To appear in *Quantitative Finance*, Vol. 00, No. 00, Month 20XX, 1–26

General closed-form basket option pricing bounds

Ruggero Caldana[†], Gianluca Fusai[*†‡], Alessandro Gnoatto[§] and Martino Grasselli[¶‖]

†Università del Piemonte Orientale, Via Perrone 18, 28100 Novara, Italy
‡ Cass Business School, City University, 106 Bunhill Row, London EC1Y 8TZ, UK
§ Mathematisches Institut der LMU, München, Germany
¶ Università degli Studi di Padova, Dipartimento di Matematica, Via Trieste 63, Padova, Italy
‖ De Vinci Finance Lab, Pole Universitaire Léonard de Vinci, 92916 Paris La Défense, France

(*Submitted November 2014*)

This article presents lower and upper bounds on the prices of basket options for a general class of continuous-time financial models. The techniques we propose are applicable whenever the joint characteristic function of the vector of log-returns is known. Moreover, the basket value is not required to be positive. We test our new price approximations on different multivariate models, allowing for jumps and stochastic volatility. Numerical examples are discussed and benchmarked against Monte Carlo simulations. All bounds are general and do not require any additional assumption on the characteristic function, so our methods may be employed also to non-affine models. All bounds involve the computation of one-dimensional Fourier transforms, hence they do not suffer from the curse of dimensionality and can be applied also to high dimensional problems where most existing methods fail. In particular we study two kinds of price approximations: an accurate lower bound based on an approximating set and a fast bounded approximation based on the arithmetic-geometric mean inequality. We also show how to improve Monte Carlo accuracy by using one of our bounds as a control variate.

Keywords: Basket option, Option pricing, Fourier inversion, Control variate

JEL Classification: C63, G13

Basket options are popular derivative contracts which are becoming increasingly widespread in many financial markets, for example equity, FX and commodity markets. Given a vector of weights $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, the basket is defined as the weighted arithmetic average of the n stock prices $S_1(t), \ldots, S_n(t)$ at time T:

$$A_n(T) = \sum_{k=1}^{n} w_k S_k(T).$$

We assume, without loss of generality, that $\sum_{k=1}^{n} w_k = 1$. A basket call option gives the holder the right, but not the obligation, to purchase the portfolio of assets at a fixed price K, known as the option's strike price. We consider European-style options, where the buyer has the right to exercise the option only at maturity T. Hence, the basket option payoff at time T is $(A_n(T) - K)^+$. Another important example is the spread option, where the payoff involves the difference of two or more underlyings, see e.g. Carmona and Durrelman (2003) and Caldana and Fusai (2013). The

[*]Corresponding author. Email: gianluca.fusai@eco.unipmn.it; gianluca.fusai.1@city.ac.uk

time t no-arbitrage fair price of the basket option is

$$C_K(t) = e^{-r(T-t)}\mathbb{E}_t\left[(A_n(T) - K)^+\right], \tag{1}$$

where the t-conditional expectation is computed with respect to a risk-neutral measure and r is a constant riskless interest rate.

A basket option is similar to an Asian claim, where the payoff is determined by the average underlying price over some predetermined period of time. In most contributions from the literature on the valuation of such products the underlying asset prices are assumed to follow lognormal processes. However, the celebrated Black and Scholes (1973) formula cannot be easily extended to the basket option case, since the lognormal distribution is not closed under summation. Several approaches have been proposed to solve the problem, including Monte Carlo simulations, tree-based methods, partial differential equations, and analytical approximations. The last category is the most appealing because most other methods are computationally expensive due to the large dimension of the problem. In addition, it is not easy to extend such methods to a non-Gaussian setting.

Under the assumption that the dynamics of the underlying follows a multivariate geometric Brownian motion, several accurate analytical approximations are available. Curran (1994) introduces the idea of a conditioning variable and conditional moment matching. In particular, he proposes a method based on conditioning on the geometric mean. Assuming Λ is a random variable correlated with A_n and satisfying $A_n \geq K$, whenever $\Lambda \geq \varkappa$ for some constant $\varkappa$, the option price is decomposed into two parts:

$$\mathbb{E}_t\left[(A_n(T) - K)^+\right] = \mathbb{E}_t\left[(A_n(T) - K)\, I(\Lambda > \varkappa)\right] + \mathbb{E}_t\left[(A_n(T) - K)^+ I(\Lambda < \varkappa)\right],$$

where $I(\cdot)$ is the indicator function, taking unit value whenever the argument is true and zero otherwise. By choosing Λ to be the geometric average, the first part can be calculated exactly. The second part can be computed approximately by means of the conditional moment matching method. A similar conditioning argument has been used by Rogers and Shi (1995), where lower and upper bounds for Asian options are derived. Since the approach for Asian options can be easily adapted to basket options and vice-versa, Thompson (1999) and Beisser (2001) extend to basket options the idea of Rogers and Shi (1995) and examine the bound

$$\mathbb{E}_t\left[(A_n(T) - K)^+\right] \geq \mathbb{E}_t\left[(\mathbb{E}\left[A_n(T)\,|\Lambda\right] - K)^+\right]. \tag{2}$$

The approximation in formula (2) can be computed in closed-form in the lognormal framework. It is a lower bound but it turns out to be very close to the true option value in many practical situations. Rogers and Shi (1995) also give an upper bound to the true value, which was later improved by Nielsen and Sandmann (2003) as

$$\mathbb{E}_t\left[(A_n(T) - K)^+\right] \leq \mathbb{E}_t\left[(\mathbb{E}\left[A_n(T)\,|\Lambda\right] - K)^+\right] + \frac{1}{2}\mathbb{E}_t\left[\mathrm{var}(A_n(T)\,|\Lambda) I(\Lambda < \varkappa)\right]^{1/2}\mathbb{E}_t\left[I(\Lambda < \varkappa)\right]^{1/2}.$$

Other bounds proposed in the literature exploit comonotonicity. In this case the central idea consists in replacing the original basket by another one, with a simpler dependence structure. The newly introduced basket involves the components of the comonotonic version of the original random vector, see for example Dhaene *et al.* (2002b) and Dhaene *et al.* (2002a). Vyncke *et al.* (2004) propose a two-moment matching approximation with a convex combination of the comonotonic lower and upper bounds for Asian options while Vanmaele *et al.* (2004) suggest a similar approximation for basket options. Deelstra *et al.* (2004) develop a general framework for pricing basket and Asian options via conditioning and derive lower and upper bounds based on comonotonic risks. The case of Asian basket option is discussed in Deelstra *et al.* (2008). All mentioned bounds are derived in the lognormal framework.

Other authors tried to approximate the basket by using the so-called moment matching method. The idea is to approximate $A_n(T)$ via $\hat{A}_n(T)$, where $\hat{A}_n(T)$ is a random variable with a suitable distribution, chosen to be "close" to the distribution of $A_n(T)$. For example, Gentle (1993) approximates the arithmetic average in the basket payoff by a geometric average. The fact that a geometric average of lognormal random variables is again lognormally distributed allows the application of a Black–Scholes-type valuation formula for pricing the approximating payoff. Vorst (1992) uses the arithmetic-geometric mean inequality to produce lower and upper bounds to the option price and proposes an approximation lying between bounds. Levy (1992) approximates the distribution of the basket by a lognormal distribution such that its first two moments coincide with those of the original distribution of the weighted sum of the stock prices. Huynh (1993) applies the Edgeworth expansion method to basket option valuation for Asian options. Milevsky and Posner (1998a) use the reciprocal Gamma distribution as an approximation for the distribution of the basket. The motivation is the fact that the distribution of correlated lognormally distributed random variables converges to a reciprocal Gamma distribution as the dimension of the basket increases, under special assumptions about the covariance structure. Milevsky and Posner (1998b) use distributions from the Johnson (1949) family as state–price densities to match the higher moments of the arithmetic mean distribution. Ju (2002) considers a Taylor expansion of the ratio of two characteristic functions: the one of the arithmetic average and the one of an approximating lognormal random variable. Such Taylor expansion is computed around zero volatility. Zhou and Wang (2008) approximate the basket distribution by a log-extended-skew-Normal distribution. Further extensions and applications are discussed by Lord (2006).

Many of the methods listed above have limited validity or scope. They may require a basket with positive weights or they may not identify the sensitivities with respect to each basket component. In this regard, Alexander and Venkatramanan (2012) derive a general analytic approximation for pricing basket options expressing each option's price as a sum of the prices of various compound exchange options. They derive an analytic approximation for the price of the compound exchange option, first under the assumption that the underlying assets of these options follow correlated lognormal processes, and then under more general assumptions for the asset price processes. The case of a basket where not all assets have positive weights ($w_k < 0$ for some k) is discussed by Borovkova *et al.* (2007), Li *et al.* (2010) and Deelstra *et al.* (2010) in a lognormal setting. Borovkova *et al.* (2007) approximate the basket distribution by using a generalized family of lognormal distributions. Li *et al.* (2010) provide an extended Kirk approximation and a second-order boundary approximation for pricing spread options on a basket. Deelstra *et al.* (2010) develop approximations formulae based on comonotonicity theory and moment matching methods for spread options, basket spread options, and Asian basket spread options.

