is a Koszul algebra (personal communication, 2009). We observe here that the Koszul assumption is unnecessary.

2. Algebraic background

Here we assemble all the facts that we will need in order to state and prove the main theorem. General references for this section are [13, Chapter I] and [2, Chapter 1].

Throughout this note, K denotes a field and A a finite-dimensional K-algebra (unital, but not necessarily commutative). 'Module' will mean left A-module. Since A is finite-dimensional, a module is finitely generated over A if and only if it is finite-dimensional over K.

Simple and indecomposable projective modules. Details for this part can be found in [7], as well as in the general references above.

A nonzero module is **simple** if it has no nontrivial submodule, and **indecomposable** if it has no nontrivial direct summand. There is a canonical bijection between the isomorphism classes of simple modules S and the isomorphism classes of indecomposable projective modules P, with S corresponding to P if and only if S is a quotient of P. (It is not an equivalence of categories.)

Choose representatives $(S_i)_{i\in I}$ of the isomorphism classes of simple modules and $(P_i)_{i\in I}$ of the isomorphism classes of indecomposable projective modules, with S_i a quotient of P_i.

Modules of both types are finitely generated (indeed, cyclic), so each vector space $\mathrm{Hom}_A(P_i, P_j)$ is finite-dimensional. Moreover, one can use either the Jordan–Hölder theorem or the Krull–Schmidt theorem to show that I is finite. Denote by $\mathbf{IP}(A)$ the category of indecomposable projective A-modules and all homomorphisms between them, which is a K-linear category. Then $\mathbf{IP}(A)$ has finite-dimensional hom-spaces and only finitely many isomorphism classes of objects.

We have $\mathrm{Hom}_A(P_i, S_j) = 0$ when $i \neq j$, since any homomorphism into a simple module is zero or surjective. It can be shown that $\mathrm{Hom}_A(P_i, S_i) \cong \mathrm{End}_A(S_i)$ as vector spaces. This is a skew field, isomorphic to K if K is algebraically closed.

Homological algebra. For each $n \geq 0$, there is a functor

$$\mathrm{Ext}^n_A \colon A\text{-}\mathbf{Mod}^{\mathrm{op}} \times A\text{-}\mathbf{Mod} \to \mathbf{Vect}. \tag{2.1}$$

One can characterise $\mathrm{Ext}^n_A(X, -)$ as the nth right derived functor of $\mathrm{Hom}_A(X, -)$, and $\mathrm{Ext}^n_A(-, Y)$ as the nth right derived functor of $\mathrm{Hom}_A(-, Y)$, but we will only need the following consequences of these characterisations.

First, $\mathrm{Ext}^0_A = \mathrm{Hom}_A$. Second, if P is projective then $\mathrm{Ext}^n_A(P, -) = 0$ for all $n > 0$. Third, Ext^n_A preserves finite direct sums in each argument. Fourth, $\mathrm{Ext}^n_A(X, Y)$ is finite-dimensional if both X and Y are. Finally, given any A-module V and short exact sequence

$$0 \to W \to X \to Y \to 0, \tag{2.2}$$

there is an induced long exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^0_A(V, W) \to \mathrm{Ext}^0_A(V, X) \to \mathrm{Ext}^0_A(V, Y) \\ \to \mathrm{Ext}^1_A(V, W) \to \mathrm{Ext}^1_A(V, X) \to \cdots , \end{aligned} \tag{2.3}$$

and dually a long exact sequence $0 \to \mathrm{Ext}^0_A(Y, V) \to \cdots$.

Assume henceforth that A has **finite global dimension** [14, Chapter 4]. This means that there exists $N \in \mathbb{N}$ such that every A-module X has a projective resolution of the form

$$0 \to Q_N \to \cdots \to Q_1 \to X \to 0. \tag{2.4}$$

When X is finite-dimensional, the projective modules Q_i can be chosen to be finite-dimensional too.

