Relationships between Archimedean copulas and Morgenstern utility functions.

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Abstract

The (additive) generator of an Archimedean copula is a strictly decreasing and convex function, while Morgenstern utility functions (applying to risk aversion decision makers) are nondecreasing and concave. In this presentation, relationships between generators and utility functions are established. For some well known Archimedean copula families, links between the generator and the corresponding utility function are demonstrated. Some new copula families are derived from classes of utility functions which appeared in the literature, and their properties are discussed. It is shown how dependence properties of an Archimedean copula translate into properties of the utility function from which they are constructed.

1 Introduction

Archimedean copulas are constructed using a one-dimensional function, the generator, which is nonincreasing and convex. Von Neumann-Morgenstern utility functions, on the other hand, are nondecreasing (decision makers prefer more to less) and concave (decision makers are risk averse). Therefore, an affine transformation of a utility function, with sign changed, could act as a generator for an Archimedean copula, subject to some additional conditions. Applying this methodology can lead to copula families that are either new or well known.

This paper examines relationships between (generators of) Archimedean copulas and Von Neumann-Morgenstern utility functions. In particular, it will be shown how properties of a utility function translate into the type of dependence induced by the Archimedean copula generated from it. For the sake of brevity, we will confine ourselves to notions of positive dependence only.

Section 2 gives a brief definition of generators of Archimedean copulas, while Section 3 elaborates on the aforementioned method of obtaining generators from utility functions.

Avérous and Dortet-Bernadet (2004) derive relationships between dependence properties of Archimedean copulas and aging properties of their generator. In Section 4, links between copula and utility function are exhibited. Section 5 considers several utility functions which appeared in the literature as examples. Particular attention will be devoted to utility functions with Decreasing Absolute Risk Aversion (DARA), a class that is widely applied in economics and decision theory. Conclusions are presented in Section 6.

2 Archimedean copulas

We define $C(\cdot,\cdot)$ to be a two dimensional copula. An Archimedean copula can be specified as:

$$C_{\varphi}(v_1, v_2) = \varphi\left(\varphi^{[-1]}(v_1) + \varphi^{[-1]}(v_2)\right); \qquad 0 \leq v_1, v_2 \leq 1, \tag{1}$$

with φ nonincreasing and convex, $\varphi(0) = 1$ and $\varphi(s) = 0$ for $s \geq s^*$ for some nonnegative s^*. The generator is strict if $\lim_{s\to\infty}\varphi(s) = 0$ (so $s^* = \infty$), and non-strict if s^* is finite. The function $\varphi^{[-1]}$ is defined as the generalized inverse of φ:

$$\varphi^{[-1]}(s) = \begin{cases} \varphi^{-1}(s) & \text{for} \quad 0 < s \leq 1 \\ s^* & \text{for} \quad s = 0 \end{cases}.$$

Remark 1 *In certain literature about Archimedean copulas, the generator is defined in terms of $\varphi^{[-1]}$ rather than φ ("inverse operator inside, rather than outside, the brackets"). However, we prefer the notation above, as it leads to somewhat simpler expressions. So for instance, the function $s \mapsto \exp[-s]$ is used as generator for the independence copula, rather than $s \mapsto -\log[s]$.*

Remark 2 *The generator, as specified in this paper, is invariant to multiplication of the argument by a positive constant. For $\epsilon > 0$, $\varphi(s)$ and $\varphi(\epsilon s)$ lead to the same copula. This often leads to simplifications.*

3 Utility functions

A utility function $\psi : I \to \mathbb{R}$, with I being a subset of $\mathbb{R}$, is of a von Neumann-Morgenstern type if it is nondecreasing and concave. In this paper we will assume $\psi'(s) > 0$ for $s \in I$. Hence, the function $-\psi : \mathbb{R} \to \mathbb{R}$ is strictly decreasing and convex. This does not mean that $-\psi$ could serve as a generator of an Archimedean copula, since in general the combination of additional requirements $\psi(0) = -1$ and $\lim_{s\to\infty}\psi(s) \geq 0$ is not satisfied.