Few results are available in the non-Gaussian setting. Flamouris and Giamouridis (2007) propose the use of a simplified jump process, namely, a Bernoulli jump process, and obtain approximate basket option valuation formulas. Xu and Zheng (2009) show that a lower bound similar to that of Rogers and Shi (1995) can also be calculated exactly in a special jump diffusion model with constant volatility and two types of Poisson jumps. An asymptotic expansion with a variance approximation and a lower bound to basket option values for local volatility jump diffusion models are studied by Xu and Zheng (2010, 2014), respectively.

In practice, it is sometimes useful to have model free pricing methods. This part of the literature considers the set of all models consistent with observed prices of vanilla options and recovers distribution free upper and lower bounds to the basket option price. The seminal paper is Bertsimas and Popescu (2002). Then, in a series of papers, Laurence and Wang (2004, 2005), Hobson *et al.* (2005b,a) and D'Aspremont and El Ghaoui (2006) derived distribution free bounds in the case of basket options with positive weights. Model free upper and lower bounds to the basket spread option are investigated in Laurence and Wang (2008). Lower and upper bounds based on comonotonicity theory are theoretically applicable to general dynamics, but research of such methods outside the lognormal setting is still in its early stage.

Another approach assumes the knowledge of the model characteristic function. In such a framework, Hurd and Zhou (2010) propose a general pricing method for a two-dimensional spread option and describe how to generalize it to a multidimensional payoff. Their pricing method is exact and based on an explicit formula for the Fourier transform of the spread option payoff in terms of the Gamma function. Lord *et al.* (2008) and Jackson *et al.* (2008) proposed a general fast Fourier transform (FFT) pricing framework for multi-asset options. All these methods require some particular assumptions on the characteristic function specification, ruling out important models such as mean-reverting models. The work of Jackson *et al.* (2008) has been later generalized by Jaimungal and Surkov (2011) to a cross-commodity modeling framework, allowing pricing for mean-reverting assets. The main drawback of all these methods is that they need an n-dimensional FFT to price an n-dimensional basket option. To this end, Leentvaar and Oosterlee (2008) propose a parallel partitioning approach to tackle the so-called curse of dimensionality when the number of underlying assets becomes large. However, they did not provide results for baskets with dimension greater than seven in their paper.

Readers interested in other basket option pricing methods, based on partial differential equations, Monte Carlo simulations, binomial trees and lattice techniques, are referred to the list of references given in Zhou and Wang (2008).

In conclusion, the existing literature on basket option approximation methods has three weak points:

(i) Many methods have limited applicability because they require the positivity of the basket weights, so they cannot deal with the basket spread option case.
(ii) Most studies are limited to the lognormal case. The study of general pricing methods is still limited.
(iii) Analytical formulas are available in the non-Gaussian case but they involve an n-dimensional FFT and, in practice, they are of little help for applications involving a large number of assets.

This article presents lower and upper bounds for the basket option price, assuming very general dynamics for the n underlyings. The only quantity we need to know explicitly is the joint characteristic function of the log-returns of the assets. All bounds are general and do not require any additional assumption on the characteristic function specification. In particular, we do not assume that the characteristic function is exponential affine with respect to the initial state of the log asset price vector. Our procedure allows the computation for a very large class of stochastic dynamics like mean reverting and non-affine models. Moreover, the basket weights are not required to be positive. Our bounds involve the computation of a univariate Fourier inversion, hence they do not suffer from the curse of dimensionality. This makes our methodologies particularly appealing for higher dimensional problems. To our knowledge, no other general method is successfully applicable to the basket option pricing problem when the basket dimension is large. In general all existing methods face unaffordable computational cost. The only feasible alternative to our approximations is Monte Carlo simulation. However, by using one of our bounds as a control variate, we can also significantly improve the accuracy of the Monte Carlo method itself. In particular we study two kinds of price approximations: an accurate lower bound based on an approximating set, and a fast bounded approximation based on the arithmetic-geometric mean inequality. We test the bounds on different models, including non Gaussian ones. Numerical examples are discussed and benchmarked against Monte Carlo simulations. The wide range of contexts in which basket option pricing problems arise means that the relevance of our result falls also beyond exotic option valuation. For example, the probability distribution of a basket is required in portfolio allocation problems as well. For such problems, a weight optimization is often required, thus a fast procedure to compute the portfolio distribution is needed.

The article is outlined as follows: Section 1 discusses an accurate lower bound based on an approximating set. Section 2 put forward a fast bounded approximation obtained through the arithmetic-geometric mean inequality. The geometric Brownian motion case is discussed in Section

3 and some non-Gaussian models are shown in Section 4. Finally, Section 5 presents numerical experiments.

1. An accurate lower bound through an approximating set

Lower bounds to spread and basket option price can be obtained by approximating the option exercise region via an event set defined through a suitable random variable. Examples in the lognormal framework are Rogers and Shi (1995), Thompson (1999), Carmona and Durrelman (2003) and Bjerksund and Stensland (2014). Extensions to some jump diffusion models are given in Xu and Zheng (2009, 2014). The contribution of this section is the original extension of this popular category of lower bounds to a characteristic function framework. Caldana and Fusai (2013) provide a similar extension, limiting their analysis to options written on the spread between two assets.

Given the set $\mathcal{A} = \{\omega \in \Omega : A_n(T) > K\}$, the value of the basket option price is

$$C_K(t) = e^{-r(T-t)}\mathbb{E}_t\left[(A_n(T) - K)^+\right] = e^{-r(T-t)}\mathbb{E}_t\left[(A_n(T) - K)\, I(\mathcal{A})\right]. \tag{3}$$

For any event set $\mathcal{G} \subset \Omega$

$$\mathbb{E}_t\left[(A_n(T) - K)\, I(\mathcal{G})\right] \leq \mathbb{E}_t\left[(A_n(T) - K)^+ I(\mathcal{G})\right] \leq \mathbb{E}_t\left[(A_n(T) - K)^+\right].$$

Applying the positive part and discounting, it follows that

$$C_K^{\mathcal{G}}(t) = e^{-r(T-t)}\mathbb{E}_t\left[(A_n(T) - K)\, I(\mathcal{G})\right]^+ \leq C_K(t), \tag{4}$$

Depending on the set $\mathcal{G}$, the value of $C_K^{\mathcal{G}}(t)$ is a lower bound to the basket option price $C_K(t)$. We define the set $\mathcal{G}$ using the geometric average $G_n(T)$ of the underlying prices,

$$G_n(T) = \prod_{k=1}^{n} S_k(T)^{w_k}.$$

Being $Y_n(T) = \ln G_n(T)$, we set $\mathcal{G} = \{\omega : Y_n(T) > \varkappa\}$. This choice, which is intuitive and technically convenient, also turns out to be very accurate. We address the choice of the parameter $\varkappa$ shortly. Let $X_k(T)$ be the log-return over the period $[t, T]$:

$$X_k(T) = \ln\left(\frac{S_k(T)}{S_k(t)}\right).$$

We assume that the risk-neutral joint characteristic function of the n stock returns is known:

$$\varphi_T(\boldsymbol{\gamma}) = \mathbb{E}_t\left[e^{\mathrm{i}\sum_{k=1}^{n}\gamma_k X_k(T)}\right], \tag{5}$$

where $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, ..., \gamma_n]$. Simple algebra shows that

$$\begin{aligned} Y_n(T) &= \sum_{k=1}^{n} w_k \ln S_k(T) \\ &= \sum_{k=1}^{n} w_k X_k(T) + \sum_{k=1}^{n} w_k \ln S_k(t) \\ &= \sum_{k=1}^{n} w_k X_k(T) + Y_n(t). \end{aligned}$$

so the joint characteristic function of the log-returns and the log-geometric average is

$$\begin{aligned} \Phi_T(\gamma_0, \boldsymbol{\gamma}, \mathbf{w}, Y_n(t)) &= \mathbb{E}_t\left[e^{\mathrm{i}\sum_{k=1}^{n}\gamma_k X_k(T) + \mathrm{i}\gamma_0 Y_n(T)}\right] \\ &= \mathbb{E}_t\left[e^{\mathrm{i}\sum_{k=1}^{n}(\gamma_k + w_k\gamma_0) X_k(T) + \mathrm{i}\gamma_0 Y_n(t)}\right] \\ &= e^{\mathrm{i}\gamma_0 Y_n(t)}\varphi_T(\boldsymbol{\gamma} + \gamma_0 \mathbf{w}) \end{aligned} \tag{6}$$

and $\boldsymbol{\gamma} + \gamma_0\mathbf{w}$ is the vector with components $\gamma_k + \gamma_0 w_k$. In particular, the characteristic function of the log-geometric average is given by $\Phi_T(\gamma_0, \mathbf{0}, \mathbf{w}, Y_n(t))$. Following Lee (2004), we denote by $\mathcal{A}_{\mathbf{X}}$ the interior of the set

$$\left\{\nu \in \mathbb{R}^n \middle|\ \mathbb{E}_t\left[e^{\mathrm{i}\sum_{k=1}^{n}\nu_k X_k(T)}\right] < \infty\right\}.$$

The explicit computation of the lower bound in (4) is given in the following proposition.

