Analytical Pricing of Discretely Monitored Asian-Style Options: Theory and Application to Commodity Markets

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Abstract

We compute an analytical expression for the moment generating function of the joint random vector consisting of a spot price and its discretely monitored average for a large class of square-root price dynamics. This result, combined with the Fourier transform pricing method proposed by Carr and Madan (1999) [Carr, P., Madan D., 1999. Option valuation using the fast Fourier transform. Journal of Computational Finance 2(4), Summer, 61-73] allows us to derive a closed-form formula for the fair value of discretely-monitored Asian-style options. Our analysis encompasses the case of commodity price dynamics displaying mean reversion and jointly fitting a quoted futures curve and the seasonal structure of spot price volatility. Four tests are conducted to assess the relative performance of the pricing procedure stemming from our formulae. Empirical results based on natural gas data from NYMEX and corn data from CBOT show a remarkable improvement over the main alternative techniques developed for pricing Asian-style options within the market standard framework of geometric Brownian motion.

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1 Introduction

Asian-style options, and other options written on alternative definitions of average prices, are effective hedging devices in commodity markets. As reported by Eydeland and Wolyniec (2003), these derivatives play an important role in price risk management performed by local delivery companies in the gas market. Moreover, oil markets often use these securities to stabilize cash flows that stem from meeting obligations to clients.

The market model for pricing Asian-style options is the geometric Brownian motion. This process is fraught with two major shortcomings. First, the assumption of normal price returns does not reflect the empirical features displayed by the vast majority of time series of commodity quotes. As illustrated by Richter and Sorensen (2000), Eydeland and Wolyniec (2003), and Regnier (2007), among others, these latter exhibit variable degrees of time varying and price dependent volatility functions. Second, there is no closed form expression for the arbitrage-free price of options on arithmetic averages of lognormally distributed prices. Time consuming price approximations must be implemented, sacrificing both precision and time in the resulting procedure (Wilmott, Dewynne, and Howison (1993)). In this respect, Fusai and Roncoroni (2008) provide a detailed comparison of alternative pricing methods for Asian-style options written on price dynamics following a geometric Brownian motion.

We start by assuming that asset price dynamics are driven by a square-root process in the spirit of Cox, Ingersoll, and Ross (1985). This process subsumes important elements characteristic of commodity price series while preserving analytical tractability. Moreover, it can easily be extended such that the resulting evolution fits the market forward curve, the time pattern of spot price volatility and mean reversion.

On the theoretical side, we derive an analytical formula for pricing discretely monitored Asian-style options in the above mentioned setting. To achieve this goal, we follow a two step procedure: first, we compute the moment generating function of the joint pair consisting of the commodity spot price $S_{n\Delta}$ at a future maturity $n\Delta$ and the weighed cumulated price $\sum_{j=0}^{n} \alpha_j S_{j\Delta}$ over the discretely monitored time horizon $\{0, \Delta, ..., n\Delta\}$; then, we apply a computational pricing approach based on the Fourier transform, as proposed by Carr and Madan (1999). The ability to compute an analytical expression for the underlying pair $\left(S_{n\Delta}, \sum_{j=0}^{n} \alpha_j S_{j\Delta}\right)$ constitutes the main theoretical result obtained in this paper in relation to the existing literature in the field. Incidentally, we note that discrete monitoring definitively represents a more realistic assumption than continuous monitoring. Moreover, it allows us to compute the transform of the joint distribution of the absolute and cumulated spot prices using a simple recursive procedure. A striking result is that the mentioned transform can be obtained using only the transform of the underlying commodity price.

We also present three important extensions of our main formula: first, we let the underlying spot price process exhibit a time dependent drift, a property allowing the resulting dynamics to recover the quoted set of forward prices;

second, we adopt a time varying volatility coefficient, a feature allowing our model to fit either the term structure of implied volatilities or a time dependent, *e.g.*, seasonal, spot price historical volatility; third, we consider spot price dynamics exhibiting mean reversion in their trend, a quality shown by some important classes of commodity prices, among which we cite agriculturals and energy-related products such as electricity and gas. These extensions represent a further theoretical innovation on the pricing of Asian-style options compared to published literature (Dassios and Nagaradjasarma (2006)).

On the empirical side, we perform four experiments aimed at assessing the absolute and relative quality of our pricing device. First, we measure the extent to which prices computed using discrete monitoring deviate from figures resulting from those obtained using formulae for the continuous monitoring case. We see that convergence is approximately linear in the number of monitoring dates. The resulting rate of convergence underpins the use of a fast, accurate method of pricing discretely monitored Asian-style options such as the one we propose herein.

Next, we compare prices obtained using the standard Black-Scholes model to the ones stemming from implementing our formulae. The former are obtained using two methods proposed in the literature on the subject. The latter are computed using a volatility assessment for our square-root model that is consistent with the volatility parameter in the geometric Brownian motion used to feed in the alternative methods mentioned above. This procedure makes our price directly comparable to the others. Our pricing device proves to be rather quick to obtain, whereas alternative pricing algorithms are always much slower to perform . Moreover, it mostly offers results which lie within the alternative methods used to approximate the option price under the market model.

Then, we measure the impact of including market information about the forward prices into the spot price dynamics for the purpose of pricing Asian-style options. We perform this analysis using quotes taken from the Natural Gas Market at NYMEX. It turns out that a non flat forward curve produces highly significant option price deviations from figures obtained in the case where such information is not accounted for by the underlying spot price model.

Finally, we assess the impact of including information about the time structure of historical volatility into our pricing device. We perform a test on corn price data quoted at CBOT. It turns out that using this information may result in significant price discrepancies compared to the quotes obtained using the market model represented by the geometric Brownian motion. Our results suggest that when pricing Asian-style options in market contexts where a seasonal component strongly affects the evolution of spot price volatility, one should include this information as precisely as possible. This remark is particularly important for several commodity markets, such as energy and agriculturals, where the time variation of volatility is significantly pronounced.

The paper is organized as follows. Section 2 derives a closed-form expression for the moment generating functions of the underlying commodity price $S_{n\Delta}$ and the pair $\left(S_{n\Delta}, \sum_{j=0}^{n} \alpha_j S_{j\Delta}\right)$. Section 3 extends these results to the case of spot price dynamics that fit a quoted forward price curve, a time varying volatility structure, and a mean reverting behavior. Section 4 relates these expressions to the Laplace transform of the fixed strike Asian-style option price and the Fourier transform of the floating strike Asian-style option price. Section 5 performs numerical experiments on gas data taken from NYMEX. Section 6 concludes with a few comments and suggestions for future development.

2 Recursive Valuation of the Underlying Pair Transform

We consider spot price dynamics driven by a simple square-root process under the (possibly selected) risk-neutral probability measure:[1]

$$\begin{aligned} dS_u &= (r-c)\,S_u du + \sigma\sqrt{S_u}dW_u, \\ S_0 &= x. \end{aligned} \tag{1}$$

Here r denotes the instantaneous short rate of interest, c is the instantaneous net spot convenience yield,[2] σ is the percentage instantaneous volatility and W represents a standard one-dimensional Brownian motion. In this section, we assume coefficients r, c and σ are all constant. Later, we derive an extension to time varying drift and volatility, allowing the model to fit the forward price curve in the market at a given date and either the term structure of implied volatilities for a given strike price or the seasonal pattern followed by the historical market price volatility.

We remark that model (1) is affine in the state variable, a property allowing us to use results taken from the vast literature on this class of models (Duffie, Pan and Singleton (2001)).

We consider a time horizon $[0, T]$ split into a number $n+1$ of Δ-spaced monitoring dates $0, \Delta, ..., n\Delta = T$. Our goal is to compute analytic formulae for fixed maturity options whose payoff structure depends on the commodity spot price at maturity $S_{n\Delta}$ or on a linear average $\mathrm{Avg}_n = \sum_{j=0}^{n} \alpha_j S_{j\Delta}$ $(\sum \alpha_j = 1)$ of the spot prices $S_0, S_\Delta, ..., S_{n\Delta}$ monitored over the contract lifetime, or on any combination of them. In the energy markets, for instance, weights α_j represent relative volumes delivered to the customer, *i.e.*, $\alpha_j = V_j / \sum_{i=1}^{n} V_i$. In particular, we consider Asian-style options with either a fixed or a floating strike price under a discrete monitoring rule. The pay-off functions for these derivatives are described in Table 1. To this end, we compute the moment generating function

[1] If the underlying market is incomplete, as is the case for nonstorable or partly storable commodities such as electricity or perishable goods, a variety of equivalent probability measures are compatible with the assumption of absence of arbitrage opportunities. In this case, the user has to adopt an appropriate method for selecting one of these pricing measures for pricing purposes. In our model setting, we assume that a measure is selected within this class at the outset.

