# The non-Hermitian Swanson model with a time-dependent metric

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## Abstract

We provide further non-trivial solutions to the recently proposed time-dependent Dyson and quasi-Hermiticity relation. Here we solve them for the generalized version of the non-Hermitian Swanson Hamiltonian with time-dependent coefficients. We construct time-dependent solutions by employing the Lewis-Riesenfeld method of invariants and discuss concrete physical applications of our results.

## I. INTRODUCTION

PT symmetric (PTS) quantum mechanics has attracted increasing attention since is was demonstrated that PTS Hamiltonians possess real spectra [1] and allow for a unitary evolution with a redefined inner product [2, 3]. Phase transitions between the regimes of unbroken and broken PT symmetry, which are a key feature in the energy spectrum are well understood to occur when two real eigenvalues coalesce to form complex conjugate pair [1]. Many interesting new results have recently emerged from the application of PTS concepts to different areas of physics, in the classical and the quantum domain, on both fronts, theoretical as well as experimental. We mention here a few, such as the design of an ultralow-threshold phonon laser [4], the demonstration of defect states [5] and beam dynamics [6] in PTS optical lattices, and the fact that the Jarzynski equality generalizes to PTS domain [7]. Reinforcing the practical features, there are optical structures described by PTS concepts that enable unprecedented control of light [8]. At a classical level, PTS properties have also been observed in a variety of experimental set-ups, ranging from quantum optics [9] to NMR [10] and superconductivity [11].

Although the grounds for treating non-Hermitian Hamiltonians using time-independent metric operators have been extensively studied and well established [12, 13], the generalization to time-dependent (TD) metric operators has raised controversy [14–17]. In Ref. [14], Mostafazadeh has demonstrated that using a TD metric operator one can not ensure the unitarity of the time-evolution simultaneously with the observability of the Hamiltonian. From this perspective, with which we agree, the authors of Refs. [15–17] fail to ensure a unitary time-evolution by insisting on the observability of the Hamiltonian. However, we have recently suggested [18] that this is not an obstacle and certainly not a no-go theorem. It is very common in the context of PTS quantum mechanics that certain operators, such as position or momentum, may become non-observable auxiliary variables and only their quasi-Hermitian counterparts can be measured. In [18] we take the view that the Hamiltonian, meaning the operator that satisfies the TD Schrödinger equation (SE), joins this set of observables in the scenario where a TD metric operator is considered. For this proposal to be meaningful the TD quasi-Hermiticity relation and TD dyson relation need to possess non-trivial solutions. When this is the case, we have unitary time evolution and well defined observables.

Here we provide new non-trivial solution to this set of equations for generalized time-dependent version of Swanson Hamiltonian [19] by solving its TDSE and by computing some observables. In order to solve the SE, we shall adapt a method presented in Ref. [20, 21] for treating TD Hermitian Hamiltonians. This method takes advantage of a unitary TD transformation on the SE, here replaced by a non-unitary transformation to conform with non-Hermitian Hamiltonians, and the diagonalization of a TD Invariant on the Lewis and Riesenfeld framework [22].

The authors in Ref. [20] pursued the solution of the SE governed by a general TD quadratic Hamiltonian in order to investigate the mechanism of squeezed states following from the nonlinear amplification terms of the Hamiltonian [23, 24]. Here, we shall focus on the technique to treat a TD non-Hermitian Hamiltonian, leaving open the possibility of further analysis of the squeezing mechanism coming from the nonlinear terms of a TD non-Hermitian Hamiltonian.

## II. NON-HERMITIAN HAMILTONIAN SYSTEMS WITH TD METRIC

Let us briefly review the scheme proposed in [18]: We consider a non-Hermitian TD Hamiltonian $H(t)$ whose associated SE, $i\partial_t |\psi(t)\rangle = H(t) |\psi(t)\rangle$, is mapped, by means of the Hermitian TD operator $\eta(t)$, into the SE $i\partial_t |\phi(t)\rangle = h(t) |\phi(t)\rangle$, where the corresponding wave functions are transformed as $|\phi(t)\rangle = \eta(t) |\psi(t)\rangle$ and the Hamiltonians are related by means of the *TD Dyson relation*

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\left[\partial_t\eta(t)\right]\eta^{-1}(t). \tag{1}$$

We set here $\hbar = 1$. The key feature in this equation is the fact that $H(t)$ is no longer quasi-Hermitian, i.e. related to $h(t)$ by means of a similarity transformation, due to the presence of the last term. Thus $H(t)$ is not a self-adjoined operator and therefore not observable. Using the Hermiticity of $h(t)$ we then derived the *TD quasi-Hermiticity relation*

$$H^\dagger(t)\rho(t) - \rho(t)H(t) = i\partial_t\rho(t), \qquad \rho(t) = \eta^\dagger\eta, \tag{2}$$

replacing the standard quasi-Hermiticity relation for a time-independent $\rho$, given by $H^\dagger\rho = \rho H$. In fact, the TD quasi-Hermiticity relation ensures the TD probability densities in the

