Complex solitons with real energies

Julia Cen and Andreas Fring

Department of Mathematics, City University London,
Northampton Square, London EC1V 0HB, UK
E-mail: julia.cen.1@city.ac.uk, a.fring@city.ac.uk

ABSTRACT: Using Hirota's direct method and Bäcklund transformations we construct explicit complex one and two-soliton solutions to the complex Korteweg-de Vries equation, the complex modified Korteweg-de Vries equation and the complex sine-Gordon equation. The one-soliton solutions of trigonometric and elliptic type turn out to be $\mathcal{PT}$-symmetric when a constant of integration is chosen to be purely imaginary with one special choice corresponding to solutions recently found by Khare and Saxena. We show that alternatively complex $\mathcal{PT}$-symmetric solutions to the Korteweg-de Vries equation may also be constructed alternatively from real solutions to the modified Korteweg-de Vries by means of Miura transformations. The multi-soliton solutions obtained from Hirota's method break the $\mathcal{PT}$-symmetric, whereas those obtained from Bäcklund transformations are $\mathcal{PT}$-invariant under certain conditions. Despite the fact that some of the Hamiltonian densities are non-Hermitian, the total energy is found to be positive in all cases, that is irrespective of whether they are $\mathcal{PT}$-symmetric or not. The reason is that the symmetry can be restored by suitable shifts in space-time and the fact that any of our N-soliton solutions may be decomposed into N separate $\mathcal{PT}$-symmetrizable one-soliton solutions.

1. Introduction

$\mathcal{PT}$-symmetrically deformed nonlinear wave equations have been found to possess various interesting properties [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In general, the $\mathcal{PT}$-symmetric deformations destroy the integrability of models with that property, although some rare cases pass the Painlevé test [5] indicating that they remain integrable. Furthermore, it was shown [4] that it is possible to construct specific $\mathcal{PT}$-symmetric deformations that preserve the supersymmetry of some models. Some $\mathcal{PT}$-symmetrically deformed nonlinear wave equations possess very intricate shock wave structures [10].

Most notably when the $\mathcal{PT}$-symmetrically deformed models are of Hamiltonian type, with densities $\mathcal{H}[u(x,t), u_x(x,t), \ldots]$ depending on some field $u(x,t)$ and its derivative, the $\mathcal{PT}$-symmetry will ensure that the energy on symmetric intervals $[-a, a]$

$$E = \int_{-a}^{a} \mathcal{H}[u(x,t)]dx = \oint_{\Gamma} \mathcal{H}[u(x,t)]\frac{du}{u_x}, \tag{1.1}$$

remains real despite the fact that the Hamiltonian density is complex [2]. The reasoning to establish this is similar to the one applied to quantum mechanical models, although the quantity E as defined in (1.1) will not play the role of the energy in the quantum theory for reduced models as explained in more detail in [8].

The simplest way to obtain complex solutions is to keep the form of the original equation intact and just take the field $u(x, t)$ to be complex by demanding that the complexified equations remain invariant under the antilinear transformation $\mathcal{PT}$: $x \rightarrow -x$, $t \rightarrow -t$, $i \rightarrow -i$ and $u \rightarrow u$ or $u \rightarrow -u$. For such a setting Khare and Saxena [12] have recently found some interesting apparently novel $\mathcal{PT}$-symmetric solutions to various types of nonlinear equations that appear to have been overlooked this far. Their approach is to start off from some well-known real solutions to these equations and then by adding a term build around that solution a suitable complex Ansatz including various constants. In many cases they succeeded to determine those constants in such a way that their expressions constitute solutions to the different types of complex nonlinear wave equations considered.

One of the purposes of this note is to demonstrate that these solutions may be derived in a more constructive, systematic and generic way. We focus here on nonlinear wave equations for which we use Hirota's direct method [13] to derive complex solutions including those of [12] as special cases. Some of the one-soliton solutions produced in this manner turn out to be $\mathcal{PT}$-symmetric, whereas the multi-soliton solutions obtained from this method break the $\mathcal{PT}$-symmetry in general. Subsequently we employ Bäcklund transformations to construct new $\mathcal{PT}$-symmetric multi-soliton solutions from some previously constructed complex solutions. For a specific case we evaluate the time-delay in the real and imaginary parts of these solutions.

Computing the energies E corresponding to our solutions we find that all of them are real irrespective of whether they are $\mathcal{PT}$-symmetric or not. While this is to be expected for the $\mathcal{PT}$-symmetric solutions, this is less obvious for the $\mathcal{PT}$-broken solutions. We will present the argument and mechanism responsible for this behaviour. Our analysis is carried out for the complex Korteweg-de Vries (KdV) equation in section 2.1, complex modified Korteweg-de Vries (mKdV) equation in section 2.2, both considered also in [12], and in addition for the complex sine-Gordon equation in section 2.3. Our conclusions are stated in section 3.

2. The construction of complex multi-soliton solutions

At first we employ here Hirota's direct method [13]. The general principle of this approach is to convert the nonlinear equation of interest into Hirota's equation of bilinear form by means of a dependent variable transformation

$$P(D_1, D_2, \ldots, D_n)\tau \cdot \sigma = 0, \tag{2.1}$$

with $P(D_1, D_2, \ldots, D_n)$ being a polynomial in the Hirota derivatives $(D_1, D_2, \ldots, D_n)$ acting on the product of the two functions τ and σ both depending on $(x_1, x_2, \ldots, x_n)$. The

general expressions for the Hirota derivatives in terms of ordinary derivatives may be obtained from the generating function

$$\tau(x_1 + y_1, \ldots, x_n + y_n)\sigma(x_1 - y_1, \ldots, x_n - y_n) = e^{y_1 D_1 + y_2 D_2 + \ldots + y_n D_n}\tau \cdot \sigma, \qquad (2.2)$$

by reading off powers in y_i. In particular, we shall require below the expressions

$$D_t \tau \cdot \sigma = \tau_x \sigma - \sigma_x \tau, \qquad (2.3)$$

$$D_x^2 \tau \cdot \sigma = \tau_{xx}\sigma - 2\tau_x\sigma_x + \tau\sigma_{xx}, \qquad (2.4)$$

$$D_x^3 \tau \cdot \sigma = \tau_{xxx}\sigma - 3\tau_{xx}\sigma_x + 3\tau_x\sigma_{xx} - \tau\sigma_{xxx}, \qquad (2.5)$$

$$D_x^4 \tau \cdot \sigma = \tau_{xxxx}\sigma - 4\tau_{xxx}\sigma_x + 6\tau_{xx}\sigma_{xx} - 4\tau_x\sigma_{xxx} + \tau\sigma_{xxxx}, \qquad (2.6)$$

$$D_x D_t \tau \cdot \sigma = \tau_{xt}\sigma + \tau_{xt}\sigma - \tau_x\sigma_t - \tau_t\sigma_x. \qquad (2.7)$$

The solution procedure is then to expand the functions τ and σ in powers of λ as $\tau = \sum_{k=0}^{\infty} \lambda^k \tau^k$, $\sigma = \sum_{k=0}^{\infty} \lambda^k \sigma^k$ and subsequently solve the *bilinear* Hirota equation order by order in λ. It turns out that one can systematically set $\tau^k = \sigma^k = 0$ for some $j \leq k$. The constant λ may then be absorbed into the τ^k and σ^k so that the terminated series constitute an *exact* solution to the Hirota equation and therefore, after re-transformation, to the original nonlinear equation.

