Selection of Decentralised Schemes and Parametrisation of the Decentralised Degenerate Compensators

N. Karcanias * **I. Meintanis** * **J. Leventides** **

* *Systems and Control Research Centre*
School of Mathematics, Computer Science and Engineering
City University London, Northampton Square, London EC1V 0HB, UK
(e-mail:N.Karcanias@city.ac.uk)
** *University of Athens, Department of Economics*
Section of Mathematics and Informatics
Pezmazoglou 8, Athens, Greece

Abstract: The design of decentralised control schemes has two major aspects. The selection of the decentralised structure and then the design of the decentralised controller that has a given structure and addresses certain design requirements. This paper deals with the parametrisation and selection of the decentralized structure such that problems such as the decentralised pole assignment may have solutions. We use the approach of global linearisation for the asymptotic linearisation of the pole assignment map around a degenerate compensator. Thus, we examine in depth the case of degenerate compensators and investigate the conditions under which certain degenerate structures exist. This leads to a parametrisation of decentralised structures based on the structural properties of the system.

Keywords: Linear multivariable systems; output feedback control; algebraic methods; geometric methods; decentralised pole assignment; decentralised control.

1. INTRODUCTION

The selection of decentralization scheme for a process is an important issue within the overall area of control systems design and control structure selection (Anderson and Linneman, 1984), (Corfmat and Morse, 1973), (McAvoy, 1983), (Siljak, 1991), (Leventides and Karcanias, 1998, 2006), (Wang, 1994). This problem has two major aspects: The selection of the decentralized structure and then the design of the decentralized controller that has a given structure and addresses certain design requirements. So far, the design of the decentralization has been addressed as an issue within the process control area where it is known as interaction analysis (McAvoy, 1983). This traditional approach is based on the selection of couplings with criterion the minimization of interactions, which may not necessarily be a design objective. On the other hand, issues of systematic design of decentralization have not been properly addressed within the mainstream control theory and design. However a number of important results have emerged which are based on the ability of a system and a given decentralization scheme to accept certain types of control solutions and are distinguished to two types (Karcanias et al., 1988), (Leventides and Karcanias, 1998, 2006):

(a) Those relying on generic models (under certain assumptions) and generic solvability of control problems;
(b) Those dealing with specific systems, decentralization schemes and exact properties.

In this paper we aim to specify forms of decentralised control structures that allow the solvability of pole assignment by constant decentralised compensators.

Both types of results may be used for developing diagnostics and a systematic methodology for selection of decentralised schemes. We specialize the results on the centralized DAP (Leventides and Karcanias, 1995),(Karcanias et al., 2013), to the case of the decentralised control problem and we use the resulting set of structural characteristics to derive diagnostics for the selection of the possible decentralisation structures. We consider the general diagnostics of the DAP framework (Karcanias and Giannakopoulos, 1984) based on exterior algebra to the case of decentralised control problems. The DAP approach deals with the study of formation of closed loop poles (under feedback compensation) and zeros (under squaring down) and hence the diagnostics of this framework relate to frequency assignment, fixed, almost-fixed frequency properties and diagnostics for non-minimum phase properties.

Our aim is to develop the parametrisation and selection of the decentralised structure such that problems such as the decentralised pole assignment may have solutions. We use the approach of global linearisation for the asymptotic linearisation of the pole assignment map around a degenerate compensator (Brockett and Byrnes, 1981). Thus, we examine in depth the case of degenerate compensators and investigate the conditions under which certain degenerate structures exist. This leads to a parametrisation of decen-

tralised structures based on the structural properties of the system. We use the notion of decentralisation characteristic and the resulting structural invariants to predict properties of the decentralised control scheme (Karcanias et al., 1988), (Leventides and Karcanias, 2006). More specifically, we use as structural criteria the ranks of decentralised Plücker matrices, the fixed and almost-fixed modes and the different types of structural decentralisation criteria. The results provide the basis for the development of a novel structural framework for screening the alternative decentralisation schemes that may be possible for a given system. The essence of the decentralisation assumption is that the controller is partially fixed (block diagonal). Central to this approach is the notion of decentralisation characteristic, which expresses the effect of decentralisation on the design problem. We aim to provide criteria for evaluating the relative advantages and disadvantages of the alternative schemes. The analysis covers both static and dynamic problems. However, we focus our study on the static output feedback compensation schemes.

The paper is structured as follows: In section 2 we define the Decentralised DAP whereas in section 3 we consider the concept of degeneracy for the decentralised case. In section 4 we develop a parametrisation of decentralised degenerate compensators and in section 5 we define the notion of structurally compatible partitions and provide an algorithm for their computation. Due to page limitations the results are stated without proof which they may be found in (Karcanias et al., 2014).