Hermitian and non-Hermitian systems to be related in the standard form

$$\left\langle \psi(t) \left| \tilde{\psi}(t) \right. \right\rangle_{\rho} = \left\langle \psi(t) \left| \rho(t) \right| \tilde{\psi}(t) \right\rangle = \left\langle \phi(t) \left| \tilde{\phi}(t) \right. \right\rangle . \tag{3}$$

With the assumption that $\rho(t)$ is a positive-definite operator, it plays the role of the TD metric and we conclude that any self-adjoined operator $o(t)$, i.e. observable, in the Hermitian system possesses a counterpart $O(t)$ in the non-Hermitian system given by

$$O(t) = \eta^{-1}(t) o(t) \eta(t), \tag{4}$$

in complete analogy to the time-independent scenario. Thus as long as the generalized equations (1) and (2) posses non-trivial solutions for $\eta(t)$ and $\rho(t)$, respectively, we have a well defined physical system with TB observables and unitary time-evolution governed by a TD non-Hermitian Hamiltonian. Albeit we have the slightly unusual feature that the TD Hamiltonian $H(t)$ does not belong to the set of observables. We should also remark that the well-known feature of the metric not being unique, see e.g. [2] and [26], will acquire here an additional ambiguity due to the fact (1) and (2) are in general nonlinear differential equations, see [18], and will therefore usually have several different types of solutions.

## III. THE GENERALIZED TIME-DEPENDENT SWANSON HAMILTONIAN

The system we wish to investigate here is related to the non-Hermitian TD Swanson Hamiltonian

$$H(t) = \omega(t) \left( a^{\dagger} a + 1/2 \right) + \alpha(t) a^2 + \beta(t) a^{\dagger 2}, \tag{5}$$

where $a$ and $a^{\dagger}$ are bosonic annihilation and creation operators, for instance of a light field mode. In comparison with time-independent case all parameters have acquired an explicit time-dependence $\omega(t), \alpha(t), \beta(t) \in \mathbb{C}$. Clearly when $\omega(t) \notin \mathbb{R}$ or $\alpha(t) \neq \beta^*(t)$ the Hamiltonian (5) is not Hermitian. It becomes PT-symmetric when demanding $\omega(t), \alpha(t), \beta(t)$ to be even functions in $t$ or generic functions of $it$.

Let us now solve the TD Dyson equation by making the following general and for simplicity Hermitian Ansatz for the Dyson map

$$\eta(a, a^{\dagger}, t) = \exp \left[ \epsilon(t) \left( a^{\dagger} a + 1/2 \right) + \mu(t) a^2 + \mu^*(t) a^{\dagger 2} \right] \tag{6}$$

$$= \exp \left[ \lambda_+(t) K_+ \right] \exp \left[ \ln \lambda_0(t) K_0 \right] \exp \left[ \lambda_-(t) K_- \right] . \tag{7}$$

We require here the variant (7) of our Ansatz to be able to compute the time-derivatives of $\eta$. The equality follows by recalling that $K_+ = a^{\dagger 2}/2$, $K_- = a^2/2$, $K_0 = (a^\dagger a/2 + 1/4)$ form an $SU(1,1)$-algebra, such that the group element in (6) can be Iwasawa decomposed according to [25]. The TD coefficients read

$$\lambda_+ = \frac{2\mu^* \sinh\Xi}{\Xi\cosh\Xi - \epsilon\sinh\Xi}, \tag{8a}$$

$$\lambda_- = \lambda_+^*, \tag{8b}$$

$$\lambda_0 = \left(\cosh\Xi - \frac{\epsilon}{\Xi}\sinh\Xi\right)^{-2}, \tag{8c}$$

where we abbreviated the argument of the hyperbolic functions to $\Xi = \sqrt{\epsilon^2 - 4|\mu|^2}$, demanding $\epsilon$ to be real and $\epsilon^2 - 4|\mu|^2 \geq 0$.

The notation may be simplified even further when introducing some new quantities. Similarly as in [26] we define $z = 2\mu/\epsilon = |z|e^{i\varphi}$ within the unit circle, such that we obtain $\Xi = \epsilon\sqrt{1-|z|^2}$. Furthermore, we define $\Phi = |z|/\Gamma_-$ with $\Gamma_\pm = 1 \pm \tilde{\Xi}\coth\Xi$, $\tilde{\Xi} = \Xi/\epsilon$, $\tilde{\Gamma}_\pm = \Gamma_\pm/\tilde{\Xi}$, and finally $\chi = \tilde{\Gamma}_+/\tilde{\Gamma}_- = 2/\Gamma_- - 1 = 2\Phi/|z| - 1$. The notation settled, the coefficients in (8a)-(8c) simplify to

$$\lambda_+ = -\Phi e^{-i\varphi}, \tag{9a}$$

$$\lambda_- = -\Phi e^{i\varphi}, \tag{9b}$$

$$\lambda_0 = \frac{1}{\tilde{\Gamma}_-^2 \sinh^2\Xi} = \Phi^2 - \chi. \tag{9c}$$

where $\sinh^2\Xi = \tilde{\Xi}^2\Phi^2/\left[|z|^2(\Phi^2-\chi)\right] = \tilde{\Xi}^2\lambda_+\lambda_-/|z|^2\lambda_0$.