[2] The net convenience yield is defined as the net convenience represented under continuous compounding. The net convenience is given by the net benefit (reward minus costs) stemming from physically holding one unit of the commodity, but not from holding a long position in a forward or futures contract written on the same commodity.

Option	Payoff
Fixed strike	$\max\left\{\mathrm{Avg}_n - K, 0\right\}$
Floating strike	$\max\left\{S_{n\Delta} - \mathrm{Avg}_n - K, 0\right\}$
	Underlying variable
Standard	$\mathrm{Avg}_n = \frac{\sum_{j=0}^n S_{j\Delta}}{n+1}$
Volume weighed	$\mathrm{Avg}_n = \sum_{j=0}^n \frac{V_j}{\sum_i V_i} S_{j\Delta}$

Table 1: Payoff functions of Asian-style options under continuous and discrete monitoring.

Option	γ	μ	α_j	m.g.f. $v_{0,x}(n,\Delta;\gamma,\mu)$
Standard European	any	0	-	$\mathbb{E}_0\left[e^{-\gamma S_{n\Delta}}\right]$
Fixed strike std.Asian	0	any	$\frac{1}{n+1}$	$\mathbb{E}_0\left[e^{-\frac{\mu}{n+1}\sum_{j=0}^n S_{j\Delta}}\right]$
Fixed strike vol.weighed	0	any	$\frac{V_j}{\sum_i V_i}$	$\mathbb{E}_0\left[e^{-\frac{\mu}{\sum_i V_i}\sum_{j=0}^n V_j S_{j\Delta}}\right]$
Floating strike std.Asian	any	$-\gamma$	$\frac{1}{n+1}$	$\mathbb{E}_0\left[e^{-\gamma\left(S_{n\Delta}-\frac{1}{n+1}\sum_{j=0}^n S_{j\Delta}\right)}\right]$

Table 2: Moment generating functions for a sample set of popular Asian-style energy derivatives.

(m.g.f.) corresponding to the joint probability density of the pair consisting of the spot price $S_{n\Delta}$ and the cumulated spot price $\sum_{j=0}^n \alpha_j S_{j\Delta}$ under the selected monitoring rule. This function is defined as:

$$(\gamma,\mu) \to v_{0,x}(n,\Delta;\gamma,\mu) \triangleq \mathbb{E}_0\left(\exp\left\{-\left[\gamma S_{n\Delta} + \mu\sum_{j=0}^n \alpha_j S_{j\Delta}\right]\right\}\right).$$

Table 2 illustrates instances of this function which correspond with traded options in the energy markets. Our main theoretical result provides a closed-form formula for the m.g.f. $v_{0,x}(n,\Delta;\gamma,\mu)$. We start by recalling the analytical expression for the m.g.f. of the underlying commodity price $S_{t+\Delta}$ given the market information available at time t as represented by the σ-algebra $\mathcal{F}_t^S$ generated by the price process up to time t.

Lemma 1 *Under spot price dynamics* (1)*, the moment generating function of*

$S_{t+\Delta}$ given the market information available at time t is given by:

$$v_{t,y}\left(1,\Delta;\gamma,0\right) \triangleq \mathbb{E}_t\left(e^{-\gamma S_{t+\Delta}}\right) = e^{-A(\Delta;\gamma)y}, \tag{2}$$

where

$$A\left(\Delta;\gamma\right) = \frac{\gamma e^{(r-c)\Delta}}{1+\frac{1}{2}\sigma^2\gamma\frac{e^{(r-c)\Delta}-1}{r-c}}. \tag{3}$$

proof. See Ingersoll (1987, pp.397-8).

The explicit dependence on t has been indicated for the sole purpose of making the formula compatible with its extension to the case of time varying coefficient as shown in the following section.

Proposition 2 *Under dynamics* (1)*, the moment generating function of the pair* $\left(S_{n\Delta}, \sum_{j=0}^{n} \alpha_j S_{j\Delta}\right)$ *given the information available at time* 0 *is given by:*

$$v_{0,x}\left(n,\Delta;\gamma,\mu\right) = e^{-\Lambda_0(\Delta;\gamma,\mu)x}, \tag{4}$$

where the function $\Lambda_j\left(\Delta;\gamma,\mu\right)$ satisfies the recursive equation:

$$\Lambda_j\left(\Delta;\gamma,\mu\right) = A\left(\Delta;\Lambda_{j+1}\left(\Delta;\gamma,\mu\right)\right) + \mu\alpha_j, \tag{5}$$

for $j = n-1, n-2, ..., 1, 0$, with starting value:

$$\Lambda_n\left(\Delta;\gamma,\mu\right) = \gamma + \mu\alpha_n. \tag{6}$$

Here A is defined as in formula (3)*.*

proof. The m.g.f. v can be equivalently expressed as:

$$v_{0,x}\left(n,\Delta;\gamma,\mu\right) = \mathbb{E}_0\left[e^{-(\gamma+\mu\alpha_n)S_{n\Delta}-\mu\sum_{i=0}^{n-1}\alpha_i S_{i\Delta}}\right].$$

By repeatedly using the *tower law* of probabilities and the analytical expression (2) for the m.g.f. $v_{0,x}\left(n,\Delta;\gamma,0\right)$ given in Lemma 1, we may write:

$$\begin{aligned}
v_{0,x}\left(n,\Delta;\gamma,\mu\right) &= \mathbb{E}_0\left[\mathbb{E}_{n-1}\left[e^{-(\gamma+\mu\alpha_n)S_{n\Delta}}\right]e^{-\mu\sum_{i=0}^{n-1}\alpha_i S_{i\Delta}}\right] \\
&= \mathbb{E}_0\left[e^{-A(\Delta;\gamma+\mu\alpha_n)S_{(n-1)\Delta}}e^{-\mu\sum_{i=0}^{n-1}\alpha_i S_{i\Delta}}\right] \\
&= \mathbb{E}_0\left[\mathbb{E}_{n-2}\left[e^{-(A(\Delta;\gamma+\mu\alpha_n)+\mu\alpha_{n-1})S_{(n-1)\Delta}}\right]e^{-\mu\sum_{i=0}^{n-2}\alpha_i S_{i\Delta}}\right] \\
&= \mathbb{E}_0\left[e^{-A(\Delta;A(\Delta;\gamma+\mu\alpha_n)+\mu\alpha_{n-1})S_{(n-2)\Delta}}e^{-\mu\sum_{i=0}^{n-2}\alpha_i S_{i\Delta}}\right] \\
&\quad \text{...(by recursion)} \\
&= e^{-\Lambda_0(\Delta;\gamma,\mu)x},
\end{aligned}$$

where $\Lambda_0\left(\Delta;\gamma,\mu\right)$ is obtained by solving for the recursive equation:

$$\Lambda_j\left(\Delta;\gamma,\mu\right) = A\left(\Delta;\Lambda_{j+1}\left(\Delta;\gamma,\mu\right)\right) + \mu\alpha_j,$$

for $j = n-1, n-2, ..., 1, 0$, starting with $\Lambda_n(\Delta; \gamma, \mu) = \gamma + \mu\alpha_n$. *Q.E.D.*

This result is important for two reasons. First, it shows that given realistic assumptions about the monitoring policy, a closed-form expression for the joint transform can be obtained with little computational effort using the method detailed in the proof. Second, no information beyond the transform of the state variable is required. In particular, there is no need to tackle the cumbersome issue of computing the joint transform of the pair $\left(S_T, \int_0^T \alpha_u S_u du\right)$ as in the continuous monitoring rule.

3 Fitting the Quoted Forward Curve and Volatility Structure

Commodity-linked derivatives should be priced consistently with all market price information available at the valuation time. In particular, traders need models which produce prices taking into account three sets of information:

1) The quoted forward/futures prices of the commodity, provided they are available;[3]

2) Possible time patterns displayed by the historical price volatility;

3) Mean reversion characterizing spot price dynamics.

These features usually reflect properties related to the physical use of the commodity for industrial or consumption processes.

