Verifying Inevitability of Oscillation in Ring Oscillators using the Deductive SOS-QE Approach

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Abstract

We present a deductive numeric-symbolic approach, using Sum of Squares programming (SOS) and Quantifier Elimination (QE), and verify that a Ring Oscillator (RO) starts oscillating from almost all random initial voltages on its nodes.

1 Introduction

ROs are an integral part of today's System-On-Chip designs. They are used for many purposes including, reference clock generation, phase interpolation, frequency translation etc. Ideally, these oscillators should start oscillating from all possible initial node voltages. Unfortunately, such ideal oscillators are impossible to design and there are always states (voltages on nodes) from where they fail to oscillate [1]. Oscillations in an RO can be pictorially shown by functions varying periodically over time, somewhat similar to a "sin" function. An another useful representation of oscillation is in the state-space where oscillatory behaviour corresponds to a periodic set of states. These two types of representations are depicted in Fig. 1. By varying design parameters, such as transistors widths and lengths, the shape and/or location of this periodic path is greatly varied in the state space as shown in Fig. 2. More importantly, this impacts the frequency/phase response of an RO. For an RO to have the desired frequency with little or no phase distortion, the trajectories must converge to the desired periodic region in the state space. A periodic set of states is said to be almost globally inevitable (AGI), if an RO eventually reaches this set, from all but a negligible dead set of voltages on its nodes. This is an important property, and in [2], researchers at Rambus, identified the failure of an even stage RO to have the global inevitability property for a subset of initial conditions and parameters. Proving that an RO starts from almost all arbitrary initial states (voltages on it nodes) is beyond the existing SPICE based simulation capabilities. This is because it requires infinite number of simulations to be carried out for establishing global inevitability of states.

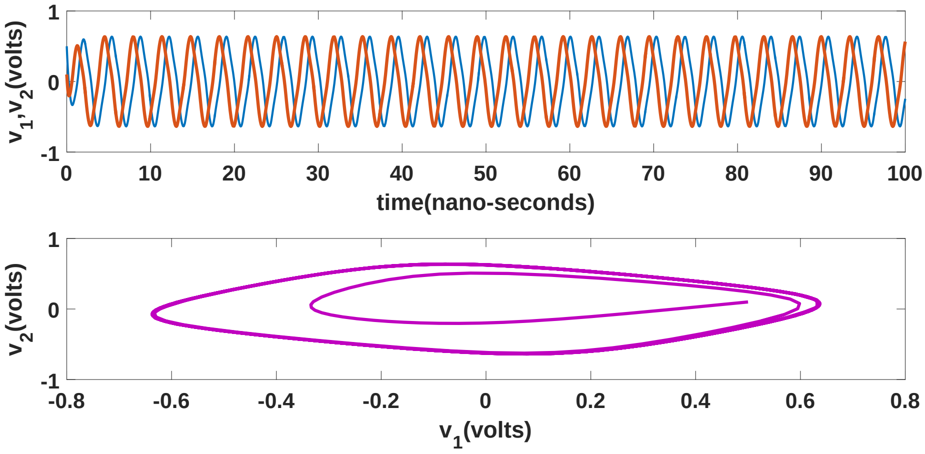

Figure 1: *Different Representation of RO Oscillations*

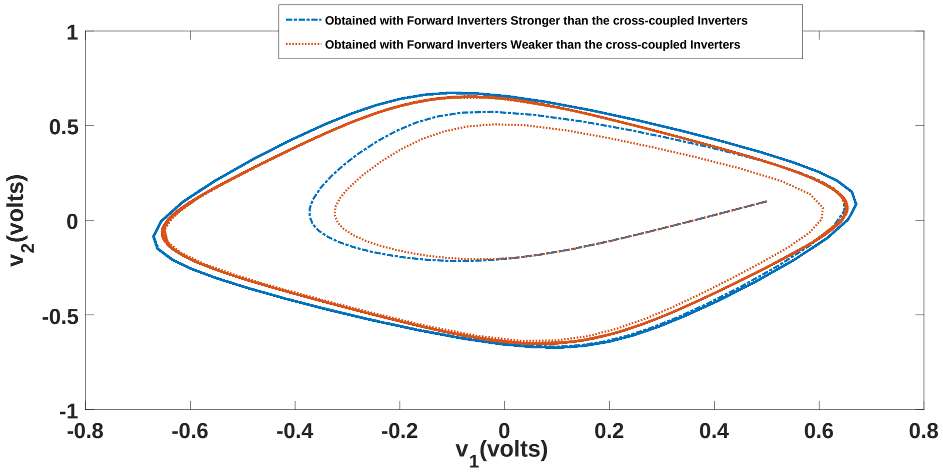

Figure 2: *Parameters variation effect on the location of the Periodic Trajectory*

Recently, there have been several efforts of using formal reachability analysis for the verification of the global inevitability property. Reachability tools model an RO as a continuous system, described by ordinary differential equations (ODEs), and use set-theoretic simulations to see whether a target set is reachable from an initial set. The inevitability property is verified by using reachability analysis iteratively. Reachability suffers from being of bounded-time nature, and since it relies on the over-approximate solutions of ODEs, is thus subjected to erroneous results. A survey of several such methods for can be found in [3]. In [1], the author showed convergence to the oscillation in an even stage RO with probability one. The author showed zero measure probability of the failure set using a cone argument. He further showed convergence to the desired limit cycle using reachability analysis. While the approach is comprehensive, it uses an expensive paper-pencil argument about the zero measure probability of the failure set. Secondly, they used the approximate but sound reachability computations, which is of bounded time nature and computationally very expensive.

