RUIN AND DEFICIT UNDER CLAIM ARRIVALS WITH THE ORDER STATISTICS PROPERTY

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We consider an insurance risk model with extended flexibility, under which claims arrive according to a point process with an order statistics (OS) property, their amounts may have any joint distribution and the premium income is accumulated following any non-decreasing, possibly discontinuous real valued function. We generalize the definition of an OS point process, assuming it is generated by an arbitrary cdf, allowing jump discontinuities which corresponds to an arbitrary (possibly discontinuous) claim arrival cumulative intensity function. The latter feature is appealing for insurance applications since it allows to consider clusters of claims arriving instantaneously. Under these general assumptions, a closed form expression for the joint distribution of the time to ruin and the deficit at ruin is derived, which remarkably involves classical Appell polynomials. Corollaries of our main result generalize previous non-ruin formulas e.g., those obtained by Ignatov and Kaishev (2000, 2004, 2006) and Lefèvre and Loisel (2009) for the case of stationary Poisson claim arrivals and by Lefèvre and Picard (2011, 2014), for OS claim arrivals.

1. Introduction. The ruin of an insurance company can be viewed as the event of its aggregate claim amount exceeding for the first time the aggregate premium income, modeled by a non-decreasing deterministic function. Therefore, ruin is equivalent to first crossing of an upper deterministic boundary by a stochastic process modeling the aggregate claim amount. There have been different stochastic models of first crossing and important contributions in the applied probability literature have been made by Zolotarev (1964) and Borovkov (1964), Kou and Wang (2003), Peskir (2007), Bernyk et al. (2008), Yang and Zhang (2001), Huzak et al. (2004), Garrido and Morales (2006), Bertoin et al. (2008), Savov (2009) and Aurzada et al. (2013) to mention only a few. The joint distribution of the first crossing time and the overshoot of a Lévy process over a fixed boundary in infinite time

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have been considered by Doney (1991), Klüppelberg et al. (2004), Doney and Kyprianou (2006), Eder and Klüppelberg (2009).

In risk theory, the first crossing time and the overshoot are interpreted as the ruin time and the deficit at ruin. Ruin time and deficit in a classical infinite time risk model, have been considered jointly through a defective renewal equation in terms of what is called Gerber-Shiu function (see Gerber and Shiu (1997, 1998). Although the literature is extensive, deriving closed form results following this approach has proved difficult (see e.g. Landriault and Willmot (2009). Recently, Ignatov and Kaishev (2014) applied a more direct approach and derived explicitly the joint distribution of the ruin time and the deficit at ruin in a finite time interval, assuming a general dependent risk model where claim arrivals form a point process with independent increments. The latter processes represent a large and flexible class including both homogeneous and non-homogeneous Poisson and negative binomial point processes. As shown by Ignatov and Kaishev (2014), the joint ruin-time-deficit distribution is elegantly expressed in terms of a new remarkable class of functions called Appell-Hessenberg functions.

The purpose of this paper is to extend these results to the case where claims arrive according to a point process with the so called *order statistics* (OS) property, or simply OS point processes, defined as follows. Consider a point process ξ, on $(0,\infty)$, with a cumulative intensity function $\Lambda((0,z]) = \Lambda(z) = E\xi[0,z] < \infty$, $\forall z \in (0,\infty)$, with $\xi[0,z]$ denoting the number of claims in $[0,z]$. The process ξ is said to have the OS property if, given n claim arrivals in a finite interval $[0,z]$, $z > 0$, the successive arrival times, $0 < T_1 < T_2 < \ldots < T_n$, coincide in distribution with the order statistics of n independent and identically distributed random variables with a cumulative distribution function, $F_z(x)$, $0 \le x \le z$, $F_z(z) = 1$.

Following the pioneering work of Nawrotzki (1962), point processes with the OS property have been studied and characterized e.g., by Holmes (1971), Westcott (1973), Crump (1975), Kallenberg (1976), Feigin (1979), Puri (1982), Liberman (1985), Huang and Shoung (1994) and Berg and Spizzichino (2000). More precisely, Crump (1975) has shown that OS processes are Markovian and that $F_z(x) = \Lambda(x)/\Lambda(z)$. It has been proven by Holmes (1971) (see also Westcott (1973)) that the only OS process with independent increments is the Poisson process. It has also been shown by Feigin (1979), that an OS process,

$$\xi \overset{a.s.}{=} \mathcal{P}\left[X\Lambda(z)\right], \tag{1}$$

where $z > 0$, $\mathcal{P}$ is a homogeneous Poisson process with unit rate and

X, an independent non-negative random variable. This result states that OS point processes are characterized, up to a time-scale transformation, by mixed Poisson processes.

In risk and ruin theory, OS processes have been applied by Willmot (1989), De Vylder and Goovaerts (1999, 2000), Lefèvre and Picard (2011, 2014) and Sendova and Zitikis (2012) to model claim arrivals. Such OS risk models are appealing since the total number of claims, $\xi(0, z]$ in $[0, z]$, denoted also by $N(z)$, can have any distribution, depending on the insurance application.

Let us note that, in all of the afore-quoted literature, it has been assumed that the OS process of interest has unit steps at the times, $T_1, \ldots, T_n$, i.e., that the underlying cdf, $F_z(x)$ is continuous. In what follows we will adopt a more general definition of an OS point process in which we allow $F_z(x)$ to be discontinuous. Since $F_z(x)$ is a distribution function, it is easy to see that the limits, $F_z(x+) = \lim_{s \downarrow x} F_z(s)$ and $F_z(x-) = \lim_{s \uparrow x} F_z(s)$ exist. If $F_z(x)$ is right-continuous, then $F_z(x) = F_z(x+)$ and if the difference $F_z(x+) - F_z(x-) \equiv F_z(x) - F_z(x-)$ differs from zero, we will say that $F_z(x)$ has a jump at x, equal to the size of that difference. Recall also that for continuous cdf's, $F_z(x-) = F_z(x+) = F_z(x)$. We can now give the following extended definition of an OS point process.

Definition 1.1. *A point process ξ, defined on $(0, \infty)$ with any possibly discontinuous cumulative intensity function $\Lambda((0, z]) = \Lambda(z) < \infty$, $\forall z \in (0, \infty)$, is said to have the order statistics (OS) property if, for every $0 < z < \infty$ and $n \geq 0$, such that $P(\xi(0, z] = n) > 0$, conditional on $\xi(0, z] = n$, the consecutive arrival times, $0 < T_1 \leq \ldots \leq T_n \leq z$, of ξ, coincide in distribution with the order statistics, $X_{1,n}, \ldots, X_{n,n}$ of n independent and identically distributed random variables, $X_1, \ldots, X_n$, with a cumulative distribution function $F_z(x) = \Lambda(x)/\Lambda(z)$, $0 \leq x \leq z$, with possible jumps, such that, $F_z(0) = 0$ and $F_z(z) = 1$, i.e., $(T_1, \ldots, T_n) \stackrel{d}{=} (X_{1,n}, \ldots, X_{n,n})$.*

Our aim in the present paper is three-fold. First, we revisit the OS risk model considered recently by Lefèvre and Picard (2011, 2014) under the assumption that $F_z(x)$ is continuous. We relax the latter assumption, and following Definition 1.1, allow $F_z(x)$ to have possible jump discontinuities at fixed instants in $[0, z]$, which is equivalent to allowing $\Lambda(x)$ to be discontinuous at these instants. This leads to extending further the flexibility of the OS risk model, allowing claims to arrive at random moments but also at fixed instants with non-zero probability, possibly forming clusters. This, is an appealing feature, both in life and non-life insurance applications (see section 3). Second, under this generalized OS risk model, we derive a closed-form expression for the joint distribution of the ruin time and deficit at ruin, given by Theorem 2.4, which covers and extends previous ruin probability formulas, due to Ignatov and Kaishev (2000, 2004, 2006), Ignatov et al. (2001), Lefèvre and Loisel (2009) and Lefèvre and Picard (2011, 2014). Furthermore, we demonstrate that our formulas, expressed in terms of a special case of what we call Appell-Hessenberg functions (see Ignatov and Kaishev 2014), are more explicit. They do not involve indicator functions and expectations of random quantities, such as $N(z)$ and the aggregate claim amount, $S_{N(z)}$, as is the case with the non-ruin probability formulas (4.1) and (4.2) of Lefèvre and Picard (2011) which, as the authors note, require further specification (see Lefèvre and Picard 2014). Third, we illustrate how the expression for the joint distribution of the ruin time and deficit can be applied in some particular cases of OS claim arrivals with both continuous and discontinuous cdf $F_z(x)$. More precisely, we revisit the three special cases considered by Crump (1975), mixed Poisson process, linear birth process with immigration equivalent to a negative binomial $N(z)$ and a linear death process implying a binomial distribution for $N(z)$. In addition we consider also the cases when $F_z(x)$ is a pure jump cdf or a cdf with jumps and continuous parts, with potential application in risk models involving claim counts panel data.