A condition equivalent to finite global dimension is that $\mathrm{Ext}^n_A = 0$ for all $n \gg 0$. For finite-dimensional A-modules X and Y, we may therefore define

$$\chi_A(X, Y) = \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}^n_A(X, Y) \in \mathbb{Z} \tag{2.5}$$

(a finite sum). This χ_A is the **Euler form** of A. We have $\chi_A(\bigoplus_r X_r, -) = \sum_r \chi_A(X_r, -)$ for any finite family (X_r) of modules, and similarly in the second argument. Moreover, the observations above imply that

$$\chi_A(P_i, P_j) = \dim \operatorname{Hom}_A(P_i, P_j) \tag{2.6}$$

for all $i, j \in I$, and that

$$\chi_A(P_i, S_j) = \begin{cases} \dim \operatorname{End}_A(S_j) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.7}$$

When K is algebraically closed, $\chi_A(P_i, S_j)$ is therefore just the Kronecker delta δ_{ij}.

Grothendieck group. The **Grothendieck group** $\mathrm{K}(A)$ is the abelian group generated by the finite-dimensional A-modules, subject to the relation $X = W + Y$ for each short exact sequence (2.2) of finite-dimensional modules. Writing $[X]$ for the class of X in $\mathrm{K}(A)$, one easily deduces that, more generally, $\sum_r (-1)^r \big[X_r\big] = 0$ for any exact sequence

$$0 \to X_1 \to \cdots \to X_n \to 0. \tag{2.8}$$

For example, take a short exact sequence (2.2) and a finite-dimensional module V. The resulting long exact sequence (2.3) has only finitely many nonzero terms (since A has finite global dimension), so the alternating sum of the dimensions of these terms is 0, giving $\chi_A(V, X) = \chi_A(V, W) + \chi_A(V, Y)$. The same holds with the arguments reversed. Thus, χ_A defines a $\mathbb{Z}$-bilinear map $\mathrm{K}(A) \times \mathrm{K}(A) \to \mathbb{Z}$.

We now show that $\mathrm{K}(A)$ is free as a $\mathbb{Z}$-module, and in fact has two canonical bases.

First, the family $\big(\big[S_i\big]\big)_{i \in I}$ generates the group $\mathrm{K}(A)$. Indeed, for any finite-dimensional A-module X, we may take a composition series

$$0 = X_n < \cdots < X_1 < X_0 = X, \tag{2.9}$$

and then $[X] = \sum_{r=1}^n \big[X_{r-1}/X_r\big]$.

Second, the family $\big(\big[P_i\big]\big)_{i \in I}$ generates $\mathrm{K}(A)$. Given a finite-dimensional A-module X, we may take a resolution (2.4) by finite-dimensional projective modules, and then $[X] = \sum_{r=1}^N (-1)^{r+1} \big[Q_r\big]$. On the other hand, each Q_r is a finite direct sum of indecomposable submodules, which are projective since Q_r is.

Finally, both $\big(\big[S_i\big]\big)$ and $\big(\big[P_i\big]\big)$ *freely* generate the abelian group $\mathrm{K}(A)$. This follows from (2.7) and the $\mathbb{Z}$-bilinearity of χ_A.

3. The result

Recall our standing conventions: A is an algebra of finite dimension and finite global dimension, over a field K which we now assume to be algebraically closed. We continue to write $(P_i)_{i \in I}$ for representatives of the isomorphism classes of indecomposable projective A-modules, and similarly $(S_i)_{i \in I}$ for the simple modules, with S_i a quotient of P_i.

The linear category $\mathbf{IP}(A)$ of indecomposable projective A-modules is equivalent to its full subcategory with objects P_i $(i \in I)$. Write $Z_A = (Z_{ij})_{i,j \in I}$ for the matrix of this finite linear category, so that $Z_{ij} = \dim \operatorname{Hom}_A(P_i, P_j)$.

We will derive our main result, Theorem 1.1, from the following basic theorem. (See e.g. [1, Proposition III.3.13(a)] for an essentially equivalent formulation.) It implies, in particular, that the matrix Z_A is invertible over the integers.