2.1 The complex Korteweg-de Vries equation

The KdV equation for the complex field $u(x,t)$ may be considered as a set of coupled equations for the real fields $p(x,t)$ and $q(x,t)$

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6\,(pq)_x + q_{xxx} = 0 \end{cases}, \qquad (2.8)$$

when taking $u = p + iq$. The coupled equations reduce to the Hirota-Satsuma [14] and Ito system [15] when setting $(pq)_x \to pq_x$ and $q_{xxx} \to 0$ in the second equation, respectively. Evidently these equations remain invariant for $\mathcal{PT}$: $x \to -x$, $t \to -t$, $i \to -i$, $u \to u$, $p \to p$, $q \to -q$. We stress here that, although there are many $\mathcal{PT}$-symmetric solutions to (2.8), not all solutions to (2.8) need to be $\mathcal{PT}$-symmetric since the symmetry could map one solution, say $u_1(x,t)$, into a new one $u_1(-x,-t) = u_2(x,t) \neq u_1(x,t)$. Unlike as in the linear quantum mechanical scenario the sum of these two solution would of course not constitute a new $\mathcal{PT}$-symmetric solution, as the KdV equation is nonlinear. In fact, it would not be a solution at all, unless $(u_1 u_2)_x = 0$.

2.1.1 Complex solutions from the Hirota method

Since the original work of Hirota [13] it is well known that the KdV equation (2.8) can be converted into Hirota's bilinear form

$$\left(D_x^4 + D_x D_t\right)\tau \cdot \tau = 0, \qquad (2.9)$$

by means of the variable transformation $u = 2(\ln \tau)_{xx}$ together with (2.6) and (2.7). Equation (2.9) is solved easily with the above mentioned expansion for τ. At order λ^0 the equation is trivially satisfied and at order λ^1 we have to solve

$$\left(D_x^4 + D_x D_t\right) (1 \cdot \tau^1 + \tau^1 \cdot 1) = 2(\tau^1)_{xt} + 2(\tau^1)_{xxxx} = 0. \tag{2.10}$$

Thus the original problem to solve a nonlinear equation has been reduced to the much simpler task of just solving a linear equation. We may now take

$$\tau^1 = e^{\eta_1} \qquad \text{with } \eta_i = k_i x + \omega_i t + \mu_i, \ k_i, \omega_i \in \mathbb{R}, \ \mu_i \in \mathbb{C}, \tag{2.11}$$

with nonlinear dispersion relation $k_1^3 + \omega_1 = 0$ to solve (2.10), stressing at this point that the constant of integration μ_1 might be complex. At order λ^2 we need to solve

$$\left(D_x^4 + D_x D_t\right)(\tau^1 \cdot \tau^1) = -\left(D_x^4 + D_x D_t\right)(1 \cdot \tau^2 + \tau^2 \cdot 1) \tag{2.12}$$

$$= -2(\tau^2)_{xt} - 2(\tau^2)_{xxxx}. \tag{2.13}$$

Using $D_x^m D_t^n (e^{k_i x + \omega_i t + \mu_i} \cdot e^{k_j x + \omega_j t + \mu_j}) = (k_i - k_j)^m (\omega_i - \omega_j)^n e^{k_i x + \omega_i t + \mu_i} e^{k_j x + \omega_j t + \mu_j}$ for $n, m \in \mathbb{N}_0$ this is easily achieved by setting $\tau^2 = 0$. Then all higher order terms vanish by setting $\tau^k = 0$ for $k > 2$. Thus an exact τ-function and corresponding solution to the KdV equation are simply

$$\tau_{\mu,\beta}(x,t) = 1 + e^{\beta x - \beta^3 t + \mu}, \quad \text{and} \quad u_{\mu,\beta}(x,t) = \frac{\beta^2}{2} \operatorname{sech}^2\left[\frac{1}{2}(\beta x - \beta^3 t + \mu)\right], \tag{2.14}$$

where we have set $\omega_1 = -\beta^3$, $k_1 = \beta$, $\mu_1 = \mu$ in order to satisfy the dispersion relation and $\lambda = 1$. The standard choices are here $\mu = 0$ and $\mu = i\pi$ giving rise to the well-known real solutions

$$u_{0,\beta}(x,t) = \frac{\beta^2}{2} \operatorname{sech}^2\left[\frac{1}{2}(\beta x - \beta^3 t)\right] \quad \text{and} \quad u_{i\pi,\beta}(x,t) = -\frac{\beta^2}{2} \operatorname{csch}^2\left[\frac{1}{2}(\beta x - \beta^3 t)\right], \tag{2.15}$$

that may also be obtained from direct integration of the KdV equation with appropriate boundary condition assuming the solutions to be travelling waves. However, it is clear that any choice for which μ is purely imaginary, i.e. $\mu = i\theta$ with $\theta \in \mathbb{R}$, would constitute a finite $\mathcal{PT}$-invariant solution. Separating this solution into its real and imaginary part we obtain

$$u_{i\theta,\beta}(x,t) = \frac{\beta^2 + \beta^2 \cos\theta \cosh(\beta x - \beta^3 t)}{\left[\cos\theta + \cosh(\beta x - \beta^3 t)\right]^2} - i\frac{\beta^2 \sin\theta \sinh(\beta x - \beta^3 t)}{\left[\cos\theta + \cosh(\beta x - \beta^3 t)\right]^2}. \tag{2.16}$$

This form also allows explicitly to identify the solutions to the coupled equation (2.8) by just reading off the real and imaginary parts. For the choice $\theta = \pm \pi/2$ this solution reduces precisely to the one found by Khare and Saxena in [12], up to an overall minus sign due to the difference in (2.8). We notice that while the $\mathcal{PT}$-invariance of $u_{\mu;\beta}(x,t)$ is apparent, the one for the corresponding τ-functions $\tau_{\mu;\beta}(x,t)$ are not immediately obvious, in fact they are not $\mathcal{PT}$-invariant. This is due to the ambiguity in those functions, as for instance $\tau(x,t) \rightarrow u_1(x,t) \exp[c_1 x + c_2 + f(t)]$ with arbitrary constants c_1, c_2 and function

$f(t)$ will give rise to the same solution $u(x,t)$ to the KdV equation. Instead of taking the standard form in (2.14) we may start from $\hat{\tau}_{\mu;\beta}(x,t) = \cosh\left[\left(\beta x - \beta^3 t + \mu\right)/2\right]$ leading also to the same $u_{\mu;\beta}(x,t)$ in (2.14). In this form the $\mathcal{PT}$-invariance is directly evident. In other words the τ-functions do not need to be $\mathcal{PT}$-symmetric in order to generate a $\mathcal{PT}$-symmetric solution for the KdV equation.