Using the relations

$$\eta(t)\begin{pmatrix} a \\ a^\dagger \end{pmatrix}\eta^{-1}(t) = \pm\frac{1}{\sqrt{\lambda_0}}\begin{pmatrix} -1 & \lambda_+ \\ -\lambda_- & \chi \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{10}$$

we obtain, after some algebra, the transformed Hamiltonian

$$\begin{aligned} h(z,\epsilon,t) &= \eta(t)H(t)\eta^{-1}(t) + i\dot{\eta}(t)\eta^{-1}(t) \\ &= W(z,\epsilon,t)(a^\dagger a + 1/2) + V(z,\epsilon,t)a^2 + T(z,\epsilon,t)a^{\dagger 2}, \end{aligned} \tag{11}$$

where the coefficient functions are

$$W(z,\epsilon,t) = -\frac{1}{\lambda_0}\left[\omega\left(\chi+\lambda_+\lambda_-\right) + 2\left(\alpha\lambda_+ + \beta\chi\lambda_-\right) - \frac{i}{2}\left(\dot{\lambda}_0 - 2\lambda_+\dot{\lambda}_-\right)\right], \tag{12a}$$

$$V(z,\epsilon,t) = \frac{1}{\lambda_0}\left(\alpha + \omega\lambda_- + \beta\lambda_-^2 + \frac{i}{2}\dot{\lambda}_-\right), \tag{12b}$$

$$T(z,\epsilon,t) = \frac{1}{\lambda_0}\left[\omega\chi\lambda_+ + \alpha\lambda_+^2 + \beta\chi^2 + \frac{i}{2}\left(\lambda_0\dot{\lambda}_+ + \lambda_+^2\dot{\lambda}_- - \lambda_+\dot{\lambda}_0\right)\right]. \tag{12c}$$

As common the overhead dot denotes derivatives with respect to time.

For the Hamiltonian $h(t)$ to be Hermitian we need to impose $W$ to be real and in addition $T = V^*$. From the first constraint we derive the equality

$$\dot{\lambda}_0 = 2\left|\omega\right|\left(\chi+\Phi^2\right)\sin\varphi_\omega + 2\Phi\left[\dot{\Phi} + 2\left|\alpha\right|\sin\left(\varphi-\varphi_\alpha\right) - 2\left|\beta\right|\chi\sin\left(\varphi+\varphi_\beta\right)\right], \tag{13}$$

while the second one leads to the coupled nonlinear differential equations

$$\begin{aligned}\dot{\Phi} &= \frac{2}{\chi-1}\left\{\left[\left|\omega\right|\Phi\sin\varphi_\omega + \left|\alpha\right|\sin\left(\varphi-\varphi_\alpha\right)\right]\left(1-\Phi^2\right) + \left|\beta\right|\left[\left(2\chi-1\right)\Phi^2-\chi^2\right]\sin\left(\varphi+\varphi_\beta\right)\right\},\\ \dot{\varphi} &= \frac{2}{\left(\chi-1\right)\Phi}\left[\left|\alpha\right|\left(1-\Phi^2\right)\cos\left(\varphi-\varphi_\alpha\right) + \left|\beta\right|\left(\Phi^2-\chi^2\right)\cos\left(\varphi+\varphi_\beta\right)\right] + 2\left|\omega\right|\cos\varphi_\omega.\end{aligned} \tag{14}$$

Here $\varphi_\alpha$, $\varphi_\beta$ and $\varphi_\omega$ are the polar angles of $\alpha$, $\beta$ and $\omega$, respectively and $\chi$ is a function of $\Phi$ and $|z|$, as defined above. Therefore, in a similar way to that in Ref. [26], we may consider $|z|$ as the only free parameter that determines the metric, with $\epsilon$ following from the relation

$$\epsilon = \frac{1}{\sqrt{1-|z|^2}}\operatorname{arctanh}\frac{\sqrt{1-|z|^2}\Phi}{\Phi-|z|} = \frac{1}{2\sqrt{1-|z|^2}}\ln\left[\frac{\left(1+\sqrt{1-|z|^2}\right)\Phi-|z|}{\left(1-\sqrt{1-|z|^2}\right)\Phi-|z|}\right], \tag{15}$$

as may be derived from the parameter $\Phi = |z|/\Gamma_-$, as defined above, which in turn depends, as well as on $\varphi$, also on the solution of the system (14) and the TD coefficients of the starting Hamiltonian (5). Evidently, a given pair $(|z|,\Phi)$, i.e., a given choice of $|z|$, this must be further corroborated by a real solution of $\epsilon$ in Eq. (15), with the argument of the arctanh (ln) being not greater than unity (greater than zero), thus demanding $|z|^2 > 2\Phi/\left(1+\Phi^2\right)$. We finally observe that $|z|$ can conveniently be considered as a time-independent parameter, constraining the time-dependence to the remaining parameters $\varphi$ and $\epsilon$.