The paper is organized as follows. In section 2, we prove our main result given by Theorem 2.4. For the purpose, we formulate and prove Lemmas 2.6, 2.7 and 2.9 (and also Proposition 2.3) which are of interest in their own right, establishing explicit and recurrent representations of Appell-Hessenberg functions. Corollaries 2.10, 2.11 and 2.12 of Theorem 2.4 give ruin formulas for important special cases. In section 3 we illustrate how the results of section 2 can be applied for some special cases of OS claim arrival processes.

2. A formula for $P(T \leq z, Y > y)$. First, let us introduce some notation and specify the ruin probability model which we will be concerned with in the sequel. The amounts of consecutive claims to an insurance company are modelled by the random variables, $W_1, W_2, \dots$ and $Y_1, Y_2, \dots$ denote their partial sums, i.e. $Y_1 = W_1, Y_2 = W_1 + W_2, \dots$. If claim severities $W_1, W_2, \dots, W_k$ are considered continuous random variables, then $\psi(w_1, \dots, w_k)$ will denote their joint density and $f(y_1, \dots, y_k)$ will denote the joint density of $Y_1, Y_2, \dots, Y_k$. Clearly, $\psi(w_1, \dots, w_k) = f(w_1, w_1 + w_2, \dots, w_1 + \dots + w_k)$ and $f(y_1, \dots, y_k) = \psi(y_1, y_2 - y_1, \dots, y_k - y_{k-1})$. In the case of discrete claim severities $W_1, W_2, \dots, W_k$, their joint probability mass function $P(W_1 = w_1, \dots, W_k = w_k)$ is denoted by $p(w_1, \dots, w_k)$.

We will further assume that claims arrive, according to an OS point process, ξ, defined as in Definition 1.1 which extends the OS property considered previously in the literature.

The cumulative premium income of the insurance company up to time t is modelled by the function $h(t)$ which is assumed a non-negative and non-decreasing real-valued function, defined on $[0, +\infty)$, such that $\lim_{t\to\infty} h(t) = +\infty$. Let us also note that the function $h(t)$ does not need to be necessarily continuous and can therefore model arrivals of lump sum premium amounts. If $h(t)$ is discontinuous, we define $h^{-1}(y) = \inf\{v : h(v) \geq y\}$.

We consider a finite time interval, $[0, z]$, where z is a fixed positive real number and express the insurance company's surplus process as $R_t = h(t) - S_t$, where $S_t = Y_{\xi(0,t]}$ is the aggregate claim amount process, and the instant of ruin, T is defined as $T := \inf\{t : 0 < t \leq z, R_t < 0\}$ or $T = \infty$ if $R_t \geq 0$ for all $0 < t \leq z$. Given ruin occurs within $[0, z]$, i.e., $T \leq z$, the deficit at ruin, Y is defined as, $Y = -R_T$. Denote by $P(T > z)$, the probability of non-ruin in $[0, z]$, i.e., $P(T > z) = P(R_t \geq 0, \forall t \in (0, z])$ and by $P(T \leq z, Y > y)$ the probability that ruin occurs before time z, with a deficit, Y, exceeding $y \geq 0$. In what follows, we will give explicit expressions for these and other related probabilities, under the assumption that the process of claim arrivals, ξ belongs to the class of point processes with the OS property, described in Definition 1.1. To reflect on the OS property of ξ, we will refer to the related risk model as an OS risk model. In order to formulate our main result, we will need to introduce a particular type of classical Appell polynomials which, belong to the wider class of Appell-Hessenberg functions considered in Ignatov and Kaishev (2014).

Definition 2.1. *For a fixed non-negative integer j, let $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z_{j+1} \equiv z$ be an arbitrary increasing sequence of positive real numbers and $p_k = F_z(z_k-) - F_z(z_{k-1}-)$, $k = 1, 2, \ldots, j$, $p_{j+1} = F_z(z_{j+1}) - F_z(z_j-) \equiv F_z(z) - F_z(z_j-)$ with $p_1 + \ldots + p_{j+1} = 1$. Define the functions $A_j(F_z(z); F_z(z_1-), \ldots, F_z(z_j-))$, $z \in (z_j, \infty)$, $j = 0, 1, 2, \ldots$ as*

$$A_j(F_z(z); F_z(z_1-), \ldots, F_z(z_j-)) = (-1)^j \det\left(\left(\delta^{(j)}_{m,l}\right)_{1\leq m,l\leq j+1}\right), \tag{2}$$

where

$$\delta^{(j)}_{m,l} = \binom{j-l+1}{m-l+1} (p_1 + \ldots + p_m)^{m-l+1},$$

$$\delta^{(j)}_{j+1,l} = (p_1 + \ldots + p_{j+1})^{j-l+1} \equiv 1^{j-l+1} \equiv 1,$$

for $1 \leq m \leq j$, $1 \leq l \leq j+1$ with

$$\binom{j-l+1}{m-l+1} \equiv 0, \quad \text{if} \quad m-l+1<0, \tag{3}$$

$$\binom{j-l+1}{m-l+1} \equiv 1 \quad \text{if} \quad m-l+1=0, \text{ and } \delta_{1,1}^{(0)} \equiv (1), \text{ for } j=0.$$

REMARK 2.2. *From (3), it follows that* $\left(\delta_{m,l}^{(j)}\right)_{1\leq m,l\leq j+1}$ *is a lower Hessenberg matrix. A matrix whose elements above or below the first subdiagonal are equal to zero (i.e., all elements,* $a_{ij}=0$ *if* $j-i>1$ *or if* $i-j>1$*) are called Hessenberg matrixes. For properties of Hessenberg matrixes and their determinants we refer to, e.g. Vein and Dale (1999). Note also that,* $A_j\left(F_z(z); F_z(z_1-)\ldots, F_z(z_j-)\right)$ *is a classical Appell polynomial of degree* j*, defined by the sequence* $F_z(z_1-)\ldots, F_z(z_j-)$*, and evaluated at* $F_z(z)=1$ *i.e.,*

$$A_j\left(F_z(z); F_z(z_1-)\ldots, F_z(z_j-)\right) = A_j\left(1; F_z(z_1-)\ldots, F_z(z_j-)\right),$$

where

$$\begin{aligned} A_0(F(z)) &= 1, \\ A_j'(F(z)) &= cA_{j-1}(F(z)), \text{ and} \\ A_j\left(F\left(z_j-\right)\right) &= 0, \end{aligned} \tag{4}$$

$j=1,2,\ldots,$ *with* c*, a constant and* $0\leq z_1\leq\ldots\leq z_j$*,* $z_j\in\mathbb{R}$*.*

Classical Appell polynomials defined above, were first shown to appear in ruin theory, in the closed form non-ruin probability formulas due to Ignatov and Kaishev (2000, 2004) in relation to the Poisson claim arrivals in a general risk model with dependence. It was shown by Ignatov and Kaishev (2000), (see Lemma 1 therein) that Appell polynomials can be represented as certain Hessenberg determinants . For further properties of classical Appell polynomials and their relation to ruin probability see Dimitrova et al. (2016). A different class of so called generalized Appell polynomials, which do not yield classical Appell polynomials was considered by Picard and Lefèvre (1997).