In this paper, we present a deductive approach to verify the AGI of oscillations in an RO. Our work is inspired by the Lyapunov theory of stability for dynamical systems [4] and uses a certificate

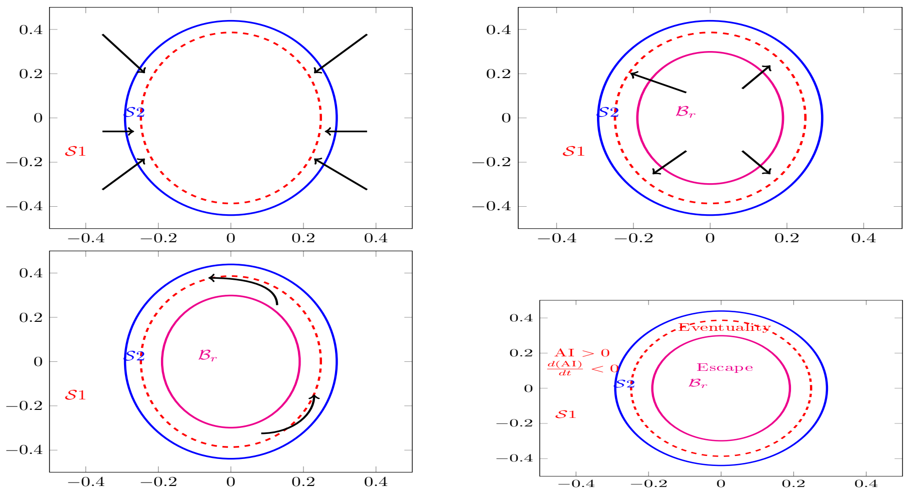

Figure 3: *Verification Strategy: Dividing the convergence of trajectories to the dashed Periodic set into several Sub-tasks*

based deductive approach to verify the inevitability of oscillations in ROs. We define the verification task as a conjunction of several sub-properties whose verification is delegated to the existence of several Lyapunov-like certificates. Construction of these certificates can be posed as first-order-formulas (FOFs) with quantifiers (universal, existential). We use a sound numeric-symbolic approach, called SOS-QE, for the verification of these FOFs. This is basically using a numeric, yet computationally efficient, SOS programming technique for the certificates construction, followed by the symbolic validation of these certificates by the QE technique. A similar technique has been used for non-linear gain analysis in [5]. In [6], the author used SOS in HOL theorem prover to verify positivity of the universally quantified polynomials. Deductive and deductive-bounded approaches have been used for the inevitability verification of a charge pump phase lock loop in [7]. To the best of our knowledge, this is the first deductive approach for the solution of the research problem posed in [2]. Being deductive, our approach does not solve the ODEs and thus avoids the conservative approximation their solutions. Furthermore, once the inevitability property is verified, it stands verified for the infinite time, unlike the bounded reachability analysis.

This paper is organized as follows: In Sec.II, we introduce the preliminaries of this paper. Sec.III illustrates verification of the inevitability of oscillation in RO. Experimental results are shown in Sec.IV. Sec.V concludes the paper.

2 Preliminaries

2.1 Verification Strategy

We use a divide-and-rule strategy and divide the verification task into several sub-tasks. To show that all trajectories converge to with an arbitrarily small distance of the periodic trajectory, we do this in three phases as shown in Fig. 3. In the first phase (top left) we show that trajectories from the set $\mathcal{S}1$