## IV. SOLUTIONS OF THE SCHRÖDINGER EQUATION FOR THE GENERALIZED TIME-DEPENDENT SWANSON HAMILTONIAN

In order to solve the SE for $H(t)$ we shall adapt to the case of TD non-Hermitian Hamiltonians a method presented in Ref. [20] for solving the SE for TD Hermitian Hamiltonians. This technique takes advantage of a TD transformation on the SE for the desired Hamiltonian, here a nonunitary transformation to conform with non-Hermitian Hamiltonians, and the diagonalization of a TD Invariant within the Lewis and Riesenfeld framework [22]. The Lewis and Riesenfeld method ensures that a solution of the SE governed by a TD Hermitian Hamiltonian $\mathcal{H}(t)$ is an eigenstate of an associated Hermitian invariant $I(t)$, defined as $\partial_t I(t) + i\left[\mathcal{H}(t), I(t)\right] = 0$, apart from a TD global phase factor. The method in Ref. [20] proposes that, instead of solving the SE for $\mathcal{H}(t)$ by deriving an invariant directly associated with this Hamiltonian, a transformation is performed on the SE for bringing the original Hamiltonian to another form which has already an associated invariant.

The authors in Ref. [20] pursued the solution of the SE governed by a general TD quadratic (Hermitian) Hamiltonian in order to investigate the mechanism of squeezed states [23, 24] following from the nonlinear amplification terms of the Hamiltonian. They thus consider the unitary squeeze operator for transforming the SE for the TD quadratic Hamiltonian, reducing it to a form associated with a linear Hamiltonian which has already an associated invariant [27]. Here, we shall focus on the method to approach a TD non-Hermitian Hamiltonian, leaving open the analysis of the squeezing mechanism coming from the nonlinear terms of a TD non-Hermitian Hamiltonian.

In the present contribution a similar strategy to that in Ref. [20] will be used, starting from the non-Hermitian $H(t)$ and then deriving the transformed Hermitian $h(t)$ through the metric operator $\eta(t)$, instead of a unitary transformation. We further identify this transformed Hamiltonian with the Hermitian quadratic one treated in Ref. [20], whose solutions have been derived. Evidently, we must disregard the linear amplification process considered in Ref. [20] since it is absent from $h(t)$. To this end, we next rewrite the coefficients of the Hermitian (11) considering the Eqs. (13) and (14). Under the Eqs. (13) and (14) we obtain the real frequency

$$W(|z|, \varphi, t) = |\omega| \cos\varphi_\omega + \frac{2\Phi}{1-\chi}\left[|\alpha| \cos\left(\varphi - \varphi_\alpha\right) - |\beta| \cos\left(\varphi + \varphi_\beta\right)\right]. \tag{16}$$

From the system (14) we obtain $V(|z|, \varphi, t) = T^*(|z|, \varphi, t) = V_R(|z|, \varphi, t) + iV_I(|z|, \varphi, t) =$

$\kappa(|z|,\varphi,t)e^{i\zeta(|z|,\varphi,t)}$, with $\kappa=\left(V_R^2+V_I^2\right)^{1/2}$, $\zeta=\arctan\left(V_I/V_R\right)$, and

$$V_R(|z|,\varphi,t)=\frac{1}{1-\chi}\left(|\omega|\,\Phi\sin\varphi_\omega\sin\varphi+|\alpha|\cos\varphi_\alpha-|\beta|\,\chi\cos\varphi_\beta\right), \tag{17a}$$

$$V_I(|z|,\varphi,t)=\frac{1}{\chi-1}\left(|\omega|\,\Phi\sin\varphi_\omega\cos\varphi-|\alpha|\sin\varphi_\alpha-|\beta|\,\chi\sin\varphi_\beta\right). \tag{17b}$$

Note that when starting with a Hermitian Hamiltonian (5), with real $\omega$ and $\alpha=\beta^*$, we verify from Eqs. (16) and Eq. (17) that $W(|z|,\varphi,t)=|\omega|$ and $V(|z|,\varphi,t)=\alpha(t)$, such that $h=H$.

The solutions of the Schrödinger equation generated by Hamiltonian (11), given in Ref. [20] as

$$|v_n(t)\rangle=U(t)\,|n\rangle\,, \tag{18}$$

define a complete set of states, $|n\rangle$ being the Fock states and $U(t)$ the unitary operator

$$U(t)=\Upsilon(t)S\left[\xi(t)\right]D\left[\theta(t)\right]R\left[\Omega(t)\right]. \tag{19}$$