Since the functions $A_j\left(F_z(z); F_z(z_1-)\ldots, F_z(z_j-)\right)$, $j=1,2,\ldots$ are values of Appell polynomials expressed as Hessenberg determinants we will more generally refer to them as Appell-Hessenberg functions. For other types of such functions see Ignatov and Kaishev (2014).

In what follows, it will some times be convenient to interchangeably use the notation $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z_{j+1}$, with $z_{j+1} \equiv z$, for the sequence $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z$. The following recurrence formula facilitates the numerical evaluation of the Appell-Hessenberg functions, $A_j\left(F_z(z); F_z(z_1-)\ldots, F_z(z_j-)\right)$.

Proposition 2.3. *For a fixed non-negative integer j, let $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z$ be an arbitrary increasing sequence of positive real numbers. For the Appell-Hessenberg functions, $A_j\left(F_z(z); F_z(z_1-)\ldots, F_z(z_j-)\right)$, defined in (2), we have*

(5)
$$A_j\left(F_z(z); F_z(z_1-)\ldots, F_z(z_j-)\right) = \sum_{i=0}^{j} \delta^{(j)}_{j+1,i+1} A_i\left(0; F_z(z_1-)\ldots, F_z(z_i-)\right), j \geq 0,$$

where $A_0\left(F_z(z)\right) \equiv 1$, $z \geq 0$ and

$$A_i\left(0; F_z(z_1-)\ldots, F_z(z_i-)\right) = -\sum_{k=0}^{i-1} \delta^{(j)}_{i,k+1} A_k\left(0; F_z(z_1-)\ldots, F_z(z_k-)\right), i \geq 1,$$

with $A_0(0) \equiv 1$.

Proof of Proposition 2.3: The proof is similar to the proof given in Ignatov and Kaishev (2000) for the case of classical Appell polynomials (see Lemma 1 therein) and is therefore omitted. □

Next, we state our main result which shows that the joint distribution of the time to ruin and the deficit at ruin in the risk model with claim arrivals following an arbitrary OS point process, ξ, from Definition 1.1 can be expressed in terms of the Appell-Hessenberg functions, $A_j\left(F_z(z); F_z(z_1)\ldots, F_z(z_j)\right)$, $j = 0, 1, 2, \ldots$.

Theorem 2.4. *The probability $P(T \leq z, Y > y)$, $0 < z < \infty$, $y \geq 0$, is given by*

(6)
$$P(T \leq z, Y > y) = P(\xi(0, z] = 1) \int_y^{\infty} \left(1 - P\left(\xi(0, h^{-1}(y_1 - y)] = 0\right)\right) f\left(y_1\right) dy_1$$
$$+ \sum_{j=2}^{\infty} P(\xi(0, z] = j) \sum_{k=1}^{j} \int \ldots \int_{C_k} \left\{A_{k-1}\left(1; F_z\left(h^{-1}(y_1)-\right), \ldots, F_z\left(h^{-1}(y_{k-1})-\right)\right)\right.$$
$$\left. - A_k\left(1; F_z\left(h^{-1}(y_1)-\right), \ldots, F_z\left(h^{-1}(y_{k-1})-\right), F_z\left(h^{-1}(y_k - y)-\right)\right)\right\} f\left(y_1, \ldots, y_k\right) dy_k \ldots dy_1,$$

where $C_k = \{(y_1, \ldots, y_k) : 0 < y_1 < \ldots < y_{k-1} \le y_k - y, y_{k-1} \le h(z)\}$*, and* $A_j(1; F_z(z_1-), \ldots, F_z(z_j-))$ *are the classical Appell polynomials evaluated at* $F_z(z) = 1$ *and defined as in (2) with* $z_1 = h^{-1}(y_1), \ldots, z_j = h^{-1}(y_j)$*,* $j = 0, 1, 2 \ldots$.

Remark 2.5. *It should be noted that for the efficient numerical evaluation of* $P(T \le z, Y > y)$*, following (6), it is essential to be able to: 1) appropriately truncate the infinite summation; 2) compute the underlying multiple integrals; 3) efficiently compute the integrand functions* $A_j(F_z(z); F_z(z_1-) \ldots, F_z(z_j-))$*. The latter can be done using recurrence formula (5). Methods for solving 1) and 2) developed in Dimitrova et al. (2016) for the special case of stationary Poisson claim arrivals could be generalized to the case of OS claim arrivals. Details of how this could be done are outside the scope of the present paper and will be considered separately.*

In order to prove Theorem 2.4 and some related corollaries, we will need the following lemmas.

Lemma 2.6. *For the real sequence* $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z_{j+1} \equiv z$*,* $A_j(F_z(z); F_z(z_1-) \ldots, F_z(z_j-))$*, defined as in (2), and* p_k *introduced in Definition 2.1, we have*

$$A_j(F_z(z); F_z(z_1-) \ldots, F_z(z_j-)) = \sum_{(g_0, \ldots, g_j) \in E(0,j)} \frac{j!}{g_0! \ldots g_j!} p_1^{g_0} \ldots p_{j+1}^{g_j} \tag{7}$$

where $E(0, j)$ *is the set of* $(j+1)$*-tuples of non-negative integers such that*

$$E(0, j) = \{(g_0, \ldots, g_j) : g_0 \le 0, g_0 + g_1 \le 1, \ldots, g_0 + \ldots + g_{j-1} \le j - 1, g_0 + \ldots + g_j = j\}, \tag{8}$$

$j \ge 0$*, and where for notational convenience we assume that* $z_{j+1} \equiv z$*.*

Proof of Lemma 2.6: We will proceed by induction. First, we verify that Lemma 2.6 holds in the cases $j = 0$ and $j = 1$. When $j = 0$, and $0 \equiv z_0 < z$, from (2), for the left-hand side of (7), we have $A_0(F_z(z)) \equiv 1$ and for the right-hand side, we have

$$\sum_{g_0 \in E(0,0)} \frac{0!}{g_0!} p_1^{g_0} = 1$$

and therefore, Lemma 2.6 holds. When $j = 1$ and $0 \equiv z_0 < z_1 < z$, from (2), for the left-hand side of (7), we have

$$A_1(F_z(z); F_z(z_1)-) = (-1) \det \begin{pmatrix} p_1 & 1 \\ p_1 + p_2 & 1 \end{pmatrix} = -\det \begin{pmatrix} p_1 & 1 \\ 1 & 1 \end{pmatrix} = 1 - p_1$$

and for the right-hand side we have that

$$\sum_{(g_0,g_1)\in E(0,1)} \frac{1!}{g_0!g_1!} p_1^{g_0} p_2^{g_1} = p_2 = 1 - p_1$$

and therefore, equality (7) is again valid. We will continue the proof by induction. We showed that Lemma 2.6 holds for $j = 0$ and $j = 1$. Assume it is true for all non-negative integers up to $j - 1$. Lemma 2.6 will be proved if we show that (7) is true also for the index j. Let us expand the determinant on the right-hand side of equality (2) with respect to its first column. We have