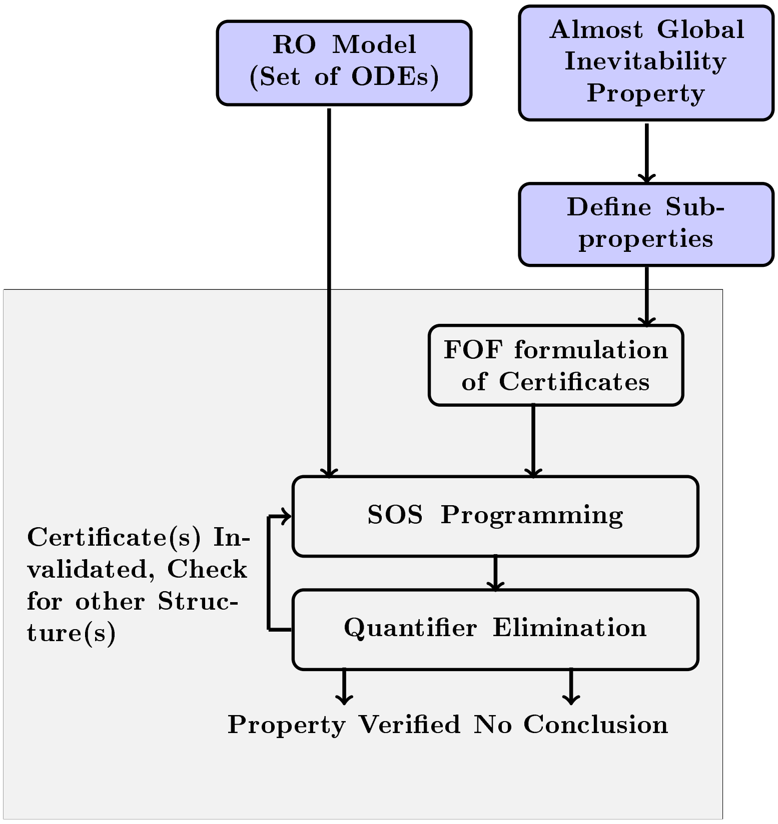

Figure 4: *AGI Property Verification Methodology*

eventually reaches $\mathcal{S}2$ and stays there forever. Note that the set $\mathcal{S}2$ is the area enclosed by the blue circular closed path whereas $\mathcal{S}1$ is the one outside it. In the second phase (top right), we show that almost all trajectories in set $\mathcal{B}_r$, defined by the area enclosed by the magenta circular line, eventually reaches to an annular region defined by the set $\mathcal{S}2 \setminus \mathcal{B}_r$. In the second stage, we also show that none of the trajectories trap in the dead-set (from where RO fails to start). Finally, we show that all trajectories in the annular region (bottom left) converge to within an arbitrarily small distance of the desired periodic trajectory, shown by the dashed red circular path in Fig. 3. For each of these three sub-tasks, we define three properties and state the AGI property as the conjunction of these three properties. Each of these sub-properties specifies the long term behavior of the trajectories of ROs in a specific sub-set of the state space which is verified by the existence of a certificate. These certificates, and their time derivatives, if exist, exhibit the characteristic of being positive (semi-positive) or negative (semi-negative) in their respective sub-sets. This scenario is depicted in the bottom right sub-figure of Fig. 3. As shown, we divide the space into three sub-sets; $\mathcal{S}1$, $\mathcal{S}2$, and $\mathcal{B}_r$. The dotted circle in the red shows the hypothetical location of the periodic trajectory (Limit Cycle) in the state space. The trajectories of an RO exhibit different long term characteristics in these three different sub-sets. We use three different certificates called, the Attractive-invariance (AI), Escape, and Eventuality to verify different sub-properties. For illustration purpose, here we have depicted the positivity/negativity of the AI certificate in the set $\mathcal{S}1$. The existence of these certificates is formulated as verification of FOFs with universal-existential quantifiers

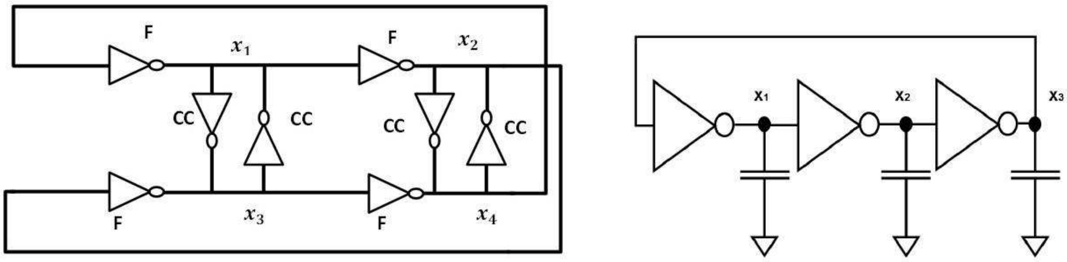

Figure 5: *Two Different Topologies of Ring Oscillators. Left: Even Stage RO, Right: Odd Stage RO*

over real polynomials. Verification of these FOFs is carried out using a numeric-symbolic technique of SOS programming and QE. The overall verification flow is shown in the Fig. 4. The existential quantification is solved by numerically finding different feasible certificates using SOS programming. To further validate these certificates, for their numerical imprecisions, symbolic analysis (QE) is carried out for each of the universally quantified formula. If a certificate is invalidated by the QE stage, a new search is made for a certificate(s) with a different structure this time. The output of our methodology results in either the AGI property being verified, or with no conclusion about its truthfulness, if a user-defined number of iterations have been exhausted.

2.2 Modelling of the Ring Oscillator

We model the RO shown in Fig. 5 as a polynomial continuous dynamical system. Let us denote by x, the vector of node voltages at the outputs of inverters. The continuous dynamical system model of an RO is a tuple $(\mathbf{X}, \mathcal{X}_{initial}, \mathbf{U}, f)$ where $\mathbf{X}$ is a set of state variables interpreted over $\mathbb{R}$, $\mathcal{X} = \mathbb{R}^{\mathbf{X}}$ is the set of all possible valuations of the variables, $\mathcal{X}_{initial} \subset \mathcal{X}$ is the set of initial conditions, $\mathbf{U}$ is the set of parameters (to model circuit capacitance, resistance, transistor parameters) interpreted over $\mathbb{R}$ with $U = \mathbb{R}^{\mathbf{U}}$ is the set of all possible parameter valuations, and

$$f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X} \tag{1}$$

is the vector field characterizing the system. We assume that the vector field f is a polynomial function of $x \in \mathcal{X}$ called a polynomial vector field. Let us denote by $\Phi(x_0, t)$ the solution of equation $\frac{d\Phi(X(t))}{dt} = f(\mathcal{X},\ \mathcal{U}),\ X(0) = x_0 \in \mathcal{X}_{init}$.