The specialized literature has examined these issues in great detail. Routledge, Seppi, and Spatt (2000) underline the impact of periodical components on the price dynamics of most commodities. Eydeland and Wolyniec (2003) show that the predictable component of electricity price dynamics is bound by weather and consumption related features. Todorova (2004) notes that oil and gas markets show seasonal components affecting expected future spot prices, while Richter and Sorensen (2000) and Lien and Koekebakker (2004) find strong evidence of seasonality effects upon agricultural commodity prices. For most commodities, mean reversion is a stylized fact empirically accepted by several studies. In energy markets, the relevance of this property may vary across products and over time within the same commodity. For instance, Bessembinder et al. (1995) find clear evidence of mean reversion across eleven commodity markets, pointing out strong patterns for agriculturals and crude oil (see also, Pindyck (2001)), and weak patterns for metals. Schwartz (1997) and Casassus and Collin-Dufresne (2005), among others, confirm the existence of a mean reversion property in crude oil, copper, gold and silver. The case of electricity markets is rather peculiar: Roncoroni (2002) and Geman and Roncoroni (2006) discover the existence of two competing mean reversion effects in most US power markets: one is the traditional smooth reversion to average prices;

[3]For the purpose of our analysis, we assume interest rates are deterministic. This amounts to treating forward and futures prices as equivalent.

the other stems from the spiky behavior of electricity spot prices during periods of capacity congestion.

Our data clearly confirm those findings. Figure 1 displays futures curves for light, sweet crude oil, natural gas, and heating oil as quoted at NYMEX on March 1, 2007, and corn as reported by CBOT on December 1, 2006. The time dependent component is plainly visible in the reported graphs. In particular, corn exhibits a clear seasonal pattern, which should be considered while pricing options on averages. This phenomenon has been extensively studied in Benth, Koekebakker, and Ollmar (2007). Figure 2 shows a periodical component affecting the spot price of corn and soybean.

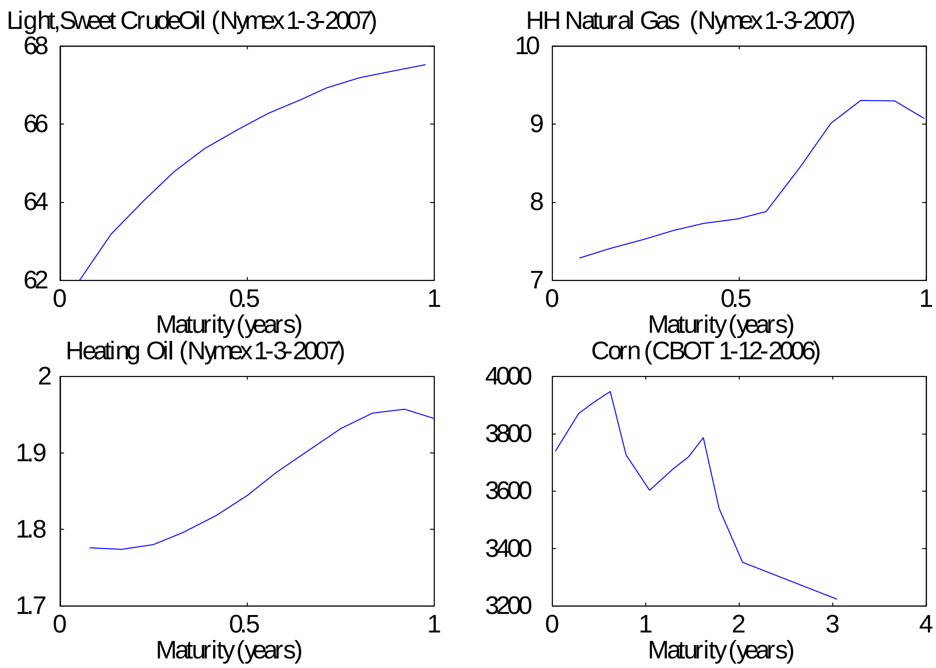

Figure 1: Futures curves for a sample of energy and agricultural commodities.

We now present a simple, yet effective method to make the spot dynamics include all price information implied by the quoted forward/futures curve, if any. This task can be achieved by letting the risk-neutral drift of spot price dynamics be time dependent. Moreover, the spot price volatility is allowed to reproduce any time pattern assigned by the user. We remark that the importance of assuming a time varying drift goes beyond the ability to fit a quoted forward/future curve. For instance, Cartea and Williams (2007) point out that gas price dynamics exhibit a time varying historical trend and market price of risk. Therefore, estimating these quantities may represent a viable alternative

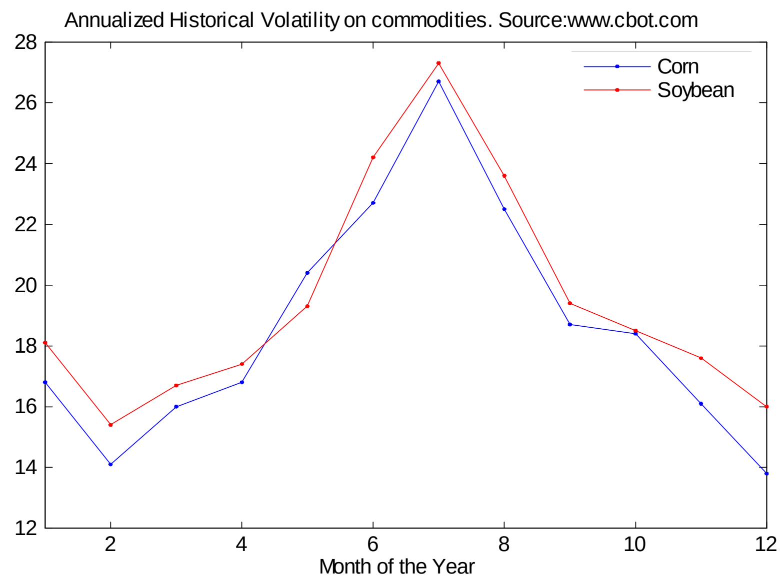

Figure 2: Seasonal patterns of historical price volatilities for two agricultural commodities.

to directly fitting the risk-neutral price drift to forward quotes. This option can be useful whenever forward/futures quotes are not available or their reliability is limited by, say, liquidity constraints.

Our starting point is the assumption that the excess θ of the risk-free interest rate r over the net spot convenience yield c is a deterministic function of time, namely:

$$\theta_t = r_t - c_t.$$

Correspondingly, spot price dynamics (1) read as:

$$\begin{aligned} dS_u &= \theta_u S_u du + \sigma_u \sqrt{S_u} dW_u, \\ S_0 &= x. \end{aligned} \tag{7}$$

We assume that forward prices $F_{0,t}$ are observed at time 0 for maturities t up to time $T = n\Delta$. Since $\mathbb{E}_0 (S_T) = x \exp \int_0^T \theta_s ds$, the matching condition reads as:

$$xe^{\int_0^T \theta_s ds} = F_{0,T}. \tag{8}$$

This condition can be equivalently written as:

$$\theta_T = \partial_T \ln \frac{F_{0,T}}{x} = \partial_T \ln F_{0,T}. \tag{9}$$

The following result extends Lemma 1 to the case of time dependent spot price drift and volatility.

Lemma 3 *Under commodity spot price dynamics* (7)*, the moment generating function of* $S_{t+\Delta}$ *given the information available at time* t *is:*

$$\nu_{t,y}^{\theta}(1,\Delta;\gamma,0) \triangleq \mathbb{E}_t\left(e^{-\gamma S_{t+\Delta}}\right) = e^{-A_t^{\theta}(\Delta;\gamma)y} \tag{10}$$

where:

$$A_t^{\theta}(\Delta;\gamma) \triangleq \frac{\gamma e^{\int_t^{t+\Delta}\theta_s ds}}{1+\frac{\gamma}{2}\int_t^{t+\Delta} ds\sigma_s^2 e^{\int_s^{t+\Delta}\theta_u du}} \tag{11}$$

$$= \frac{\gamma\frac{F_{0,t+\Delta}}{F_{0,t}}}{1+\frac{\gamma}{2}F_{0,t+\Delta}\int_t^{t+\Delta}\frac{\sigma_s^2}{F_{0,s}}ds}, \tag{12}$$

and $y = S(t)$*.*

proof. See Fusai, Marena, and Roncoroni (2007).

We remark that expression (12) may be essential to the numerical effectiveness of our pricing mechanism. At a first sight, formula (9) suggests that a model fitting a forward curve ought to be derived following a two step procedure: first, a continuous forward curve $F = (F_{0,t})_{0\leq t\leq T}$ is computed by interpolating a set of market quotes; then, a corresponding drift θ_T is calculated by numerically evaluating the first order derivative of F. The quality of the resulting assessment can be undermined by the lack of stability that characterizes any differentiation procedure. This property may lead to unreasonable patterns for the time varying drift, a phenomenon known as the "Sydney opera house effect" (Rebonato (1996)). Formula (12) states that we do not need to take this path so long as only integrals of θ_T are required to compute the moment generating function in question. In particular, no differentiation is required for pricing purposes.

The result stated in the lemma allows us to extend the statement in Proposition 2 to the case of time dependent drift and volatility.