2. THE DECENTRALISED POLE ASSIGNMENT PROBLEM

2.1 Problem Statement

Consider linear multivariable systems described by the proper rational transfer function matrix $G(s) \in \mathbb{R}^{m\times p}$ of McMillan degree n. We assume that we have a $k-$channel decentralisation scheme, where $k \leqslant \min(m,p)$, defined by the $k-$partition of the input, output vectors $\underline{u} \in \mathbb{R}^p$ and $\underline{y} \in \mathbb{R}^m$. For a given pair (m,p) we also define the sets of integers, introduced by partitioning of m, p as:

$$\{m\} = \{m_i,\, m_i \geqslant 1,\ \sum_{i=1}^{k} m_i = m\}$$

$$\{p\} = \{p_i,\, p_i \geqslant 1,\ \sum_{i=1}^{k} p_i = p\}$$

where it is also assumed that $m_i \geqslant p_i,\ \forall i \in \tilde{k}$. The set $\mathcal{I}_D = (\{m\},\{p\};k)$ will be called a *decentralisation index*. If local feedback laws of the following type

$$\underline{u}_i(s) = C_i(s)\underline{y}_i(s),\ i = 1,2,\ldots,k$$

are applied to each channel, then the closed-loop transfer function is given by $G(s)[I + C(s)G(s)]^{-1}$, with $C(s) = diag\{C_1(s),\ldots,C_k(s)\} \in \mathbb{R}^{p\times m}(s)$, $C_i(s) \in \mathbb{R}^{p_i\times m_i}(s)$ representing the controller transfer function matrix. It is well known that the closed-loop pole polynomial of this feedback system is given by

$$p(s) = det\{[A(s),B(s)]\cdot\begin{bmatrix} D(s) \\ N(s)\end{bmatrix}\} = det\{H(s)\cdot M(s)\}$$

where, $A(s)^{-1}B(s)$ is a left coprime MFD for $C(s)$ and $N(s)D(s)^{-1}$ is a right coprime MFD of $G(s)$ and $M(s) = [D^t(s),N^t(s)]^t$, $H(s) = [A(s),B(s)]$ are the composite descriptions of $G(s)$ and $C(s)$ respectively. It is easy to see that the structured controller matrix $H(s)$ can be written as $\left[\bar{A}(s);B(\bar{s})\right]$, where $\bar{A}(s) = bl.diag\{A_1(s),\ldots,A_k(s)\}$ and $\bar{B}(s) = bl.diag\{B_1(s),\ldots,B_k(s)\}$. If we also partition $D(s)$, $N(s)$ in a compatible way, by a simple reordering of the blocks we may define the following problems:

(i) The *dynamic decentralised pole assignment problem*: Given an arbitrary symmetric set of poles by $p(s)$, solve the structured determinantal equation

$$p(s) = det\left\{bl.diag\left[H_1(s),\ldots,H_k(s)\right]\cdot\begin{bmatrix} M_1(s) \\ \vdots \\ M_k(s)\end{bmatrix}\right\}$$
$$= det\left\{H_{dec}(s)\cdot M_{dec}(s)\right\} \quad (1)$$

with respect to the decentralised controller $H_{dec}(s)$, where $H_i(s) = [A_i(s),B_i(s)]$ and $M_i(s) = [D_i^t(s),N_i^t(s)]^t$.

(ii) The *static decentralised pole assignment problem*: Given an arbitrary symmetric set of poles, solve the structured determinantal equation

$$p(s) = \det\{\begin{bmatrix} I_{p_1} & 0 & 0 & 0 & H_1 & 0 & 0 & 0 \\ 0 & I_{p_2} & 0 & 0 & 0 & H_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & I_{p_k} & 0 & 0 & 0 & H_k \end{bmatrix}\begin{bmatrix} D_1(s) \\ \vdots \\ D_k(s) \\ N_1(s) \\ \vdots \\ N_k(s)\end{bmatrix}\}$$
$$= \det\left\{[I_p;H_{dec}]\cdot M_{dec}(s)\right\} \quad (2)$$

with respect to the constant structured matrix $[I_p;H_{dec}]$.

Remark 1. It is well known that every dynamic DAP can be formulated to an equivalent constant DAP (Karcanias, 2013) and hence the overall study will be focused to the static version of the problem.

□

3. DEGENERACY OF THE DECENTRALISED OUTPUT FEEDBACK CONFIGURATION

We start by giving the following definition:

Definition 2. (Degeneracy of the feedback configuration). A decentralised controller $H_{dec}(s)$ is *degenerate* if the closed-loop system is not well posed, i.e if

$$det\{H_{dec}(s)\cdot M_{dec}(s)\} \equiv 0 \quad (3)$$

holds true.

□

The conditions for existence of dynamic decentralised degenerate controllers are summarized below (Leventides and Karcanias, 2006). Let us denote by $\mathcal{M} = col.span\{M(s)\}$.