(9)

$$A_j\left(F_z(z); F_z(z_1-), \ldots, F_z(z_j-)\right) = (-1)^j \det\left(\left(\delta^{(j)}_{m,l}\right)_{1\le m,l\le j+1}\right) = (-1)^j \left(\begin{pmatrix} j \\ 1 \end{pmatrix} p_1^1 A_{1,1} + \begin{pmatrix} j \\ 2 \end{pmatrix} (p_1 + p_2)^2 A_{2,1} + \ldots + \begin{pmatrix} j \\ j \end{pmatrix} (p_1 + \ldots + p_j)^j A_{j,1} + (p_1 + \ldots + p_{j+1})^j A_{j+1,1}\right),$$

where $A_{k,1} = (-)^{k+1} \det \Delta_{k,1}$ is the cofactor of the element, $\delta^{(j)}_{k,1}$, $1 \le k \le j + 1$ on the k-th row and the 1-st column of $\left(\delta^{(j)}_{m,l}\right)_{1\le m,l\le j+1}$ and $\Delta_{k,1}$ is a sub-matrix, obtained by deleting the k-th row and 1-st column of $\left(\delta^{(j)}_{m,l}\right)_{1\le m,l\le j+1}$. For, $(1 < k \le j)$, we can express the matrix $\Delta_{k,1}$, in a block-matrix form as

$$\Delta_{k,1} = \begin{pmatrix} \delta_{1,1} & \delta_{1,2} \\ \delta_{2,1} & \delta_{2,2} \end{pmatrix}, \tag{10}$$

where $\delta_{1,1}$ is a $(k-1)\times(k-1)$ unit lower triangular matrix, i.e. with ones on the main diagonal and zeros in the upper triangle, $\delta_{1,2}$ is a $(k-1)\times(j-k+1)$ matrix of zeros, and $\delta_{2,2}$ is a $(j - k + 1) \times (j - k + 1)$ matrix for fixed k, $1 < k \le j$, obtained from $\left(\delta^{(j)}_{m,l}\right)_{1\le m,l\le j+1}$, applying in it the following formal substitutions

$$j \to j - k; p_1 \to p_1 + \ldots + p_{k+1}; p_2 \to p_{k+2}; \ldots; p_{j-k} \to p_j; p_{j-k+1} \to p_{j+1}.$$

Similarly, for $k = 1$, the matrix $\Delta_{1,1}$ is defined by $\left(\delta^{(j)}_{m,l}\right)_{1\le m,l\le j+1}$, applying in it the following substitutions

$$j \to j - 1; \quad p_1 \to p_1 + p_2; \quad p_2 \to p_3; \ldots; p_{j-1} \to p_j; p_j \to p_{j+1}.$$

Since $\delta_{1,1}$ is a unit lower triangular matrix,

$$\det(\delta_{1,1}) = 1, \tag{11}$$

whereas by the induction assumption, for the determinant of $\delta_{2,2}$, we have

$$(-1)^{j-k}\det(\delta_{2,2}) = \sum_{(g_0,\dots,g_{j-k})\in E(0,j-k)} \frac{(j-k)!}{g_0!\dots g_{j-k}!}(p_1+\dots+p_{k+1})^{g_0}\, p_{k+2}^{g_1}\dots p_{j+1}^{g_{j-k}}. \tag{12}$$

Hence, from (10), (11) and (12) for $1 \le k \le j$, we have

$$\begin{aligned}\det(\Delta_{k,1}) &= \det(\delta_{1,1})\det(\delta_{2,2})\\ &= (-1)^{j-k}\sum_{(g_0,\dots,g_{j-k})\in E(0,j-k)} \frac{(j-k)!}{g_0!\dots g_{j-k}!}(p_1+\dots+p_{k+1})^{g_0}\, p_{k+2}^{g_1}\dots p_{j+1}^{g_{j-k}}.\end{aligned} \tag{13}$$

If $k = j$, then, from (13)

$$\det(\Delta_{j,1}) = 1, \tag{14}$$

whereas, for $k = j+1$ we have

$$\det(\Delta_{j+1,1}) = 1. \tag{15}$$

From (13), (14) and (15), for the cofactors $A_{k,1}$, $1 \le k \le j$ and $A_{j+1,1}$ we have

$$\begin{aligned}A_{k,1} &= (-1)^{k+1}\det(\Delta_{k,1})\\ &= (-1)^{j+1}\sum_{(g_0,\dots,g_{j-k})\in E(0,j-k)} \frac{(j-k)!}{g_0!\dots g_{j-k}!}(p_1+\dots+p_{k+1})^{g_0}\, p_{k+2}^{g_1}\dots p_{j+1}^{g_{j-k}},\end{aligned} \tag{16}$$

and

$$A_{j+1,1} = (-1)^{j+2}. \tag{17}$$

Substituting (16) and (17) back in (9), we obtain

(18)

$$A_j\left(F_z(z);F_z(z_1-),\dots,F_z(z_j-)\right)=(-1)^j\det\left(\left(\delta^{(j)}_{m,l}\right)_{1\le m,l\le j+1}\right)$$

$$=(p_1+\ldots+p_{k+1})^j-\left(\left(\begin{array}{c}j\\1\end{array}\right)p_1^1\sum_{(g_0,\dots,g_{j-1})\in E(0,j-1)}\frac{(j-1)!}{g_0!\ldots g_{j-1}!}(p_1+p_2)^{g_0}p_3^{g_1}\ldots p_{j+1}^{g_{j-1}}\right.$$

$$-\left(\begin{array}{c}j\\2\end{array}\right)(p_1+p_2)^2\sum_{(g_0,\dots,g_{j-2})\in E(0,j-2)}\frac{(j-2)!}{g_0!\ldots g_{j-2}!}(p_1+p_2+p_3)^{g_0}p_4^{g_1}\ldots p_{j+1}^{g_{j-2}}$$

$$-\ldots-\left(\begin{array}{c}j\\k\end{array}\right)(p_1+\ldots+p_k)^k\sum_{(g_0,\dots,g_{j-k})\in E(0,j-k)}\frac{(j-k)!}{g_0!\ldots g_{j-k}!}(p_1+\ldots+p_{k+1})^{g_0}p_{k+2}^{g_1}\ldots p_{j+1}^{g_{j-k}}$$

$$\left.-\ldots-\left(\begin{array}{c}j\\j\end{array}\right)(p_1+\ldots+p_j)^j\sum_{g_0\in E(0,0)}\frac{0!}{g_0!}(p_1+\ldots+p_{j+1})^{g_0}\right),$$

$$=\sum_{(g_0,\dots,g_j)\in E(0,j)}\frac{j!}{g_0!\ldots g_j!}p_1^{g_0}\ldots p_{j+1}^{g_j},$$

where the last equality, in (18) follows after some tedious but straightforward algebra, which is therefore omitted. This completes the proof of Lemma 2.6. □

LEMMA 2.7. *Let $0\equiv z_0<z_1<z_2<\ldots<z_j<z$ be a sequence of positive real numbers and for a fixed z, let $0<T_1\le T_2\le\ldots\le T_j\le z$ be the consecutive points of an OS point process ξ, defined as in Definition 1.1. We have*

$$P\left(T_1>z_1,\dots,T_j>z_j\right)=\sum_{(g_0,\dots,g_j)\in E(0,j)}\frac{j!}{g_0!\ldots g_j!}p_1^{g_0}\ldots p_{j+1}^{g_j}, \tag{19}$$

where $E(0,j)$ is the set of $(j+1)$-tuples of non-negative integers, defined in (8), and p_k, $k=1,2,\dots,j+1$ are defined in Definition 2.1.

REMARK 2.8. *When $j=0$, the left-hand side of (19) should be interpreted as the conditional probability of non-ruin, given that there are zero claims in $(0,z]$, i.e., that $\xi(0,z]=0$. Clearly, this conditional probability is equal to one, and in this case equality (19) is still valid since substituting $j=0$ on the left-hand side also gives one.*

Proof of Lemma 2.7: The proof will be based on interpreting, $P\left(T_1 > z_1, \dots, T_j > z_j\right)$ as the probability of non-crossing within $[0, z]$ of an upper deterministic boundary,

$$h(x) = \begin{cases} 0 & \text{for } x \in [0, z_1), \\ 1 & \text{for } x \in [z_1, z_2), \\ \quad \dots & \\ (j-1) & \text{for } x \in [z_{j-1}, z_j), \\ j & \text{for } x \in [z_{j-1}, z_j], \end{cases}$$

by the trajectory of an OS counting process, ξ, defined as in Definition 1.1. It is easy to see that every trajectory of ξ, i.e., the number of points of the random point set $X_1, \dots, X_j$, which occur in the interval $[0, x]$, for $0 < x \le z$, coincide in distribution with $jF_j(x)$, i.e., $\xi[0, x] \stackrel{d}{=} jF_j(x)$, where $F_j(x)$ is the empirical distribution function based on the sample $X_1, \dots, X_j$. Furthermore, it can directly be seen that the event

$$\{\omega : T_1(\omega) > z_1, \dots, T_j(\omega) > z_j\} = \{\omega : \xi\,[0, x] \le h(x), \text{ for every } x \in (0, z]\}\,,$$

i.e., the probability of non-crossing $h(x)$, is equal to $P\left(T_1 > z_1, \dots, T_j > z_j\right)$.