However, generators of Archimedean copulas can be constructed from affine transformations of utility functions. An important measure for risk perception in utility theory is the degree of absolute risk aversion, defined by Pratt (1964) as

$$r_\psi(s) = -\frac{\psi''(s)}{\psi'(s)} \geq 0, \quad s \in \mathbb{R}. \tag{2}$$

(The subscript ψ in $r_\psi$indicates that the degree of absolute risk aversion is related to the utility function). We define $u(s) = \alpha + \beta\psi(s)$, with α and β real, $\beta > 0$. It is easy to verify that $r_u(s) = r_\psi(s)$, so risk perception is invariant up to an affine transformation. Then for $s \geq 0$, $\max[-u(s), 0]$ could serve as a generator of an Archimedean copula, provided that: a) $u(0) = -1$; b) $\lim_{s\longrightarrow\infty} u(s) \geq 0$. Applying the first condition gives $\alpha = -1 - \beta\psi(0)$. The corresponding generator will then be

$$\varphi(s) = \max[1 + \beta(\psi(0) - \psi(s)), 0], \quad s \geq 0. \tag{3}$$

A necessary condition for strictness of the generator is that $\lim_{s\to\infty}\psi(s) = \psi(\infty) < \infty$. Then satisfaction of the second condition requires $\beta \geq (\lim_{s\to\infty}\psi(s) - \psi(0))^{-1}$, and a strict generator is obtained for $\beta = (\lim_{s\to\infty}\psi(s) - \psi(0))^{-1}$, reducing (3) to

$$\varphi(s) = \frac{\psi(\infty) - \psi(s)}{\psi(\infty) - \psi(0)}, \quad s \geq 0. \tag{4}$$

In all other cases, and also in all cases with $\lim_{s\to\infty}\psi(s) = \infty$, the generator is not strict. The inverse of the generator is

$$\varphi^{[-1]}(s) = \psi^{-1}\left(\psi(0) + \frac{1-s}{\beta}\right).$$

Remark 3 *Obviously, we can only derive generators in this way for $\psi(0)$ well defined and finite. This condition is e.g. not met for the widely applied utility functions $\psi(s) = \log s$, and $\psi(s) = -s^{1-\gamma}$ with $\gamma > 1$.*

Three observations can be made regarding the role of the parameter β:

1. As defined in Nelsen (2006), the upper tail dependence coefficient, denoted by λ_u, is

$$\lambda_u = 2 - \lim_{s \downarrow 0} \frac{1 - \varphi(2s)}{1 - \varphi(s)} = 2 - \lim_{s \downarrow 0} \frac{\psi(2s) - \psi(0)}{\psi(s) - \psi(0)},$$

implying that λ_u does not depend on β.

2. According to Nelsen (2006), a sufficient condition for the copula generated by (3) to be negatively ordered in terms of β (in the sense that $C_{\beta_1} \leq C_{\beta_2}$ for $\beta_1 > \beta_2$, where C_{β_i} indicates the copula with parameter β_i, $i \in \{1, 2\}$) is that $\left(\varphi_{\beta_1}^{[-1]}\right)'(s) \Big/ \left(\varphi_{\beta 2}^{[-1]}\right)'(s)$ is nondecreasing for $s \in (0, 1)$ (here $\varphi_{\beta_i}^{[-1]}(s)$ indicates the inverse generator with parameter β_i, $i \in \{1, 2\}$). We have that

$$\begin{aligned}
&\partial \frac{\left(\varphi_{\beta_1}^{[-1]}\right)'(s) \Big/ \left(\varphi_{\beta 2}^{[-1]}\right)'(s)}{\partial s} \\
= \; & \left(\psi^{-1}\right)'\left(\psi(0) + \frac{1-s}{\beta_1}\right)\left(\psi^{-1}\right)'\left(\psi(0) + \frac{1-s}{\beta_2}\right) \\
& \left(\frac{r^{-1}\left(\psi(0) + \frac{1-s}{\beta_2}\right)}{\beta_2} - \frac{r^{-1}\left(\psi(0) + \frac{1-s}{\beta_1}\right)}{\beta_1}\right),
\end{aligned} \tag{5}$$

defining

$$r^{-1}(x) = \frac{\left(\psi^{-1}\right)''(x)}{\left(\psi^{-1}\right)'(x)}, \quad x \in (\psi(0), \psi(\infty)).$$

Note that $r^{-1}(x) \geq 0$, since $\left(\psi^{-1}\right)'(x) \geq 0$ and $\left(\psi^{-1}\right)''(x) \geq 0$ for all $x \in (\psi(0), \psi(\infty))$ (the inverse of a nondecreasing and concave function is nondecreasing and convex). Hence, (5) is nonnegative for $r^{-1}\left(\psi(0) + \frac{1-s}{\beta}\right) \Big/ \beta$ decreasing in β. It will transpire that all the copulas obtained from the generators derived as above in this paper are negatively ordered in β.