Proposition 3. A polynomial matrix

$$H_{dec}(s) = bl.diag\left[H_1(s),\cdots,H_k(s)\right]$$

corresponds to a degenerate compensator of the feedback configuration, if and only if, either of the following equiv-

alent conditions holds true:
(i) There exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s) \in \mathcal{M}$ such that

$$H_{dec}(s) \cdot \underline{m}(s) = 0,\ \forall s \in \mathbb{C}$$

(ii) There exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s) \in \mathcal{M}$, which if partitioned (conformally with the decentralised controller) into the set of $(m_i+p_i)\times 1$ polynomial vectors, we have that:

$$H_i(s) \cdot m_i(s) = 0$$

for $\underline{m}_i(s) \in \mathcal{M}_i,\ i = 1, \ldots, k$.

□

When constant structured matrices are considered as gains, we may define for any given $\mathcal{I}_D$, the corresponding composite output feedback constant decentralised gain as

$$[I_p; H_{dec}] = \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & H_1 & 0 & 0 & 0 \\ 0 & I_{p_2} & 0 & 0 & 0 & H_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & I_{p_k} & 0 & 0 & 0 & H_k \end{bmatrix}$$

where, $H_i \in \mathbb{R}^{p_i \times m_i},\ \forall i \in \tilde{k}$.

Definition 4. We define as a *Decentralised Generalised Gain* (DGG) the matrix:

$$\begin{bmatrix} H_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H_k \end{bmatrix} = \begin{bmatrix} A_1, B_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k, B_k \end{bmatrix}$$

and such a gain will be called a *constant* Decentralised Degenerate Gain (D-DG) if and only if

$$\det\{H_{dec} \cdot M_{dec}(s)\} \equiv 0 \tag{4}$$

□

For a generator $\underline{m}(s) \in \mathcal{M}$ and a decentralisation index $\mathcal{I}_D$, the $\underline{m}^*(s)$ will denote the corresponding permuted vector. The family of all generators that lead to degenerate gains will be denoted by $\mathcal{D}$. The conditions for existence of constant D-DG for a given $\underline{m}(s) \in \mathcal{D}$ are discussed next.

Theorem 5. For a system with dimensions (n, m, p) and decentralisation index $\mathcal{I}_D = (\{m\}, \{p\}; k)$, let $\underline{m}(s) \in \mathcal{D}$ and denote by

$$\underline{m}^*(s) = P^* \underline{e}_\delta(s) \in \mathcal{M}^*$$

the corresponding permuted generator vector and let us consider $P^* \in \mathbb{R}^{(p+m)\times(\delta+1)}$ partitioned into $k-$blocks, according to $\mathcal{I}_D$, as indicated below:

$$P^* = \begin{bmatrix} P_1 \\ \vdots \\ P_i \\ \vdots \\ P_k \end{bmatrix} \begin{matrix} \\ \updownarrow p_1 + m_1 \\ \\ \updownarrow p_i + m_i \\ \\ \updownarrow p_k + m_k \end{matrix} \tag{5}$$

If $\mathcal{L}_{\underline{m}^*}$ is the $\underline{m}^*(s)-$DDG family, then $\mathcal{L}_{\underline{m}^*}$ contains decentralised gains with $\mathcal{I}_D-$characteristic if and only if:

$$m_i \geqslant rank(P_i),\ \forall i \in \tilde{k}$$

and this family is defined by

$$\mathcal{L}_{\underline{m}^*} = \{H_{dec} : H_{dec} = bl.diag\{\cdots; H_i; \cdots\} :\ H_i P_i = 0\}$$

where, $rank(H_i) = P_i,\ \forall i \in \tilde{k}$.

□

Proposition 6. Given that

$$rank\{P_i\} \leqslant rank\{P\} \leqslant \delta + 1$$

where, $\delta = \partial\{m(s)\}$, it follows that a sufficient condition for the existence of a decentralised degenerate gain in $\mathcal{L}_{\underline{m}^*}$, or equivalently $\mathcal{L}_{\underline{m}}$, is that:

$$m_i \geqslant \delta + 1,\ \forall i \in \tilde{k}. \tag{6}$$

Obviously, the smaller the degree of $\underline{m}(s)$, easier it is to find decentralised degenerate solutions.

□

4. PARAMETRISATION OF THE DECENTRALISED DEGENERATE COMPENSATORS

In this section we will define the Gain Degeneracy Set $< \mathcal{L} >$ and its structural properties which are related with the presence of decentralized elements. Without loss of generality we consider the case where $m \geqslant p$ for a given generator vector denoted as $\underline{m}(s) \in \mathcal{D}$ and we study under which conditions the set $\mathcal{L}_{\underline{m}}$ contains at least one decentralised element.