On the other hand, let us interpret, $X_1, \dots, X_j$ as the consecutive random placement of j independent points on the interval $[0, z]$, which is partitioned into $j+1$ consecutive intervals $[0, z_1), [z_1, z_2), \dots [z_j, z)$. We can view these intervals as urns, and if for example, $X_i(\omega) \in [z_{l-1}, z_l)$, we will say that in this urn model, the j-th point (or particle) has been placed in the l-th urn. It is well known, that the probability to have g_0 particles in the first urn, g_1 particles in the second urn and so on, g_{j-1} particles in the j-th and g_j particles in the last, $(j+1)$-th urn is given by the multinomial formula

$$\frac{j!}{g_0! \dots g_j!} p_1^{g_0} \dots p_{j+1}^{g_j},$$

where, $0 \le g_i \le j$, $i = 0, \dots, j$ and $g_0 + g_1 + \dots + g_j = j$.

It can directly be checked that,

$$P\left(T_1 > z_1, \dots, T_j > z_j\right) = \sum_{(g_0, \dots, g_j) \in E(0,j)} \frac{j!}{g_0! \dots g_j!} p_1^{g_0} \dots p_{j+1}^{g_j}.$$

This completes the proof of Lemmas 2.7. □

The following lemma, which directly follows from lemmas 2.6 and 2.7 gives a probabilistic interpretation of the Appell-Hessenberg function, $A_j\left(F_z(z); F_z(z_1-) \dots, F_z(z_j-)\right)$.

LEMMA 2.9. *Let* $0 \equiv z_0 < z_1 < z_2 < \ldots < z_j < z$ *be a sequence of positive real numbers and for a fixed* z*, let* $0 < T_1 \leq T_2 \leq \ldots \leq T_j \leq z$ *be the consecutive points of an OS point process* ξ*, defined as in Definition 1.1. We have*

$$P\left(T_1 > z_1, \ldots, T_j > z_j\right) = A_j\left(F_z(z); F_z(z_1-), \ldots, F_z(z_j-)\right),$$

where $A_j\left(F_z(z); F_z(z_1-), \ldots, F_z(z_j-)\right)$ *is defined as in (2).*

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4: Applying the formula of total probability, we have

$$P(T \leq z, Y > y) = \sum_{j=1}^{\infty} P\left(\xi\left(0, z\right] = j\right) P\left(T \leq z, Y > y \mid \xi\left(0, z\right] = j\right) \tag{20}$$

For the conditional probability on the right-hand side of (20), we have

$$\begin{aligned} &P\left(T \leq z, Y > y \mid \xi\left(0, z\right] = j\right) \\ &= P\left(\bigcup_{k=1}^{j}\left\{\left(\bigcap_{l=1}^{k-1}\left(h\left(T_l\right) > Y_l\right)\right) \cap \left(h\left(T_k\right) + y < Y_k\right)\right\}\right), \end{aligned} \tag{21}$$

where we assume that, for $k = 1$, $\overset{k-1}{\underset{l=1}{\cap}}\left(h\left(T_l\right) > Y_l\right)$ is the sure event, i.e. $\overset{0}{\underset{l=1}{\cap}}\left(h\left(T_l\right) > Y_l\right) \equiv \Omega$.

Continuing (21), we have

$$P\left(T \leq z, Y > y \mid \xi\left(0, z\right] = j\right) = \sum_{k=1}^{j} P\left(\left(\bigcap_{l=1}^{k-1}\left(h\left(T_l\right) > Y_l\right)\right) \cap \left(h\left(T_k\right) + y < Y_k\right)\right), \tag{22}$$

where we have used the fact that the events are disjoint. Indeed, if we take two consecutive events,

$$\left(\bigcap_{l=1}^{k-1}\left(h\left(T_l\right) > Y_l\right)\right) \cap \left(h\left(T_k\right) + y < Y_k\right) \tag{23}$$

and

$$\left(\bigcap_{l=1}^{k}\left(h\left(T_l\right) > Y_l\right)\right) \cap \left(h\left(T_{k+1}\right) + y < Y_{k+1}\right), \tag{24}$$

then it is easy to see that the event $(h(T_k) + y < Y_k)$ from (23) and the event $(h(T_k) > Y_k)$ from (24) are independent for $y \geq 0$. Hence, the events in (23) and (24) are independent. In view of (22), we can equivalently rewrite equality (20) as

(25)
$$P(T \leq z, Y > y) = P(\xi(0,z] = 1)\, P\left(T_1 < h^{-1}(Y_1 - y)\right)$$
$$+ \sum_{j=2}^{\infty} P(\xi(0,z] = j)\, P\left(\left(\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}(Y_l)\right)\right) \cap \left(T_k < h^{-1}(Y_k - y)\right)\right),$$

where $(T_1 < h^{-1}(Y_1 - y))$ is the event of ruin at the first claim with deficit at least y and

$$G_k = \left(\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}(Y_l)\right)\right) \cap \left(T_k < h^{-1}(Y_k - y)\right), k = 2, 3, \ldots,$$

is the event of survival after the first $k-1$ claims have arrived and ruin at the k-th claim with deficit at least y. Let us now transform the probabilities in (25). By means of conditional probabilities, we have

$$\begin{aligned} P\left(T_1 < h^{-1}(Y_1 - y)\right) &= \int_0^{+\infty} P\left(T_1 < h^{-1}(y_1 - y)\right) f(y_1)\, dy_1 \\ &= \int_y^{+\infty} P\left(T_1 < h^{-1}(y_1 - y)\right) f(y_1)\, dy_1, \\ &= \int_y^{+\infty} \left(1 - P\left(T_1 > h^{-1}(y_1 - y)\right)\right) f(y_1)\, dy_1, \\ &= \int_y^{+\infty} \left(1 - P\left(\xi(0, h^{-1}(y_1 - y)] = 0\right)\right) f(y_1)\, dy_1 \end{aligned} \tag{26}$$

where the range of integration is $y_1 \in [y, +\infty]$ since for $0 < y_1 < y$, $h^{-1}(y_1 - y) = 0$ and $(T_1 < h^{-1}(y_1 - y)) \equiv (T_1 < 0)$, becomes the impossible event $\emptyset$. Let us now simplify $P(G_k)$, $k = 2, 3, \ldots$. We have

(27)
$$P(G_k) = \int_{0 \leq y_1 \leq \ldots \leq y_k} \!\!\ldots\! \int P\left(\left(\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}(y_l)\right)\right) \cap \left(T_k < h^{-1}(y_k - y)\right)\right) f(y_1, \ldots, y_k)\, dy_1 \ldots dy_k$$
$$= \int_{\substack{0 \leq y_1 \leq \ldots \leq y_{k-1} \leq y_k - y, \\ y_{k-1} \leq h(x)}} \!\!\ldots\! \int P\left(\left(\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}(y_l)\right)\right) \cap \left(T_k < h^{-1}(y_k - y)\right)\right) f(y_1, \ldots, y_k)\, dy_1 \ldots dy_k$$

where in the last equality in (27) we have cut off the domain of integration, $0 \le y_1 \le \ldots \le y_k$, where the conditional probability is 0. Setting $C_k = \{(y_1, \ldots, y_k) : 0 \le y_1 \le \ldots \le y_{k-1} \le y_{k-1} \le y_k - y, y_{k-1} \le h(z)\}$ and using the identity $P(A \cap B) = P(A) - P\left(A \cap \bar{B}\right)$, from (27) we obtain

(28)

$$P\left(G_k\right) = \int \underset{C_k}{\ldots} \int \left\{ P\left[\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}\left(y_l\right)\right)\right] \right.$$
$$\left. - P\left[\left(\bigcap_{l=1}^{k-1} \left(T_l > h^{-1}\left(y_l\right)\right)\right) \cap \left(T_k > h^{-1}\left(y_k - y\right)\right)\right]\right\} f\left(y_1, \ldots, y_k\right) dy_1 \ldots dy_k$$

From (25),(26) and (28), applying Lemma 2.9 to the probabilities on the right-hand side of (28) we obtain the asserted formula (6). □

The following corollaries of Theorem 2.4 give explicitly formulas for the joint distribution of the ruin time and deficit and for the finite time probability of ruin, under an OS claim arrival process defined as in Definition 1.1.