Let us next construct a two-soliton solution. As a starting point we take

$$\tau^1 = e^{\eta_1} + e^{\eta_2}, \tag{2.17}$$

which naturally solves (2.10) with nonlinear dispersion relations $k_i^3 + \omega_i = 0$ for $i = 1, 2$. At order λ^2 we determine from (2.12) that

$$\tau^2 = \gamma e^{\eta_1 + \eta_2}. \tag{2.18}$$

with $\gamma = (\alpha - \beta)^2/(\alpha + \beta)^2$. The equation resulting at order λ^3

$$\left(D_x^4 + D_x D_t\right)\left(1 \cdot \tau^3 + \tau^1 \cdot \tau^2 + \tau^2 \cdot \tau^1 + \tau^3 \cdot 1\right) = 0, \tag{2.19}$$

is solved by τ^1 and τ^2 given in (2.17) and (2.18) when setting $\tau^3 = 0$. Once again all higher order equations are also satisfied when setting $\tau^k = 0$ for $k \geq 3$, so that with $\lambda = 1$

$$\tau = 1 + e^{\eta_1} + e^{\eta_2} + \gamma e^{\eta_1 + \eta_2}, \tag{2.20}$$

becomes an exact solution to the Hirota equation (2.9), with μ_i as defined in (2.11) possibly being complex. Translating the τ-function back to the u-variable we obtain thee two-soliton solution

$$u_{\mu,\nu;\alpha,\beta}^H(x,t) = \frac{2\alpha^2 e^{\alpha x - \alpha^3 t + \mu} + 2\beta^2 e^{\beta x - \beta^3 t + \nu} + 4(\alpha - \beta)^2 e^{\alpha x - \alpha^3 t + \mu + \beta x - \beta^3 t + \nu}}{(1 + e^{\alpha x - \alpha^3 t + \mu} + e^{\beta x - \beta^3 t + \nu} + \gamma e^{\alpha x - \alpha^3 t + \mu + \beta x - \beta^3 t + \nu})^2} \tag{2.21}$$

$$+ \frac{2\gamma\left(\alpha^2 e^{\alpha x - \alpha^3 t + \mu + 2(\beta x - \beta^3 t + \nu)} + \beta^2 e^{2(\alpha x - \alpha^3 t + \mu) + \beta x - \beta^3 t + \nu}\right)}{(1 + e^{\alpha x - \alpha^3 t + \mu} + e^{\beta x - \beta^3 t + \nu} + \gamma e^{\alpha x - \alpha^3 t + \mu + \beta x - \beta^3 t + \nu})^2}.$$

Notice that (2.21) is not $\mathcal{PT}$-symmetric, even for the real solution when taking $\mu = \nu = 0$. It is evident that further multi-soliton solutions constructed by means of the Hirota method will also not be $\mathcal{PT}$-symmetric. However, as we will show in section 2.2 that does not mean that all multi-soliton solutions have broken $\mathcal{PT}$-symmetry. Moreover, it will turn out that despite having broken $\mathcal{PT}$-symmetry their corresponding energies are real. In the next subsection we shall demonstrate that $\mathcal{PT}$-symmetric multi-soliton solutions may be constructed from Bäcklund transformations instead.

2.1.2 Complex solutions from Bäcklund transformations

Converting the KdV equation (2.8) into an equation for the quantity w, defined via $u = w_x$, the KdV-Bäcklund transformations are well known to relate two different solutions u, w and u', w' as

$$w_x + w_x' = \kappa - \frac{1}{2}(w - w')^2, \tag{2.22}$$

$$w_t + w_t' = (w - w')(w_{xx} - w_{xx}') - 2[w_x^2 + w_x w_x' + (w_x')^2]. \tag{2.23}$$

A "nonlinear superposition principle" is then obtained by relating four different solutions as $w_0 \xrightarrow{\kappa_1} w_1$, $w_0 \xrightarrow{\kappa_2} w_2$, $w_1 \xrightarrow{\kappa_2} w_{12}$ and $w_2 \xrightarrow{\kappa_1} w_{12}$. Using the corresponding four versions of (2.22) all differentials may be eliminated, such that one can construct a new solution w_{12} to the KdV equation from three known solutions w_0, w_1 and w_2 as

$$w_{12} = w_0 + 2\frac{\kappa_1 - \kappa_2}{w_1 - w_2}. \tag{2.24}$$

With $w_{\mu;\beta}(x,t) = \beta \tanh\left[\frac{1}{2}(\beta x - \beta^3 t + \mu)\right]$, resulting from $u_{\mu;\beta}(x,t)$ in (2.14), we identify $\kappa = \beta^2/2$ from (2.22) when taking $w = w_{0;\beta} = 0$ and $w' = w_{\mu;\beta}$. A new solution to the KdV equation is therefore

$$w_{\mu,\nu;\alpha,\beta} = \frac{\alpha^2 - \beta^2}{w_{\mu;\alpha} - w_{\nu;\beta}}, \tag{2.25}$$

with corresponding wavefunction

$$u^B_{\mu,\nu;\alpha,\beta}(x,t) = \frac{\alpha^2 - \beta^2}{2} \frac{\beta^2 \operatorname{sech}^2\left[\frac{1}{2}(\beta x - \beta^3 t + \nu)\right] - \alpha^2 \operatorname{sech}^2\left[\frac{1}{2}(\alpha x - \alpha^3 t + \mu)\right]}{\left[\alpha \tanh\left[\frac{1}{2}(\alpha x - \alpha^3 t + \mu)\right] - \beta \tanh\left[\frac{1}{2}(\beta x - \beta^3 t + \nu)\right]\right]^2}. \tag{2.26}$$
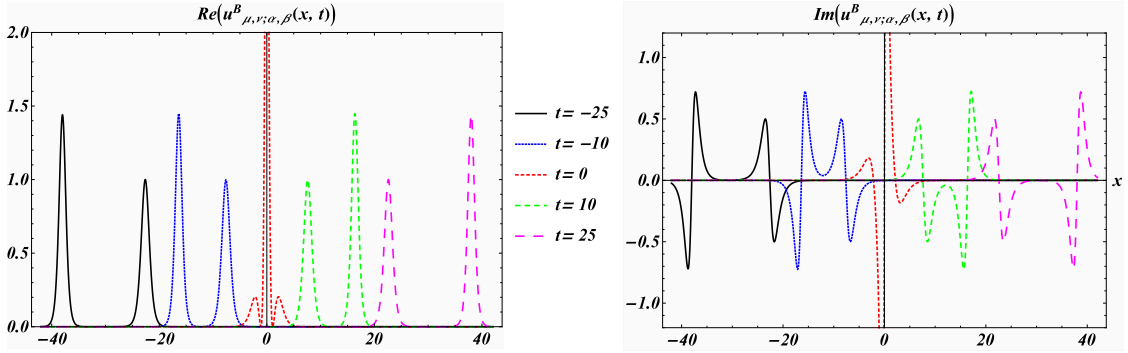

Figure 1: $\mathcal{PT}$-symmetric KdV two-soliton solution with $\alpha = 6/5$, $\beta = 1$ and $\mu = \nu = i\pi/2$.