Theorem 3.1. *The inverse of the matrix Z_A is the 'Euler matrix' $E_A = (E_{ij})_{i,j\in I}$, given by $E_{ij} = \chi_A(S_j, S_i)$.*

Proof. Since $\big([P_i]\big)_{i\in I}$ and $\big([S_i]\big)_{i\in I}$ are both bases for the $\mathbb{Z}$-module $\mathrm{K}(A)$, there is an invertible matrix $C_A = (C_{ij})_{i,j\in I}$ over $\mathbb{Z}$ such that, writing $C_A^{-1} = (\overline{C}_{ij})$,

$$[P_j] = \sum_{k\in I} C_{kj}[S_k], \tag{3.1}$$

$$[S_j] = \sum_{k\in I} \overline{C}_{kj}[P_k] \tag{3.2}$$

for all $j \in I$. Since K is algebraically closed, equation (2.7) states that $\chi_A(P_i, S_j) = \delta_{ij}$. Applying $\chi_A(P_i, -)$ to each side of (3.1) therefore gives $\chi_A(P_i, P_j) = C_{ij}$, which by (2.6) is equivalent to $Z_{ij} = C_{ij}$. On the other hand, applying $\chi_A(-, S_i)$ to each side of (3.2) gives $E_{ij} = \overline{C}_{ij}$. Hence $Z_A = C_A$ and $E_A = C_A^{-1}$. □

The matrix $C_A = Z_A$ is known as the **Cartan matrix** of A ([3], [4, §5], [5]). Explicitly, C_{ij} is the multiplicity of S_i as a composition factor of P_j.

We now deduce Theorem 1.1. By definition, $|\mathbf{IP}(A)|$ is the sum of the entries of Z_A^{-1}. Hence by Theorem 3.1 and the $\mathbb{Z}$-bilinearity of χ_A,

$$|\mathbf{IP}(A)| = \sum_{i,j\in I} \chi_A(S_j, S_i) = \chi_A\left(\bigoplus_{j\in I} S_j, \bigoplus_{i\in I} S_i\right) = \chi_A(S, S), \tag{3.3}$$

completing the proof.

Example 3.2. Let Q be a finite acyclic quiver (directed graph). Then Q consists of a finite set I of vertices together with, for each $i, j \in I$, a finite set $Q(i, j)$ of arrows from i to j. The **path algebra** A of Q is defined as follows. As a vector space, it is generated by the paths in Q, including the zero-length path e_i on each vertex i. Multiplication is concatenation of paths where that is defined, and zero otherwise. We write multiplication in the same order as composition, so that if α is a path from i to j and β is a path from j to k then $\beta\alpha$ is a path from i to k. The identity is $\sum_{i\in I} e_i$. That Q is finite and acyclic guarantees that A is of finite dimension and finite global dimension.

Path algebras of quivers are very well-understood (e.g. [13, Chapter I]). The simple and indecomposable projective A-modules are indexed by the vertex-set I. The indecomposable projective module P_i corresponding to vertex i is the submodule of the A-module A spanned by the paths beginning at i. It has a unique maximal submodule N_i, spanned by the paths of nonzero length beginning at i, and the corresponding simple module $S_i = P_i/N_i$ is one-dimensional.

Using the facts listed in Section 2, we can compute the Euler form of A. For each $i, j \in I$, the short exact sequence

$$0 \to N_i \to P_i \to S_i \to 0 \tag{3.4}$$

gives rise to a long exact sequence

$$0 \to \mathrm{Ext}^0_A(S_i, S_j) \to \mathrm{Ext}^0_A(P_i, S_j) \to \mathrm{Ext}^0_A(N_i, S_j) \to \cdots . \tag{3.5}$$

Observing that $N_i = \bigoplus_{k\in I} P_k^{Q(i,k)}$, we deduce from (3.5) that

$$\mathrm{Ext}^n_A(S_i, S_j) = \begin{cases} K^{\delta_{ij}} & \text{if } n = 0,\\ K^{Q(i,j)} & \text{if } n = 1,\\ 0 & \text{if } n \geq 2. \end{cases} \tag{3.6}$$