Proposition 4 *Under spot price dynamics* (7)*, the moment generating function of the pair* $\left(S_{n\Delta}, \sum_{j=0}^{n}\alpha_j S_{j\Delta}\right)$ *given the information available at time* 0 *is:*

$$v_{0,x}^{\theta}(n,\Delta;\gamma,\mu) = e^{-\Lambda_0^{\theta}(\Delta;\gamma,\mu)x}, \tag{13}$$

where the function $\Lambda_j^{\theta}(\Delta;\gamma,\mu)$ *satisfies the recursive equation:*

$$\Lambda_j^{\theta}(\Delta;\gamma,\mu) = A_{j\Delta}^{\theta}\left(\Delta;\Lambda_{j+1}^{\theta}(\Delta;\gamma,\mu)\right) + \mu\alpha_j,$$

for $j = n-1, n-2, ..., 1, 0$*, with starting value:*

$$\Lambda_n^{\theta}(\Delta,\gamma,\mu) = \gamma + \mu\alpha_n.$$

Here A^{θ} *is defined as in formula* (11)*.*

proof. See Fusai, Marena, and Roncoroni (2007).

We finally extend the results above to the case of commodity price dynamics exhibiting mean reversion to a time varying trend:

$$dS_u = \beta\left(\eta_u - S_u\right)du + \sigma_u\sqrt{S_u}dW_u. \tag{14}$$

Again, the spot price volatility is allowed to follow any time pattern of interest and the drift term η is selected such that the model fits the forward/futures price curve quoted in the market, i.e.,

$$E_0\left(S_T\right) = F_{0,T}. \tag{15}$$

Since

$$E_0\left(S_T\right) = e^{-\beta T}x + \beta\int_0^T e^{-\beta(T-s)}\eta_s ds,$$

a simple differentiation with respect to the variable T leads to an expression for the fitting drift term:

$$\eta_T = F_{0,T} + \frac{1}{\beta}\partial_T F_{0,T}. \tag{16}$$

The following result extends Lemma 3 to the case of a mean reverting spot price process with time varying coefficients.

Lemma 5 *Under commodity spot price dynamics* (14)*, the moment generating function of $S_{t+\Delta}$ given the information available at time t is:*

$$\nu_{t,y}^{\beta}\left(1,\Delta;\gamma,0\right) \triangleq \mathbb{E}_t\left(e^{-\gamma S_{t+\Delta}}\right) = e^{-A_t^{\beta}(\Delta;\gamma)y - B_t^{\beta}(\Delta;\gamma)},$$

where

$$A_t^{\beta}\left(\Delta;\gamma\right) = \frac{\gamma e^{-\beta\Delta}}{1 + \frac{\gamma}{2}\int_t^{t+\Delta}\sigma_s^2 e^{-\beta(t+\Delta-s)}ds}, \tag{17}$$

$$B_t^{\beta}\left(\Delta;\gamma\right) = \gamma F_{0,T} - F_{0,t}A_t^{\beta}\left(\Delta;\gamma\right) - \frac{1}{2}\int_t^{t+\Delta} F_{0,s}\sigma_s^2 A_s^{\beta}\left(\Delta;\gamma\right)^2 ds, \tag{18}$$

and $y = S\left(t\right)$.

proof. See Fusai, Marena, and Roncoroni (2007).

The result stated in proposition 4 can be extended as follows:

Proposition 6 *Under spot price dynamics* (14)*, the moment generating function of the pair $\left(S_{n\Delta}, \sum_{j=0}^{n}\alpha_j S_{j\Delta}\right)$ given the information available at time* 0 *is:*

$$v_{0,x}^{\beta}\left(n,\Delta;\gamma,\mu\right) = e^{-\Lambda_0^{\beta}(\Delta;\gamma,\mu)x - \sum_{j=0}^{n-1} B_{j\Delta}^{\beta}\left(\Delta;\Lambda_{j+1}^{\beta}(\Delta;\gamma,\mu)\right)}, \tag{19}$$

where the function $\Lambda_j^{\beta}\left(\Delta;\gamma,\mu\right)$ satisfies the recursive equation:

$$\Lambda_j^{\beta}\left(\Delta;\gamma,\mu\right) = A_{j\Delta}^{\beta}\left(\Delta;\Lambda_{j+1}^{\beta}\left(\Delta;\gamma,\mu\right)\right) + \mu\alpha_j,$$

for $j = n-1, n-2, ..., 0$, with starting value:

$$\Lambda_n^{\beta}(\Delta, \gamma, \mu) = \gamma + \mu \alpha_n.$$

Here $A_{j\Delta}^{\beta}$ is defined as in formula (17) *and B_t^{β} is given by expression* (18).

proof. See Fusai, Marena, and Roncoroni (2007).

4 Asian-Style Option Price Transforms

We now relate the results obtained in the previous section to the option pricing problem stated in the introduction. We need to consider two cases separately: 1) plain vanilla and fixed strike Asian-style options; 2) floating strike Asian-style option.

In the former, the underlying variables are the commodity spot price $S_{n\Delta}$ and the average price $\sum \alpha_j S_{j\Delta}$. These quantities are positive valued random variables, a property allowing us to use the Laplace transform of the option price.

In the case of floating strike, the underlying variable is represented by the difference $S_{n\Delta} - \sum \alpha_j S_{j\Delta}$, a quantity that may assume positive as well as negative values. This fact leads us to adopt a computational approach based on the Fourier transform in the spirit of Carr and Madan (1999).

We restrict our analysis to the case of standard Asian-style options.[4]

4.1 Case 1: Fixed strike

We consider a contingent claim paying off $(Y_T - k)^+$ dollars at time T, where k is the strike and Y is a nonnegative Markovian stochastic process. This form includes plain vanilla calls ($Y_T = S_{n\Delta}$) and standard fixed strike Asian-style options ($Y_T = \sum_{j=0}^{n} \alpha_j S_{j\Delta}$) struck at k. As before, x denotes the time 0 spot price S_0.

The time 0 arbitrage-free option price seen as a function of the strike price k reads as:

$$k \to C_{0,x}^{T}(k) = e^{-rT} \int_0^{+\infty} (y-k)^+ f_{Y_T}(y)\, dy = e^{-rT} \int_k^{+\infty} (y-k) f_{Y_T}(y)\, dy,$$

where f_{Y_T} denotes the risk-neutral probability density of Y_T.

Provided that the m.g.f. of Y_T exists, then we can define the Laplace transform L of the option price $C_{0,x}^{T}(k)$ with respect to the strike price k as:

$$\lambda \to \mathcal{L}\left[C_{0,x}^{T}(\cdot)\right](\lambda) = c_{0,x}^{T}(\lambda) \triangleq \int_0^{+\infty} e^{-\lambda k} C_{0,x}^{T}(k)\, dk.$$

[4] Minor modifications to what follows allow the user to derive expressions for volume-weighed options.

Simple calculations lead to the following analytical explicit expression:

$$c_{0,x}^{T}(\lambda) = e^{-rT}\left(\frac{\mathbb{E}_0\left[e^{-\lambda Y_T}\right]}{\lambda^2} + \frac{\mathbb{E}_0(Y_T)}{\lambda} - \frac{1}{\lambda^2}\right).$$

Provided that $c_{0,x}^{T}(\lambda)$ has abscissa of convergence with real part λ_0, the option price may be recovered as:

$$C_{0,x}^{T}(k) = \mathcal{L}^{-1}\left[c_{0,x}^{T}(\cdot)\right](k) = \lim_{R\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{a-\sqrt{-1}R}^{a+\sqrt{-1}R} c_{0,x}^{T}(\lambda)\, e^{k\lambda} d\lambda, \tag{20}$$

where the real number $a > \lambda_0$ must be selected such that all the singularities of the image function $c_{0,x}^{T}$ are located to the left-hand side of the vertical line $\lambda = \lambda_0$.