Notice that, unlike for $u^H_{\mu,\nu;\alpha,\beta}$, for real values of μ and ν the denominator of $u^B_{\mu,\nu;\alpha,\beta}$ vanishes at certain values for x and t. Thus complex values for μ and ν can be used to regularize this expression. We observe further that while the $\mathcal{PT}$-symmetric one-soliton solutions may formally be obtained simply from complex shifts in space or time from one basic solution $u_{0;\beta}(x,t)$, neither the broken $\mathcal{PT}$-symmetric two-soliton u^H nor the $\mathcal{PT}$-symmetric two-soliton u^B is obtainable from a known two-soliton solution in this simple manner when $\mu \neq \nu$. However, we may use real shifts in space or time to restore the $\mathcal{PT}$-symmetry for the broken $\mathcal{PT}$-symmetric one-soliton solution $u_{\mu_r + i\mu_i}$, with $\mu_r, \mu_i \in \mathbb{R}$, as

$$u_{\mu_r + i\mu_i;\beta}\left(x - \frac{\mu_r}{\beta}, t\right) = u_{\mu_r + i\mu_i;\beta}\left(x, t + \frac{\mu_r}{\beta^3}\right) = u_{i\mu_i;\beta}(x,t). \tag{2.27}$$

To achieve this restoration for the broken $\mathcal{PT}$-symmetric two-soliton solution we require a simultaneous shift in space and time

$$u^B_{\mu_r + i\mu_i, \nu_r + i\nu_i;\alpha,\beta}\left(x + \frac{\beta^3 \mu_r - \alpha^3 \nu_r}{\alpha^3 \beta - \alpha\beta^3}, t + \frac{\beta\mu_r - \alpha\nu_r}{\alpha^3 \beta - \alpha\beta^3}\right) = u^B_{i\mu_i, i\nu_i;\alpha,\beta}(x,t). \tag{2.28}$$

In figure 1 we display the two-soliton solution $u^B_{\mu,\nu;\alpha,\beta}$ for a $\mathcal{PT}$-symmetric choice of the parameters $\mu = \nu$. We observe $\mathrm{Re}[u^B(x,t)] = \mathrm{Re}[u^B(-x,-t)]$ and also $\mathrm{Im}[u^B(x,t)] = -\mathrm{Im}[u^B(-x,-t)]$. The real part exhibits the typical features of a two-soliton scattering, that is being separated into two one-soliton solutions in the past and regaining the original shapes with exchanged positions in the future, with a time-delay as the only residual effect. For the complex solutions this behaviour is now accompanied by a smooth scattering structure for the imaginary part. In as similar fashion as in [16, 17, 18] we compute the time-delay for the real and imaginary parts as

$$\lim_{t\to\pm\infty} \mathrm{Re}\left[u^B_{i\pi/2,i\pi/2;\alpha,\beta}(x,t)\right] \sim \mathrm{Re}\left[u_{i\pi/2;\beta}(x,t\mp\Delta_\beta)\right] + \mathrm{Re}\left[u_{i\pi/2;\alpha}(x,t\pm\Delta_\alpha)\right], \ (2.29)$$

$$\lim_{t\to\pm\infty} \mathrm{Im}\left[u^B_{i\pi/2,i\pi/2;\alpha,\beta}(x,t)\right] \sim \mathrm{Im}\left[u_{i\pi/2;\beta}(x,t\mp\Delta_\beta)\right] - \mathrm{Im}\left[u_{i\pi/2;\alpha}(x,t\pm\Delta_\alpha)\right], \ (2.30)$$

where the time shifts are given by

$$\Delta_x = \frac{1}{x^3}\ln\left[\frac{\alpha+\beta}{\alpha-\beta}\right]. \tag{2.31}$$

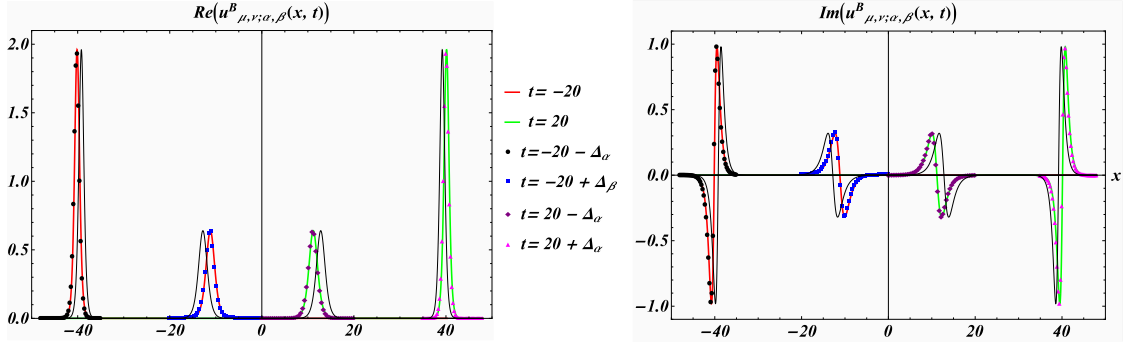We confirm our analytic results by numerical computations displayed in figure 2.

Figure 2: Time delay in the KdV complex two-soliton scattering $u^B_{\mu,\nu;\alpha,\beta}$ for $\mu = \nu = i\pi/2$, $\alpha = 7/5$ and $\beta = 4/5$. The scattered solitons are the one-soliton solutions at the shifted times as indicated in the legend and the thin solid black lines are the unshifted one-soliton solutions at time $t = -20, -20, 20, 20$ from the left to the right.

We observe a perfect match between the two-soliton solutions and the Δ_x-shifted one-soliton solution in the real as well as in the imaginary part. The faster soliton, i.e. the one related to α in our choice of parameters, in the two-soliton solution is shifted to the left in the past and to the right in the future. These shifts are in the opposite direction for the slower soliton related to β. The details of the derivation for (2.29), (2.30), (2.31) together with a some further analysis are presented elsewhere [19].