Corollary 2.10. *In the case of discrete claim amounts $W_1, W_2, \ldots$ with joint probability mass function $P_{w_1,\ldots,w_k} = P\left(W_1 = w_1, \ldots, W_k = w_k\right)$, $k = 1, 2, \ldots,$ we have*

(29)

$$P(T \le z, Y > y) = P\left(\xi(0, z] = 1\right)\left(1 - \sum_{w_1=1}^{m-1} P_{w_1} - \sum_{w_1=m}^{l} P_{w_1} \times P\left(\xi(0, h^{-1}(w_1 - y)) = 0\right)\right)$$
$$+ \sum_{j=2}^{l} P\left(\xi(0, z] = j\right) \sum_{k=1}^{j} \sum_{(w_1,\ldots,w_k)\in\tilde{C}_k} P_{w_1,\ldots,w_k}$$
$$\times \left\{A_{k-1}\left(1; F_z\left(h^{-1}\left(w_1\right)-\right), \ldots, F_z\left(h^{-1}\left(w_1 + \ldots + w_{k-1}\right)-\right)\right)\right.$$
$$\left. - A_k\left(1; F_z\left(h^{-1}\left(w_1\right)-\right), \ldots, F_z\left(h^{-1}\left(w_1 + \ldots + w_k - y\right)-\right)\right)\right\}$$

where $m = [y] + 1$,

$$l = \begin{cases} [h(z) + y] & \textit{if } h(z) + y \textit{ is not integer,} \\ h(z) + z - 1 & \textit{if } h(z) + y \textit{ is integer,} \end{cases}$$

with $[\cdot]$ *denoting the integer part,* $\tilde{C}_k = \{(w_1, \ldots, w_k) : 1 \le w_i, i = 1, \ldots, k, y < w_k, w_1 + \ldots + w_{k-1} \le h(z)\}$*, and* $A_k(1; F_z(z_1-), \ldots, F_z(z_k-))$ *the classical Appell polynomials evaluated at* $F_z(z) = 1$ *and defined as in (2).*

Let us note that (29) is an explicit and exact expression, involving only finite summations and is therefore appealing for numerical purposes. We now give some useful special cases of our main result.

COROLLARY 2.11. *The following expression for the probability of ruin* $P(T < z)$ *follows directly from Theorem 2.4.*

(30)

$$
\begin{aligned}
&P(T < z) = P(T < z, Y > 0)\\
&\quad = P(\xi(0,z] = 1) \int_0^\infty \left(1 - P\left(\xi(0, h^{-1}(y_1)] = 0\right)\right) f(y_1)\, dy_1\\
&\quad + \sum_{j=2}^{\infty} P(\xi(0,z] = j) \sum_{k=1}^{j} \int \ldots \int_{D_k} \Big\{ A_{k-1}\left(1; F_z\left(h^{-1}(y_1)-\right), \ldots, F_z\left(h^{-1}(y_{k-1})-\right)\right)\\
&\quad - A_k\left(1; F_z\left(h^{-1}(y_1)-\right), \ldots, F_z\left(h^{-1}(y_{k-1})-\right), F_z\left(h^{-1}(y_k)-\right)\right)\Big\} f(y_1, \ldots, y_k)\, dy_k \ldots dy_1,
\end{aligned}
$$

where $D_k = \{(y_1, \ldots, y_k) : 0 < y_1 < \ldots < y_{k-1} \le y_k, y_{k-1} \le h(z)\}$*, and* $A_j(1; F_z(z_1), \ldots, F_z(z_j))$*, are the classical Appell polynomials evaluated at* $F_z(z) = 1$ *and defined as in (2) with* $z_1 = h^{-1}(y_1), \ldots, z_j = h^{-1}(y_j)$*,* $j = 0, 1, 2 \ldots$*.*

The following corollary of Theorem 2.4, covers and extends the non-ruin formulas (4.1) and (4.2) and (4.8) obtained by Lefèvre and Picard (2011) for the particular case of an OS risk model with continuous $F_z(z)$.

COROLLARY 2.12. *For the probability of non-ruin* $P(T > z)$*, assuming the OS claim arrival process from Definition 1.1, we have*

(31)

$$
\begin{aligned}
&P(T > z) = P(\xi(0,z] = 0)\\
&\quad + \sum_{j=1}^{\infty} P(\xi(0,z] = j) \Bigg(1 - \sum_{k=1}^{j} \int \ldots \int_{D_k} \Big\{ A_{k-1}\left(1; F_z\left(h^{-1}(y_1)-\right), \ldots, F_z\left(h^{-1}(y_{k-1})-\right)\right)\\
&\quad - A_k\left(1; F_z\left(h^{-1}(y_1)-\right), \ldots, F_z\left(h^{-1}(y_{k-1})-\right), F_z\left(h^{-1}(y_k)-\right)\right)\Big\} f(y_1, \ldots, y_k)\, dy_k \ldots dy_1\Bigg),
\end{aligned}
$$

Proof of Corollary 2.12: From Corollary 2.11, multiplying both sides of (30) by -1 and adding one on each side gives

$$
\begin{aligned}
&1-P(T<z) \\
&\quad = 1-P(\xi(0,z]=1)\int_0^{\infty}\left(1-P\left(\xi(0,h^{-1}(y_1)]=0\right)\right) f\left(y_1\right) dy_1 \\
&\quad -\sum_{j=2}^{\infty} P(\xi(0,z]=j)\sum_{k=1}^{j}\int\underset{D_k}{\cdots}\int\Big\{A_{k-1}\left(1;F_z\left(h^{-1}(y_1)-\right),\dots,F_z\left(h^{-1}(y_{k-1})-\right)\right) \\
&\quad -A_k\left(1;F_z\left(h^{-1}(y_1)-\right),\dots,F_z\left(h^{-1}(y_{k-1})-\right),F_z\left(h^{-1}(y_k)-\right)\right)\Big\} f\left(y_1,\dots,y_k\right) dy_k\dots dy_1,
\end{aligned}
\tag{32}
$$

Applying the identity $1\equiv\sum_{j=0}^{\infty}P(\xi(0,z]=j)$ to express the unity on the right-hand side of (32), after some elementary algebra of summing up the factors (one of which equal to 1) multiplying each of the terms $P(\xi(0,z]=j)$, we obtain

$$
\begin{aligned}
&P(T>z)=1-P(T<z) \\
&\quad = P(\xi(0,z]=0)+P(\xi(0,z]=1)\left(1-\int_0^{\infty}\left(1-P\left(\xi(0,h^{-1}(y_1)]=0\right)\right) f\left(y_1\right) dy_1\right) \\
&\quad +\sum_{j=2}^{\infty} P(\xi(0,z]=j)\Bigg(1-\sum_{k=1}^{j}\int\underset{D_k}{\cdots}\int\Big\{A_{k-1}\left(1;F_z\left(h^{-1}(y_1)-\right),\dots,F_z\left(h^{-1}(y_{k-1})-\right)\right) \\
&\quad -A_k\left(1;F_z\left(h^{-1}(y_1)-\right),\dots,F_z\left(h^{-1}(y_{k-1})-\right),F_z\left(h^{-1}(y_k)-\right)\right)\Big\} f\left(y_1,\dots,y_k\right) dy_k\dots dy_1,\Bigg)
\end{aligned}
\tag{33}
$$

The asserted equation (31) now follows noting that the second term on the right-hand side of (33) can be added to the sum, for $j=1$. □

Let us note that Corollaries 2.11 and 2.12 generalize previous ruin probability formulas of Ignatov and Kaishev (2000, 2004, 2006) obtained for the Poisson case.