Proposition 1.1 *Let $\delta > 0$ and assume that $\{\mathbf{e}_k, \delta\mathbf{w} + \mathbf{e_k}\} \in \mathcal{A}_{\mathbf{X}}$, $\forall k = 1, \cdots, n$, for $\mathbf{e}_k$ denoting the k-th element of the canonical basis in $\mathbb{R}^n$. A lower bound to the basket option price is given by the following formula*

$$C_K^{\mathcal{G}}(t) = \max_{\varkappa \in \mathbb{R}} C_K^{\mathcal{G}}(t, \varkappa), \tag{7}$$

where

$$C_K^{\mathcal{G}}(t, \varkappa) = \left(e^{-\delta\varkappa - r(T-t)}\frac{1}{\pi}\int_0^{+\infty} e^{-\mathrm{i}\gamma\varkappa}\Psi_T(\gamma;\delta)d\gamma\right)^+, \tag{8}$$

$$\Psi_T(\gamma;\delta) = \frac{1}{\mathrm{i}\gamma + \delta}\left[\sum_{k=1}^{n} w_k S_k(t)\,\Phi_T(\gamma - \mathrm{i}\delta, -\mathrm{i}\mathbf{e}_k, \mathbf{w}, Y_n(t)) - K\Phi_T(\gamma - \mathrm{i}\delta, \mathbf{0}, \mathbf{w}, Y_n(t))\right]. \tag{9}$$

Proof: See Appendix A. □

Some remarks are in order about the above formula. First, the computation of the lower bound requires an univariate Fourier transform inversion and an optimization with respect to the parameter $\varkappa$. The damping factor $\exp(-\delta\varkappa)$, for $\delta > 0$, is introduced in (8) to ensure the existence of the Fourier transform, as Carr and Madan (2000) do. However the numerical inversion is not restricted to this approach and can be performed by using alternative representations, as in Lewis (2000) and Lee (2004). In particular, by following Lee (2004), it is possible to write the lower bound for any

$\delta \in \mathbb{R}$.

Second, if the characteristic function Φ_T is explicitly known, then the Fourier transform of the lower bound can be expressed in closed form as well in terms of the complex function Ψ_T. The integral in (8) can be easily computed using standard numerical quadratures (e.g. `NIntegrate` in Mathematica or `quadgk` in Matlab) or via an FFT algorithm.

The third remark is relative to the characteristic function. The only requirement we set on it is its availability. In particular, we do not require the characteristic function to be exponential affine with respect to the initial value of the state variables. In contrast to this, existing Fourier-based methods for basket options are limited to affine models. In addition, no assumption on the sign of basket weights is introduced in our case.

The fourth remark is about the optimal value of $\varkappa = \varkappa^*$. Figure 1 shows a typical shape for $C_K^{\mathcal{G}}(t, \varkappa)$, as a function of the parameter $\varkappa$.

Our lower bound requires the maximization of $C_K^{\mathcal{G}}(t, \varkappa)$. In practice, the optimization can be accelerated by using a one-dimensional FFT to bound the optimization interval and to guess the starting optimization value $\varkappa^{start}$. Therefore we adopt a two-step strategy which results in a significant time saving:

Step 1 – Bounding the search domain We compute formula (8) via FFT and we obtain $C_K^{\mathcal{G}}(t, \varkappa)$ on an equally spaced grid $\{\varkappa_1, \ldots, \varkappa_M\}$. Then we perform a grid search to find $\varkappa_m$ such that

$$\varkappa_m = \arg\max_{\varkappa \in \{\varkappa_1, \ldots, \varkappa_M\}} C_K^{\mathcal{G}}(t, \varkappa),$$

i.e. an estimate of the lower bound on such a grid. Since $C_K^{\mathcal{G}}(t, \varkappa_m)$ is the best approximation we can get via FFT, we select $\varkappa_m$ as the starting point of the optimization routine in the second step. Extensive numerical tests show that the target function is unimodal, so the maximum of $C_K^{\mathcal{G}}(t, \varkappa)$ should lie in the interval $[\varkappa_{m-1}, \varkappa_{m+1}]$. If the maximum is not unique (i.e. it is achieved on two different points of the grid), we restrict the optimization to the interval delimited by these two values and we use their average as starting point.

Step 2 – Constrained optimization We perform an optimization for the integral in (8) to all

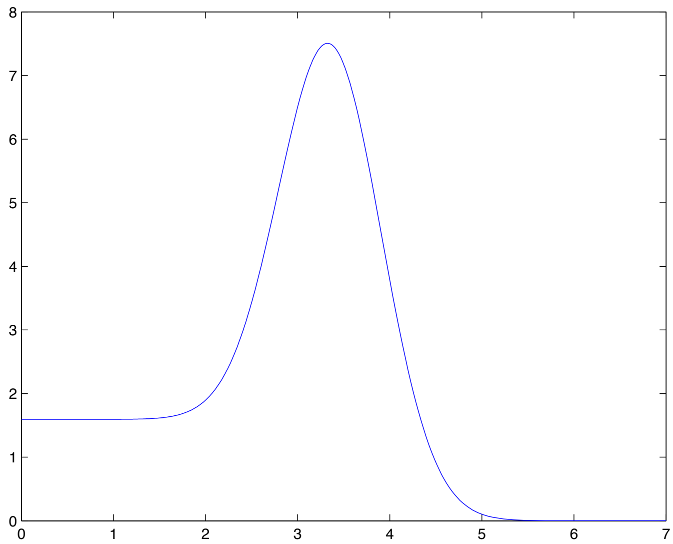

Figure 1. Lower bound $C_K^{\mathcal{G}}(t, \varkappa)$ as a function of the parameter $\varkappa$ for a mean reverting jump diffusion model. The basket is composed by four assets. Parameter values are as in Table 4 and strike price $K = 30$.

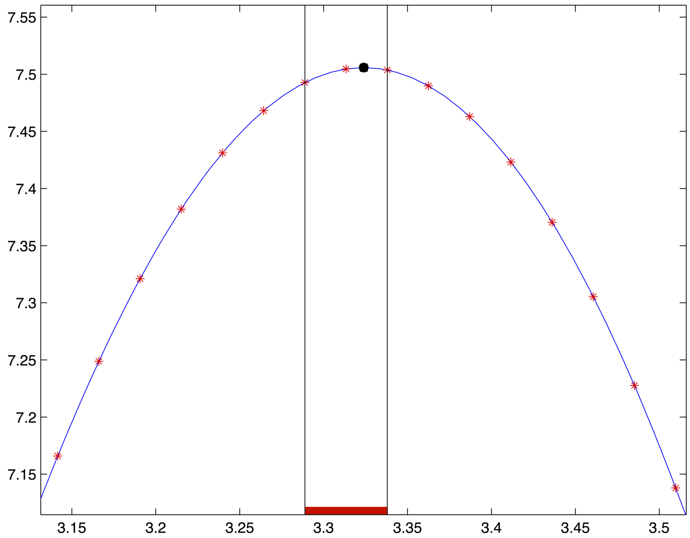

Figure 2. Optimization procedure for a mean reverting jump diffusion model. The basket is composed by four assets. Parameter values are as in Table 4 and strike price $K = 30$. The blue line indicates the values $C_K^{\mathcal{G}}(t, \varkappa)$ as a function of the parameter $\varkappa$. The red markers refer to values obtained via FFT. The black marker indicates the optimized lower bound $C_K^{\mathcal{G}} = 7.5059$. The search domain is restricted to the red segment. The true price estimated via Monte Carlo simulation is 7.5768.