The set of coefficient matrices $\{P_i \in \mathbb{R}^{(p_i+m_i)\times(\delta+1)},\ i \in \tilde{k}\}$ corresponding to a given $\underline{m}(s) \in \mathcal{D}$ and for a given decentralisation scheme will be denoted in short by $\{P; \mathcal{I}_D\}$ and referred to as $\mathcal{I}_D-$partition of P. If the conditions (6) are satisfied for this set, then $\{P; \mathcal{I}_D\}$ will be called an $\mathcal{I}_D-$*Compatible Partition* ($\mathcal{I}_D-$CP); clearly, for a given $\underline{m}(s) \in \mathcal{D}$ there may be more than one $\mathcal{I}_D-$CPs and the family of all such compatible $\mathcal{I}_D-$partitions will be denoted by $\{P; [\mathcal{I}_D]\}$. The search for all $\mathcal{I}_D-$CPs for a given $\underline{m}(s)$ and then for the whole family $\mathcal{D}$ will be considered subsequently. Before we start this investigation we give a general result on the parametrisation expression of the corresponding families.

Proposition 7. Let $\underline{m}(s) = P \cdot \underline{e}_\delta(s) \in \mathcal{D}$ and $\mathcal{N} = \mathcal{N}_\ell\{P\}$, where $\tau = \dim\{\mathcal{N}\} \geqslant p$. The following properties hold true:

(i) If $N \in \mathbb{R}^{\tau\times(m+p)}$ is a basis matrix for $\mathcal{N}$, then the family $\mathcal{L}_{\underline{m}}$ is defined by:

$$\mathcal{L}_{\underline{m}} = \{L : LP = 0,\ L \in \mathbb{R}^{p\times(p+m)},\ rank\{L\} = p\}$$

and parametrically all L are defined in terms of a free parameter $T \in \mathbb{R}^{p\times\tau}$ by

$$L = TN :\ rowsp\{T\} \cap \mathcal{N}_\ell\{N\} = \{0\}$$

Furthermore, if $N = [N_1, N_2]$, $N_1 \in \mathbb{R}^{\tau\times p}$, then the subfamily of $\mathcal{L}_{\underline{m}}$ of all regular-type gains, $\mathcal{L}^r_{\underline{m}}$, is provided by

$$L = TN,\ |TN_1| \neq 0.$$

(ii) If $\mathcal{I}_D$ is a CP of P and $\{P_i \in \mathbb{R}^{(p_i+m_i)\times(\delta+1)}, i \in \tilde{k}\}$ is the corresponding set, $\mathcal{N}_i = \mathcal{N}_\ell\{P_i\}$, and $N_i \in \mathbb{R}^{\tau_i\times(p_i+m_i)}$ is a basis matrix for $\mathcal{N}_i$, then the decentralized subfamily of $\mathcal{L}_{\underline{m}}$ is defined by:

$$\mathcal{L}_{\underline{m}}(\mathcal{I}_D) = \{H : H = bl.diag\{H_i,\ i \in \tilde{k}\},$$
$$H_i \in \mathbb{R}^{p_i\times(p_i+m_i)},\ rank\{H_i\} = p_i,\ H_i P_i = 0\}$$

and parametrically, all H_i are defined in terms of $T_i \in \mathbb{R}^{p_i\times\tau_i}$ by:

$$H_i = T_i N_i :\ rowsp\{T_i\} \cap \mathcal{N}_\ell\{N\} = \{0\}$$

Furthermore, the sub-family of $\mathcal{L}_{\underline{m}}(\mathcal{I}_D)$ of all regular $\mathcal{I}_D-$decentralised solutions is given by

$$H_i = T_i N_i,\ |T_i N_{i1}| \neq 0,\ \forall i \in \tilde{k}$$

where, $N_i = [N_{i1}, N_{i2}]$, $N_{i1} \in \mathbb{R}^{\tau_i \times p_i}$.

□

5. THE SET OF STRUCTURALLY COMPATIBLE PARTITIONS

In this section we are going to examine the following problem: For a given generator element $\underline{m}(s) \in \mathcal{D}$, define the family of all $\mathcal{I}_D-$compatible partitions. The existence of a family of $\mathcal{I}_D-$CP is established below.

Proposition 8. Let $\underline{m}(s) = P \cdot \underline{e}_\delta(s) \in \mathcal{D}$ of a system with dimensions (n, p, m), $r = rank\{P\} \leqslant \delta + 1$ and let $\bar{k}$ be the integer defined by

$$\bar{k} = \max\{k \in \mathbb{Z}_{>0} : k \leqslant m/r\}$$

If $\bar{k} \geqslant 2$, then for any $k : 2 \leqslant k \leqslant \bar{k}$ there exist $\mathcal{I}_D-$CP, $\mathcal{I}_D = (\{m\}, \{p\}; k)$, defined by certain $k-$partitions of m, p and satisfying the following conditions:

$$m_i \geqslant r, \quad m_i \geqslant p_i,\ \forall i = 1, 2, \ldots, k. \tag{7}$$

□

For any generator vector $\underline{m}(s) \in \mathcal{D}$ corresponding to a system with $(n, p, m)-$dimensions and with $r = rank\{P\}$, Proposition 8 defines a set of non-trivial ($k \neq 1$) $\mathcal{I}_D-$CP, which are independent from the numerical values of the corresponding partitioned matrices P_i. Such a set will be denoted by $\{\mathcal{I}_D; \underline{m}\}$ and referred to as the set of *Structurally Compatible Partitions* (SCP) to $\underline{m}(s)$, since its description does not depend on the values of elements of P, but only on (m, p, r) numbers. It is clear that there is a need for generating all possible compatible partitions in a systematic algorithmic way and this is what it is considered next. We first give some useful notation and definitions.