Here $S\left[\xi(t)\right]=\exp\left\{\left[\xi(t)a^{\dagger 2}-\xi^*(t)a^2\right]/2\right\}$ is the squeeze operator, with $\xi(t)=r(t)e^{i\phi(t)}$ defining the squeeze parameters, which follow from another set of coupled nonlinear differential equations

$$\dot{r}(t)=-2\kappa(t)\sin\left[\zeta(t)-\phi(t)\right], \tag{20a}$$

$$\dot{\phi}(t)=-2W(t)-4\kappa(t)\coth\left[2r(t)\right]\cos\left[\zeta(t)-\phi(t)\right]. \tag{20b}$$

where $D\left[\theta(t)\right]=\exp\left[\theta(t)a^\dagger-\theta^*(t)a\right]$ is the displacement operator and $\theta(t)$ satisfies the equation $i\dot{\theta}(t)=\Omega(t)\theta(t)$, with

$$\Omega(t)=W(t)+2\kappa(t)\tanh r(t)\cos\left[\zeta(t)-\phi(t)\right]. \tag{21a}$$

Finally, $R\left[\Omega(t)\right]=\exp\left[-i\varpi(t)a^\dagger a\right]$ is the rotation operator, with $\varpi(t)=\int_0^t\Omega(t')dt'$, and $\Upsilon(t)=\exp\left(-i\varpi(t)/2\right)$ is a global phase factor.

Having the wave vectors in Eq. (18), we directly obtain the solutions of the Schrödinger equation generated by Hamiltonian (5), given by

$$|\psi_n(t)\rangle=\eta^{-1}(t)\,|v_n(t)\rangle=\eta^{-1}(t)U(t)\,|n\rangle\,. \tag{22}$$

For a generic superposition $|\psi(t)\rangle=\sum_n c_n\,|\psi_n(t)\rangle$ it follows that

$$|\psi(t)\rangle=\eta^{-1}(t)V(t)\,|v(0)\rangle\,, \tag{23}$$

with the evolution operator

$$V(t) = U(t)U^{\dagger}(0) = \Upsilon(t)S\left[\xi(t)\right]D\left[\theta(t)\right]R\left[\Omega(t)\right]S^{\dagger}\left[\xi(0)\right]D^{\dagger}\left[\theta(0)\right],. \tag{24}$$

At this point it is worth mentioning a theorem which can be straightforwardly adapted from Ref. [20] to the context of TD non-Hermitian quantum mechanics: *If $I(t)$ is an invariant associated with an non-Hermitian Hamiltonian $H(t)$, then $I_\eta(t) = \eta(t)I(t)\eta^{-1}(t)$ is also an invariant but associated with the transformed Hermitian Hamiltonian $h(t)$, both invariants $I(t)$ and $I_\eta(t)$ sharing the same eigenvalue spectrum. Moreover, the Lewis and Riesenfeld phase is invariant under the transformation $\eta(t)$.* It is not difficult to see that this theorem fully supports the solutions presented in Eqs. (22) and (23).

Before analyzing the observables associated with the pseudo-Hermitian $H(t)$, it is worth addressing two particular cases: when the coefficients of $H(t)$ are real TD functions and when considering a time-independent metric operator.

### A. On the solutions for the TD coupled differential equations (14), (20) and (28)

Before addressing particular cases where the coefficients of the Hamiltonian (5) are real TD functions and/or a time-independent metric operator is considered, we add a few comment on the coupled equations ruling the evolution of the metric parameters $\Phi$ and $\varphi$ [Eqs. (14) and (28)] and the squeezing parameters $r$ and $\phi$ [Eq. (20)]. As advanced in Ref. [20], despite its time dependence, the system (20) can be solved analytically, by quadrature, under particular constraints linking together its TD functions and thus leaving a lower degree of arbitrariness. Some solutions for system (20) have been presented in Ref. [20], and reasoning by analogy with this reference it will be possible to find analytical solutions for the systems (14) and (28), at least for some specific demands on the TD functions. For example, considering a real TD function

$$\omega(t) \equiv \frac{\dot{f}(t)}{2} + 2\left|\beta\right|\Phi\cos\left(\varphi - \varphi_\alpha\right) \tag{25}$$

and $\varphi_\beta(t) = -\varphi_\alpha(t)$, we eliminate the parameter time from the system (14), to obtain, with $\varsigma(t) = \varphi(t) + f(t)$ and a constant $v = \varphi_\alpha(t) + f(t)$, the first order differential equation

$$\frac{d\Phi}{d\varsigma} = \frac{\Phi}{\tan\left(\varsigma - v\right)}, \tag{26}$$

whose integration leads to a constant of motion and thus to the solutions for $\Phi$ and $\varphi$.

## V. PARTICULAR CASES

### A. The generalized TD Swanson's Hamiltonian with real coefficients $\omega(t), \alpha(t), \beta(t)$

When considering the TD coefficients $\omega(t), \alpha(t), \beta(t)$ to be real functions instead of complex ones, the equations in Sections III and IV considerably simplify. Let us start by demanding $h(t)$ in Eq. (11) to be Hermitian. By imposing $W$ to be real we now obtain

$$\dot{\lambda}_0 = 2\Phi\left[\dot{\Phi} + 2\left(\alpha - \beta\chi\right)\sin\varphi\right], \tag{27}$$

while the imposition $T = V^*$ leads to the simplified coupled nonlinear differential equations

$$\dot{\Phi} = \frac{2}{\chi - 1}\left\{\alpha\left(1 - \Phi^2\right) + \beta\left[\left(2\chi - 1\right)\Phi^2 - \chi^2\right]\right\}\sin\varphi, \tag{28a}$$

$$\dot{\varphi} = 2\omega - \frac{2}{\left(1 - \chi\right)\Phi}\left[\alpha\left(1 - \Phi^2\right) + \beta\left(\Phi^2 - \chi^2\right)\right]\cos\varphi. \tag{28b}$$