$\varkappa$ in the range $[\varkappa_{m-1}, \varkappa_{m+1}]$. We assume that the numerical quadrature is performed using a grid with N integration points. Given the integration grid and a maturity T, we notice that all evaluations of the function Ψ_T in (9) do not depend on the variable $\varkappa$ over which the optimization is performed. Hence it is possible to evaluate and store all instances of this function computed on the quadrature nodes and then use the stored values in the optimization.

Figure 2 shows the two-step procedure with reference to a mean reverting jump diffusion model.

The lower bound can also be used for the computations of Greeks. The envelope theorem guarantees that changes in the optimizer of the objective do not contribute to the change in the objective function, see Takayama (1974) page 160. Therefore, assuming that interchange of differentiation and integration is allowed, the first-order sensitivity of the basket option price to a change in the spot price of a generic asset is given by

$$\frac{\partial C_K^{\mathcal{G}}(t, \varkappa^*)}{\partial S_k} = I\left(\int_0^{+\infty} e^{-\mathrm{i}\gamma\varkappa^*}\Psi_T(\gamma;\delta)d\gamma \geq 0\right) e^{-\delta\varkappa^* - r(T-t)}\frac{1}{\pi}\int_0^{+\infty} e^{-\mathrm{i}\gamma\varkappa^*}\frac{\partial \Psi_T(\gamma;\delta)}{\partial S_k}d\gamma,$$

Similar formula can be computed for other Greeks.

Finally, the main point concerning formula (7) is that the approximated option price is always obtained through the optimization of a univariate Fourier inversion. The computational cost of the method is $O(n^2 N + M\log(M))$, and increases quadratically with the number of assets n composing the basket. Therefore our technique does not suffer from the curse of dimensionality as it happens for many other Fourier inversion methods proposed in the literature (Hurd and Zhou (2010), Lord *et al.* (2008), Jackson *et al.* (2008) and Jaimungal and Surkov (2011)). These methods provide an exact solution but, requiring a multivariate FFT, they have a cost that is of order $O(nN^n\log(N))$. Due to their computational cost, they are not applicable to the basket option problem when the basket dimension is high. Indeed, the largest dimension of the basket we found in the literature seven and the result is obtained by means of a parallel partitioning approach, see Leentvaar and

Oosterlee (2008).

Looking at the remaining literature, the model free bounds of Hobson *et al.* (2005b) and Laurence and Wang (2008) are the only bounds so general to cover all the models and the basket sizes we are interested in. Such model free bounds require one to compute prices of European call and put on each underlying and solve an optimization problem. However numerical experiments, available upon request, show that their performance in terms of computational speed and accuracy is very poor with respect to the pricing problem under examination.

Our method guarantees faster approximations. Basket options can be easily priced also for high dimensions, and a broader class of problems can be considered. Up to our knowledge, the only feasible alternative to our approximations is Monte Carlo simulation. However, using the bound $C_K^{\mathcal{G}}(t)$ as a control variate[1], we can also improve the accuracy of a Monte Carlo method. Indeed, we rewrite Eq. (1) as

$$C_K(t) = C_K^{\mathcal{G}}(t) + e^{-r(T-t)}\mathbb{E}_t\left[(A_n(T) - K)^+\right] - e^{-r(T-t)}\mathbb{E}_t\left[(A_n(T) - K)\, I(\mathcal{G})\right]^+ .$$

We calculate $C_K^{\mathcal{G}}(t)$ via formula (7) on the optimal approximating set $\mathcal{G}$ and we use Monte Carlo simulation to compute the two expected values, which are highly correlated. In this way the simulation error is considerably reduced. Our formula provides a ready-to-use control variate estimate that allows us to improve the accuracy of Monte Carlo simulations. The accuracy of our lower bound as well as our control variate are proved via extensive numerical tests on a battery of different models in section 5.

2. A fast bounded approximation through the arithmetic-geometric mean inequality

We discuss here new upper and lower bounds to the basket option price and we propose a price approximation lying between such bounds, exploiting the so-called geometric-arithmetic mean inequality. This consists in a generalization of the Vorst (1992) approach to a characteristic function framework, allowing the basket weights to be negative.

Denoting J^{pos} and J^{neg} the sets of the indices corresponding to the positive and negative weights respectively, the basket can be rewritten as

$$A_n(T) = \sum_{k \in J^{pos}} w_k S_k(T) - \sum_{k \in J^{neg}} |w_k| S_k(T) = c^{pos} A_n^{pos}(T) - c^{neg} A_n^{neg}(T), \tag{10}$$

where

$$A_n^{pos}(T) = \frac{\sum_{k\in J^{pos}} w_k S_k(T)}{\sum_{k\in J^{pos}} w_k}, \quad A_n^{neg}(T) = \frac{\sum_{k\in J^{neg}} |w_k| S_k(T)}{\sum_{k\in J^{neg}} |w_k|}, \tag{11}$$

and

$$c^{pos} = \sum_{k\in J^{pos}} w_k, \quad c^{neg} = \sum_{k\in J^{neg}} |w_k|. \tag{12}$$

We define $\mathbf{w}^{pos}$ the vector having as k-th component $w_k^{pos} = w_k / \sum_{k\in J^{pos}} w_k$, when $k \in J^{pos}$ and 0 when $k \in J^{neg}$. Similarly, we define $\mathbf{w}^{neg}$ the vector having $w_k^{neg} = |w_k| / \sum_{k\in J^{neg}} |w_k|$ in the kth

[1] See e.g. Glasserman (2003).

position when $k \in J^{neg}$ and 0 when $k \in J^{pos}$. We also define

$$G_n^{pos}(T) = \prod_{k \in J^{pos}} S_k(T)^{w_k^{pos}}, \qquad G_n^{neg}(T) = \prod_{k \in J^{neg}} S_k(T)^{w_k^{neg}},$$

$Y_n^{pos}(T) = \ln G_n^{pos}(T)$ and $Y_n^{neg}(T) = \ln G_n^{neg}(T)$.

Assuming $K > 0$[1], we can now provide upper and lower bounds to the basket option price. We also obtain an approximation lying between such bounds, in this way generalizing Vorst (1992).

PROPOSITION 2.1 *A lower bound $L_K^{AG}(t)$, an upper bound $U_K^{AG}(t)$ and an approximation $C_K^{AG}(t)$ to the basket option value (1), such that $L_K^{AG}(t) \leq C_K^{AG}(t) \approx C_K(t) \leq U_K^{AG}(t)$, are obtained as*

$$\begin{aligned} L_K^{AG}(t) &= e^{-r(T-t)}\mathbb{E}_t[(c^{pos}G_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K)^+] + \\ &\quad c^{neg}e^{-r(T-t)}\{\mathbb{E}_t[G_n^{neg}(T)] - \mathbb{E}_t[A_n^{neg}(T)]\}, \qquad (13) \\ U_K^{AG}(t) &= e^{-r(T-t)}\mathbb{E}_t[(c^{pos}G_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K)^+] + \\ &\quad c^{pos}e^{-r(T-t)}\{\mathbb{E}_t[A_n^{pos}(T)] - \mathbb{E}_t[G_n^{pos}(T)]\}, \qquad (14) \\ C_K^{AG}(t) &= e^{-r(T-t)}\mathbb{E}_t[(c^{pos}G_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K^*)^+], \qquad (15) \end{aligned}$$

where

$$K^* = K - \mathbb{E}_t[c^{pos}A_n^{pos}(T)] + \mathbb{E}_t[c^{pos}G_n^{pos}(T)] + \mathbb{E}_t[c^{neg}A_n^{neg}(T)] - \mathbb{E}_t[c^{neg}G_n^{neg}(T)]. \qquad (16)$$

Proof: See Appendix B. □

In the spirit of Vorst (1992), approximation $C_K^{AG}(t)$ replaces the random variable $c^{pos}A_n^{pos}(T) - c^{neg}A_n^{neg}(T) - K$ with $c^{pos}G_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K^*$ in the basket spread option payoff. Then the strike price K^* is corrected as in formula (16), so that the unbiasedness condition on the first moment is guaranteed.