Using the fact that the Laplace inverse transform of $1/\lambda$ is 1, and that of $1/\lambda^2$ is k, the option price can be written as:

$$C_{0,x}^{T}(k) = e^{-rT}\left(\mathcal{L}^{-1}\left[\frac{\mathbb{E}_0\left[e^{-\lambda Y_T}\right]}{\lambda^2}\right](k) + \mathbb{E}_0(Y_T) - k\right).$$

Note that the expected values $E_0\left[e^{-\lambda Y_T}\right]$ and $E_0(Y_T)$ can easily be computed as reported in the following table:

	Y_T	**Dynamics' coefficients**	$\mathbb{E}_0\left[e^{-\lambda Y_T}\right]$	$\mathbb{E}_0(Y_T)$
Vanilla Call	$S_{n\Delta}$	Constant(1) Varying (7) Mean rev.(14)	$v_{0,x}(n,\Delta;\gamma,0)$ $v_{0,x}^{\theta}(n,\Delta;\gamma,0)$ $v_{0,x}^{\beta}(n,\Delta;\gamma,0)$	$xe^{(r-c)T}$ $F_{0,T}$ $F_{0,T}$
Asian Options	$\sum_{j=1}^{n}\alpha_j S_{j\Delta}$	Constant (1) Varying (7) Mean rev.(14)	$v_{0,x}(n,\Delta;0,\mu)$ $v_{0,x}^{\theta}(n,\Delta;0,\mu)$ $v_{0,x}^{\beta}(n,\Delta;0,\mu)$	$\sum_{j=0}^{n}\alpha_j xe^{(r-c)j\Delta}$ $\sum_{j=0}^{n}\alpha_j F_{0,j\Delta}$ $\sum_{j=0}^{n}\alpha_j F_{0,j\Delta}$

where expressions for $v_{0,x}$, $v_{0,x}^{\theta}$, and $v_{0,x}^{\beta}$ are given in formulae (4), (13), and (19), respectively, and $F_{0,t}$ denotes the commodity forward price prevailing in the market at time 0.

The Asian-style option price can be represented as:

$$\begin{aligned} C_{0,x}^{T}(k) &= e^{-rt}\left(\frac{1}{2\pi\sqrt{-1}}\int_{a_l-\sqrt{-1}\infty}^{a_l+\sqrt{-1}\infty} e^{\mu k}\frac{v_{0,x}(\Delta,n;0,\mu)}{\mu^2}d\mu\right. \\ &\left.\sum_{j=0}^{n}\alpha_j F_{0,j\Delta} - k\right) \end{aligned} \tag{21}$$

where a_l is located to the right-hand side of the real part of the largest singularity of the Laplace transform, *i.e.*, $a_l > 0$.

Whenever the analytical inversion of the transform is no longer possible, numerical evaluation is required. We use the Fourier-Euler algorithm proposed by Abate and Whitt (1992) for this purpose.[5]

4.2 Case 2: Floating strike

For the sake of brevity, we restrict our attention to price dynamics (1) with constant coefficients. The other cases can easily be worked out by following the path described in the previous paragraph.

We set $Y_T = S_{n\Delta} - \sum \alpha_j S_{j\Delta}$ and consider the option payoff $(\omega Y_T - k)^+$ where ω can be either 1 or -1 and both Y_T and k are now allowed to assume negative values as well as positive ones. Consequently, it is more convenient to work with the characteristic function, *i.e.*, the Fourier transform, of the random variable ωY_T

$$
\begin{aligned}
\phi^{\omega}_{0,x}(n, \Delta; \gamma) &= \mathbb{E}_0\left[e^{\sqrt{-1}\gamma\omega Y_T}\right] \\
&= v_{0,x}\left(\Delta, n; -\omega\sqrt{-1}\gamma, \omega\sqrt{-1}\gamma\right). \qquad (22)
\end{aligned}
$$

Given these quantities, we obtain an expression for the Fourier transform of the options price with respect to k. Since the pricing function $C^{T,\omega}_{0,x}(k)$ is not square integrable on $\mathbb{R}$, we need to include a dampening factor represented, *e.g.*, by an exponentially decreasing function, and consider the quantity $C^{T,\omega}_{0,x}(k; a) \triangleq C^{T,\omega}_{0,x}(k)\, e^{ak}$, where a is a suitable positive-valued constant as in Eydeland and Geman (1995) and Carr and Madan (1999). Then, we compute the corresponding Fourier transform as:

$$
\begin{aligned}
\widehat{c}^{T,\omega}_{0,x}(\gamma; a) &= e^{-rT}\int_{-\infty}^{+\infty} e^{\sqrt{-1}\gamma k} e^{ak} \int_{k}^{+\infty} (y-k)\, f_{\omega Y_T}(y)\, dy dk \\
&= -e^{-rT}\frac{\phi^{\omega}_{0,x}\left(n, \Delta; \gamma - \sqrt{-1}a\right)}{\left(\gamma - \sqrt{-1}a\right)^2}.
\end{aligned}
$$

The standard floating strike Asian-style option price reads as follows:

$$
\begin{aligned}
C^{T,\omega}_{0,x}(k) &= -e^{-ak}e^{-rT}\frac{1}{\pi}\int_{0}^{+\infty} e^{-\sqrt{-1}\gamma k}\frac{\phi^{\omega}_{0,x}\left(n, \Delta; \gamma - \sqrt{-1}a\right)}{\left(\gamma - \sqrt{-1}a\right)^2} d\gamma \qquad (23) \\
&= -e^{-ak}e^{-rT}\frac{1}{\pi}\int_{0}^{+\infty} e^{-\sqrt{-1}\gamma k}\frac{v_{0,x}\left(\Delta, n; \omega\left(\sqrt{-1}\gamma + a\right), -\omega\left(\sqrt{-1}\gamma + a\right)\right)}{\left(\gamma - a\sqrt{-1}\right)^2} d\gamma.
\end{aligned}
$$

where the last equality directly follows from formula (22).

We finally perform a numerical inversion using the fast Fourier transform algorithm. Carr and Madan (1999) show that the Fourier transform of the option delta and gamma can be obtained by differentiating the Fourier transform of the option price with respect to the standing spot price x.

[5] This algorithm has been implemented in Mathematica 5.2® using the following parametric setting: $A_l = 18.4, m = 25, n = 15$. We refer to Abate and Whitt (1992) for details on this notation.

5 Numerical results

We perform a few numerical experiments on the pricing formulae derived in the previous sections.

Our first test aims at assessing the discrepancy of prices stemming from the alternative assumptions of a discrete *vs.* continuous monitoring rule. For barrier options, Fusai, Abrahams, and Sgarra (2006) showed that price differences can be very large in spite of a relatively high monitoring frequency. As for Asian-style options, we conduct a test under spot price dynamics (1), with short rate of interest $r = 0.04$, instantaneous convenience yield $c = 0$, volatility coefficient $\sigma = 0.7$, and starting price $x = 1$. We assume that monitoring occurs over a one-year period ($T = 1$) and we compute Asian-style option prices for varying strikes and a varying number of monitoring dates. Calculations are performed using the pricing formula (23). The continuously monitored option price is calculated using the analytical formula for the joint moment generating function of the pair $\left(S\left(T\right), \int_0^T S\left(u\right) du\right)$ as is reported in Lamberton and Lapeyre (1996):

$$v_{0,x}^{\text{continuous}}\left(\gamma,\mu\right) = \mathbb{E}_0\left(\exp\left\{-\left[\gamma S\left(t\right) + \mu\int_0^t S\left(u\right) du\right]\right\}\right) = e^{-A(t;\gamma,\mu)S_0},$$

where:

$$A\left(t;\gamma,\mu\right) = \frac{\gamma\left(\lambda - r + c + \left(\lambda + r - c\right)\exp\left(t\lambda\right)\right) + 2\mu\left(\exp\left(t\lambda\right) - 1\right)}{\sigma^2\gamma\left(\exp\left(t\lambda\right) - 1\right) + \lambda + r - c + \left(\lambda - r + c\right)\exp\left(t\lambda\right)}, \tag{24}$$

and $\lambda = \sqrt{\left(r - c\right)^2 + 2\mu\sigma^2}$.

Table 3 compares floating strike Asian-style option prices obtained for a number $n = 12, 25, 50, 100$, and 250 of monitoring dates over the contract lifetime. The strike price varies from a minimum value of -0.05 dollars to a maximum value of 0.05 dollars, with a price step of 0.01 dollars. These values reflect realistic differences between the underlying spot price at maturity and the average spot price since the contract outset. The last column reports prices using continuous monitoring. As expected, price differences between discrete and continuous monitoring rules decrease as long as the number of monitoring dates increases.