3. Assuming that the assumption of $r^{-1}\left(\psi(0) + \frac{1-s}{\beta}\right) \Big/ \beta$ decreasing in β (as in Observation 2 above) holds, Theorem 4.4.7 of Nelsen (2006) can be applied to find out if this family includes W (Fréchet-Höffding's lower bound) as a limiting member for $\beta \to \infty$. Applying this Theorem 4.4.7 leads to

$$\lim_{\beta \to \infty} \frac{\varphi^{[-1]}(s)}{\left(\varphi^{[-1]}\right)'(t)} = \lim_{\beta \to \infty} \frac{\psi^{-1}\left(\psi(0) + \frac{1-s}{\beta}\right)}{\left(\psi^{-1}\right)'\left(\psi(0) + \frac{1-t}{\beta}\right)\beta^{-1}}.$$

Using de l'Hopital's rule gives

$$\lim_{\beta\to\infty}\frac{\varphi^{[-1]}(s)}{\left(\varphi^{[-1]}\right)'(t)} = \lim_{\beta\to\infty}\frac{\left(\psi^{-1}\right)'\left(\psi(0)+\frac{1-s}{\beta}\right)(s-1)\beta^{-2}}{\left(\psi^{-1}\right)'\left(\psi(0)+\frac{1-t}{\beta}\right)\beta^{-2}+\left(\psi^{-1}\right)''\left(\psi(0)+\frac{1-t}{\beta}\right)(1-t)\beta^{-3}}$$

$$= (s-1)\lim_{\beta\to\infty}\frac{\left(\psi^{-1}\right)'\left(\psi(0)+\frac{1-s}{\beta}\right)}{\left(\psi^{-1}\right)'\left(\psi(0)+\frac{1-t}{\beta}\right)}\frac{1}{1+r^{-1}\left(\psi(0)+\frac{1-t}{\beta}\right)(1-t)\beta^{-1}}.$$

For $\left(\psi^{-1}\right)'(\psi(0))\neq 0$, this limit equals $s-1$, which means that W is then obtained as a limiting member for $\beta\to\infty$.

4 Relationships between properties of utility functions and properties of generators

Using concepts from reliability theory, Avérous and Dortet-Bernadet (2004) derive several relationships between type of dependence of a copula, and aging properties of the generator, which is in fact a survival function. Given the expressions (3) and (4), these aging characteristics translate into properties of the corresponding utility function. In this Section, links between type of dependence of copulas and behavior of corresponding utility functions will be investigated.

For the sake of brevity, we will restrict ourselves to concepts of positive dependence. All notions of positive dependence that appeared in the literature, including the weakest one of Positive Quadrant Dependence (PQD) as defined by Lehmann (1966), require the generator to be strict.

For this reason we will focus on strict generators. It should be stated that most applications in the literature are based on copulas with a strict generator. A non strict generator implies that $C(u_1,u_2)=0$ for some $u_1,u_2>0$. It can sometimes be hard to justify that two events have a nonzero chance of happening individually, but cannot happen jointly. Furthermore, applying the pseudomaximum likelihood method as in Genest et al. (1995) requires the copula to be absolutely continuous, which is implied by a strict generator.

In the sequel, we consider two continuous random variables X and Y, and either an Archimedean distribution copula C_φ with generator φ defined in (1) such that $\Pr[X\leq x,Y\leq y]=C_\varphi(\Pr[X\leq x],\Pr[Y\leq y])$, or an Archimedean survival copula defined as $\widehat{C}_{\widehat{\varphi}}$ with generator $\widehat{\varphi}$ such that $\Pr[X>x,Y>y]=\widehat{C}_{\widehat{\varphi}}(\Pr[X>x],\Pr[Y>y])$. The notation $\widehat{\psi}$ refers to the Morgenstern utility function, from which the generator $\widehat{\varphi}$ is constructed, just as in (4).