We observe that pricing formulae (13), (14) and (15) depend on the value of a call option written on the difference between $c^{pos}G_n^{pos}(T)$ and $c^{neg}G_n^{neg}(T)$, that is

$$e^{-r(T-t)}\mathbb{E}_t[(c^{pos}G_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K)^+].$$

Therefore, we need the pricing of a spread option, that can be easily performed via the Hurd and Zhou (2010) method or through the approximation in Caldana and Fusai (2013), that we recall in Appendix C. These methods require the joint characteristic function of $[\ln(c^{pos}G_n^{pos}(T)), \ln(c^{neg}G_n^{neg}(T))]^\intercal$, that we state here for the sake of completeness:

$$\mathbb{E}_t[e^{\ln(\mathrm{i}\gamma_1 c^{pos}G_n^{pos}(T)) + \mathrm{i}\gamma_2 \ln(c^{neg}G_n^{neg}(T))}] = e^{\mathrm{i}\gamma_1 \ln(c^{pos}G_n^{pos}(t)) + \mathrm{i}\gamma_2 \ln(c^{neg}G_n^{neg}(t))}\varphi_T(\gamma_1 \mathbf{w}^{pos} + \gamma_2 \mathbf{w}^{neg}).$$

When the basket is strictly positive, $G_n(T) = G_n^{pos}(T)$ and $c^{pos} = 1$ by assumptions. We have then the following corollary:

COROLLARY 2.2 *When the basket is strictly positive, a lower bound $L_K^{AG}(t)$, an upper bound $U_K^{AG}(t)$ and an approximation $C_K^{AG}(t)$ for the basket option value (1), such that $L_K^{AG}(t) \leq C_K^{AG}(t) \approx$*

[1]The strike price $K \leq 0$ leads to $\mathbb{E}_t[(A_n(T) - K)^+] = \mathbb{E}_t[(A_n(T)] - K$ for a positive basket.

$C_K(t) \le U_K^{AG}(t)$, *are obtained as*

$$L_K^{AG}(t) = e^{-r(T-t)}\mathbb{E}_t[(G_n(T)-K)^+], \tag{17}$$

$$U_K^{AG}(t) = e^{-r(T-t)}\mathbb{E}_t[(G_n(T)-K)^+] + e^{-r(T-t)}\{\mathbb{E}_t[A_n(T)] - \mathbb{E}_t[G_n(T)]\}, \tag{18}$$

$$C_K^{AG}(t) = e^{-r(T-t)}\mathbb{E}_t[(G_n(T)-K^*)^+], \tag{19}$$

where

$$K^* = K - \mathbb{E}_t[A_n(T)] + \mathbb{E}_t[G_n(T)]. \tag{20}$$

Proof: We consider Proposition (2.1) with $G_n^{pos}(T) = G_n(T)$, $c^{pos} = 1$ and $c^{neg} = 0$. □

When weights are positive, pricing formulae (17),(18) and (19) require the pricing of a call option written on $G_n(T)$, rather than pricing a spread option. The call price is easily computed via Fourier inversion as

$$e^{-r(T-t)}\mathbb{E}_t[(G_n(T)-K)^+] = \frac{e^{-\delta \ln K - r(T-t)}}{\pi}\int_0^\infty e^{-\mathrm{i}\gamma \ln K}\Psi_T^G(\gamma;\delta)d\gamma, \tag{21}$$

where the characteristic functions Ψ_T^G of $\ln G_n(T)$ is

$$\Psi_T^G(\gamma;\delta) = \frac{\Phi_T(\gamma - \mathrm{i}(\delta+1), \mathbf{0}, \mathbf{w}, Y_n(t))}{\delta^2 + \delta - \gamma^2 + \mathrm{i}\gamma(2\delta+1)}, \tag{22}$$

and the parameter δ tunes the damping factor.

Lower and upper bounds $L_K^{AG}(t)$ and $U_K^{AG}(t)$ in formulae (13), (14), (17) and (18) provide an interval for the approximation error of $C_K^{AG}(t)$. Depending on the expected differences between the arithmetic and the geometric average, such an interval may be small or wide.

The main advantage of pricing based on the arithmetic-geometric mean inequality is its computational speed. In the positive basket case, the computation of the bounded approximation requires one-dimensional integrations. The basket spread option case requires two-dimensional FFTs using the Hurd and Zhou (2010) method, or one-dimensional integrations using the approximation in Caldana and Fusai (2013). The computation of $L_K^{AG}(t)$, $C_K^{AG}(t)$ and $U_K^{AG}(t)$ is very fast, regardless of the basket dimension. Its computational cost is linearly increasing in the number of assets, rather than quadratically as for the lower bound of section 1.

Proposition 2.1 and Corollary 2.2 provide very general bounded approximations, that can be computed when the model characteristic function is known. Numerical experiments in Section 5 show that the approximation $C_K^{AG}(t)$, based on the arithmetic-geometric mean inequality, is in general less accurate than the lower bound $C_K(t)$ discussed in Section 1. However the former is much faster than the latter and computing $L_K^{AG}(t)$, $C_K^{AG}(t)$ and $U_K^{AG}(t)$ may be very useful in all applications involving a large number of underlyings.

3. The geometric Brownian motion case

This section discusses in greater detail the geometric Brownian motion case and the explicit computation of the previously introduced price approximations.

We consider a multivariate Black–Scholes model. The dynamics are given by

$$d\mathbf{S}(t) = Diag(\mathbf{S}(t))\left((r\mathbf{1} - \mathbf{q})dt + \sqrt{\Sigma}d\mathbf{W}(t)\right), \tag{23}$$

where r is the risk-free rate, $\mathbf{q}$ is the vector of dividend yields for each asset, $\mathbf{1}$ is a vector whose entries are all equal to one, Σ is the covariance matrix, and $\mathbf{W}$ is an n-dimensional Brownian motion. The risk-neutral joint characteristic function of the n stock returns in the geometric Brownian motion case is

$$\varphi_T(\boldsymbol{\gamma}) = e^{\mathrm{i}\boldsymbol{\gamma}^\intercal \mathbf{m}(T-t) - \frac{1}{2}\boldsymbol{\gamma}^\intercal \Sigma \boldsymbol{\gamma}(T-t)}, \tag{24}$$

where

$$\mathbf{m} = r\mathbf{1} - \mathbf{q} - \frac{1}{2} Vec(\Sigma_{kk}) \tag{25}$$

and $Vec(\cdot)$ is the vectorization operator. From (6), the joint characteristic function of the log-returns and the log-geometric average is

$$\Phi_T\left(\gamma_0, \boldsymbol{\gamma}, \mathbf{w}, Y_n(t)\right) = e^{\mathrm{i}\gamma_0 Y_n(t) + \mathrm{i}(\boldsymbol{\gamma}^\intercal + \gamma_0 \mathbf{w}^\intercal)\mathbf{m}(T-t) - \frac{1}{2}(\boldsymbol{\gamma}^\intercal + \gamma_0 \mathbf{w}^\intercal)\Sigma(\boldsymbol{\gamma} + \gamma_0 \mathbf{w})(T-t)}. \tag{26}$$

Expression (26) can be used to compute the proposed lower and upper bounds; however, in the geometric Brownian motion setting, all formulas can be explicitly computed, see details in Appendix D. The lower bound $C_K^{\mathcal{G}}(t)$ given in formula (8) becomes

$$C_K^{\mathcal{G}}(t) = \max_{\varkappa} e^{-r(T-t)} \left(\sum_{k=1}^n w_k S_k(t) e^{(r-q_k)(T-t)} \mathcal{N}(a_k\sqrt{T-t} - d) - K\mathcal{N}(-d) \right)^+ \tag{27}$$

where

$$d = \frac{\varkappa - \mathbf{w}^\intercal \left(\ln(\mathbf{S}(t)) + \mathbf{m}(T-t)\right)}{\sigma^* \sqrt{T-t}}, \qquad a_k = \frac{\sum_{j=1}^n w_j \Sigma_{kj}}{\sigma^*}, \qquad \sigma^* = \sqrt{\mathbf{w}^\intercal \Sigma \mathbf{w}}$$

and we indicate with $\mathcal{N}(\cdot)$ the standard Normal distribution function. The following value for $\varkappa$ is a good starting point to implement the maximization in (27)

$$\varkappa^{start} = \sigma^* \frac{K - \sum_{k=1}^n w_k S_k(t) e^{(r-q_k)(T-t)}}{\sum_{k=1}^n w_k a_k S_k(t) e^{(r-q_k)(T-t)}} + \sum_{k=1}^n w_k \left(\ln S_k(t) + \left(r - q_k - \frac{\Sigma_{kk}}{2} \right)(T-t) \right).$$

The expectations $\mathbb{E}_t[A_n(T)]$ and $\mathbb{E}_t[G_n(T)]$ are

$$\mathbb{E}_t[A_n(T)] = \sum_{k=1}^n w_k S_k(t) e^{(r-q_k)(T-t)} \quad \text{and} \quad \mathbb{E}_t[G_n(T)] = G(t) e^{\left(\mathbf{w}^\intercal \mathbf{m} + \frac{\sigma^{*2}}{2}\right)(T-t)}.$$

In the positive basket case, call option values involved in bounds through the arithmetic-geometric mean inequality can be obtained using a Black–Scholes formula. Indeed, each asset price $S_k(T)$ is lognormally distributed. Since $G(T)$ is a product of lognormally distributed variables, it is also lognormally distributed

$$G(T) \sim \mathcal{LN}(\ln(G(t)) + \mathbf{w}^\intercal \mathbf{m}(T-t), \sigma^{*2}(T-t))$$

and clearly also $e^{X_k(T)}$ is lognormally distributed

$$e^{X_k(T)} \sim \mathcal{LN}(m_k(T-t), \Sigma_{kk}(T-t)).$$

Then, given a lognormal distributed random variable $Z \sim \mathcal{LN}(\hat{\mu}, \hat{\sigma}^2)$,

$$\mathbb{E}[(Z-K)^+] = e^{\hat{\mu}+\hat{\sigma}^2/2}\mathcal{N}\left(\frac{\hat{\mu}-\ln(K)+\hat{\sigma}^2}{\hat{\sigma}}\right) - K\mathcal{N}\left(\frac{\hat{\mu}-\ln(K)}{\hat{\sigma}}\right).$$

Trivial modifications occur to compute the bounded approximation for basket spread options, see Caldana and Fusai (2013).