Table 4 reports results obtained for fixed strike Asian-style options. The number of monitoring dates is identical to that of floating strike options, while the strike ranges between a minimum value of 0.90 dollars to a maximum value of 1.10 dollars in increments of 0.05 dollars. These numbers reflect realistic figures for the price of the underlying commodity at the contract expiration. Again, convergence of the discretely monitored option price to the continuously monitored one is almost linear in the monitoring frequency ($1/n$), as illustrated in Figure (3).[6]

Our second test aims at measuring the price differences between options prices using the simplest specification of our square-root model (1) and those

[6] We note, however, that the speed of convergence for barrier options is even slower. In this case, Fusai, Abrahams, and Sgarra (2006) have shown that this figure is in the order of $1/\sqrt{n}$.

k	n=12 Price	n=12 %Diff	n=25 Price	n=25 %Diff	n=50 Price	n=50 %Diff	n=100 Price	n=100 %Diff	n=250 Price	n=250 %Diff	cts Price
-0.05	0.16897	-1.58	0.17033	-0.78	0.17099	-0.40	0.17133	-0.20	0.17153	-0.08	0.17167
-0.04	0.16320	-1.64	0.16456	-0.81	0.16523	-0.41	0.16557	-0.21	0.16577	-0.08	0.16591
-0.03	0.15752	-1.70	0.15889	-0.85	0.15955	-0.43	0.15989	-0.22	0.16010	-0.09	0.16024
-0.02	0.15193	-1.77	0.15330	-0.88	0.15397	-0.45	0.15431	-0.23	0.15452	-0.09	0.15466
-0.01	0.14643	-1.83	0.14780	-0.91	0.14847	-0.46	0.14881	-0.23	0.14902	-0.09	0.14916
0	0.14102	-1.91	0.14239	-0.95	0.14306	-0.48	0.14341	-0.24	0.14362	-0.10	0.14376
0.01	0.13570	-1.98	0.13708	-0.99	0.13775	-0.50	0.13810	-0.25	0.13831	-0.10	0.13845
0.02	0.13048	-2.06	0.13186	-1.03	0.13253	-0.52	0.13288	-0.26	0.13309	-0.11	0.13323
0.03	0.12535	-2.15	0.12673	-1.07	0.12741	-0.54	0.12775	-0.27	0.12796	-0.11	0.12810
0.04	0.12033	-2.23	0.12170	-1.11	0.12238	-0.57	0.12272	-0.29	0.12293	-0.12	0.12308
0.05	0.11540	-2.33	0.11677	-1.16	0.11745	-0.59	0.11779	-0.30	0.11800	-0.12	0.11814

Table 3: Prices of floating strike Asian-style options for varying strike prices K and monitoring dates n. Column %Diff reports the percentage difference with respect to the continuous formula 100*(D-C)/C. Experiments are conducted using the following parametric setting: r = 0.04, σ = 0.7, x = 1, t = 1, ω = -1.

k	n=12 Price	n=12 %Diff	n=25 Price	n=25 %Diff	n=50 Price	n=50 %Diff	n=100 Price	n=100 %Diff	n=250 Price	n=250 %Diff	cts Price
0.9	0.21279	-1.37	0.21428	-0.68	0.21501	-0.35	0.21538	-0.17	0.21560	-0.07	0.21575
0.95	0.18659	-1.58	0.18810	-0.79	0.18883	-0.40	0.18920	-0.20	0.18943	-0.08	0.18958
1	0.16282	-1.80	0.16432	-0.90	0.16505	-0.46	0.16542	-0.23	0.16565	-0.09	0.16580
1.05	0.14140	-2.03	0.14287	-1.01	0.14359	-0.51	0.14395	-0.26	0.14418	-0.10	0.14433
1.1	0.12223	-2.27	0.12365	-1.13	0.12434	-0.57	0.12470	-0.29	0.12492	-0.12	0.12506

Table 4: Prices of fixed strike Asian-style options for varying strike prices K and monitoring dates n. Column %Diff reports the percentage difference with respect to the continuous formula 100*(D-C)/C. Experiments are conducted using the following parametric setting: r = 0.04, σ = 0.7, x = 1, t = 1.

resulting from implementing alternative numerical methods proposed in the existing literature for spot price dynamics driven by a geometric Brownian motion, which represents the currently adopted market model. Specifically, we computed figures using the following methods:

1. Geman and Yor (1993) Laplace transform inverse;

2. Turnbull and Wakeman (1991) and Levy (1992) lognormal approximation.

Table 5 shows prices obtained for strikes ranging from a minimum value of 90 dollars to a maximum value of 110 dollars in increments of 5 dollars and geometric Brownian motion volatilities σ_{GBM} ranging from 0.1 to 0.5 units in increments of 0.1 units. Option values under the square-root model are obtained by selecting a volatility coefficient in model (1) leading to a European-style call option price matching the one obtained using the geometric Brownian motion corresponding to the selected σ_{GBM}.[7]

We find that prices obtained using our square-root model accurately approximate quotes stemming from the market model. This fact constitutes a major result of our analysis since the method we provided for in the preceding section allows us to compute Asian-style option prices in real time, whereas numerical approximation for the geometric Brownian motion case requires intensive calculations and much greater computational time. Moreover, we find that our quotes mostly lie between those obtained by the two methods used to approximate the options price under the market model. This result is quite robust across the examined spectrum of parameters, the only case where market model prices exceed our quotes being the one related to deeply out-of-the-money options.

Our third test consists of assessing the importance of including information about the currently quoted forward curve into the spot price dynamics for the purpose of pricing Asian-style options. We apply the method detailed in section 3 to the forward price curve of Natural Gas as quoted at NYMEX on March 1, 2007, which we report in Table 6. Figure 4 shows the curve dynamics during March 2007. Each curve is obtained by interpolating the observed quotes using a cubic spline. The short rate $-$ spot convenience yield discrepancy can be assessed by setting:

$$e^{(r-c)(T-0)} = \frac{F_{0,T}}{F_{0,0}},$$

where $F(0,0)$ stands for the underlying spot price as obtained by curve extrapolation beyond the shortest quoted time to maturity. The integral appearing in formula (12) can be computed using the Mathematica 5.2© built-in function `NIntegrate`. We obtain a value $r - c = 0.230238$. Since the 1-year US Swap rate, which we adopt as a proxy for the short rate standing in the US market on March 1, 2007, is 5.25% *per annum*, the corresponding net convenience yield for gas turns out to be quite considerable.

[7] We consider an at-the-money call option struck at 1 Euro at time 0, with a residual lifetime equal to 1 year.

K	σ_{GBM}	Option Prices in the GBM case Inverse Laplace	Lognormal	σ_{SR}	Option price (sr)
90	0.1	15.39763	15.39906	0.97411	15.39890
95	0.1	11.09410	11.12196	0.98711	11.12064
100	0.1	7.02774	7.03485	0.99966	7.03086
105	0.1	3.61263	3.61869	1.01180	3.61456
110	0.1	1.41362	1.41080	1.02356	1.41070
90	0.2	15.64176	15.66533	1.94761	15.65206
95	0.2	11.79376	11.82644	1.97361	11.80735
100	0.2	8.40884	8.44079	1.99870	8.42045
105	0.2	5.64186	5.66133	2.02298	5.64572
110	0.2	3.55561	3.55639	2.04649	3.54928
90	0.3	16.51291	16.59082	2.91994	16.54281
95	0.3	13.14669	13.22661	2.95891	13.17443
100	0.3	10.20983	10.27814	2.99654	10.22873
105	0.3	7.73830	7.78408	3.03292	7.74352
110	0.3	5.73012	5.74800	3.06817	5.71995
90	0.4	17.80010	17.95199	3.89057	17.84950
95	0.4	14.78577	14.92999	3.94251	14.82609
100	0.4	12.13320	12.25606	3.99265	12.15829
105	0.4	9.84443	9.93595	4.04114	9.85037
110	0.4	7.90501	7.96003	4.08812	7.89053
90	0.5	19.30572	19.55391	4.86178	19.37724
95	0.5	16.55052	16.78229	4.92736	16.60820
100	0.5	14.10024	14.30234	4.99086	14.13906
105	0.5	11.94600	12.10904	5.05249	11.96311
110	0.5	10.07128	10.18997	5.11247	10.06599

Table 5: Prices of fixed strike Asian-style options for varying strikes K under four alternative valuation devices: 1) Geman and Yor (1993) Laplace inverse transform; 2) Turnbull and Wakeman (1991) and Levy (1992) lognormal approximation. Experiments are conducted using the following parametric setting: $r = 0.15$, $\sigma = 0.7$, $x = 100$, $t = 1$.

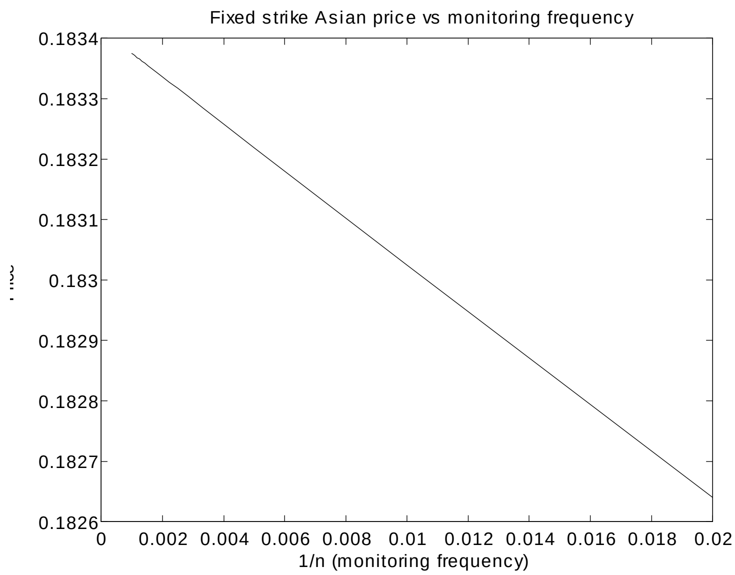

Figure 3: Price of a fixed strike Asian-style option versus the monitoring frequency $(1/n)$. Experiments are conducted using the parametric setting of Table (5) , with $K = 0.95$.