Apart from PQD, we will consider SI (Stochastically Increasing) (also from Lehmann, 1966) and both LTD (Left Tail Decreasing) and RTI (Right Tail Increasing) (from Esary and Proschan, 1972) as notions of dependence. The definitions are as below:

Definition 4 *(X,Y) is PQD* $\Leftrightarrow \Pr[X\leq x,Y\leq y]\geq\Pr[X\leq x]\Pr[Y\leq y]$

Definition 5 *Y is LTD in X* $\Leftrightarrow \Pr[Y\leq y\,|X\leq x]$ *is nonincreasing in x for all y.*

Definition 6 *Y is RTI in X* $\Leftrightarrow \Pr[Y>y\,|X>x]$ *is nondecreasing in x for all y.*

Definition 7 *Y is SI in X* $\Leftrightarrow \Pr[Y\leq y\,|X=x]$ *is nonincreasing in x for all y.*

As pointed out in Avérous and Dortet-Bernadet (2004), SI implies LTD and RTI, each of which in turn imply PQD. This can also be shown by using (conditional) hazard functions. Assuming X and Y to be continuous random variables, we define the unconditional hazard functions μ_Y in the usual way

$$\mu_X(x) = \frac{\frac{\partial}{\partial x}\Pr[X \le x]}{\Pr[X > x]}; \quad \mu_Y(y) = \frac{\frac{\partial}{\partial y}\Pr[Y \le y]}{\Pr[Y > y]}.$$

Furthermore, we define some conditional hazard functions. For instance, we define

$$\mu_Y(y\,|X = x) = \frac{\frac{\partial}{\partial y}\Pr[Y \le y\,|X = x]}{\Pr[Y > y\,|X = x]}, \tag{6}$$

as the conditional hazard function of Y at y given $X = x$. In a similar way, we define the conditional hazard functions X at x given $Y = y$ as

$$\mu_X(x\,|Y = y) = \frac{\frac{\partial}{\partial x}\Pr[X \le x\,|Y = y]}{\Pr[X > x\,|Y = y]}.$$

Likewise, we define the conditional hazard functions $\mu_Y(y\,|X \le x)$ and $\mu_Y(y\,|X = x)$ and so on. This leads to the following propositions, proofs of which are straightforward. Note that X and Y can be interchanged.

Proposition 8 *(X, Y) is PQD $\Leftrightarrow \mu_Y(y\,|X > x) \le \mu_Y(y) \le \mu_Y(y\,|X \le x)$ for all x and y.*

Proposition 9 *Y is LTD in $X \Leftrightarrow \mu_Y(y\,|X \le x) \ge \mu_Y(y\,|X = s)$ for all x and s with $x < s$.*

Proposition 10 *Y is RTI in $X \Leftrightarrow \mu_Y(y\,|X > x) \le \mu_Y(y\,|X = s)$ for all x and s with $s < x$.*

Proposition 11 *Y is SI in $X \Leftrightarrow \mu_Y(y\,|X = x)$ is nonincreasing in x for all y.*

The representation in terms of hazard functions also shows that SI is closely related to the notion of long-term dependence as defined in Hougaard (2000).

Definition 12 *Let X and Y be continuous random variables representing lifetimes. Then X and Y exhibit long-term dependence if $\mu_X(x\,|Y = y)$ is constant or decreasing as a function of $y \in [0, x]$ (or alternatively, if $\mu_Y(y\,|X = x)$ is constant or decreasing as a function of $x \in [0, y]$).*

When comparing definitions, one sees that SI requires the conditional hazard function $\mu_Y(y\,|X = x)$ to be nonincreasing also for $x > y$. This condition is not required for long-term dependence.

Remark 13 *Whether or not long-term dependence is a desirable feature in a model is a different question. As discussed in Spreeuw (2006), long-term dependence seems to be a realistic assumption in many applications of reliability theory. For coupled lives, on the other hand, the presumption of long-term dependence seems dubious. The "broken heart syndrome", experienced in some empirical studies, indicates that bereaved lives whose partner died recently have a higher mortality than those who lost their partner years ago.*

The following proposition shows the connection between the dependence properties of either the distribution copula C_φ or the survival copula $\widehat{C}_{\widehat{\varphi}}$, and the risk perception properties of the utility functions ψ and $\widehat{\psi}$, respectively.