4. Non-Gaussian price models

This section presents several price models on which we analyze the performance of our novel bounds. For each model, we give a brief description and we provide the risk-neutral joint characteristic function of asset log-returns $\varphi_T(\boldsymbol{\gamma})$. The joint characteristic function of the log-returns and the log-geometric average $\Phi_T(\gamma_0, \boldsymbol{\gamma}, \mathbf{w}, Y_n(t))$ used in the bounds computation is then immediately obtained via formula (6).

4.1. *A jump diffusion stock price model*

Let us consider a generalization to an n-dimensional case of the two-dimensional jump diffusion process with asymmetric Laplace distributed jump size discussed in Huang and Kou (2006), with reference to the pricing of two dimensional barrier options in equity markets.

The components of the stock price vector, for $k = 1, \dots, n$, have the form

$$S_k(t) = S_k(0)\exp\left[\left(r - q_k - \frac{\sigma_k^2}{2} - \lambda\varkappa_k - \lambda_k\varkappa_{Z_k}\right)t + \sigma_k W_k(t) + \sum_{m_Z=1}^{N_k(t)} Z_k(m_Z) + \sum_{m_Y=1}^{N(t)} Y_k(m_Y)\right], \quad (28)$$

where $\sigma_k > 0$, for $k = 1, \dots, n$, and W_k, W_j are risk-neutral Brownian motions with instantaneous correlation $\rho_{kj}, |\rho| < 1$, for $k, j = 1, \dots, n$. In addition, $\sum_{m_Z=1}^{N_k(t)} Z_k(m_Z)$, for $k = 1, \dots, n$, are n univariate compound Poisson processes driven by the Poisson processes N_k with intensity rate λ_k. This jump component is unique to each stock and describes the idiosyncratic shocks for that particular asset only. The idiosyncratic jump sizes Z_k are independently and identically distributed according to an asymmetric Laplace distribution $\mathcal{AL}(\alpha_{kk}, \xi_{kk}^2)$. The model also allows for macroeconomic shocks described by

$$\sum_{m_Y=1}^{N(t)} \mathbf{Y}(m_Y) = \left(\sum_{m_Y=1}^{N(t)} Y_1(m_Y), \dots, \sum_{m_Y=1}^{N(t)} Y_n(m_Y)\right)^{\intercal},$$

which is a n-dimensional compound Poisson process with intensity rate λ. Under the risk-neutral measure $\mathbb{Q}$ the jump sizes $\mathbf{Y}$ are assumed to be independently and identically distributed according to a multivariate asymmetric Laplace distribution $\mathcal{MAL}(\alpha, \Sigma_Y)$, where $\alpha = (\alpha_1, \dots, \alpha_n)^{\intercal}$ and $\Sigma_{\mathbf{Y}}$ is an $n \times n$ matrix whose elements are defined as

$$(\Sigma_{\mathbf{Y}})_{k,j} = \xi_k \xi_j \rho_{kj}^Y, \qquad k, j = 1, \dots, n.$$

Finally, the quantities $\varkappa_k$ and $\varkappa_{Z_k}, k = 1, \dots, n$, in (28) are, respectively,

$$\varkappa_k = \int_{\mathbb{R}^2} [e^{y_k} - 1] m_{\mathbb{Q}}(dy) = \int_{\mathbb{R}} [e^{y_k} - 1] m_{\mathbb{Q}}(dy_k) = \frac{1}{1 - \alpha_k - \xi_k^2/2} - 1,$$

$$\varkappa_{Z_k} = \int_{\mathbb{R}} [e^{z_k} - 1] m_{\mathbb{Q}}(dz_k) = \frac{1}{1 - \alpha_{kk} - \xi_{kk}^2/2} - 1.$$

Proposition 4.1 *The joint characteristic function of the log-returns for the asymmetric Laplace jump diffusion model is*

$$\varphi_T(\boldsymbol{\gamma}) = \exp\Big[(T-t)\Big(\mathrm{i}\boldsymbol{\gamma}^\intercal\boldsymbol{\eta} - \tfrac{1}{2}\boldsymbol{\gamma}^\intercal\Sigma\boldsymbol{\gamma} + \tfrac{\lambda}{1-\mathrm{i}\boldsymbol{\gamma}^\intercal\alpha+\boldsymbol{\gamma}^\intercal\Sigma_{\mathbf{Y}}\boldsymbol{\gamma}/2} - \lambda + \\ \textstyle\sum_{k=1}^n \Big(\tfrac{\lambda_k}{1-\mathrm{i}\gamma_k\alpha_{kk}+\gamma_k^2\xi_{kk}^2/2} - \lambda_k\Big)\Big)\Big], \tag{29}$$

where $(\Sigma)_{k,j} = \sigma_k\sigma_j\rho_{k,j}$ *and* $\eta_k := r - q_k - \sigma_k^2/2 - \lambda\varkappa_k - \lambda_k\varkappa_{Z_k}, k = 1, \ldots, n.$

Proof: Straightforward generalization of Huang and Kou (2006) to the n-dimensional case. □

4.2. *Mean-reverting jump diffusion model*

The third model is a mean-reverting jump diffusion that generalizes the model proposed by Hambly *et al.* (2009) to describe the electricity spot price in energy markets. For $k = 1, \ldots, n$, the spot price process $S_k(t)$ is defined as the exponential of the sum of three components: a deterministic function $f_k(t)$, a Gaussian Ornstein–Uhlenbeck process $X_k(t)$, and a mean-reverting process with a jump component $Y_k(t)$:

$$\begin{aligned} &S_k(t) = \exp\left(f_k(t) + X_k(t) + Y_k(t)\right), \\ &dX_k = -\alpha_k X_k(t)dt + \sigma_k dW_k, \\ &dY_k = -\alpha_k Y_k(t-)dt + J_k^+ dN_k^+ - J_k^- dN_k^-. \end{aligned}$$

The parameter σ_k is strictly positive and W_k is a risk-neutral Brownian motion. We assume a speed of mean reversion $\alpha_k > 0$ for both the diffusion process $X_k(t)$ and the jump process $Y_k(t)$. The Brownian motions W_k and W_j have instantaneous correlation $\rho_{kj}, |\rho_{kj}| < 1$ for $k \neq j$ and equal to 1 for $k = j$. We denote with N_k^+ and N_k^- Poisson processes with intensity λ_k^+ and λ_k^-, respectively, and describe the positive and negative jump arrivals separately. The terms J_k^+ and J_k^- are independent identically distributed random variables representing the jump size and we assume they are exponentially distributed with parameters $0 < \mu_k^+ < 1$ and $\mu_k^- > 0$, respectively. We denote with $\boldsymbol{\eta}(T)$ the vector having elements

$$\eta_k(T) = (X_k(t) + Y_k(t))(e^{-\alpha_k(T-t)} - 1) + f_k(T) - f_k(t)$$

and $\Sigma(T)$ the matrix having elements

$$\Sigma_{kj}(T) = \rho_{kj}\frac{\sigma_k\sigma_j}{\alpha_k + \alpha_j}\left(1 - e^{-(\alpha_k+\alpha_j)(T-t)}\right).$$

Assuming independence between the jump processes we get the following result:

Proposition 4.2 *The joint characteristic function of the log-returns for the mean reverting jump* *diffusion model is*

$$\varphi_T(\boldsymbol{\gamma}) = \exp\left[\mathrm{i}\boldsymbol{\gamma}^\intercal\boldsymbol{\eta}(T) - \frac{1}{2}\boldsymbol{\gamma}^\intercal\Sigma(T)\boldsymbol{\gamma} + \sum_{k=1}^{n}\frac{\lambda_k^+}{\alpha_k}\ln\left(\frac{1-\mathrm{i}\mu_k^+\gamma_k e^{-\alpha_k(T-t)}}{1-\mathrm{i}\mu_k^+\gamma_k}\right) + \sum_{k=1}^{n}\frac{\lambda_k^-}{\alpha_k}\ln\left(\frac{1+\mathrm{i}\mu_k^-\gamma_k e^{-\alpha_k(T-t)}}{1+\mathrm{i}\mu_k^-\gamma_k}\right)\right]$$