As seen in figure 3 the qualitative behaviour does not change in the broken regime, with the only difference that two solutions for some specific values t' and $-t'$ are no longer symmetric around $x = 0$, similarly as for the solution u^B. Taking μ different from ν we can modulate the shapes of the different solutions as displayed in figure 4.
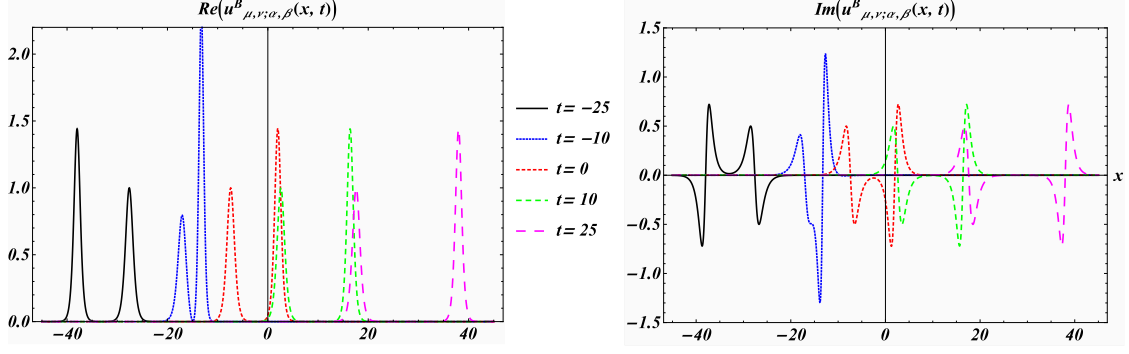
Figure 3: Broken $\mathcal{PT}$-symmetric KdV two-soliton solution with $\alpha = 6/5$, $\beta = 1$, $\mu = 5 + i\pi/2$ and $\nu = i\pi/2$.
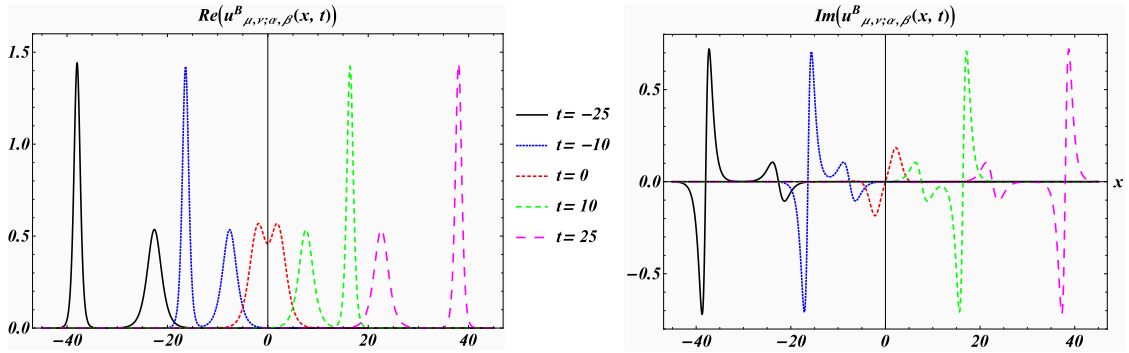

Figure 4: $\mathcal{PT}$-symmetric KdV two-soliton solution from different types of one-solitons with $\alpha = 6/5$, $\beta = 4/5$, $\mu = i\pi/6$ and $\nu = i\pi/2$.

2.1.3 Real energies from $\mathcal{PT}$-symmetric and broken $\mathcal{PT}$-symmetric solutions

Having obtained various types of solutions, we will now compute the corresponding energies resulting from the expression (1.1). The Hamiltonian density leading to the KdV equation in the form (2.8) is given by

$$\mathcal{H}(u, u_x) = -u^3 + \frac{1}{2} u_x^2. \tag{2.32}$$

From the KdV Bäcklund transformation (2.22) with $w' = 0$ and the observation that $(u_{\mu;\beta})_x / u_{\mu;\beta} = -w_{\mu;\beta} = -\beta \tanh\left[\frac{1}{2}(\beta x - \beta^3 t + \mu)\right]$, we derive the identity $(u_{\mu;\beta})_{xx} = (u_{\mu;\beta})_x^2 / u_{\mu;\beta} - u_{\mu;\beta}^2$. These relations allow us to write the Hamiltonian density as

$$\mathcal{H}\left[u_{\mu;\beta}, (u_{\mu;\beta})_x\right] = \left\{ \frac{1}{10} \left[\frac{(u_{\mu;\beta})_x}{u_{\mu;\beta}}\right]^5 + \frac{1}{2} \left[\frac{(u_{\mu;\beta})_x}{u_{\mu;\beta}}\right]^2 (u_{\mu;\beta})_x + u_{\mu;\beta}(u_{\mu;\beta})_x \right\}_x. \tag{2.33}$$

The corresponding energy then simply results to

$$E_{\mu;\beta} = \int_{-\infty}^{\infty} \mathcal{H}[u_{\mu;\beta}, (u_{\mu;\beta})_x]dx = \frac{1}{2}\left[\frac{1}{5}\left[\frac{(u_{\mu;\beta})_x}{u_{\mu;\beta}}\right]^5 + \frac{(u_{\mu;\beta})_x^3}{u_{\mu;\beta}^2} + (u_{\mu;\beta}^2)_x\right]_{-\infty}^{\infty} = -\frac{\beta^5}{5}, \tag{2.34}$$

when using asymptotically vanishing boundary conditions for the wave function and its derivative $\lim_{x \to \pm\infty} u_{\mu;\beta} = \lim_{x \to \pm\infty} (u_{\mu;\beta})_x = 0$ together with $\lim_{x \to \pm\infty} (u_{\mu;\beta})_x / u_{\mu;\beta} = \mp\beta$. Notice that as long as β is real this energy is real at all times t, irrespective of whether $u_{\mu;\beta}$ is $\mathcal{PT}$-symmetric or not. The reason is simple: Taking μ to be of the form $\mu = \mu_r + i\mu_i$, the $\mathcal{PT}$-symmetry of $\mathcal{H}$ is broken when $\mu_r \neq 0$. However, a simple shift in time or space, as explained in (2.27), will restore the $\mathcal{PT}$-symmetry of the integrand. Both type of shifts are permitted, as the shift in x can be absorbed in the limits of the integral and the shift in t is allowed since $\mathcal{H}$ is a conserved quantity in time. As argued before, having a $\mathcal{PT}$-symmetric integrand the complex part does not contribute to the overall value of $E_{\mu;\beta}$.

For the two-soliton solutions $u^{H,B}_{\mu,\nu;\alpha,\beta}(x,t)$, we compute numerically that the total energy is the sum of the individual one-soliton solutions

$$E^{H,B}_{\mu,\nu} = \int_{-\infty}^{\infty} \mathcal{H}\left[u^{H,B}_{\mu,\nu;\alpha,\beta}(x,t), \left(u^{H,B}_{\mu,\nu;\alpha,\beta}(x,t)\right)_x\right] dx = E_{\mu;\beta} + E_{\nu;\alpha} = -\frac{\alpha^5 + \beta^5}{5}. \quad (2.35)$$

Once again we notice that we obtain real energies also for the $\mathcal{PT}$-symmetrically broken scenario. In this case we can restore the $\mathcal{PT}$-symmetry by a simultaneous shift in x and t as explained in (2.28). While this explains the reality of the spectrum, it does not yet account for the concrete values in (2.35). However, as we have seen in (2.29) and (2.30) for one specific case, asymptotically the two-soliton solution separates into two from each other isolated one-soliton solutions, in both the real and imaginary part. These one-soliton solutions contribute separately to the total energy, which is the same value at all times. As the latter argument applies to any N-soliton solution we expect their energies to be the sum of all their N asymptotic individual one-soliton solutions. However this still needs verification [19].