Proposition 14 *i) C_φ or $\widehat{C}_{\widehat{\varphi}}$ is PQD*
$\iff (\psi(\infty) - \psi(s))(\psi(\infty) - \psi(t)) \leq (\psi(\infty) - \psi(s+t))(\psi(\infty) - \psi(0))$
or $\left(\widehat{\psi}(\infty) - \widehat{\psi}(s)\right)\left(\widehat{\psi}(\infty) - \widehat{\psi}(t)\right) \leq \left(\widehat{\psi}(\infty) - \widehat{\psi}(s+t)\right)\left(\widehat{\psi}(\infty) - \widehat{\psi}(0)\right)$, *respectively;*
ii) C_φ is LTD $\iff \log[\psi(\infty) - \psi(s)]$ *is convex in s;*
iii) $\widehat{C}_{\widehat{\varphi}}$ is RTI $\iff \log\left[\widehat{\psi}(\infty) - \widehat{\psi}(s)\right]$ *is convex in s;*
iv) C_φ or $\widehat{C}_{\widehat{\varphi}}$ is SI $\iff r_\psi(s) = -\frac{\psi''(s)}{\psi'(s)}$ *is nonincreasing in s or* $r_{\widehat{\psi}}(s) = -\frac{\widehat{\psi}''(s)}{\widehat{\psi}'(s)}$ *is nonincreasing in s, respectively.*

Proof. i) and iv) Follows from the proofs of Proposition 1 in Avérous and Dortet-Bernadet (2004), in connection with Equation (4).

ii) Follows from the proof of Proposition 3 in Avérous and Dortet-Bernadet (2004), in connection with Equation (4).

iii) Observe that $\widehat{C}_{\widehat{\varphi}}$ has the RTI property if and only if

$$\frac{\widehat{\varphi}\left(\widehat{\varphi}^{[-1]}(s) + \widehat{\varphi}^{[-1]}(t)\right)}{s} \geq \frac{\widehat{\varphi}\left(\widehat{\varphi}^{[-1]}(s') + \widehat{\varphi}^{[-1]}(t)\right)}{s'} \quad \forall 0 < t < 1; \quad \forall 0 < s < s' < 1.$$

Following the proof of Proposition 3 in Avérous and Dortet-Bernadet (2004), it follows that $-\log\widehat{\varphi}(s)$ is concave in s, and hence $\log\left[\widehat{\psi}(\infty) - \widehat{\psi}(s)\right]$ is convex in s. ■

Remark 15 *For survival copulas (of the Archimedean type) Spreeuw (2006) shows that long term dependence is equivalent to $r_{\widehat{\psi}}(s)$ nonincreasing in s.*

There seems to be general consensus in the economic literature that the coefficient of absolute risk aversion should be decreasing (or at least nonincreasing) in terms of wealth. Arguments in favor of this property were already given in Arrow (1971) and Pratt (1964). For this reason, most utility functions share the property of Decreasing Absolute Risk Aversion (DARA). This means that several utility functions can be used to construct copulas with the SI property, provided that $\psi(\infty)$ is finite.

Most examples of utility functions as given in the next Section, do feature DARA. As we shall see, the generators constructed from some utility functions belong to well established copula families, but new generators do arise as well.

5 Examples

5.1 Classical cases

5.1.1 Linear utility

Linear utility is equivalent to risk neutrality. The corresponding generator of an Archimedean copula is $\varphi(s) = \max[1 - \beta s, 0]$ reducing to $\max[1 - s, 0]$, (since, as stated above, a generator determines a copula, up to a constant positive factor) being the generator of the Fréchet-Höffding lower bound copula $C(v_1, v_2) = \max[v_1 + v_2 - 1, 0]$. Linear utility corresponds to $r(s) \equiv 0$.

5.1.2 Constant Absolute Risk Aversion (CARA)

CARA functions correspond to $\psi(s) = -\exp[-\gamma s]$, $\gamma > 0$. They derive their name from $r(s) \equiv \gamma$, being independent of s. The corresponding generator of an Archimedean copula is therefore $\varphi(s) = \max[1 - \beta(1 - \exp[-s]), 0]$, requiring $\beta \geq 1$, generating the copula $C(v_1, v_2) = \max\left[\frac{1}{\beta}v_1v_2 + \left(1 - \frac{1}{\beta}\right)(v_1 + v_2 - 1), 0\right]$ which is Family 4.2.7 of Table 4.1 as in Nelsen (2006). The family is negatively ordered in β. The generator is strict only if $\beta = 1$, giving the independence copula.