Proof: Generalization of Hambly *et al.* (2009) to the n-dimensional case. □

4.3. *FX stochastic volatility model*

To provide a coherent framework for the evaluation of FX basket options, we consider the model by De Col *et al.* (2013). The idea is to introduce a foreign exchange market featuring n currencies. De Col *et al.* (2013) start by considering the value of each of these currencies in units of an artificial currency that can be viewed as a universal numéraire. The model is initially introduced under the risk neutral measure defined by the artificial currency. In this setting, $S^{0,i}(t)$ denotes the value at time t of one unit of the currency i in terms of the artificial currency. Each artificial exchange rate $S^{0,i}$ is modeled via a multi-variate stochastic volatility model with d independent square-root components, $\mathbf{V}(t) \in \mathbb{R}^d$. According to De Col *et al.* (2013), the dimension d can be chosen according to the specific problem. A further assumption is that the stochastic volatility components are *common* between the different $S^{0,i}$. Formally, we write

$$\frac{dS^{0,i}(t)}{S^{0,i}(t)} = (r^0 - r^i)dt - (\mathbf{a}^i)^\top\sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{Z}(t), \qquad i = 1,\dots,n; \tag{30}$$

$$dV_k(t) = \varkappa_k(\theta_k - V_k(t))dt + \xi_k\sqrt{V_k(t)}dW_k(t), \qquad k = 1,\dots,d; \tag{31}$$

where $\varkappa_k, \theta_k, \xi_k \in \mathbb{R}$ are parameters in a square root dynamics. $\sqrt{\mathrm{Diag}(\mathbf{V})}$ denotes the diagonal matrix with the square root of the elements of the vector $\mathbf{V}$ in the main diagonal. In each monetary area i, the money-market account, based on the deterministic risk free rate r^i, satisfies the following ODE

$$dB^i(t) = r^iB^i(t)dt, \qquad i = 1,\dots,n.$$

Finally, De Col *et al.* (2013) assume an orthogonal correlation structure between the stochastic drivers

$$d\langle Z_k, W_h\rangle(t) = \rho_k\delta_{kh}dt, \qquad k,h = 1,\dots,d, \tag{32}$$

together with $d\langle Z_k, Z_h\rangle(t) = \delta_{kh}dt$ and $d\langle W_k, W_h\rangle(t) = \delta_{kh}dt$.

The philosophy behind this approach is that each exchange rate is driven by several independent noises Z_k ($k = 1,..,d$), each with an independent stochastic variance factor V_k, to which Z_k is partially correlated via ρ_k. The vectors $\mathbf{a}^i$ ($i = 1,\dots,n$) describe how much each of the different volatilities contributes to the dynamics of $S^{0,i}$.

Following De Col *et al.* (2013), we now turn our attention to the exchange rate $S^{i,j}$ between two different currencies, say i and j. We set by definition $S^{i,j} = S^{0,j}/S^{0,i}$.

The resulting dynamics of $S^{i,j} = \left(S^{i,j}(t)\right)_{t\geq 0}$ under the $\mathbb{Q}^i$ risk neutral measure is shown to be equal to

$$dS^{i,j}(t) = S^{i,j}(t)\left((r^i - r^j)dt + (\mathbf{a}^i - \mathbf{a}^j)^\top \sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{Z}^{\mathbb{Q}^i}(t)\right). \tag{33}$$

The valuation of vanilla FX options can be performed using standard Fourier based techniques, see De Col *et al.* (2013). In the sequel, we provide the characteristic function of the log-returns in order to approximate FX basket options.

Proposition 4.3 *The joint characteristic function of the log-returns in the model of De Col et al. (2013) is given by*

$$\varphi_T(\gamma) = e^{A(\tau)+\sum_{k=1}^d B_k(\tau)V_k(t)}, \tag{34}$$

where, for $\tau = T - t$ we have

$$\begin{aligned}
A(\tau) &= \sum_{j=1,j\neq i}^{n}\left(r^i - r^j\right)\mathrm{i}\gamma_j\tau + \sum_{k=1}^{d}\frac{\varkappa_k\theta_k}{\xi_k^2}\left[(Q_k - d_k)\,\tau - 2\log\frac{1 - c_k e^{-d_k\tau}}{1 - c_k}\right],\\
B_k(\tau) &= \frac{Q_k - d_k}{\xi_k^2}\frac{1 - e^{-d_k\tau}}{1 - c_k e^{-d_k\tau}},\\
d_k &= \sqrt{Q_k^2 - 4R_kP_k},\\
c_k &= \frac{Q_k - d_k}{Q_k + d_k},\\
P_k &= \frac{1}{2}\sum_{j,l=1,j,l\neq i}^{n}\mathrm{i}\gamma^j\mathrm{i}\gamma^l\left(a_k^i - a_k^j\right)\left(a_k^i - a_k^l\right) - \frac{1}{2}\sum_{j=1,j\neq i}^{n}\left(a_k^i - a_k^j\right)^2\mathrm{i}\gamma^j,\\
Q_k &= \varkappa_k - \sum_{j=1,j\neq i}^{n}\mathrm{i}\gamma^j B_k(\tau)\left(a_k^i - a_k^j\right)\rho_k\xi_k,\\
R_k &= \frac{1}{2}\xi_k^2.
\end{aligned}$$

Proof. See De Col *et al.* (2013). □

4.4. *The WASC model*

The Wishart Affine Stochastic Correlation (WASC) model, introduced by Da Fonseca *et al.* (2007), is a model which is applicable to different asset classes whenever a realistic and analytically tractable description of instantaneous correlations among state variables is required see e.g. Escobar *et al.* (2012). It describes an n-dimensional vector of assets $(S_1(t), \ldots, S_n(t))^\top$, $t \geq 0$ according to the following dynamics:

$$d\mathbf{S}(t) = Diag\left(\mathbf{S}(t)\right)\left(r\mathbf{1}dt + \sqrt{\Sigma(t)}dZ(t)\right), \tag{35}$$

where $Z(t) \in \mathbb{R}^n$ is a vector Brownian motion and the returns' covariance matrix $\Sigma(t)$ evolves according to the following matrix SDE:

$$d\Sigma(t) = \left(\alpha Q^\top Q + M\Sigma(t) + \Sigma(t)M^\top\right) dt + \sqrt{\Sigma(t)}dW(t)Q + Q^\top dW(t)^\top \sqrt{\Sigma(t)}, \tag{36}$$

$$\Sigma(0) \in S_n^+, \tag{37}$$

where S_n^+ denotes the cones of positive semidefinite matrices endowed with the scalar product given by the trace operator applied to the matrix product. In the dynamics above, $M, Q \in GL(n)$ and we assume that M has negative eigenvalues, so as to ensure the stationarity of the process. Furthermore, we assume $\alpha \geq n - 1$, see Cuchiero *et al.* (2011).

The asymmetric correlation effects are modeled by introducing the following correlation structure among Brownian motions:

$$dZ(t) = \sqrt{1 - \rho^\top \rho}dB(t) + dW(t)\rho \tag{38}$$

where $\rho \in \mathbb{R}^n$, with $\rho \in [-1, 1]^n$ and $\rho^\top \rho \leq 1$. The model belongs to the class of multidimensional stochastic volatility models. In the following, we follow the approach of Grasselli and Tebaldi (2008) and Da Fonseca *et al.* (2007) and report their result on the joint Fourier transform of assets' returns, that we adapt to our setting.

Proposition 4.4 *Let $\tau := T - t$. Given a real vector $\gamma \in \mathbb{R}^n$, the characteristic function of the WASC model is given by*

$$\varphi_T(\boldsymbol{\gamma}) = \exp\left\{A(\tau) + Tr\left[B(\tau)\Sigma(t)\right]\right\}$$

where

$$B(\tau) = B_{22}(\tau)^{-1}B_{21}(\tau),$$

$$\begin{pmatrix} B_{11}(\tau) & B_{12}(\tau) \\ B_{21}(\tau) & B_{22}(\tau) \end{pmatrix} = \exp\tau \begin{pmatrix} M + \mathrm{i}Q^\top \rho\left(\boldsymbol{\gamma}^\top\right) & -2Q^\top Q \\ \Lambda & -\left(M + \mathrm{i}Q^\top \rho\left(\boldsymbol{\gamma}^\top\right)\right)^\top \end{pmatrix},$$

$$\Lambda = -\frac{1}{2}\left((\boldsymbol{\gamma})\left(\boldsymbol{\gamma}^\top\right) + \mathrm{i}\sum_{j=1}^n (\boldsymbol{\gamma})_j\, e_{jj} \right),$$

$$A(\tau) = -\frac{\alpha}{2}\left[\log(B_{22}(\tau)) + \tau\left(M + \mathrm{i}Q^\top \rho\left(\boldsymbol{\gamma}^\top\right)\right)^\top\right] + \mathrm{i}\boldsymbol{\gamma}^\top \mathbf{1} r\tau,$$

where e_{jj} denotes a matrix with a unique non zero entry along the main diagonal equal to one on the position jj.