Again, $|z|$ can be taken as the only free parameter that determines the metric, with $\epsilon$ following from Eq. (15). To further identify the transformed Hermitian $h(t)$ with the quadratic Hamiltonian whose SE is solved in Ref. [20], we rewrite the coefficients of $h(t)$ considering the Eqs. (27) and (28). We thus obtain the real frequency

$$W(|z|, \varphi, t) = \omega + \frac{2\Phi}{1 - \chi}\left[\alpha - \beta\right]\cos\varphi, \tag{29}$$

and the simplified real function

$$V(|z|, \varphi, t) = T(|z|, \varphi, t) = \kappa(|z|, \varphi, t) = \frac{\alpha - \beta\chi}{1 - \chi}. \tag{30a}$$

From the above equations the solutions presented in Eqs. (22) and (23) follow straightforwardly.

## B. The generalized TD Swanson's Hamiltonian with a time-independent metric operator

When a time-independent metric operator is considered, the coefficients of the transformed Hamiltonian $h(z,\epsilon,t)$ simplify to

$$W(z,\epsilon,t) = -\frac{1}{\lambda_0}\left[\omega\left(\chi+\lambda_+\lambda_-\right)+2\left(\alpha\lambda_+ + \beta\chi\lambda_-\right)\right], \tag{31a}$$

$$V(z,\epsilon,t) = \frac{1}{\lambda_0}\left(\alpha+\omega\lambda_- + \beta\lambda_-^2\right), \tag{31b}$$

$$T(z,\epsilon,t) = \frac{1}{\lambda_0}\left(\omega\chi\lambda_+ + \alpha\lambda_+^2 + \beta\chi^2\right), \tag{31c}$$

For $h$ to be Hermitian we again impose $W$ to be real and $T = V^*$. The first constraint leads to the relation

$$|\omega|\left(\chi+\Phi^2\right)\sin\varphi_\omega + 2\Phi\left[|\alpha|\sin\left(\varphi-\varphi_\alpha\right) - |\beta|\,\chi\sin\left(\varphi+\varphi_\beta\right)\right] = 0, \tag{32}$$

while the latter gives rise to the equations

$$|\omega|\left(1-\chi\right)\Phi\cos\varphi_\omega - |\alpha|\left(1-\Phi^2\right)\cos\left(\varphi-\varphi_\alpha\right) + |\beta|\left(\chi^2-\Phi^2\right)\cos\left(\varphi+\varphi_\beta\right) = 0, \tag{33a}$$

$$|\omega|\left(1+\chi\right)\Phi\sin\varphi_\omega + |\alpha|\left(1+\Phi^2\right)\sin\left(\varphi-\varphi_\alpha\right) - |\beta|\left(\chi^2+\Phi^2\right)\sin\left(\varphi+\varphi_\beta\right) = 0. \tag{33b}$$

From Eqs. (32) and (33b) we obtain the relation

$$|\alpha|\left(1-\Phi^2\right)\sin\left(\varphi-\varphi_\alpha\right) = |\beta|\left(\chi^2-\Phi^2\right)\sin\left(\varphi+\varphi_\beta\right), \tag{34}$$

which, together with Eq. (33a), gives us

$$\sin\left(\varphi-\varphi_\alpha\right) = \frac{|\beta|\left(\chi^2-\Phi^2\right)}{|\omega|\left(1-\chi\right)\Phi\cos\varphi_\omega}\sin\left(\varphi_\alpha+\varphi_\beta\right), \tag{35a}$$

$$\sin\left(\varphi+\varphi_\beta\right) = \frac{|\alpha|\left(1-\Phi^2\right)}{|\omega|\left(1-\chi\right)\Phi\cos\varphi_\omega}\sin\left(\varphi_\alpha+\varphi_\beta\right). \tag{35b}$$

By substituting Eq. (35) back into Eq. (32), we finally obtain the equation

$$|z|\,\Phi^3 + \left(2-|z|^2\right)\Phi^2 - 3\,|z|\,\Phi + |z|^2 = 0, \tag{36}$$

whose roots enable us to compute $\varphi$ from Eq. (35) and then $\epsilon$ from the relation given in Eq. (15). Here, the real frequency $W(|z|,\varphi,t)$ and the complex $V(|z|,\varphi,t) = T^*(|z|,\varphi,t)$ still follow from Eqs. (16) and (17), respectively, with time-independent $z$ and $\epsilon$.