Hence, writing $E = \coprod_{i,j\in Q} Q(i,j)$ for the set of arrows of Q,

$$\mathrm{Ext}^n_A(S, S) = \begin{cases} K^{|I|} & \text{if } n = 0,\\ K^{|E|} & \text{if } n = 1,\\ 0 & \text{if } n \geq 2. \end{cases} \tag{3.7}$$

It follows that $\chi_A(S, S) = |I| - |E|$, which is the Euler characteristic (in the elementary sense) of the quiver Q.

On the other hand, each path from vertex j to vertex i induces a homomorphism $P_i \to P_j$ by composition, and in fact every homomorphism $P_i \to P_j$ is a unique linear combination of homomorphisms of this form. Hence Z_{ij} is the number of paths from j to i in Q.

So in the case at hand, Theorem 1.1 states that if we take an acyclic quiver Q, form the matrix whose (i, j)-entry is the number of paths from j to i, invert this matrix, and sum its entries, the result is equal to the Euler characteristic of Q. This was also shown directly as Proposition 2.10 of [8].

4. Some remarks

Arbitrary base fields. The assumption that the base field is algebraically closed is needed for the simple form of the duality formula, $\chi_A(P_i, S_j) = \delta_{ij}$. Otherwise, equation (2.7) only gives

$$\chi_A(P_i, S_j) = \begin{cases} d_j & \text{if } i = j,\\ 0 & \text{if } i \neq j, \end{cases} \tag{4.1}$$

where $d_j = \dim \mathrm{End}_A(S_j)$. Then, applying $\chi_A(P_i, -)$ to (3.1) yields $Z_{ij} = d_i C_{ij}$, while applying $\chi_A(-, S_i)$ to (3.2) yields $E_{ij} = d_i \overline{C}_{ij}$. Therefore, writing $Z_A^{-1} = (\overline{Z}_{ij})$, we get

$$\overline{Z}_{ij} = d_i^{-1} E_{ij} d_j^{-1}, \tag{4.2}$$

which generalises Theorem 3.1. We can then sum (4.2) to generalise Theorem 1.1 as follows:

$$|\mathbf{IP}(A)| = \chi_A(\widetilde{S}, \widetilde{S}), \tag{4.3}$$

where $\widetilde{S} = \bigoplus_{i\in I} d_i^{-1} S_i$, which may be regarded as a formal module or we may note that (4.3) only depends on the class $[\widetilde{S}] = \sum_{i\in I} d_i^{-1}[S_i] \in \mathrm{K}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The determinant of the Cartan matrix. The fact that, when A has finite global dimension, the Cartan matrix C_A is invertible over $\mathbb{Z}$ or, equivalently, is unimodular, i.e. $\det C_A = \pm 1$, is an old observation of Eilenberg [4, §5]. On the other hand, the 'Cartan determinant conjecture' that, in fact, $\det C_A = 1$ is still unsolved in general, although it is confirmed in many cases; see [5] for a survey.

An easy example is when A is (Morita equivalent to) a quotient of the path algebra of an acyclic quiver, in which case A is necessarily finite dimensional and of finite global dimension. In this case $C_A = Z_A$ can be made upper triangular with 1s on the diagonal, so it certainly has $\det C_A = 1$. As another example, Zacharia [15] showed that the conjecture holds whenever A has global dimension 2.

It is not hard to give an example of an algebra A for which $C_A = Z_A$ is not even invertible over $\mathbb{Q}$: e.g. the quiver algebra given by a single n-cycle, with all paths of length n set to 0, has $C_{ij} = Z_{ij} = 1$ for all $i, j \in I$. Inevitably, this algebra does not have finite global dimension.

In this example, it is in fact still possible [9, §1] to define the magnitude of $\mathbf{IP}(A)$, and indeed $|\mathbf{IP}(A)| = 1$. However, it is less clear how one might find a homological interpretation of this.

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