Definition 1 (Equilibrium state)**.**

A state $x_e \in \mathcal{X}$ is called an equilibrium of the RO, iff $f(x_e) = 0$.

Definition 2 (Attractive Invariance of a set)**.**

A set $\mathcal{X}_I$ is invariant iff $\forall x_0 : x_0 \in \mathcal{X}_I,\ \Phi(x_0, t) \in \mathcal{X}_I$ for all t. It is called attractive invariant (AI) iff $\forall x_0 : x_0 \in \mathcal{X} \setminus \mathcal{X}_I,\ \lim_{t \to b}\ \Phi(x_0, t) \in \mathcal{X}_I,\ b \in \mathbb{R}_{\geq 0}$.

Definition 3 (Limit Cycle)**.**

A set $\gamma \subset \mathcal{X}$ is called a Limit cycle, iff $\forall x_0 : x_0 \in \gamma$, $\Phi(x_0, T) = x_0$ for $T > 0$, and for all $0 < t < T$, $\Phi(x_0, t) \neq x_0$. This is an invariant set.

Definition 4 (Inevitability of the Limit cycle)**.**

The Limit cycle γ is said to be inevitable, iff $\forall x_0 : x_0 \in \mathcal{X}_{initial},\ y \in \gamma,\ r > 0,\ b \in \mathbb{R}_{\geq 0}$,

$$\lim_{t \to b} \|\Phi(x_0, t) - y\| \leq r \tag{2}$$

Assumption 1.

In this work, we assume that the location of γ in the state space is known.

For a practically feasible RO, there are states in $\mathbb{R}^n$ from where it fails to start and reaches the limit cycle γ [1]. For example, Equilibrium is one such state from where an RO can not start. We call the set of all such states the “Dead set".

Definition 5 (Dead Set)**.**

A set of states is called a dead set denoted by $\mathcal{X}_{dead}$, such that $\forall x_0 : x_0 \in \mathcal{X}_{dead},\ \lim_{t \to \infty} \|\Phi(x_0, t) - x_e\| = 0$. Here x_e is an equilibrium state.

Definition 6 (AGI of Oscillation in RO)**.**

The Limit cycle $\gamma \subset \mathcal{X}$, is said to be “Almost Globally Inevitable", iff $\forall x_0 : x_0 \in \{\mathcal{X} \setminus \mathcal{X}_{dead}\},\ y \in \gamma,\ r > 0,\ b \in \mathbb{R}_{\geq 0}$,

$$\lim_{t \to b} \|\Phi(x_0, t) - y\| \leq r \tag{3}$$

In this paper, we consider two different topologies of ROs, i.e., odd stage and even stage RO as depicted in Fig. 5. While we treat each individual node voltage as a state variable for the Odd stage RO, we use the strategy suggested in [1] for the even stage RO, and divide its operation into differential and common modes. We denote the node voltages of the even stage RO by $x(0, j)$ and $x(1, j)$ for $j = 0, 1...n-1$. Here n is the number of stages. For the even stage RO in Fig. 5, $x(0, 0) = \mathcal{X}_1,\ x(0, 1) = \mathcal{X}_2,\ x(1, 0) = \mathcal{X}_3,\ x(1, 1) = \mathcal{X}_4$. The voltages $x(0, j)$ and $x(1, j)$ form the differential pair whose differential component is $x(0, j) - x(1, j)$, and the common mode component is $x(0, j) + x(1, j)$. The even stage RO while operating normally, has its oscillation manifested in the differential mode, whereas the common mode settles to the constant zero value. If we assume that inverters are identical then, $\forall j : j \in [0, n-1],\ \forall x : x \in \mathcal{X}$ such that $x(0, j) = x(1, j),$ we have $\Phi(x, t) = x_e$ as $t \to \infty$. This means that the set $\{x(0, j) = x(1, j), \forall j : j \in [0, n-1]\} \in \mathcal{X}_{dead}$. Similarly, for odd stage RO, if $x_1 = x_2 = x_3$, then, $lim_{t \to \infty} \Phi(x, t) = x_e$.

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2.3 RO properties verification using Lyapunov-like Certificates

To verify the AGI of the limit cycle γ, we use several Lyapunov-like certificates in different subsets of the state space of the RO, Fig. 3. To show attractive invariance of a set, a Lyapunov-like certificate has been presented in [8].

Lemma 1.