2.2 The complex modified Korteweg-de Vries equation

Using the variable transformation $v = \hat{w}_x$ the mKdV equation can be written in the two equivalent forms

$$v_t + 24v^2 v_x + v_{xxx} = 0 \quad \Leftrightarrow \quad \hat{w}_t + 8\hat{w}_x^3 + \hat{w}_{xxx} = 0. \quad (2.36)$$

Unlike the KdV equation, the mKdV equation allows for two alternative types of $\mathcal{PT}$-symmetries $\mathcal{PT}_\pm$: $x \to -x$, $t \to -t$, $i \to -i$, $v \to \pm v$. With the further substitution $\hat{w} = \arctan(\tau/\sigma)$ the latter equation in (2.36) can be converted into Hirota's bilinear form [20]

$$\left(D_t + D_x^3\right)\tau \cdot \sigma = 0, \qquad \text{and} \qquad D_x^2\left(\tau \cdot \tau + \sigma \cdot \sigma\right) = 0, \quad (2.37)$$

when using the relations (2.3)-(2.7). Taking now $\sigma = 1$ the equations (2.37) reduce to

$$\tau_t + \tau_{xxx} = 0, \qquad \text{and} \qquad \tau\tau_{xx} - \tau_x^2 = 0. \quad (2.38)$$

The exact solutions to these equations with corresponding solution to the mKdV equation (2.36) are

$$\tau_{\mu;\beta}(x,t) = e^{\beta x - \beta^3 t + \mu}, \quad \text{and} \quad v_{\mu;\beta}(x,t) = \frac{\beta}{2}\operatorname{sech}\left[\beta x - \beta^3 t + \mu\right]. \quad (2.39)$$

It is well known that the mKdV and the KdV equation are related by a Miura transformation. Here we find that the solutions (2.14) and (2.39) to the KdV equations and mKdV equation (2.8) and (2.36), respectively, are related as

$$u_{\mu\pm i\frac{\pi}{2};\beta}(x,t) = 4v_{\mu;\beta}^2(x,t) \pm i2\partial_x v_{\mu;\beta}(x,t). \tag{2.40}$$

This means for instance that the real solution $v_{0;\beta}(x,t)$ to the mKdV equation leads inevitably to the complex $\mathcal{PT}$-symmetric solutions $u_{\pm i\frac{\pi}{2};\beta}(x,t)$ for the KdV equation. Thus we have obtained yet another way to derive the solutions reported in [12]. The complex part simply results from scaling the more familiar transformation $u = v^2 + v_x$, that relates the mKdV with nonlinear term $-6v^2 v_x$ to the KdV equation with nonlinear term $+6uu_x$, to the present forms (2.8) and (2.36).

The latter argument may also be applied to solutions in terms of Jacobi elliptic functions[1]. Starting with the shifted known solution to the mKdV equation

$$\hat{v}_{\mu;\beta}(x,t) = \frac{\beta}{2}\,\mathrm{dn}\left[\beta x - \beta^3 t(2-m) + \mu, m\right], \tag{2.41}$$

we obtain from (2.40) the corresponding solution to the KdV equation

$$\hat{u}_{\mu;\beta}^{\pm}(x,t) = \beta^2\,\mathrm{dn}^2\left[\hat{z}, m\right] \pm im\beta^2\,\mathrm{cn}\left[\hat{z}, m\right]\mathrm{sn}\left[\hat{z}, m\right], \tag{2.42}$$

where we abbreviated the argument $\hat{z} := \beta x - \beta^3 t(2-m) + \mu$. The elliptic parameter is denoted by m as usual. Likewise from the shifted known solution to the mKdV equation

$$\tilde{v}_{\mu;\beta}(x,t) = \frac{\beta}{2}\sqrt{m}\,\mathrm{cn}\left[\beta x - \beta^3 t(2m-1) + \mu, m\right] \tag{2.43}$$

we construct

$$\tilde{u}_{\mu;\beta}^{\pm}(x,t) = m\beta^2\,\mathrm{cn}^2\left[\tilde{z}, m\right] \pm i\sqrt{m}\beta^2\,\mathrm{dn}\left[\tilde{z}, m\right]\mathrm{sn}\left[\tilde{z}, m\right], \tag{2.44}$$

with $\tilde{z} := \beta x - \beta^3 t(2m-1) + \mu$. Thus the solutions $\hat{v}_{0;\beta}(x,t)$ and $\tilde{v}_{0;\beta}(x,t)$ to the mKdV equation, which could be real for specific values, lead to the complex $\mathcal{PT}$-symmetric solution for the KdV equation reported in [12], see for instance also [8, 22] for construction procedures of such type of sultions. It is clear that this is only one possibility as other choices for purely imaginary μ also respect the $\mathcal{PT}$-symmetry.

2.2.1 Real energies from $\mathcal{PT}$-symmetric and broken $\mathcal{PT}$-symmetric solutions

Next we compute the energy resulting from the mKdV Hamiltonian density leading to the equation of motion (2.36) after variation

$$\mathcal{H}(v, v_x) = -2v^4 + \frac{1}{2}v_x^2. \tag{2.45}$$

For the solution $v_{\mu;\beta}$ in (2.39) we compute the energy

$$E_{\mu;\beta} = \int_{-\infty}^{\infty}\mathcal{H}\left[v_{\mu;\beta}(x,t), (v_{\mu;\beta}(x,t))_x\right]dx = -\frac{\beta^3}{12}, \tag{2.46}$$

---

[1]We use here the conventions of [21].

which has the same properties as the energy of the KdV one-soliton, that is being real for all values of μ. The elliptic solutions have the two periods $4K(m)/\beta$ and $i4K(1-m)/\beta$ in x with $K(m)$ denoting the elliptic integral of the first kind. Thus we have to restrict the domain of integration in (1.1) in order to obtain finite energies. For the solution $\hat{v}_{\mu;\beta}$ in (2.39) we compute the real energies

$$\hat{E}_{\mu;\beta} = \int_{-2K(m)/\beta}^{2K(m)/\beta} \mathcal{H}\left[\hat{v}_{\mu;\beta}(x,t),(\hat{v}_{\mu;\beta}(x,t))_x\right]dx \tag{2.47}$$

$$= \frac{\beta^3}{24}\left[(m-2)E\left[\mathrm{am}\left(4K(m)|m\right),m\right] + 4K(m)(m-1)\right],$$

where $\mathrm{am}\,(u|m)$ denotes the amplitude of the Jacobi elliptic function and $E\left[\phi,m\right]$ the elliptic integral of the second kind. Similarly for the solution $\tilde{v}_{\mu;\beta}$ in (2.43) we find

$$\tilde{E}_{\mu;\beta} = \int_{-2K(m)/\beta}^{2K(m)/\beta} \mathcal{H}\left[\tilde{v}_{\mu;\beta}(x,t),(\tilde{v}_{\mu;\beta}(x,t))_x\right]dx \tag{2.48}$$

$$= \frac{\beta^3}{24}\left[(1-2m)E\left[\mathrm{am}\left(2K(m)|m\right),m\right] - 4K(m)(3m^2-4m+1)\right].$$

We observe that $\lim_{m\to 1}\hat{E}_{\mu;\beta} = \lim_{m\to 1}\tilde{E}_{\mu;\beta} = 2E_{\mu;\beta}$. For the same reason as for the hyperbolic solutions all energies are real, irrespective of whether the Hamiltonian densities are $\mathcal{PT}$-symmetric or not.