3. $P(T\le z, Y>y)$ for some special cases of the OS claim arrival process ξ . In this section we consider applications of our main result given by Theorem 2.4, which cover the two important special cases of OS claim arrival processes, i.e., when $F_z(x)$ is assumed continuous, or discontinuous. In the first case there are three models considered previously in the literature, which arise under the assumption of stationary transition probabilities of the OS process, namely, the (mixed) Poisson, negative binomial and binomial models (see e.g. Crump (1975) and Lefèvre and Picard (2011, 2014). In the second case of $F_z(x)$ with possible jump discontinuities, we consider purely discrete time OS point processes with claims arriving at some fixed instants, forming a sequence of positive real numbers, and general OS point processes with arrivals both at fixed and random instants. To the best of our knowledge such OS claim arrival models have not been considered in the risk and ruin literature.

3.1. *$P(T \leq z, Y > y)$ for OS processes generated by a continuous cdf $F_z(x)$.* When ξ has stationary transition probabilities, i.e., when $P(\xi(0,s] = j \mid \xi(0,t] = i)$, with $t < s$ and $i \leq j$, do not depend on s and t, it has been established by Crump (1975) that the OS process ξ must be one of the three processes: either a homogeneous Poisson process or a linear birth process with immigration, or a linear death process. We will revisit these three special cases of OS claim arrivals which have also been considered by Lefèvre and Picard (2011, 2014), who give particular expressions for the non-ruin probability. The latter expressions are directly covered and generalized, applying Corollary 2.12 and Theorem 2.4, i.e., evaluating $P(T > z)$ and $P(T \leq z, Y > y)$, substituting in (31) and (6) the specific expressions for $P(\xi(0,s] = j)$ and $F_z(z)$, for each of the three cases, as follows.

(a) If ξ is a Poisson process with rate λ, then $F_z(x) = x/z$, $0 \leq x \leq z$, which corresponds to $X \equiv 1$ and $\Lambda(z) \equiv \lambda z$ in representation (1). The latter holds also in the mixed Poisson case, randomizing λ;

(b) if ξ is a linear birth process with immigration, with birth rate $b > 0$ and immigration rate $a \geq 0$, then $\xi(0,z]$ has a negative binomial distribution, with parameters, a/b and $1 - e^{-bz}$ and $F_z(x) = (e^{bx} - 1)/(e^{bz} - 1)$;

(c) if ξ is the number of deaths in a linear death process with initial population size, ρ (positive integer), and death rate $d > 0$, then $\xi(0,z]$ has a binomial distribution with parameters, ρ and $1 - e^{-dz}$ and $F_z(x) = (1 - e^{-bx})/(1 - e^{-bz})$.

3.2. *$P(T \leq z, Y > y)$ for OS processes generated by a cdf $F_z(x)$ with possible jump discontinuities .* Let us note that OS processes defined as in Definition 1.1 form a new interesting class, whose characterization is outside the scope of this paper and will be considered separately. Here, our purpose will be to give examples of such OS processes of relevance to modeling insurance claim arrivals in the context of ruin.

First we consider the case when there are claim counts, $\xi_1, \xi_2, \ldots$ at some fixed instants $0 < t_1 < t_2 < \ldots$ which could be observed regularly e.g. monthly or annually, for say, j periods of time. Such data, called longitudinal (or panel) data is typically collected by insurance companies at individual and portfolio levels. In such cases it is common to seek for appropriate discrete distributions in order to model the marginal and the joint distribution of the r.v.s $\xi_1, \ldots, \xi_j$. Different count data models have been proposed, based on integer-valued (count) time series and random effect Poisson or negative binomial distributions (see e.g., Boucher et al. 2007 and Shi and Valdez 2014). Our purpose here will be to demonstrate, based on a simple example, that the general OS processes introduced by Definition 1.1 can also be used to model claim counts data in the context of ruin probability.

Define the sequence of independent Poisson distributed random variables, $\xi_i \in \mathcal{P}(\mu_i)$, $i = 1, 2, \ldots$ where ξ_1 is attached to t_1, ξ_2 to t_2 and so on, and the point (claim count) process $\xi(0, z] = \xi_1 \mathbb{1}_{(0,x]}(t_1) + \ldots + \xi_j \mathbb{1}_{(0,x]}(t_j)$, where

$$\mathbb{1}_{(0,x]}(u) = \begin{cases} 0 & \text{if } u > x, \\ 1 & \text{if } 0 < u \leq x. \end{cases}$$

We will define its underlying cdf $F_z(x)$, for $z \geq t_1$ since for $0 < z < t_1$, with probability one there will be no claims in the interval $(0, z]$ and it does not make sense to define $F_z(x)$, also because it does not appear in formula (6) of Theorem 2.4 and its corollaries. Assume that $t_j \leq z < t_{j+1}$. Then

$$F_z(x) = \begin{cases} 0 & \text{for } x < t_1, \\ \frac{\mu_1}{\mu_1 + \ldots + \mu_j} & \text{for } t_1 \leq x < t_2, \\ \ldots & \\ \frac{\mu_1 + \ldots + \mu_{j-1}}{\mu_1 + \ldots + \mu_j} & \text{for } t_{j-1} \leq x < t_j, \\ 1 & \text{for } \min(z, t_j) \leq x \end{cases} \tag{34}$$

and it can be seen that ξ is an OS process, where the claim clusters, $\xi_1, \ldots, \xi_j$ are concentrated at the points $0 < t_1 < \ldots < t_j$, respectively. Therefore, one can directly apply Theorem 2.4 and its corollaries to obtain various ruin related quantities. We will demonstrate how the non-ruin probability can be computed using Corollary 2.12, based on the following simple example.

Example 3.1. *Let $0 < t_1 < t_2 \leq z$. Then from (34) for $F_z(x)$, we have*

$$F_z(x) = \begin{cases} 0 & \text{for } -\infty < x < t_1, \\ \frac{\mu_1}{\mu_1 + \mu_2} & \text{for } t_1 \leq x < t_2, \\ 1 & \text{for } t_2 \leq x < +\infty. \end{cases}$$

Take the premium income function to be,

$$h(x) = \begin{cases} 0.5 & \text{for } x < t_1, \\ 1.5 & \text{for } t_1 \le x < t_2, \\ 2.5 & \text{for } t_2 \le x, \end{cases} \tag{35}$$

and the partial claim sums, $Y_1, Y_2, \ldots$ *to be the integers* $1, 2, \ldots,$ *respectively. Applying equation (31) of Corollary 2.12, for the non-ruin probability,* $P(T > z)$*, we have*

(36)

$$\begin{aligned}
P(T > z) &= P(\xi(0,z] = 0) + P(\xi(0,z] = 1)\left(1 - \left[A_0(1) - A_1\left(1; F_z\left(h^{-1}(1)-\right)\right)\right]\right) \\
&+ P(\xi(0,z] = 2)\left(1 - \left[A_0(1) - A_1\left(1; F_z\left(h^{-1}(1)-\right)\right) + A_1\left(1; F_z\left(h^{-1}(1)-\right)\right)\right.\right. \\
&- \left.\left. A_2\left(1; F_z\left(h^{-1}(1)-\right), F_z\left(h^{-1}(2)-\right)\right)\right]\right) \\
&= P(\xi(0,z] = 0) + P(\xi(0,z] = 1)\left(1 - \left[1 - A_1\left(1; F_z\left(t_1-\right)\right)\right]\right) \\
&+ P(\xi(0,z] = 2)\left(1 - \left[1 - A_2\left(1; F_z\left(t_1-\right), F_z\left(t_2-\right)\right)\right]\right) \\
&= \exp\left(-(\mu_1 + \mu_2)\right) + (\mu_1 + \mu_2)\exp\left(-(\mu_1 + \mu_2)\right)\left(1 - \left[1 - A_1\left(1; 0\right)\right]\right) \\
&+ \frac{(\mu_1 + \mu_2)^2}{2!}\exp\left(-(\mu_1 + \mu_2)\right)\left(1 - \left[1 - A_2\left(1; 0, \frac{\mu_1}{\mu_1 + \mu_2}\right)\right]\right) \\
&= \exp\left(-(\mu_1 + \mu_2)\right)\left(1 + \mu_1 + \mu_2 + \frac{\mu^2}{2} + \mu_1\mu_2\right),
\end{aligned}$$

where in the last equality we have used the fact that, $A_1(1; 0) = 1$ *and that* $A_2\left(1; 0, \frac{\mu_1}{\mu_1+\mu_2}\right) = \left(1 - \frac{\mu_1^2}{(\mu_1+\mu_2)^2}\right)$*,(c.f. (2)).*