If there exist a polynomial with real coefficients $V : \mathcal{X} \to \mathbb{R}$, $\epsilon > 0$ and a minimum $\eta > 0$ such that,

1. $V(x) > 0, \ \forall x \in \mathbb{R}^n \setminus 0,$
2. $\{V(x) \leq 1\} \subseteq \{q(x) \leq \eta\},$
3. $\{V(x) \geq 1\} \subseteq \{\frac{\partial V}{\partial x}(x).f(x,u) \leq -\epsilon\},$

then the set $\mathcal{S}2 := \{V(x) \leq 1\}$ is an AI set for an RO with a vector field given in Eq. 1, and it is contained in the set $\{q(x) \leq \eta\}$ where $q : \mathcal{X} \to \mathbb{R}$.

Proof. . See [8]. □

In the above Lemma 1, the set $\{q(x) \leq \eta\}$ is used for optimization purposes and the parameter η is minimized so that this set contains the desired AI set $\mathcal{S}2 := \{V(x) \leq 1\}$. Inside the AI set $\mathcal{S}2$, trajectories can reach either, the dead set $\mathcal{X}_{dead}$, or to within a small distance of the limit cycle γ (shown in dotted red in Fig. 3). Let us define a set, $\mathcal{B}_r = V(x) \leq r, \ 0 < r < 1$ (shown in magenta in Fig. 3). To show that trajectories starting in the set $\mathcal{B}_r$ are not trapped in the dead set $\mathcal{X}_{dead}$, and eventually escape to the set $\mathcal{S}2 \setminus \mathcal{B}_r$, we introduce an Escape certificate.

Lemma 2.

For a compact set $\mathcal{B}_r \subset \mathcal{S}2$, if there is a differentiable Escape certificate, $\mathcal{E} : \mathcal{X} \to \mathbb{R}$, such that

1. $\mathcal{E}(x) = 0 \ \forall x : x \in \mathcal{X}_{dead},$
2. $\mathcal{E}(x) > 0 \ \forall x : x \in \mathcal{B}_r \setminus \mathcal{X}_{dead},$
3. $\frac{\partial \mathcal{E}}{\partial x}(x).f(x,u) > 0 \ \forall x : x \in \mathcal{B}_r \setminus \mathcal{X}_{dead},$

then $\forall x : x \in \{\mathcal{B}_r \setminus \mathcal{X}_{dead}\}, \ lim_{t \to \infty} x(t) \notin \mathcal{B}_r$.

Proof. See [4, Chapter4]. □

To show that trajectories in the set $\mathcal{S}2 \setminus \mathcal{B}_r$ eventually reach to within a close distance of the limit cycle γ, we use the Eventuality certificate presented in [9]. Let us we have a set $\mathcal{X}_{LC}$, such that, $\|y - x\| \leq \alpha, \ \forall x \in \mathcal{X}_{LC}, \ y \in \gamma, \ \alpha > 0$.

Theorem 1.

If there exists a differentiable certificate of eventuality $E : \mathcal{X} \to \mathbb{R}$ satisfying the following conditions

1. $E(x) \leq 0 \; \forall x \in (\mathcal{S}2 \setminus \mathcal{B}_r) \setminus \mathcal{X}_{dead}$,
2. $E(x) > 0 \; \forall x \in Cl(\partial\mathcal{S}2 \setminus \partial\mathcal{X}_{LC})$,
3. $\frac{\partial E}{\partial x}(x).f(x,u) < 0 \; \forall x \in Cl(\mathcal{S}2 \setminus \mathcal{X}_{LC})$,

then for all initial conditions $x_0 \in \mathcal{S}2 \setminus \mathcal{B}_r$, the trajectory $x(t)$ satisfies, $x(T) \in \mathcal{X}_{LC}$, for some $T \geq 0$ and for all $t \in [0, T]$, $x(t) \in \mathcal{X}$. Here Cl and ∂ denote closure and boundary of a closed set respectively.

Proof. . See [9]. □

For the common mode of the even stage RO, we further show that common mode voltages settle down to zero in the steady state. We verify this using the Lyapunov certificate restated for the common mode in Th. 2.

Theorem 2.
For the continuous dynamical system with a vector field given in Eq. 1, and with the state vector replaced by $x = \{x(0,0)+x(1,0), x(0,1)+x(1,1), .., x(0,n-1)+x(1,n-1)\}$, let us we assume an invariant set $\mathcal{X}_{com}$, which we call the common-mode state space. Note that this set is invariant due to the fact that node voltages are bounded by the supply voltage. If there exist a Lyapunov certificate $\mathcal{L} : \mathcal{X} \to \mathbb{R}$ such that,

$$\mathcal{L}(x) > 0, \; \forall x \in \{\mathcal{X}_{com} \setminus \{0\}\}, \; \mathcal{L}(0) = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial x}(x).f(x,u) < 0, \; \forall x \in \{\mathcal{X}_{com} \setminus \{0\}\} \tag{5}$$

then the set $\{x = 0\}$ is asymptotically stable, and $\forall x \in \mathcal{X}_{com}, \; \lim_{t\to\infty} \Phi(x,t) = 0$.