Notation: If $1 \leqslant k \leqslant n$, then $\tilde{G}_{k,n}$ denotes the totality of non-increasing sequences of k integers chosen from $\{1, 2, \ldots, n\}$ and $S_{k,n}$ is the totality of sequences of k integers chosen from $\{1, 2, \ldots, n\}$. For instance, $\tilde{G}_{2,3}$ is the set $\{(3,3), (3,2), (2,2), (3,1), (2,1), (1,1)\}$. Furthermore, $\prod_n$ designates the symmetric group of permutations on the set $\{1, 2, \ldots, n\}$. If $\{i_1, i_2, \ldots, i_k\} \in S_{k,n}$, then $\pi(i_1, i_2, \ldots, i_k) = \{\pi(i_1), \pi(i_2), \ldots, \pi(i_k)\}$ denotes the corresponding permutation of $\{i_1, i_2, \ldots, i_k\}$ for every $\pi \in \prod_k$. $\tilde{G}_{k,n}$ has $\binom{n+k-1}{k}$ sequences in it, where $S_{k,n}$ has n^k sequences in it.

Definition 9. On sequences of $S_{k,n}$ the following operations are defined:

(i) Let $\alpha_1 = (i_1, i_2, \ldots, i_k)$, $\alpha_2 = (j_1, j_2, \ldots, j_k)$, $\alpha_1, \alpha_2 \in S_{k,n}$. We define their $*$ product as the set:

$$\alpha_1 * \alpha_2 = (i_1, i_2, \ldots, i_k) * (j_1, j_2, \ldots, j_k) \triangleq \mathfrak{X}(\alpha_1, \alpha_2)$$

where,

a) If for some $\mu \in \tilde{k}$, $i_\mu < j_\mu$, then $\mathfrak{X}(\alpha_1, \alpha_2) = \emptyset$

b) If $i_\mu \geqslant j_\mu$, $\forall \mu \in \tilde{k}$ then we define:

$$\mathfrak{X}(\alpha_1, \alpha_2) = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$$

(ii) If $\alpha_1 = (i_1, i_2, \ldots, i_k)$, $\alpha_2 = (j_1, j_2, \ldots, j_k)$, $\alpha_1, \alpha_2 \in S_{k,n}$, then we define as the $[\cdot, \cdot]$ product the set:

$$\begin{aligned}[\alpha_1, \alpha_2] &= \bigcup_{\forall \pi, \pi' \in \prod_k} \pi(i_1, i_2, \ldots, i_k) * \pi'(j_1, j_2, \ldots, j_k) \\ &= \bigcup_{\forall \pi, \pi' \in \prod_k} \mathfrak{X}(\pi\alpha_1, \pi'\alpha_2)\end{aligned}$$

where, $\pi\alpha_1 = \pi(i_1, i_2, \ldots, i_k)$, $\pi'\alpha_2 = \pi(j_1, j_2, \ldots, j_k)$ are the permutations of α_1, α_2 corresponding to $\pi, \pi' \in \prod_k$ respectively.

(iii) Consider now two sets $\mathcal{A} = \{\alpha_1, \ldots, \alpha_\mu : \alpha_i \in S_{k,n}\}$, $\mathcal{B} = \{\beta_1, \ldots, \beta_\nu :\ \beta_i \in S_{k,n}\}$. We define as the $[\cdot, \cdot]$ product of the two sets the set:

$$[\mathcal{A}, \mathcal{B}] = \{[\alpha_i, \beta_j]\,;\ \forall \alpha_i \in \mathcal{A}, \forall \beta_j \in \mathcal{B}\}$$

□

To demonstrate the operations defined above we give the following example:

Example 1: i) Let $\alpha_1 = (4, 2, 1)$, $\alpha_2 = (2, 2, 1) \in S_{3,4}$. For the various permutations we have:

$$\begin{aligned}(4,2,1) * (2,2,1) &= \{(4,2), (2,2), (1,1)\} \\ (2,4,1) * (2,2,1) &= \{(2,2), (4,2), (1,1)\} \\ (2,1,4) * (2,2,1) &= \emptyset \\ (4,1,2) * (2,2,1) &= \emptyset\end{aligned}$$