5.1.3 Constant Relative Risk Aversion (CRRA)

As stated in Pratt (1964), there are three cases of utility functions with CRRA, i.e. $s\ r(s)$ is constant (and therefore DARA):

$$\psi(s) = \begin{cases} s^{1-\gamma} & \text{if } \ 0 < \gamma = s\ r(s) < 1 \\ \log s & \text{if } \quad s\ r(s) = 1 \\ -s^{-(\gamma-1)} & \text{if } \quad \gamma = s\ r(s) > 1. \end{cases} .$$

CRRA utility functions are widely applied in the economic literature. But in spite of the DARA property, one cannot derive a generator of a copula that is SI (or features a weaker type of positive dependence). It is only in the first case that the utility function $\psi(s)$ is well defined for $s = 0$. This gives $\varphi(s) = \max\left[1 - s^{1-\gamma}, 0\right]$, which is Family (4.2.2) of Table 4.1 as in Nelsen (2006). This generator is not strict, since $\varphi(1) = 0$.

5.2 The HARA family

This family (Hyperbolic Absolute Risk Aversion), which contains several utility functions discussed above as special cases, has been introduced in Merton (1971). It is specified as:

$$\psi(s) = \frac{1-\gamma}{\gamma}\left(\frac{s}{1-\gamma} + \epsilon\right)^{\gamma}; \quad \gamma \notin \{0, 1\}; \frac{s}{1-\gamma} + \epsilon > 0; \ \epsilon = 1 \text{ if } \gamma = -\infty.$$

This utility function has risk aversion coefficient $r(s) = \left(\frac{s}{1-\gamma} + \epsilon\right)^{-1}$. Given that the utility function must be well-defined for $s = 0$, $\epsilon \geq 0$ is required, with strict inequality for $\gamma < 0$ or $\gamma > 1$. For $\epsilon = 0$ (requiring $0 < \gamma < 1$) we get the CRRA case already seen before. Otherwise, the generator $\varphi(s) = \max\left[1 + \beta\frac{1-\gamma}{\gamma}\left(1 - \left(\frac{s}{1-\gamma} + 1\right)^{\gamma}\right), 0\right]$ is obtained. The case $\gamma \to 1$ involves linear utility (and hence W), while $\gamma \to \pm\infty$ leads to CARA (and therefore Family 4.2.7 of Table 4.1 as in Nelsen, 2006). Strictness of the generator requires $\gamma < 0$, implying DARA, leading to $\varphi(s) = (s+1)^{\gamma}$ being the generator of the Clayton copula, which is a standard example of an SI copula.

5.3 The expo power utility

This family has been introduced in Saha (1993). A closely related family (which is essentially the same) is considered in Xie (2000). The expo-power utility function is as given below:

$$\psi(s) = \alpha - \exp[-\delta s^{\gamma}]; \quad \alpha > 1; 0 < \gamma \leq 1 \quad \text{and } \delta > 0.$$

We have $r(s) = \frac{1-\alpha+\alpha\delta s^{\gamma}}{s}$, which is decreasing in s. We obtain $\varphi(s) = \max[1 - \beta(1 - \exp[-s^{\gamma}]), 0]$. The family is negatively ordered in both γ and β. For $\beta \to \infty$, the generator reduces to the one for Family (4.2.2) of Table 4.1 as in Nelsen (2006) (see above, for CRRA). The generator is strict for $\beta = 1$, leading to the Gumbel-Hougaard copula, which is also a standard SI case.

5.4 Other examples of Decreasing Absolute Risk Aversion (DARA) as in Pratt (1964)

Pratt developed a few more examples, that will be discussed below.