2.3 The complex sine-Gordon equation

The quantum field theory version of the complex sine-Gordon model has been studied for some time [23, 24, 25, 26, 27, 28]. Here we demonstrate that its classical version also admits interesting $\mathcal{PT}$-symmetric solutions with similar properties to those constructed in the previous subsections. We consider the equation in the form

$$\phi_{xt} = \sin\phi, \tag{2.49}$$

using light-cone variables, which we still call x and t with a slight abuse of notation. We observe that this equation admits various symmetries for $\mathcal{PT}_{\pm}^{(n)}$: $x \to -x$, $t \to -t$, $i \to -i$, $\phi \to \pm\phi + n2\pi$ with $n \in \mathbb{Z}$, with $\mathcal{PT}_{-}^{(n)}$ and $\mathcal{PT}_{+}^{(0)}$ squaring to 1 as expected for a proper $\mathcal{PT}$-symmetry. In [29] Hirota showed that the sine-Gordon equation (2.49) can be converted into the bilinear form

$$D_xD_t\tau \cdot \sigma = \tau \cdot \sigma, \qquad \text{and} \qquad D_xD_t\tau \cdot \tau = D_xD_t\sigma \cdot \sigma, \tag{2.50}$$

when using the relations (2.3)-(2.7) and the transformation $\phi = 4\arctan(\tau/\sigma)$. Taking $\sigma = 1$ these equations reduce to

$$\tau_{xt} = \tau \qquad \text{and} \qquad \tau\tau_{xt} = \tau_x\tau_t. \tag{2.51}$$

The exact solutions to these equations and therefore the corresponding solutions to the sine-Gordon equation (2.49) are easily found. For instance, we obtain the well-known kink solution as

$$\tau_{\mu;\beta}(x,t) = e^{\beta x+t/\beta+\mu}, \quad \text{and} \quad \phi_{\mu;\beta}(x,t) = 4\arctan\left[e^{\beta x+t/\beta+\mu}\right]. \tag{2.52}$$

Recalling that $\arctan z = -\arctan z^{-1} \pm \pi/2$ for $\text{Re}\, z \gtrless 0$, we note that the solution for $\mu = i\theta$ with $\theta \in \mathbb{R}$ is $\mathcal{PT}_{-}^{(\pm)}$-symmetric. Let us separate off the real and imaginary parts of the solution for these values of μ by using the well-known relation $\arctan z = -i/2 \ln\left[(i - z)/(i + z)\right]$. For the principle value of the logarithm we obtain

$$\phi_{i\theta;\beta}(x, t) = 2 \arg\left[\frac{-\sinh\varphi + i\cos\theta}{\cosh\varphi + \sin\theta}\right] - i \ln\left[\frac{\sinh^2\varphi + \cos^2\theta}{(\cosh\varphi + \sin\theta)^2}\right], \tag{2.53}$$

where we abbreviated $\varphi = \beta x + t/\beta$. Using the relation between the argument function and the arctan function equation (2.53) can be converted into the more practical form

$$\phi_{i\theta;\beta}(x, t) = \begin{cases} 4 \arctan\left[\frac{\sqrt{\sinh^2\varphi + \cos^2\theta} + \sinh\varphi}{\cos\theta}\right] - i \ln\left[\frac{\sinh^2\varphi + \cos^2\theta}{(\cosh\varphi + \sin\theta)^2}\right] & \text{for } \theta \neq \pm\frac{\pi}{2} \\ -i \ln\left[\frac{\sinh^2\varphi}{(\cosh\varphi \pm 1)^2}\right] & \text{for } \theta = \pm\frac{\pi}{2} \end{cases}. \tag{2.54}$$

The real part is $\mathcal{PT}_{-}^{(\pm)}$-symmetric and the imaginary part respects a $\mathcal{PT}_{-}^{(0)}$-symmetry, such that overall $\phi_{i\theta;\beta}$ is $\mathcal{PT}_{-}^{(\pm)}$-symmetric. As depicted in figure 5 for $\theta \neq \pm\pi/2$ the real part of the solution constitutes a kink solution accompanied by a one-soliton solution in the imaginary part. For $\theta = \pm\pi/2$ the real part of the solution vanishes and the imaginary part becomes a cusp type solution as can be found for instance in [30].
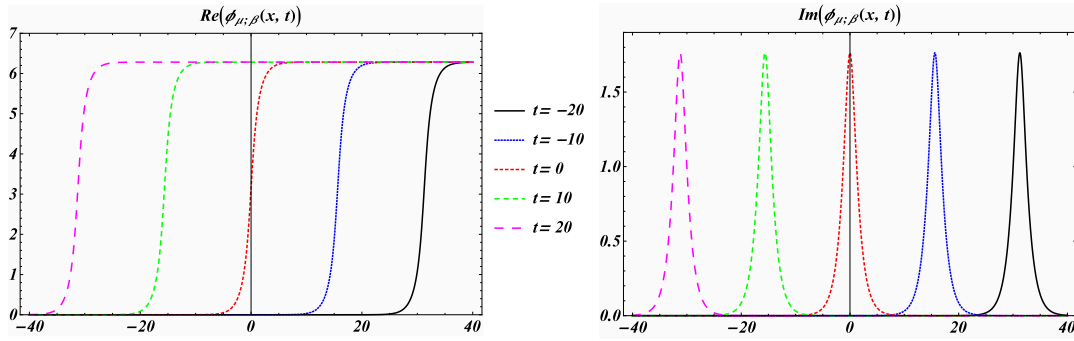

Figure 5: Complex sine-Gordon solution with kink as real part and soliton as imaginary part at different values of time for $\alpha = 4/5$ and $\mu = i\pi/4$.

Let us now construct a two-soliton solution from the sine-Gordon complex solitons using the Bäcklund transformation which associates two different types of solutions ϕ and ϕ' via the two solutions

$$\frac{\phi_x + \phi'_x}{2} = \frac{1}{\kappa} \sin\left[\frac{\phi_x - \phi'_x}{2}\right], \qquad \text{and} \qquad \frac{\phi_t - \phi'_t}{2} = \kappa \sin\left[\frac{\phi_x + \phi'_x}{2}\right]. \tag{2.55}$$

In this case the "nonlinear superposition principle" relates four solutions ϕ_0, ϕ_1, ϕ_2, ϕ_3 as

$$\tan\left[\frac{\phi_3 - \phi_0}{4}\right] = \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \tan\left[\frac{\phi_1 - \phi_2}{4}\right]. \tag{2.56}$$

Taking now $\phi' = 0$, $\phi = \phi_{\mu;\alpha}$ we identify the constant in (2.55) as $\kappa = 1/\alpha$. Then taking $\phi_1 = \phi_{\mu;\alpha}$, $\phi_2 = \phi_{\nu;\beta}$ and $\phi_3 = \phi_{\mu,\nu;\alpha,\beta}$ equation (2.56) leads to the new complex two-

solution solution

$$\phi_{\mu,\nu;\alpha,\beta} = 4\arctan\left[\frac{\beta+\alpha}{\beta-\alpha}\tan\left(\frac{\phi_{\mu;\alpha}-\phi_{\nu;\beta}}{4}\right)\right]. \quad (2.57)$$