Proof. Similar to [4]. □

2.4 SOS Programming and QE

We formulate our verification methodology as a conjunction of several FOFs having polynomial equations, inequalities, quantifiers $\{\forall, \exists\}$ and boolean operators $\{\wedge, \vee, \neg, \rightarrow, \text{etc}\}$. There are algorithms that can in principle generate quantifier free formulas from a universal-existential quantified FOF over the real fields (See [6] and the references therein). However, they are complex and only work for small academic problems. Showing positivity of a real polynomial, SOS uses a sufficient but incomplete criterion of establishing the decomposition of the polynomial into a sum of squares of polynomials [10]. A sufficient condition for a multivariate polynomial $p(x)$ to be non-negative everywhere is that it can be decomposed as a sum of squares of polynomials, i.e., $p(x) = \sum_{i=1}^{m} p_i^2(x), \; p_i(x) \in \mathcal{R}_n$. We denote the set of polynomials in n variables with real coefficients by $\mathcal{P}_n$. A subset of this set is the set of SOS polynomials in n variables denoted by $\mathcal{S}_n$.

3 Verification of AGI of Oscillation in RO

3.1 Formulation of the Verification problem

We formulate the verification of the AGI property as the conjunction of different sub-properties, corresponding to the three sub-figures in Fig. 3, defined below.

Property 1.

$\forall x(0) : x(0) \in \mathcal{S}1,\ lim_{t\to b}\ x(t) \in \mathcal{S}2,\ b \in \mathbb{R}_{\geq 0}$.

Property 2.

$\forall x(0) : x(0) \in \mathcal{B}_r,\ lim_{t\to\infty}\ (x(t) \notin \mathcal{X}_{dead}\ \wedge\ x(t) \in \mathcal{S}2 \setminus \mathcal{B}_r)$.

Property 3.

$\forall x(0) : x(0) \in \mathcal{S}2 \setminus \mathcal{B}_r,\ lim_{t\to b}\ \|y - x(t)\| \leq \alpha,\ y \in \gamma,\ b \in \mathbb{R}_{\geq 0},\ \alpha > 0$.

We define a fourth property characterizing the common mode behavior of the even stage RO in the invariant set $\mathcal{X}_{com}$.

Property 4.

$\forall x(0) : x(0) \in \mathcal{X}_{com},\ \lim_{t\to\infty}\ x(t) = 0$.

If we denote the almost global inevitability property by φ, Property.1 by $\varphi 1$, Property.2 by $\varphi 2$, Property.3 by $\varphi 3$, and Property.4 by $\varphi 4$, then $\varphi = \varphi 1 \wedge \varphi 2 \wedge \varphi 3$, for odd stage RO, and , $\varphi = \varphi 1 \wedge \varphi 2 \wedge \varphi 3 \wedge \varphi 4$, for even stage RO. A trajectory $x(t)$ of the odd stage RO satisfies φ, iff, it satisfies $\varphi 1$ in $\mathcal{S}1$, $\varphi 2$ in $\mathcal{B}_r$, and $\varphi 3$ in $\mathcal{S}2 \setminus \mathcal{B}_r$, i.e.,

$\forall x : x \in \mathcal{X},\ x \models \varphi \iff (x \models \varphi 1\ \forall x : x \in \mathcal{S}1)\ \wedge (x \models \varphi 2\ \forall x : x \in \mathcal{S}2)\ \wedge (x \models \varphi 3\ \forall x : x \in \mathcal{S}2 \setminus \mathcal{B}_r)$.

Similarly, for en even stage RO,

$\forall x : x \in \mathcal{X},\ x \models \varphi \iff (x \models \varphi 1\ \forall x : x \in \mathcal{S}1)\ \wedge (x \models \varphi 2\ \forall x : x \in \mathcal{B}_r)\ \wedge (x \models \varphi 3\ \forall x : x \in \mathcal{S}2 \setminus \mathcal{B}_r) \wedge (x \models \varphi 4\ \forall x : x \in \mathcal{X}_{com})$.

3.2 SOS-QE approach to verify AGI of oscillation

Here we present the formalization and verification of the Property.1 using a SOS-QE approach and a similar approach is used for the verification of other sub-properties. We define the conditions of Lemma. 1

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by the following FOF.

$$\psi_0 := \exists p^{\mathcal{P}} : \psi_1$$

$$\psi_1 := \forall x^{\mathcal{X}} : \psi_2$$

$$\psi_2 := \Bigg[(x \neq 0 \implies V(p,x) > 0) \wedge$$

$$\{(1 - V(p,x) \geq 0) \implies (\eta - q(x)) \geq 0\} \wedge \{(V(p,x) - 1 \geq 0) \implies (\frac{\partial V}{\partial x}(p,x).f(x,u) \leq -\epsilon)\} \Bigg]$$

Here $p \in \mathcal{P}$ represents the coefficients space of the certificate V. A sufficient condition for the verification of property $\varphi 1$ is stated in the following theorem,

Theorem 3.

If there is a feasible certificate $V(x)$, fulfilling the conditions in Lemma. 1, then, $(x \models \psi_0 \iff x \models \varphi 1)$, $\forall x(0) \in \mathcal{S}1$, and $\mathcal{S}2 = V(x) \leq 1$.