5.4.1 Equation (37a) of Pratt

This concerns the case

$$\psi(s) = \arctan[s+\delta]; \quad \delta \geq 0.$$

Actually, Pratt imposed the restriction $\delta \geq 1$ to ensure that the utility function features DARA, but a valid generator is obtained for $0 \leq \delta < 1$ as well. We get as generator $\varphi(s) = \max[1+\beta(\arctan[\delta] - \arctan[s+\delta]), 0]$, $s \geq 0$, requiring $\beta \geq \frac{1}{\frac{1}{2}\pi - \arctan\delta}$. Like all other types considered in this note, this family is negatively ordered in β, with W attained for $\beta \to \infty$. The generator is strict if $\beta = \frac{1}{\frac{1}{2}\pi - \arctan\delta}$, reducing the generator to $\varphi(s) = \frac{\frac{1}{2}\pi - \arctan[s+\delta]}{\frac{1}{2}\pi - \arctan\delta}$. Kendall's tau is in that case $\frac{4\delta(\pi - 2\arctan[\delta]) - 2}{(\pi - 2\arctan[\delta])^2} + 1$, which is increasing in δ with lower bound $1 - \frac{8}{\pi^2} \approx 0.18943$ (for $\delta = 0$) and upper bound $\frac{1}{3}$ (for $\delta \to \infty$) so the range of dependence is limited. This utility function features DARA (implying SI for the copula) for $\delta \geq 1$ (the restriction imposed by Pratt) while convexity of $\log[\psi(\infty) - \psi(s)]$ (implying LTD or RTI) is obtained for $\delta \geq 0.35735$.

5.4.2 Equation (37b) of Pratt

This concerns the case

$$\psi(s) = \ln\left[1 - (s+\delta)^{-1}\right]; \quad \delta \geq 1.$$

We get

$$\varphi(s) = \max\left[1 + \beta\ln\left[\frac{(\delta-1)(s+\delta)}{\delta(s+\delta-1)}\right], 0\right], \quad s \geq 0.$$

It is required that $\beta \geq \left(\ln\left[\frac{\delta}{\delta-1}\right]\right)^{-1}$. This family is also negatively ordered in β, with Fréchet-Höffding's lower bound reached for β approaching infinity. The generator is strict if $\beta = \left(\ln\left[\frac{\delta}{\delta-1}\right]\right)^{-1}$ leading to $\varphi(s) = \frac{\ln[s+\delta] - \ln[s+\delta-1]}{\ln[\delta] - \ln[\delta-1]}$, $s \geq 0$. Then Kendall's tau is equal to $4\frac{2-(2\delta-1)\ln\left[\frac{\delta}{\delta-1}\right]}{\left(\ln\left[\frac{\delta}{\delta-1}\right]\right)^2} + 1$, varying in value between 1 (for $\delta \downarrow 0$) and $\frac{1}{3}$(for $\delta \to \infty$). The case $\delta \to \infty$ leads to $\varphi(s) = (s+1)^{-1}$ which is the generator of the copula $C(v_1, v_2) = \frac{v_1 v_2}{v_1 + v_2 - v_1 v_2}$.

5.4.3 Equation (39) of Pratt

This concerns the case

$$\psi(s) = -c_1 e^{-\gamma s} - c_2 e^{-\delta s}; \quad \gamma, c_1, c_2, \delta > 0.$$

We get

$$\varphi(s) = \max\left[1 - \beta\left(c_1\left(1 - e^{-s}\right) - c_2 e^{-\delta s}\right)\right], \quad s \geq 0.$$

This is an example of a generator that has no analytical inverse, so the analysis that can be performed is limited. It is required that $\beta \geq (c_1 + c_2)^{-1}$. The strict generator gives

$$\varphi(s) = \frac{c_1}{c_1 + c_2} e^{-s} + \frac{c_2}{c_1 + c_2} e^{-\delta s},$$

being the Laplace transform of a two point frailty distribution with probabilities $\frac{c_1}{c_1+c_2}$ at point 1 and $\frac{c_2}{c_1+c_2}$ and point δ. Since the utility function is DARA, and $\psi(\infty)$ is finite, the copula generated is SI. This is no surprise given that all shared frailty distributions constitute long term dependence, as pointed out in Spreeuw (2006). Writing $c = \frac{c_1}{c_1+c_2}$, we obtain $c(1-c)\left(\frac{d-1}{d+1}\right)^2$ as the expression for Kendall's tau, taking a minimum of zero for $c = 0$, $c = 1$ or $d = 1$. Comonotonicity is obtained for $c = 0.5$ and $d \to \infty$.

6 Conclusion

A flexible family of Archimedean copulas has been presented that can cover a large range of dependence, including countermonotonicity. In most examples, a strict generator is contained as a special case. Given the general consensus in economic theory that utility functions should feature Decreasing Absolute Risk Aversion, the connection between this property and the Stochastic Increasing notion of the corresponding copula is particularly useful.

Most examples concentrate on strict generators, as this is a requirement for any notion of positive dependence. In the future, we intend to study (3) in a more general sense, considering negative dependence as well.

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