Delivery Month	Trading Termination	Time to maturity (years)	Settlement Price (US$)
April	28-Mar-07	0.0740	7.288
May	26-Apr-07	0.1534	7.405
June	29-May-07	0.2438	7.52
July	27-Jun-07	0.3233	7.635
August	27-Jul-07	0.4055	7.73
September	29-Aug-07	0.4959	7.785
October	26-Sep-07	0.5726	7.88
November	29-Oct-07	0.6630	8.45
December	28-Nov-07	0.7452	9.01
January	27-Dec-07	0.8247	9.3
February	29-Jan-08	0.9151	9.295
March	27-Feb-08	0.9945	9.075
April	27-Mar-08	1.0725	7.71
May	28-Apr-08	1.1601	7.59
June	27-May-08	1.2394	7.64
July	26-Jun-08	1.3215	7.71
August	29-Jul-08	1.4118	7.77
September	27-Aug-08	1.4911	7.82

Table 6: Forward prices on Natural Gas quoted at NYMEX on March 1, 2007.

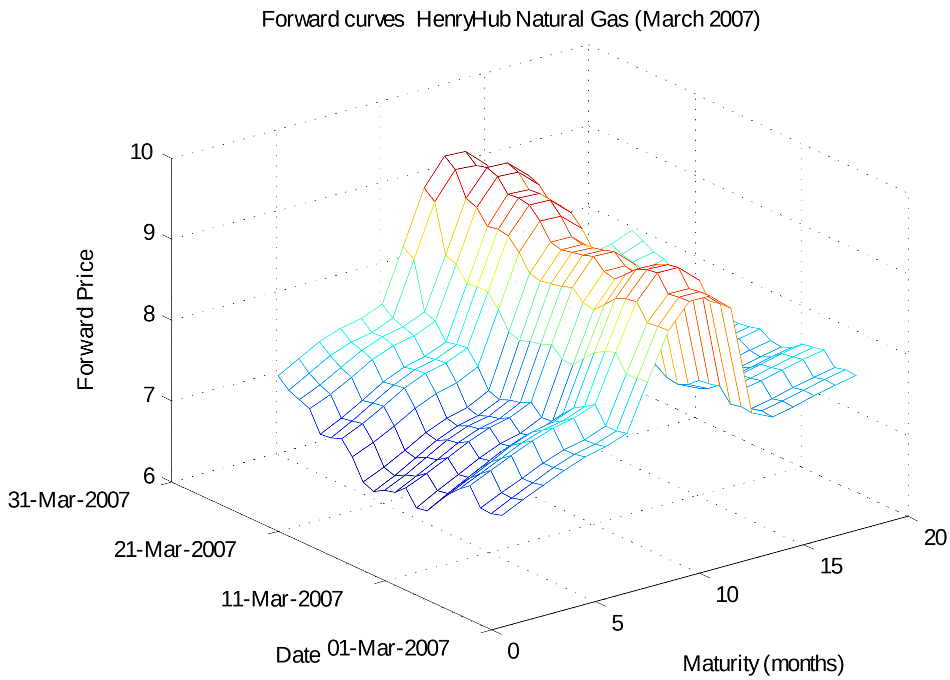

Figure 4: Natural gas forward curve evolution between March 1 and March 31, 2007.

Figure (5) exhibits a plot of the density function of the difference $S_{n\Delta} - \left(\sum S_{j\Delta}\right) / (n+1)$ corresponding to $\sigma = 0.7$, $x = 7.1409$, $r - c = 0.230238, r = 0.0525$, and $n = 50$.

Table (7) reports differentials between the floating strike Asian-style option price computed irrespectively of the market forward curve and the one obtained using spot dynamics fitting the gas curve under investigation. Option values are calculated for a number $n = 5, 12, 25, 50, 100$, and 250 of monitoring dates over the contract lifetime. As before, the strike price varies from a minimum value of -0.05 dollars to a maximum value of 0.05 dollars, with a price step of 0.01 dollars.

Results clearly show that including the information embedded in the forward quotes may have a significant impact on the option price. Moreover, the price discrepancy between the two cases increases with the monitoring frequency. We note that the extent of this effect may depend on the shape of the forward curve as well. This evidence calls for including forward market quotes into option pricing devices whenever a market forward curve displays a high degree of variability across the time to maturity spectrum. This requirement may be particularly compelling for commodity markets, in particular energy markets, where the

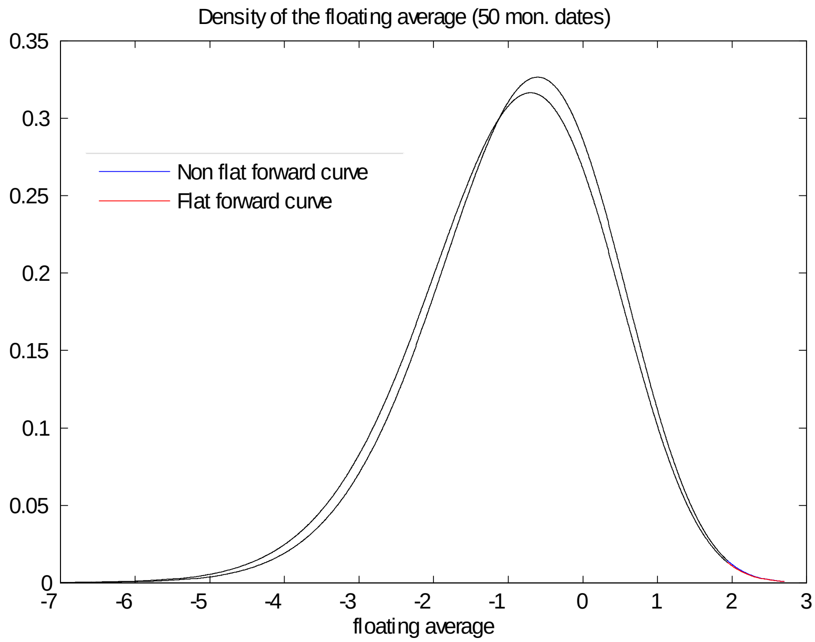

Figure 5: Plot of the probability density function of the difference $S_{n\Delta} - \sum S_{j\Delta}/(n+1)$ under the assumptions of flat and actual forward curve for the price of gas.

seasonal character of the underlying spot price translates into a corresponding component in the forward curve shape.

Our fourth, and final, test allows us to examine the effect of a seasonal historical volatility structure over the fair value of Asian-style options. We consider the forward curve on corn as quoted on December 1, 2006 at CBOT. Values across all delivery months are reported in Table (8), where we also indicate the exact day of trading termination and the time to maturity of the contract as expressed in year units. We compute a single annualized historical average volatility for each month in the calendar year. Averaging occurs over the observation period covering the years ranging from 1980 to 2006. The resulting figures are reported in the first row of Table 9. From these volatilities of logarithmic returns, we derive the corresponding figures for the volatilities entering the square-root dynamics (1). Conversion can be performed by solving the following equation in the unknown σ_{SR}:

$$\sigma_{GBM} x = \sigma_{SR} \sqrt{x},$$

where the standing spot price can be obtained by extrapolating the forward curve beyond the shortest maturity for which a quote is available, namely the