By going through all the possible permutation it is clear that either we get $\emptyset$, or the identical sets previously defined and since repetitions and $\emptyset$ do not count in the union we have:

$$\mathfrak{X}(a_1, a_2) = [(4,2,1), (2,2,1)] = \{(4,2), (2,2), (1,1)\}$$

ii) Let $\alpha_1 = (4, 3, 1)$, $\alpha_2 = (3, 2, 1) \in S_{3,4}$. For the various permutations we have:

$$\begin{aligned}(4,3,1) * (3,2,1) &= \{(4,3), (3,2), (1,1)\} \\ (4,3,1) * (2,3,1) &= \{(4,2), (3,3), (1,1)\} \\ (4,3,1) * (2,1,3) &= \emptyset \\ (4,3,1) * (1,3,2) &= \emptyset \\ (4,3,1) * (1,2,2) &= \emptyset \\ &\vdots\end{aligned}$$

Going through all the permutations it can be seen that:

$$\mathfrak{X}(a_1, a_2) = \{(4,3), (3,2), (1,1); (4,2), (3,3), (1,1)\}$$

□

5.1 Algorithm for Computing SCP

From the definitions given so far, it can readily proved the following result.

Theorem 10. For every system with (n, p, m) dimensions and any generator $\underline{m}(s) = P \cdot \underline{e}_\delta(s) \in \mathcal{D}$ with $r = rank\{P\} \leqslant \delta + 1$, the set of all Structurally Compatible Partitions of the (m, p) pair is given by

$$\{m, p; r\} = \mathfrak{X}(m, p; r) = \bigcup_{k=1}^{k=\bar{k}} [\mathfrak{X}(m), \mathfrak{X}(p)] \tag{8}$$

□

For a given $\underline{m}(s) = P \cdot e_\delta(s) \in \mathcal{D}$ with $r = rank\{P\} \leqslant \delta + 1$, the set of all SCP of the pair (m, p), $m \geqslant p$, $\{m, p; r\}$ may be constructed in a systematic algorithmic way as shown below in Algorithm 1.

Algorithm 1 Algorithm for Computing the Set of SCP

1: Compute the number of possible channels, i.e.

$$\bar{k} = \min\{p, \max\{k \in \mathbb{Z}_{>0} : k \leqslant m/r\}\}$$

The range of values: $1 \leqslant k \leqslant \bar{k}$, denotes the possible orders of decentralisation and if $\bar{k} \geqslant 2$ then proceed to the generation of the SCP of m, p; otherwise, if $\bar{k} = 1$, the only SCP is the trivial one, i.e. the centralised.

2: Compute the non-trivial candidate partitions of m and for each $k : 2 \leqslant k \leqslant \bar{k}$ perform the following steps:

(2.1) Compute the integer: $\tau = m - rk \in \mathbb{Z}_{\geqslant 0}$

(2.2) Define all possible $k-$partitions of τ over $\mathbb{Z}_{\geqslant 0}$

$$\sum\nolimits_k (m) \triangleq \{(i_1, i_2, \ldots, i_k), \; i_1 + \ldots + i_k = \tau,\, i_1 \geqslant i_2 \geqslant \ldots \geqslant i_k \geqslant 0\}$$

(2.3) For every $(i_1, i_2, \ldots, i_k) \in \sum_k (m)$ define the set of $k-$partitions of m represented in an ordered form:

$$\mathfrak{X}_k(m) \triangleq \{(k_1, k_2, \ldots, k_k) = (r + i_1, \ldots, r + i_k)\}$$

3: Compute the non-trivial candidate partitions of p and for each $k : 2 \leqslant k \leqslant \bar{k}$ perform the following steps:

(3.1) Compute the integer $\upsilon = p - k \in \mathbb{Z}_{\geqslant 0}$

(3.2) Define all possible $k-$partitions of υ over $\mathbb{Z}_{\geqslant 0}$

$$\sum\nolimits_k (p) \triangleq \{(j_1, j_2, \ldots, j_k), \; j_1 + \ldots + j_k = \upsilon,\, j_1 \geqslant j_2 \geqslant \ldots \geqslant j_k \geqslant 0\}$$

(3.3) For every $(j_1, j_2, \ldots, j_k) \in \sum_k (p)$ define the set of $k-$partitions of p represented in an ordered form:

$$\mathfrak{X}_k(p) \triangleq \{(\ell_1, \ell_2, \ldots, \ell_k) = (r + j_1, \ldots, r + j_k)\}$$

4: For every $k : 2 \leqslant k \leqslant \bar{k}$ define the product of the $\mathfrak{X}_k(m)$, $\mathfrak{X}_k(p)$ sets

$$[\mathfrak{X}_k(m), \mathfrak{X}_k(p)] \triangleq \{[(k_1, k_2, \ldots, k_k), (\ell_1, \ell_2, \ldots, \ell_k)], \; \forall (k_1, k_2, \ldots, k_k) \in \mathfrak{X}_k(m),\, \forall (\ell_1, \ell_2, \ldots, \ell_k) \in \mathfrak{X}_k(p)\}$$

as well as compute the set

$$\mathfrak{X}(m, p; r) = \bigcup_{k=1}^{\bar{k}} [\mathfrak{X}(m), \mathfrak{X}(p)] \tag{9}$$

5.2 Examples

We may demonstrate the algorithmic construction of all possible SCP with the following example.