This solution exhibits the same kind of symmetry properties as the one-soliton solutions as we observe in figure 6. When $\mu \neq i\pi/2$, $\nu \neq i\pi/2$ the real part consists of a kink-kink scattering and the imaginary part of a two soliton scattering.
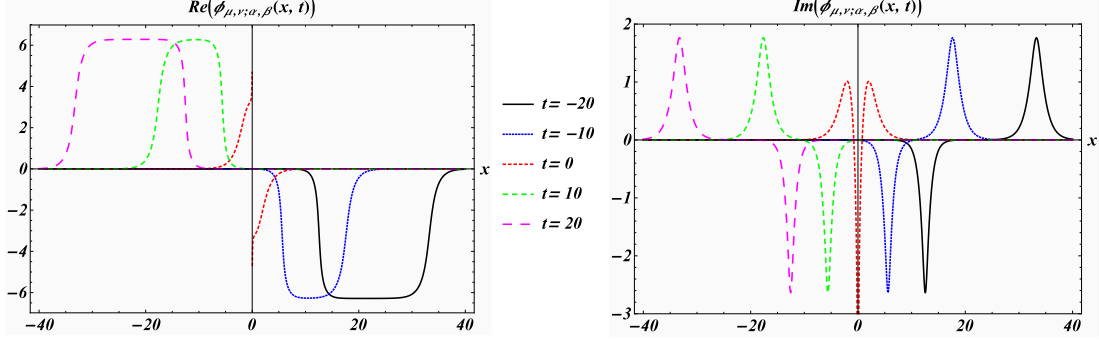

Figure 6: Complex sine-Gordon solution with kink-kink scattering in the real part and soliton-soliton scattering in the imaginary part at different values of time for $\alpha = 6/5$, $\beta = 4/5$, $\mu = i\pi/3$ and $\nu = i\pi/4$.

As computed in (2.54), when $\mu = i\pi/2$ the kink solution in the real part vanishes and the soliton solution in the imaginary part degenerates into a cusp. Choosing $\mu = \nu = i\pi/2$ we observe a two cusps scattering in the imaginary part. These features are depicted in figure 7.
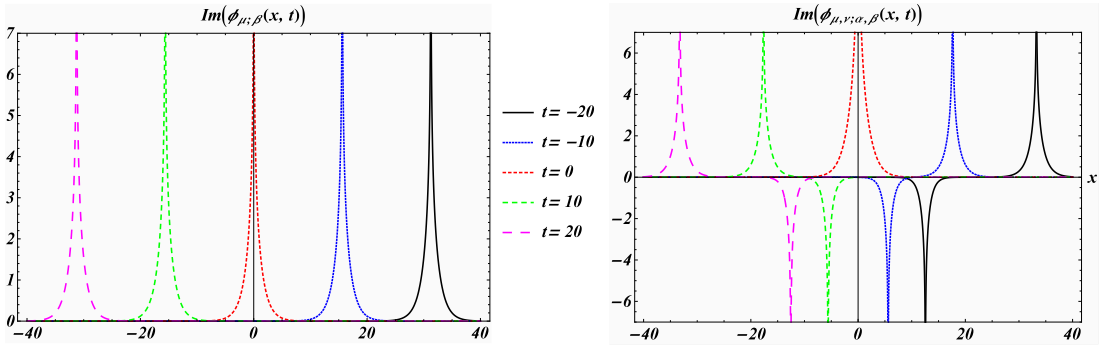

Figure 7: Complex sine-Gordon travelling one-cusp solution with $\beta = 4/5$, $\mu = i\pi/2$ and two-cusp scattering solution for $\alpha = 6/5$, $\beta = 4/5$ and $\mu = i = \nu = i\pi/3$ at different values of time.

2.3.1 Real energies from $\mathcal{PT}$-symmetric and broken $\mathcal{PT}$-symmetric solutions

The Hamiltonian density for the sine-Gordon equation is well-known, see e.g. [31]. When converted to light-cone variables it reads

$$\mathcal{H}(\phi,\phi_x,\phi_t) = \frac{1}{4}\left(\phi_x^2 + \phi_t^2\right) + 1 - \cos(\phi). \quad (2.58)$$

From this expression we compute real energies for all times t and any values μ to

$$E^{SG}_{\mu;\beta} = \int\limits_{-\infty}^{\infty} \mathcal{H}\left[\phi_{\mu;\beta}, (\phi_{\mu;\beta})_x, (\phi_{\mu;\beta})_t\right] dx = \frac{(1+\beta^2)^2}{\beta^2} \int\limits_{-\infty}^{\infty} \text{sech}^2(\beta x + t/\beta + \mu) dx = \frac{2(1+\beta^2)^2}{\beta^3}.$$

(2.59)

Once again the imaginary parts of $\mathcal{H}$ do not contribute as the are already or, by suitable shifts, can be made $\mathcal{PT}$-symmetric. Numerically we also confirm that the energy of the two-soliton solution is the sum of the individual one-soliton solutions

$$E^{SG}_{\mu,\nu;\alpha,\beta} = \int\limits_{-\infty}^{\infty} \mathcal{H}\left[\phi_{\mu,\nu;\alpha,\beta}, (\phi_{\mu,\nu;\alpha,\beta})_x, (\phi_{\mu,\nu;\alpha,\beta})_t\right] dx = E^{SG}_{\mu;\alpha} + E^{SG}_{\mu;\beta},$$

(2.60)

at all times t and any values of μ and ν.

3. Conclusions

Using various techniques, such as Hirota's direct method, Bäcklund and Miura transformations, we have constructed complex one and two-soliton solutions to the complex KdV, mKdV and sine-Gordon equations. Some of the solutions turned out to be $\mathcal{PT}$-symmetric, whereas others have broken $\mathcal{PT}$-symmetry, as for instance the two-soliton solution obtained from Hirota's method. Nonetheless, despite the fact that the corresponding Hamiltonian densities are non-Hermitian, all solutions were found to lead to real energies. While this was to be expected [2] for the $\mathcal{PT}$-symmetric solution, it is less obvious why this should be the case for the broken scenario. However, as we have shown any of our one-soliton solution may be converted into a $\mathcal{PT}$-symmetric one-soliton solution by suitable shifts in time or space and any of our two-soliton solution may be converted into a $\mathcal{PT}$-symmetric two-soliton solution by suitable simultaneous shifts in time and space. Since the value of the energy is insensitive to any of these shifts it must therefore be real. Moreover, when considering the asymptotic behaviour of N-soliton solutions we conjecture that one might be able to use of the fact that they separate into N different one-soliton solutions with possible shifts in time, with each of them contributing a real value to the overall energy. As we have seen in section 2.1.2 this is certainly correct for the KdV two-soliton solution when $\mu = \nu = i\pi/2$, but in order to establish this in more generality we need to investigate in more detail the effect of the time-delay the for different values of μ and ν and especially the cases $N > 2$ [19].

The above mechanism explains well why certain complex soliton solutions posses real energies. Here we have not allowed complex dispersion relations, i.e. keeping our parameters α, β real, or permitted complex parameters occurring directly in the nonlinear wave equations. In fact, also for those scenarios it was found [8] that broken $\mathcal{PT}$-symmetric solutions with non-Hermitian Hamiltonian densities may lead to real energies, although in a much more constrained setting. The mechanism responsible for the reality of the energy in those cases is still unclear, but we believe that the studies presented here will also shed light onto those situations.

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