### 1. *A time-independent metric operator with real TD coefficients* $\omega(t), \alpha(t), \beta(t)$

When a time-independent metric operator is considered together with real TD parameters $\omega(t), \alpha(t), \beta(t)$, it follows from Eq. (32) that $\varphi = 0$ and from Eq. (33a) we derive the equation

$$\left(|\alpha| - |\beta|\right)\Phi^2 + |\omega|\left(1-\chi\right)\Phi - |\alpha| + |\beta|\chi^2 = 0 \tag{37}$$

which leads to the relation

$$\frac{\tanh(2\Xi)}{\tilde{\Xi}} = \frac{\alpha-\beta}{\alpha+\beta-z\omega}, \tag{38}$$

and, consequently, to the metric parameter

$$\epsilon = \frac{1}{2\sqrt{1-|z|^2}} \operatorname{arctanh} \frac{\left(|\alpha| - |\beta|\right)\sqrt{1-|z|^2}}{|\alpha| + |\beta| - |z|\,|\omega|} \tag{39a}$$

$$= \frac{1}{4\sqrt{1-|z|^2}} \ln \frac{|\alpha| + |\beta| - |z|\,|\omega| + \left(|\alpha| - |\beta|\right)\sqrt{1-|z|^2}}{|\alpha| + |\beta| - |z|\,|\omega| - \left(|\alpha| - |\beta|\right)\sqrt{1-|z|^2}}. \tag{39b}$$

However, a time-independent metric brings about the constraint on the TD parameters of the Hamiltonian

$$\frac{|\dot{\alpha}| + \left|\dot{\beta}\right| - |z|\,|\dot{\omega}| + \left(|\dot{\alpha}| - \left|\dot{\beta}\right|\right)\sqrt{1-|z|^2}}{|\alpha| + |\beta| - |z|\,|\omega| + \left(|\alpha| - |\beta|\right)\sqrt{1-|z|^2}} = \frac{|\dot{\alpha}| + \left|\dot{\beta}\right| - |z|\,|\dot{\omega}| - \left(|\dot{\alpha}| - \left|\dot{\beta}\right|\right)\sqrt{1-|z|^2}}{|\alpha| + |\beta| - |z|\,|\omega| - \left(|\alpha| - |\beta|\right)\sqrt{1-|z|^2}}, \tag{40}$$

where we have assume a time-independent $|z|$ as the only free parameter that determines the metric, with $\epsilon$ following from Eq. (41). The existence of a real solution for $\epsilon$ demands the argument of the arctanh (ln) to be not greater than unity (to be greater than zero), and consequently, there is no real solution for $|z| \in [|z_-|, |z_+|]$, with

$$|z_\pm| = \frac{\left(|\alpha| + |\beta|\right)|\omega| \pm \left(|\alpha| - |\beta|\right)\left(|\omega|^2 - 4|\alpha|\,|\beta|\right)}{|\omega|^2 + \left(|\alpha| - |\beta|\right)^2}. \tag{41}$$

The roots $|z_\pm|$ present the same form as those in Ref. [26], the difference here being that $|\omega|$, $|\alpha|$, and $|\beta|$ are TD functions instead of constant parameters, additionally constrained by Eq. (40), thus placing an additional difficulty for the observance of the requirements for a real solution for $\epsilon$. Finally, when we identify the Hamiltonian $h(z, \epsilon, t)$ with the Hermitian quadratic one in ([20]) we obtain for $W(|z|, \varphi, t)$ and $V(|z|, \varphi, t) = T^*(|z|, \varphi, t) = \kappa(|z|, \varphi, t)e^{i\zeta(|z|,\varphi,t)}$, the same expressions as in Eqs. (29) and (30a), respectively, with time-independent $|z|$ and $\epsilon$.

## VI. OBSERVABLES

### A. The generalized TD Swanson Hamiltonian

Considering the observables for the generalized TD Swanson Hamiltonian, we start by focusing on the derivation of all the Hermitian operators on the continuous variety of Hilbert spaces $\mathcal{H}_z$ for any $|z| \in [-1, 1]$. As argued in [18], the Hamiltonian $H$ itself is not one of the Hermitian operator due the presence of the gauge-like term in Eq. (1). Using Eq. (15) to rewrite the metric operator in Eq. (7) in the form [26, 28]

$$\eta(t) = \left(\frac{\left(1+\sqrt{1-|z|^2}\right)\Phi - |z|}{\left(1-\sqrt{1-|z|^2}\right)\Phi - |z|}\right)^{\frac{a^\dagger a + \frac{1}{2}\left(za^2 + z^* a^{\dagger 2}\right) + \frac{1}{2}}{2\sqrt{1-|z|^2}}} \tag{42a}$$

$$= \left(\frac{\left(1+\sqrt{1-|z|^2}\right)\Phi - |z|}{\left(1-\sqrt{1-|z|^2}\right)\Phi - |z|}\right)^{\frac{(1-|z|\cos\varphi)p^2 + (1+|z|\cos\varphi)\omega^2 x^2 - |z|\omega\sin\varphi\{x,p\}}{4\omega\sqrt{1-|z|^2}}}, \tag{42b}$$

which we use to solve the quasi-Hermiticity condition $O^\dagger(t)\mu(t) = \mu(t)O(t)$. Given (42), we only find the observables