Example 2: Consider a system with $p = 6$ inputs and $m = 8$ outputs and let $\underline{m}(s) = P \cdot \underline{e}_2(s) \in \mathcal{D}$ with $r = rank\{P\} = 2$. Finding the set of all structurally compatible partitions involves the steps:

Step (1): Compute $\bar{k}$, i.e.

$$\bar{k} = \min\{6, \max\{k : k \leqslant 8/2 = 4\}\} = 4$$

Thus, the set of possible orders of decentralisation are:

$$\bar{k} = 1, 2, 3, 4$$

Steps (2),(3): Compute the non-trivial partitions of m, p:

- For $\bar{k} = 2$:
  Partitions of m: $\tau = 8 - 2 \cdot 2 = 4$
  $$\left.\begin{array}{l}(4,0) \to (6,2)\\(3,1) \to (5,3)\\(2,2) \to (4,4)\end{array}\right\} \to \mathfrak{X}_2(m) = \{(6,2), (5,3), (4,4)\}$$
  Partitions of p: $\upsilon = 6 - 2 \cdot 1 = 4$
  $$\left.\begin{array}{l}(4,0) \to (5,1)\\(3,1) \to (4,2)\\(2,2) \to (3,3)\end{array}\right\} \to \mathfrak{X}_2(p) = \{(5,1), (4,2), (3,3)\}$$
- For $\bar{k} = 3$:
  Partitions of m: $\tau = 8 - 3 \cdot 2 = 2$
  $$\left.\begin{array}{l}(2,0,0) \to (4,2,2)\\(1,1,0) \to (3,3,2)\end{array}\right\} \to \mathfrak{X}_3(m) = \{(4,2,2), (3,3,2)\}$$
  Partitions of p: $\upsilon = 6 - 3 \cdot 1 = 3$
  $$\left.\begin{array}{l}(3,0,0) \to (4,1,1)\\(2,1,0) \to (3,2,1)\\(1,1,1) \to (2,2,2)\end{array}\right\} \to$$
  $$\mathfrak{X}_3(p) = \{(4,1,1), (3,2,1), (2,2,2)\}$$
- For $\bar{k} = 4$:
  Partitions of m: $\tau = 8 - 4 \cdot 2 = 0$
  $$\mathfrak{X}_4(m) = \{(2,2,2,2)\}$$
  Partitions of p: $\upsilon = 6 - 4 \cdot 1 = 2$
  $$\left.\begin{array}{l}(2,0,0,0) \to (3,1,1,1)\\(1,1,0,0) \to (2,2,1,1)\end{array}\right\} \to$$
  $$\mathfrak{X}_4(p) = \{(3,1,1,1), (2,2,1,1)\}$$

Step (4): Compute the SCPs. We have the following *decentralisation cases*:

Case $\bar{k} = 1$: $(m, p) \equiv (8, 6)$, i.e. the trivial centralised case.

Case $\bar{k} = 2$: Compute $[\mathfrak{X}_2(8), \mathfrak{X}_2(6)]$, i.e.

$$\begin{aligned}[\mathfrak{X}_2(8), \mathfrak{X}_2(6)] = \{&[(6,2),(5,1)]\,;[(6,2),(4,2)]\,;[(6,2),(3,3)]\,;\\ &[(5,3),(5,1)]\,;[(5,3),(4,2)]\,;[(5,3),(3,3)]\,;\\ &[(4,4),(5,1)]\,;[(4,4),(4,2)]\,;[(4,4),(3,3)]\}\end{aligned}$$

which gives the following:

$$\begin{aligned}[(6,2),(5,1)] &= \{(6,5),(2,1)\}\\ [(6,2),(4,2)] &= \{(6,4),(2,2)\}\\ [(6,2),(3,3)] &= \emptyset\\ [(5,3),(5,1)] &= \{(5,5),(3,1)\}\\ [(5,3),(4,2)] &= \{(5,4),(3,2)\}\\ [(5,3),(3,3)] &= \{(5,3),(3,3)\}\\ [(4,4),(5,1)] &= \emptyset\\ [(4,4),(4,2)] &= \{(4,4),(4,2)\}\\ [(4,4),(3,3)] &= \{(4,3),(4,3)\}\end{aligned}$$