Following the sufficiency conditions in Th. 3, we verify $\varphi 1$ using the mixed SOS-QE approach. We start with a SOS program searching for the AI certificate $V(x)$ such that it satisfies the conditions in Lemma. 1. Note that every condition on $V(x)$ in Lemma. 1 is a positivity/negativity condition which can be formulated as a SOS condition. Furthermore, we need these conditions to be satisfied in different sets which is encoded using a sound mathematical technique called the S-procedure [10]. A SOS program incorporating these conditions is given below.

$$\text{minimize} \quad \eta$$

$$\text{subject to}$$

(i) $V(0) = 0$,

(ii) $\left[V(x) - \epsilon - \sum_{k=1}^{n} s_1^k(x) g_k(x) \right] \in \mathcal{S}_n$,

(iii) $\left[(\eta - q(x)) - s_2(x)(1 - V(x)) \right] \in \mathcal{S}_n$,

(iv) $\left[(-\epsilon - \frac{\partial V}{\partial x}(x).f(x,u)) - s_3(x)(V(x) - 1) - \sum_{k=1}^{n} s_4^k(x) g_k(x) - \sum_{j=1}^{m} s_5^j(x) a_j(u) \right] \in \mathcal{S}_n$,
$\forall x \in \mathcal{X}, \{s_1^k, s_2, s_3, s_4^k, s_5^j\} \in \mathcal{S}_n,\ \forall k \in \{1,..,n\},\ \forall j \in \{1,..,m\}, \epsilon > 0, \eta > 0$.

Here $V(x),\ s_1^k,\ s_2,\ s_3,\ s_4^k,\ s_5^j$, are polynomials of degree d.

In this SOS program, constraint (ii) enforces positive definiteness on the certificate $V(x)$ by introducing a small positive number ϵ. This constraint has to be satisfied in the state space $\mathcal{X}$ defined as,

$\mathcal{X} = \{x \in \mathbb{R}^n \ : g_k(x) \geq 0, \text{ for } k \in \{1,..,n\}\}$. Constraint (iii) ensures that $\{V(x) \leq 1\} \subseteq \{q(x) \leq \eta\}$. Constraint (iv) incorporates the set inclusion $\{V(x) \geq 1\} \subseteq \{\frac{\partial V}{\partial x}.f(x,u) \leq \epsilon\}$. This constraint also ensures that parameters u belong to the set, $\{a_j(u) \geq 0, \text{ for } j \in \{1,..,m\}\}$. The above SOS program, if feasible, returns a certificate of attractive invariance $V(x)$ with its parameters $p \in \mathcal{P}$ fixed within a limited numerical precision. We further verify the validity of this certificate using symbolic QE. Note that in QE, coefficients are represented in $\mathbb{Q}^n$. Using QE, we check the truth value of the negation of the formula ψ_1, since the existential quantifier has already been eliminated by the SOS program. On refutation of $\neg\psi_1$, we conclude, $(x \models \varphi 1 \iff x \models \psi_0)$, $\forall x \in \mathcal{S}1$. If either the SOS program is infeasible for a certificate $V(x)$, or the QE tool returns a true valuation for the formula $\neg\psi_1$, we repeat the process by increasing the degree of the certificate $V(x)$. If we still can't get the desired certificate, we conclude inconclusiveness about the truth value of $\varphi 1$.

4 Experimental Evaluation

We used a degree-7 least-square polynomial model characterizing the input-output non-linear behavior of an inverter. We obtained this approximation of the inverter model by running MATLAB simulation using the Schichman-Hodges MOS transistor models. Note that, in this model, we take into account the effect of transistor widths/lengths on the slope of the inverter output. We used YALMIP [11] solver within MATLAB for SOS programming, and REDLOG [12], for QE on a 2.6 GHZ Intel Core i5 machine with 4 GB of memory. For an odd RO, we were able to compute a degree-4 AI certificate. The AI set, marked by the level set $V(x) \leq 1$, is shown in the Fig. 6. Inside the AI set, we showed trajectories escape the set $V \leq r$, by computing a degree-2 Escape certificate. Similarly, the convergence of the trajectories to within a small distance of the limit cycle has been shown by computing a degree-4 Eventuality certificate in the set $\{V \leq 1 \wedge V \geq r\}$. Time taken by the SOS solver to compute these certificates is listed in the second column of Table. 1. Verification of these certificates in REDLOG, given its ability of how large a formula it can handle, has been divided into the verification of the individual clauses of the FOFs benefiting from its disjunctive normal form (DNF). Since we were interested in the negation of FOFs in the DNF, we verified whether each clause was “false". The verification times of the QE are listed in the third column of Table. 1. For AI and Escape certificates, REDLOG successfully verified the negation of their universally quantified FOFs. A time-out was reported by the REDLOG tool for all clauses of the eventuality FOF of the odd RO. The reason for these time-outs is the set, an intersection of two level curves of the AI certificate, that puts an additional burden on the solver resulting in its time out. To overcome this issue, we instead, conservatively over-under approximate the set $\{V \leq 1 \wedge V \geq r\}$, by a quadratic polynomial, and construct the Eventuality certificate for this new set. This solved our problem and REDLOG has been able to verify the Eventuality certificate in this conservative approximation of