K	n=5 Flat	n=5 Non Flat	n=5 %Diff	n=12 Flat	n=12 Non Flat	n=12 %Diff	n=25 Flat	n=25 Non Flat	n=25 %Diff
-0.05	0.140308	0.14598	4.04232	0.147329	0.158948	7.88637	0.150324	0.164002	9.09867
-0.04	0.138022	0.14359	4.03411	0.145002	0.156444	7.89113	0.147981	0.161458	9.10732
-0.03	0.135762	0.141227	4.02569	0.142701	0.153968	7.8957	0.145663	0.158941	9.1158
-0.02	0.133529	0.138893	4.01705	0.140426	0.151519	7.90008	0.143371	0.156452	9.1241
-0.01	0.131322	0.136585	4.00819	0.138177	0.149098	7.90428	0.141104	0.15399	9.13223
0	0.129141	0.134305	3.9991	0.135953	0.146705	7.90829	0.138864	0.151556	9.14018
0.01	0.126986	0.132052	3.98979	0.133755	0.144338	7.91212	0.136648	0.149149	9.14795
0.02	0.124857	0.129826	3.98026	0.131583	0.141999	7.91574	0.134458	0.146769	9.15554
0.03	0.122753	0.127627	3.97049	0.129436	0.139686	7.91918	0.132294	0.144416	9.16294
0.04	0.120675	0.125455	3.9605	0.127314	0.137401	7.92241	0.130154	0.142089	9.17016
0.05	0.118623	0.123309	3.95027	0.125218	0.135142	7.92545	0.128039	0.13979	9.17718

K	n=50 Flat	n=50 Non Flat	n=50 %Diff	n=100 Flat	n=100 Non Flat	n=100 %Diff	n=250 Flat	n=250 Non Flat	n=250 %Diff
-0.05	0.15179	0.166207	9.49791	0.152543	0.167265	9.65085	0.153002	0.167885	9.72734
-0.04	0.149438	0.163646	9.50782	0.150187	0.164697	9.66123	0.150644	0.165314	9.73795
-0.03	0.147112	0.161113	9.51756	0.147857	0.162157	9.67143	0.148311	0.162769	9.74839
-0.02	0.144812	0.158608	9.52713	0.145553	0.159645	9.68147	0.146005	0.160253	9.75866
-0.01	0.142537	0.15613	9.53653	0.143274	0.157159	9.69134	0.143723	0.157763	9.76875
0	0.140288	0.15368	9.54575	0.141021	0.154701	9.70103	0.141467	0.155301	9.77868
0.01	0.138064	0.151256	9.5548	0.138793	0.152271	9.71055	0.139237	0.152866	9.78843
0.02	0.135866	0.14886	9.56367	0.13659	0.149867	9.71989	0.137032	0.150458	9.79801
0.03	0.133693	0.14649	9.57236	0.134413	0.14749	9.72906	0.134851	0.148077	9.80741
0.04	0.131545	0.144148	9.58086	0.13226	0.14514	9.73804	0.132696	0.145722	9.81663
0.05	0.129421	0.141832	9.58918	0.130132	0.142816	9.74684	0.130566	0.143395	9.82566

Table 7: Prices of floating strike Asian options for varying strikes K and monitoring dates under the assumption of non flat forward curve. Experiments are conducted using the following parametric setting: $r = 0.0525$, $\sigma = 0.7$, $x = 100$, $t = 1$, $r - c = 0.230238$, ω=-1.

Delivery Month	Trading Termination	Time to maturity (years)	Settlement Price ($cent/bushel)
December	15-Dec-06	0.0384	3740
March	15-Mar-07	0.2849	3870
May	15-May-07	0.4521	3912
July	15-Jul-07	0.6192	3947
September	15-Sep-07	0.7890	3727
December	15-Dec-07	1.0384	3602

Table 8: Forward prices on Corn as quoted at CBOT on December 2, 2006.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
σ_{GBM}	0.168	0.141	0.16	0.168	0.204	0.227	0.267	0.225	0.187	0.184	0.161	0.138
σ_{SR}	10.23	8.59	9.74	10.23	12.42	13.82	16.26	13.70	11.39	11.20	9.80	8.40

Table 9: Historical volatility of corn quotes and implied square-root volatilities.

one corresponding to the December contract. (This procedure leads to a value for x equal to 3707.88 dollars.) The set of square-root volatilities is exhibited in the second row of Table 9. Armed with this information, we may start performing our analysis and computing Asian-style option prices for 1-year maturity contracts using the following three methodologies:

a) Setting a flat volatility curve by selecting a constant $\sigma_{(a)}$ that leads to the same integrated denominator in formula (12) as the one resulting from using the market (time dependent) volatility structure. This amounts to setting $\sigma_{(a)}$ such that:

$$\sigma_{(a)}^2 \int_0^T \frac{1}{F_{0,s}} ds = \int_0^T \frac{\sigma_s^2}{F_{0,s}} ds.$$

We then obtain $\sigma_{(a)} = 11.2131$ points.

b) Setting a flat volatility curve at the level of the unique constant $\sigma_{(b)}$ that leads to an option price matching the price obtained using the time varying volatility structure in the market. For this purpose, we consider a standard fixed strike Asian-style option struck at-the-money under a 5-period discrete monitoring rule. We obtain an option price equal to 178.154 dollars and a corresponding flat volatility $\sigma_{(b)}$ equal to 10.8396 points.

c) Using the market volatility structure.

Table 10 reports prices across varying levels of moneyness and number of monitoring dates. Specifically, the ratio K/x ranges from 0.9 to 1.1 in increments of 0.5 units and monitoring is allowed to occur $12, 25, 50, 100, 250,$ and 1000 times over the 1-year long contract lifetime. Percentage differences refer to the discrepancy between the option prices obtained using either of the flat volatilities ($\sigma_{(a)}$ or $\sigma_{(b)}$) and the one resulting from using the market volatility structure.

We note the importance of price differences when including the exact volatility structure observed in the market compared to the market standard of quoting option prices using a single volatility coefficient. This effect is particularly significant for out-of-the money options. Compared to method a), method b) provides a better approximation of actual prices. However, its concrete implementation requires that one know the price of the options in advance or, alternatively, that one calculate the option price using the market volatility structure. Consequently, to date, the extent of its use seems to be rather limited.

6 Conclusions

It is commonly held that pricing Asian-style options under the market model represented by a geometric Brownian motion is a difficult task. First, no closed-

κ/x	n	$\sigma_{\text{a}=11.2131}$	%Diff	$\sigma_{\text{b}=10.8396}$	%Diff	Non Flat Vol. Curve
0.9	12	460.72	0.59	458.62	0.13	458.03
0.95	12	314.93	1.45	311.33	0.29	310.43
1	12	196.08	3.02	191.37	0.54	190.34
1.05	12	109.74	5.49	104.93	0.87	104.03
1.1	12	54.71	9.03	50.80	1.23	50.18
0.9	25	467.01	0.63	464.89	0.17	464.10
0.95	25	320.97	1.54	317.34	0.40	316.09
1	25	201.39	3.20	196.64	0.77	195.14
1.05	25	113.89	5.83	109.01	1.29	107.61
1.1	25	57.53	9.61	53.51	1.95	52.48
0.9	50	470.06	0.64	467.93	0.19	467.05
0.95	50	323.89	1.58	320.26	0.44	318.85
1	50	203.98	3.28	199.21	0.87	197.50
1.05	50	115.92	5.97	111.01	1.48	109.39
1.1	50	58.92	9.86	54.85	2.27	53.64
0.9	100	471.57	0.65	469.43	0.20	468.51
0.95	100	325.35	1.60	321.71	0.46	320.22
1	100	205.27	3.32	200.49	0.92	198.68
1.05	100	116.95	6.04	112.01	1.57	110.28
1.1	100	59.63	9.98	55.53	2.42	54.22
0.9	250	472.49	0.66	470.35	0.20	469.40
0.95	250	326.24	1.61	322.59	0.48	321.06
1	250	206.06	3.34	201.28	0.94	199.40
1.05	250	117.57	6.08	112.63	1.62	110.83
1.1	250	60.06	10.05	55.95	2.51	54.58
0.9	1000	472.95	0.66	470.81	0.20	469.85
0.95	1000	326.68	1.62	323.04	0.48	321.48
1	1000	206.46	3.35	201.67	0.96	199.76
1.05	1000	117.89	6.10	112.94	1.65	111.11
1.1	1000	60.28	10.08	56.16	2.55	54.76

Table 10: Prices of 1-year fixed strike Asian-style options on corn under three volatility curve assessments: a) flat volatility matching the integrated forward weighed volatility; b) flat volatility matching a key Asian-style option price; c) market volatility structure.

form expression exists for the fair option value. Second, information embedded within the standing market forward curve is neglected during the valuation process. Finally, prices are derived irrespectively of the seasonal path exhibited by the spot price volatility or mean reversion properties.

We propose a solution to these issues by analytically pricing discretely monitored Asian-style options written on a spot price whose dynamics are driven by a generalized square-root process. Our method is particularly useful for pricing options in commodity markets where traders must quickly produce quotes compatible with the market view expressed in terms of forward prices, the seasonal trend shown by the spot price volatility, and with a mean reversion behavior.

Extensive numerical experiments on gas and corn data suggest that price discrepancies between figures obtained using a more accurate pricing device and those resulting from the market model may be very significant.

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