Therefore, all the $\bar{k} = 2-$SCP are:

$$\begin{aligned}[\mathfrak{X}_2(8), \mathfrak{X}_2(6)] = \{&\{(6,5),(2,1)\};\{(6,4),(2,2)\};\\ &\{(5,5),(3,1)\};\{(5,4),(3,2)\};\{(5,3),(3,3)\};\\ &\{(4,4),(4,2)\};\{(4,3),(4,3)\}\}\end{aligned}$$

Case $\bar{k} = 3$: Compute $[\mathfrak{X}_3(8), \mathfrak{X}_3(6)]$, i.e.

$$\begin{aligned}[\mathfrak{X}_3(8), \mathfrak{X}_3(6)] = \{&[(4,2,2),(4,1,1)];[(4,2,2),(3,2,1)];\\ &[(4,2,2),(2,2,2)];[(3,3,2),(4,1,1)];\\ &[(3,3,2),(3,2,1)];[(3,3,2),(2,2,2)]\}\end{aligned}$$

which is analysed to the following

$$\begin{aligned}[(4,2,2),(4,1,1)] &= \{(4,4),(2,1),(2,1)\}\\ [(4,2,2),(3,2,1)] &= \{(4,3),(2,2),(2,1)\}\\ [(4,2,2),(2,2,2)] &= \{(4,2),(2,2),(2,2)\}\\ [(3,3,2),(4,1,1)] &= \emptyset\\ [(3,3,2),(3,2,1)] &= \{(3,3),(3,2),(2,1)\}\\ [(3,3,2),(2,2,2)] &= \{(3,2),(3,2),(2,2)\}\end{aligned}$$

Hence, all $\bar{k} = 3-$ SCP are:

$$[\mathfrak{X}_3(8), \mathfrak{X}_3(6)] = \{\{(4,4),(2,1),(2,1)\}; \{(4,3),(2,2),(2,1)\}; \{(4,2),(2,2),(2,2)\}; \{(3,3),(3,2),(2,1)\}; \{(3,2),(3,2),(2,2)\}\}$$

Case $\bar{k} = 4$: Compute $[\mathfrak{X}_4(8), \mathfrak{X}_4(6)]$, i.e.

$$[\mathfrak{X}_4(8), \mathfrak{X}_4(6)] = \{[(2,2,2,2),(3,1,1,1)]; [(2,2,2,2),(2,2,1,1)]\}$$

which gives

$$[(2,2,2,2),(3,1,1,1)] = \emptyset$$
$$[(2,2,2,2),(2,2,1,1)] = \{(2,2),(2,2),(2,1),(2,1)\}$$

and thus all $\bar{k} = 4-$SCPs are:

$$[\mathfrak{X}_4(8), \mathfrak{X}_4(6)] = \{(2,2),(2,2),(2,1),(2,1)\}$$

Finally, the set $\{m, p;\, r\}$ is:

$$\{8,6;\, 2\} = \{[(8,6)]\,;\, [\mathfrak{X}_2(8), \mathfrak{X}_2(6)]\,;\, [\mathfrak{X}_3(8), \mathfrak{X}_3(6)]\,;\, [\mathfrak{X}_4(8), \mathfrak{X}_4(6)]\}$$

□

It should be pointed that the study of properties on a given $\underline{m}(s) \in \mathcal{D}$ depends only on its degree, rank and (p, m) number (inputs, outputs) and not on n (states). In fact, we may state:

Remark 11. The structure of the set of all structurally compatible partitions $\{m, p; r\}$ depends on the three numbers m, p and r, from which the only numerically dependent parameter is r. Given that $r \leqslant \delta + 1$, a subset of $\{m, p; r\}$ which is entirely numerically independent is the set $\{m, p; \delta + 1\}$. The number of states n, apart from affecting the possible values of r, or δ does not explicitly enter into the shaping of the $\{m, p; r\}$ set.

□

6. CONCLUSION

In this paper the development of criteria for selection of the decentralisation structure as a core step in the design of decentralised controllers was considered. We have further developed the concept of real degeneracy associated with the standard feedback configuration and a proper system for the case of decentralised control schemes. The motivation comes from the very important property that degenerate feedback gains may be used for the linearisation of the multi-linear nature pole assignment map. We have considered the existence of decentralised degenerate compensators and this have led to criteria, based on structural diagnostics that can guarantee the existence of such controllers and also provide some parametrisation of them. The emphasis has been on defining possible degenerate feedback solutions which are systems model parameter independent and defining the families of systems for which the families of degenerate feedback solutions is non-trivial. Further work is needed in characterising the families for which there exist regular degenerate feedbacks (avoiding infinite gains) and defining the conditions for the existence of such decentralisation scheme. Also it is required to define the additional conditions that guarantee a full rank differential of the pole assignment map at the selected decentralised degenerate controller. This is directly linked to the solvability of the decentralised pole assignment problem. The results were mainly presented for the static output feedback problem and their extension to dynamic decentralised schemes is also part of the future work.

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