$$O(t) = (1 - |z|\cos\varphi)\, p^2 + (1 + |z|\cos\varphi)\, \omega^2 x^2 - |z|\, \omega \sin\varphi \,\{x, p\}, \tag{43}$$

demonstrating that neither the position $x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right)$ nor the momentum $p = i\sqrt{\frac{\omega}{2}}\left(a^\dagger - a\right)$ operators remain Hermitian as they are in the standard $L^2$-metric, with regard to the TD $\eta(t)$-metric even for particular choices of $|z|$. Using the relation $O(t) = \eta^{-1}(t) o \eta(t)$ together with Eq. (14) we may compute the quasi-Hermitian position $X(t)$ and momentum $P(t)$ operators

$$X(t) = \frac{1}{|z|\sqrt{\Phi^2 - \chi}}\left\{\left[(1 - i\,|z|\sin\varphi)\,\Phi - |z|\right] x + \frac{i}{\omega}\,(1 - |z|\cos\varphi)\,\Phi p\right\}, \tag{44a}$$

$$P(t) = \frac{1}{|z|\sqrt{\Phi^2 - \chi}}\left\{\left[(1 + i\,|z|\sin\varphi)\,\Phi - |z|\right] p - i\omega\,(1 + |z|\cos\varphi)\,\Phi x\right\}, \tag{44b}$$

corroborating the conclusion we have drawn from Eq. (43).

### B. Particular cases

The observables computed above in Eqs. (43) and (44) also apply to the cases where real coefficients $\omega(t), \alpha(t), \beta(t)$ are assumed and when a time-independent metric operator is considered, the difference being that $\Phi$ and $\varphi$ now follow, instead of Eq. (14), from the coupled Eqs. (28) in the former case, and from Eqs. (35) and (36) in the latter case. However, when a time-independent metric operator is considered simultaneously with real coefficients $\omega(t), \alpha(t), \beta(t)$, the Hermitian observables in Eq. (43) and those in Eq. (44) simplify to

$$O(t) = (1-|z|)\,p^2 + (1+|z|)\,\omega^2 x^2, \tag{45a}$$

$$\begin{aligned} X(t) &= \frac{1}{|z|\sqrt{\Phi^2-\chi}}\left[(\Phi-|z|)\,x + \frac{i}{\omega}(1-|z|)\,\Phi p\right], \\ &= \cosh(\Xi)\,x + \frac{i}{\omega}\frac{(1-|z|)}{\sqrt{1-|z|^2}}\sinh(\Xi)\,p, \end{aligned} \tag{45b}$$

$$\begin{aligned} P(t) &= \frac{1}{|z|\sqrt{\Phi^2-\chi}}\left[(\Phi-|z|)\,p - i\omega\,(1+|z|)\,\Phi x\right], \\ &= \cosh(\Xi)\,p - i\omega\frac{(1+|z|)}{\sqrt{1-|z|^2}}\sinh(\Xi)\,x. \end{aligned} \tag{45c}$$

The Eqs. (45) are exactly of the same form as those in Ref. [26], the difference being that here we have TD parameters. Therefore, when considering the Hamiltonian (5) with time-independent real parameters together with a time-independent metric operator, it is straightforward to verify that all the above derivations are in complete agreement with those in [26].

## VII. CONCLUSION

We have studied a generalized Swanson Hamiltonian allowing for TD complex coefficients and a TD metric operator. We treated the model within the framework introduced in Ref. [18] where, despite the lack of the observability of the non-Hermitian Hamiltonian under a TD metric operator, their associated observables are computed as in the case where a time-independent metric is considered. To solve the SE for the generalized TD Swanson's Hamiltonian we have adapted a technique presented in Ref. [20] which relies on the Lewis and Riesenfeld TD invariants. Apart from deriving the solutions of the SE for our TD non-Hermitian Hamiltonian we have thus computed their associated observables, analyzing particular cases where a time–independent metric operator is considered and TD real coefficients are assumed for the non-Hermitian Hamiltonian.

From the results presented here we may next explore some interesting applications such as the generation of squeezing from a non-Hermitian parametric oscillator. Moreover, our TD Hamiltonian can be also considered to describe the non-Hermitian dynamical Casimir effect, and thus the rate of particles creations resulting, for example, from the accelerated movement of a cavity mirror can also be computed. The results for the generation of squeezing and the rate of photon creation derived from a non-Hermitian quadratic Hamiltonian can then be compared with the well-known results coming from the Hermitian Hamiltonians, thus delivering more timely hints on the physics of non-Hermitian Hamiltonians.

As another application motivated by this work is the possibility of engineering effective non-Hermitian Hamiltonians within trapped ions, circuit or cavity QED, NMR and other systems presenting great flexibility of handling its internal interactions. By mastering not only the technique for treating non-Hermitian Hamiltonians, but also for constructing non-Hermitian interactions, we may seek to contribute with the implementation of processes such as quantum simulation and quantum logical implementation, bringing additional ingredients to the usual Hermitian quantum mechanics.

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