Remark 3.2. *For simplicity and practical relevance, we have assumed in (35) that premiums are collected at the times* $0, t_1, t_2$*, but since* t_1, t_2 *are also the instants of the claim arrivals* ξ_1 *and* ξ_2*, the last expression in (36) does not depend on* t_1, t_2 *and* z*. Let us note that, in general,* $P(T > z)$ *will depend on the instants* $t_1 < t_2 < \ldots < t_j < z$*. To simplify the calculations, we have also assumed unit claim amounts, but in general, the expression for* $P(T > z)$ *will be more complex, involving the multiple integration with respect to the joint density* $f(y_1, \ldots, y_k)$*, following (31).*

Second, consider an OS point process, ξ with a continuous component and a pure jump component in the underlying cdf, $F_z(x)$, i.e., $\xi(0, z] = \eta(0, x] + \xi_1 \mathbb{1}_{(0,x]}(t_1) + \ldots + \xi_j \mathbb{1}_{(0,x]}(t_j)$, where $\eta(0, x]$ is a Poisson process with unit rate, defined on $(0, \infty]$ and independent of the Poisson random variables,

$\xi_1, \dots, \xi_j$, $\xi_i \in \mathcal{P}(\mu_i)$, $i = 1, \dots j$, assumed also mutually independent. By construction, ξ is an OS process with independent increments. It could be particularly suitable for applications, especially when data comes from two (or more) independent insurance portfolios (lines of business), among which one with claim frequency data at fixed instants $0 < t_1 < t_2 < \dots < t_j$ (e.g. annual observations) and a second one with data at policy level of the instants of claiming. In view of the Solvency II requirements, it would be instructive to be able to evaluate the probability of non-ruin in a finite time interval, $(0, z]$, due to claims coming from all lines of business. Without loss of generality, we will illustrate this, based on the following example.

EXAMPLE 3.3. *Let* $0 < t_1 < t_2 \le z$. *Then, the cumulative intensity function,* $\Lambda(x)$ *is*

$$\Lambda(x) = \begin{cases} x & \text{for } 0 < x < t_1, \\ x + \mu_1 & \text{for } t_1 \le x < t_2, \\ x + \mu_1 + \mu_2 & \text{for } t_2 \le x \le z, \end{cases}$$

and the related cdf

$$F_z(x) = \begin{cases} 0 & \text{for } -\infty < x < 0, \\ x/(z + \mu_1 + \mu_2) & \text{for } 0 \le x < t_1, \\ (x + \mu_1)/(z + \mu_1 + \mu_2) & \text{for } t_1 \le x < t_2, \\ (x + \mu_1 + \mu_2)/(z + \mu_1 + \mu_2) & \text{for } t_2 \le x \le z, \\ 1 & \text{for } z < x < +\infty. \end{cases}$$

As in Example 3.1, take the premium income function to be, given by (35) and the partial claim sums, $Y_1, Y_2, \dots$ *to be the integers* $1, 2, \dots$, *respectively.*

Applying equation (31) of Corollary 2.12, similarly as in (36), we have

(37)

$$
\begin{aligned}
P(T > z) &= P(\xi(0,z] = 0) + P(\xi(0,z] = 1)A_1\left(1; F_z\left(h^{-1}(1)-\right)\right) \\
&+ P(\xi(0,z] = 2)A_2\left(1; F_z\left(h^{-1}(1)-\right), F_z\left(h^{-1}(2)-\right)\right) \\
&= \exp\left(-(z+\mu_1+\mu_2)\right) + (z+\mu_1+\mu_2)\exp\left(-(z+\mu_1+\mu_2)\right)A_1\left(1; \frac{t_1}{z+\mu_1+\mu_2}\right) \\
&+ \frac{(z+\mu_1+\mu_2)^2}{2!}\exp\left(-(z+\mu_1+\mu_2)\right)A_2\left(1; \frac{t_1}{z+\mu_1+\mu_2}, \frac{t_2+\mu_1}{z+\mu_1+\mu_2}\right) \\
&= \exp\left(-(z+\mu_1+\mu_2)\right)\Bigg(1 + z(1+\mu_1+\mu_2+\frac{z}{2}-t_1) + t_1t_2 - \frac{t_2^2}{2} - \mu_1t_2 \\
&+\mu_1\mu_2 + \frac{\mu_2^2}{2} - \mu_2t_1 + \mu_1 + \mu_2 - t_1\Bigg),
\end{aligned}
$$

where in the last equality we have used (2) to evaluate $A_1\left(1; \frac{t_1}{z+\mu_1+\mu_2}\right)$ *and* $A_2\left(1; \frac{t_1}{z+\mu_1+\mu_2}, \frac{t_2+\mu_1}{z+\mu_1+\mu_2}\right)$.

REMARK 3.4. *Note that in contrast to (36) from Example 3.1, expression (37), depends on the instants* t_1, t_2, z*, which is due to the continuous time component,* η*, in* ξ*.*

REMARK 3.5. *The last expression in (37), can be verified through direct but tedious calculations. Thus, for the event,* $\{T > z\}$ *we have*

(38)

$$
\begin{aligned}
\{T > z\} = &\{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap \{\eta(0,z] = 0\} \cup \{\xi_1 = 1\} \cap \{\xi_2 = 0\} \cap \{\eta(0,z] = 0\} \\
&\cup \{\xi_1 = 0\} \cap \{\xi_2 = 1\} \cap \{\eta(0,z] = 0\} \cup \{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap \{\eta(0,z] = 1\} \\
&\cup \{\xi_1 = 0\} \cap \{\xi_2 = 2\} \cap \{\eta(0,z] = 0\} \cup \{\xi_1 = 1\} \cap \{\xi_2 = 0\} \cap \{\eta(0,t_2] = 0\} \cap \{\eta(t_2,z] = 1\} \\
&\cup \{\xi_1 = 0\} \cap \{\xi_2 = 1\} \cap \{\eta(0,t_1] = 0\} \cap \{\eta(t_1,z] = 1\} \\
&\cup \{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap \{\eta(0,t_1] = 0\} \cap \{\eta(t_1,t_2] = 1\} \cap \{\eta(t_2,z] = 1\} \\
&\cup \{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap \{\eta(0,t_2] = 0\} \cap \{\eta(t_2,z] = 2\}
\end{aligned}
$$

Applying probability on both sides of (38) we have

(39)

$$\begin{aligned}P(T > z) &= \exp(-\mu_1)\exp(-\mu_2)\exp(-z) + \mu_1\exp(-\mu_1)\exp(-\mu_2)\exp(-z)\\ &+ \exp(-\mu_1)\mu_2\exp(-\mu_2)\exp(-z) + \exp(-\mu_1)\exp(-\mu_2)z\exp(-z)\\ &+ \exp(-\mu_1)\frac{\mu_2^2}{2}\exp(-\mu_2)\exp(-z) + \mu_1\exp(-\mu_1)\exp(-\mu_2)\exp(-t_2)(z-t_2)\exp(-(z-t_2))\\ &+ \exp(-\mu_1)\mu_2\exp(-\mu_2)\exp(-t_1)(z-t_1)\exp(-(z-t_1))\\ &+ \exp(-\mu_1)\exp(-\mu_2)\exp(-t_1)(t_2-t_1)\exp(-(t_2-t_1))(z-t_2)\exp(-(z-t_2))\\ &+ \exp(-\mu_1)\exp(-\mu_2)\exp(-t_2)\frac{(t_2-z)^2}{2}\exp(-(t_2-z)),\end{aligned}$$

which, after some trivial algebraic transformations leads to the last expression in (37).

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