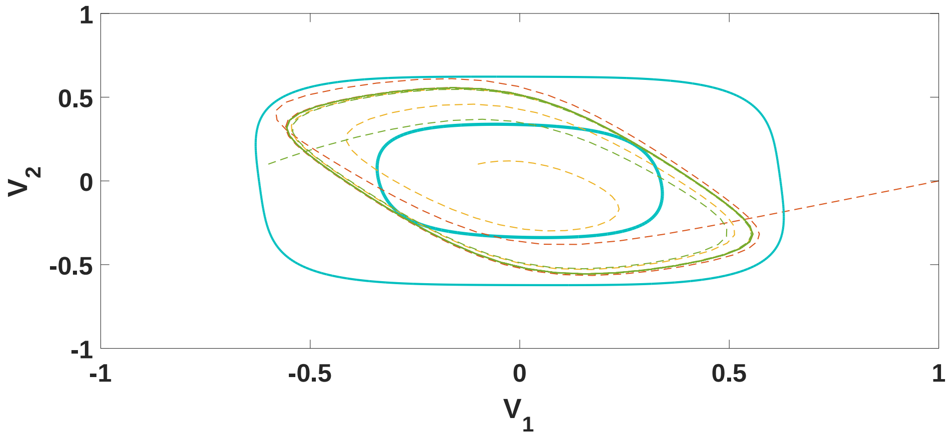

Figure 6: *ODD RO AI Set, $\{r <= V <= 1\}$: Annular region between Solid green plots, Trajectories: Dashed plots*

Certificate	YALMIP-SOS Time(Sec)	REDLOG-QE Time(Sec)
Attractive Invariants	824.8 (Degree 4)	Clause 1 =0.219 Clause 2 =0.047 Clause 3 =8.222
Escape	6.3 (Degree 2)	Clause 1 = 0.060 Clause 2 = 0.026 Clause 3 = 0.320
Eventuality	31.5 (Degree 4)	Clause 1 = 0.070 Clause 2 = 0.025 Clause 3 = 0.636

Table 1: *ODD RO Inevitability Verification Time*

the set $\{V \leq 1 \wedge V \geq r\}$. Note that this conservatism is to approximate the annular set $\{V \leq 1 \wedge V \geq r\}$ and does not add to the overall conservatism of our methodology.

Similarly, for even stage RO, we computed a degree-10 AI, a degree-4 Escape, a degree-6 Eventuality and a degree-4 Lyapunov certificate. Computation times are given in Table. 2 and the AI set is shown in Fig. 7.

Though verifying the property, using SOS-QE approach, needs user input, it offers a comparable computation time to [1]. They have reported approximately 22000 seconds for the complete verification of the even stage RO, whereas our accumulative time for the even stage RO is approximately 6500 seconds. Even if we add the time of all the instances for which we got an infeasible certificate, our computation time is still not more than half of what has been reported in [1]. This is in addition to our methodology being less conservative and applicable to infinite horizon.

5 Conclusion

Results show the effectiveness of our approach to verifying the complex AGI property of a real world analog circuit. We have verified the AGI using the Lyapunov based deductive method which is not only applicable to infinite time, but also avoids explicitly solving ODEs and is thus less conservative than the reachability approaches.

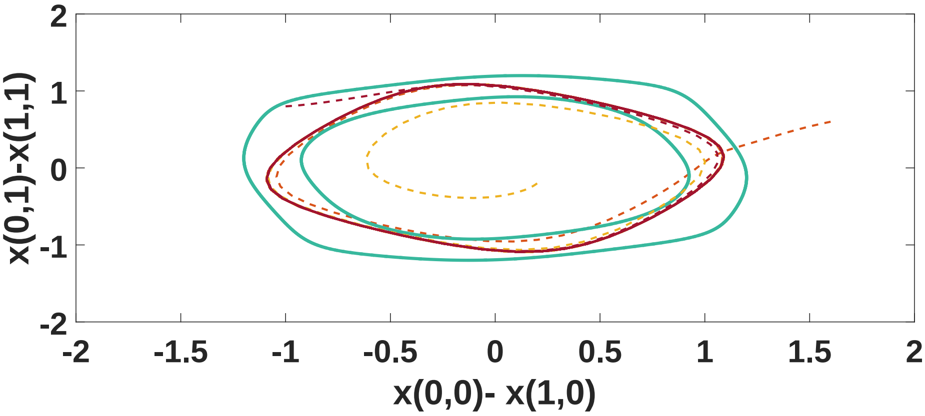

Figure 7: *Even RO AI Set, $\{r <= V <= 1\}$: Annular region between Solid green Plot, Trajectories: Dashed*

Certificate	YALMIP-SOS Time(Sec)	REDLOG-QE Time(Sec)
Attractive Invariants	6127.6 (Degree 10)	Clause 1 =5.24 Clause 2 =0.33 Clause 3 =1.56
Escape	320.6757 (Degree 4)	Clause 1 = 0.01 Clause 2 = 0.30 Clause 3 = 2.50
Eventuality	4128.8 (Degree 6)	Clause 1 = 0.349 Clause 2 = 0.300 Clause 3 = 0.615
Lyapunov	55.24(Degree 4)	Clause 1 = 0.02 Clause 2 = 0.75 Clause 3 = 0.57

Table 2: *Even RO Inevitability Verification Time*

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