Proof. See Da Fonseca *et al.* (2007). □

5. Numerical results

This section discusses numerical results with reference to the models we introduced in the previous section. Numerical experiments were coded and implemented in Matlab version 7.14.0 on an Intel Core i5 2.40 GHz machine running under Mac OS X with 4 GB physical memory. We compute the fair value of basket option contracts, spanning different strike prices, for the geometric Brownian motion case and for each non-Gaussian model presented in Section 4. Numerical results are reported in Tables (2–11). Results for the positive basket case are given in Tables (2–6). Results for the basket spread option are in Tables (7–11).

At first we have to select a benchmark. To this aim we consider Monte Carlo simulation, using the lower bound $C_K^{\mathcal{G}}(t)$ as a control variate, as described in Section 1. In this way the simulation error is considerably reduced. The Monte Carlo simulation for the geometric Brownian motion has been implemented by sampling from the lognormal distribution. For the jump diffusion model we sample from the Gaussian noise and the jump components. The other three models have been simulated through the Euler-Maruyama discretization scheme. The number of simulations is chosen depending on the model, as indicated in each table caption. Columns labeled with C.I. length give the length of the 95% mean-centered Monte Carlo confidence interval.

Table 1 compares the control variate (MC) and the crude (MC^{cr}) Monte Carlo for each model. Model parameters are set as in Tables 2–6 and the option strike price K is chosen to be close to at the money. The confidence interval length of each Monte Carlo simulation is also provided. Using the lower bound $C_K^{\mathcal{G}}(t)$ as a control variate in the simulation, the standard error and therefore the confidence interval of the crude Monte Carlo estimate are significantly reduced. For example the length of the confidence interval in the geometric Brownian motion model is reduced from 8.2416×10^{-2} to 3.3707×10^{-3}. A substantial reduction is also obtained for all other models. For example in the FX stochastic volatility model the confidence interval length is reduced from 1.2348×10^{-1} to 8.8989×10^{-4}.

The bounds for the geometric Brownian motion in Table 2 and 7 are computed by exploiting the explicit formulas of section 3. For non-Gaussian models, integrals involved in all lower and upper bound computations are evaluated by means of a Gauss–Kronrod quadrature rule, using Matlab's built-in function `quadgk`. The optimization involved in the computation of $C_K^{\mathcal{G}}(t)$ is performed via the Matlab function `fminunc` for the geometric Brownian motion and using `fminbnd` in remaining cases. Spread options in formulae (13), (14) and (15) have been evaluated using formula (C2) in Appendix C. For all computations involving a Fourier inversion, we used a damping parameter $\delta = 0.75$. The bottom line of each table shows the average CPU time of a single option price evaluation. The quantity is measured in seconds and depends on the model and the pricing method.

The best performances in terms of accuracy are generally obtained by the lower bound $C_K^{\mathcal{G}}(t)$, that outperforms the approximation $C_K^{AG}(t)$ in most cases. The accuracy of the approximations depends on the basket distribution and is affected by the basket size and the presence of negative weights. In particular, the lower bound $C_K^{\mathcal{G}}(t)$ is usually more accurate when the basket is small and weights are positive. Indeed, when the basket weights are strictly positive (Tables 2–6), the bound based on the approximating set argument $C_K^{\mathcal{G}}(t)$ always outperforms the arithmetic-geometric mean inequality approximation $C_K^{AG}(t)$. In the basket spread case (Tables 7–11), i.e. when weights can assume negative values, the lower bound $C_K^{\mathcal{G}}(t)$ still provides good results but it is sometimes outperformed by the bounded approximation $C_K^{AG}(t)$. In particular, $C_K^{AG}(t)$ performs better than $C_K^{\mathcal{G}}(t)$ for out of the money options in Tables 7 and 6 and for in the money options in Table 8. In the pure spread option case in Table 10, the bounded approximation becomes exact and it is always more accurate than $C_K^{\mathcal{G}}(t)$.

Depending on expected differences between the arithmetic and the geometric average, the interval between lower and upper bounds $[U_K^{AG}(t) - L_K^{AG}(t)]$ can be very small or very wide. For example, it is null in the spread option case of Table 10, providing an exact result. It is very large and practically useless for basket spread options in table 8. In general, the true price is closer to $U_K^{AG}(t)$ when the option is in the money, while it is closer to $L_K^{AG}(t)$ when the option is out of the money.

We consider now the GBM model and we compare Table 2 with the results obtained for different approximation methods in Krekel *et al.* (2004) (see tables reported in that paper). Using the same parameter setting, the lower bound $C_K^{\mathcal{G}}(t)$ is as accurate as the best methods for a positive basket (Beisser (2001) and Ju (2002)). Concerning the basket spread option case, we compare Table 7 with the results obtained for different approximation methods in Deelstra *et al.* (2010) (see tables reported in that paper). On the same parameter setting, the lower bound $C_K^{\mathcal{G}}(t)$ is less accurate than the best methods they considered for a basket spread option (Borovkova *et al.* (2007) and Deelstra *et al.* (2010)).

The MRJD model, presented in Tables (4) and (6), is a nice example of the generality of our bounds. Other methods such as Hurd and Zhou (2010), Lord *et al.* (2008) and Jackson *et al.* (2008) cannot cope with this model, because they require assumptions on the model characteristic function that rule out mean reverting models.

As an example of application, we tested two models calibrated to current market data. Results in Tables (5) and (10) refer to the stochastic volatility model for currencies of Section 4.3. We take the parameters from Table 1 in De Col *et al.* (2013), which features the result of a calibration performed on market data as of July 23rd 2010. Taking the perspective of a Japanese investor who seeks protections against fluctuations of both EUR-JPY and USD-JPY, we evaluate the payoff

$$\left(w_1 S^{USD,JPY}(T) + w_2 S^{EUR,JPY}(T) - K\right)^+ .$$

Results in Table (6) refers to a two-asset basket option on FTSE and Eurostoxx 50 modeled with the WASC process of Section 4.4. The parameters refer to a calibration performed on August 20^{th} 2008, as in Da Fonseca and Grasselli (2011).

The computational cost of our price approximation methods varies depending on the performed tests. Computations are extremely fast for the GBM, where no numerical integration is required and in addition we are able to choose a good starting point for the optimization problem. As the complexity of the model characteristic function increases, the computational cost increases as well. In Tables 3 and 8 we tested our methods by considering large baskets consisting of ten and twenty assets. The price approximation is always obtained at a reasonable time cost. Both methodologies involve the computation of one dimensional integrals, and hence they do not suffer from the curse of dimensionality, as opposed to the approach of Hurd and Zhou (2010), Lord *et al.* (2008), Jackson *et al.* (2008) and Jaimungal and Surkov (2011). All these latter require an n-dimensional quadrature, and cannot be used in the practice when the basket dimension is high.

Table 12 investigates the relation between CPU time and basket size, as we increase the number of assets up to 100. We select a basket with positive weights under the jump diffusion model of Section 4.1. We consider the two price approximations $C_K^{\mathcal{G}}(t)$ and $C_K^{AG}(t)$ for an increasing basket size n. Column CPU$^{\mathcal{G}}$ provides the CPU time for the lower bound in formula (7). CPU times for the approximation based on the arithmetic-geometric mean inequality of formula (19) are given in column CPU^{AG}. The computation of the approximation $C_K^{AG}(t)$ is much faster than the lower bound $C_K^{\mathcal{G}}(t)$ because its complexity is linear in the basket dimension and it does not require any optimization. This approximation can be very useful for pricing basket options written on an high number of underlying assets, in particular when the model characteristic function valuation is computationally expensive. On the other hand, the lower bound $C_K^{\mathcal{G}}(t)$ is generally more accurate and it is also reasonably fast because its CPU time grows quadratically in the basket dimension. In our experiments, the price for a basket on 100 underlyings is computed in only 15.53 seconds. Considering basket spread options does not change the above considerations. Monte Carlo simulation in such a case turns out to be very slow and, given the same time budget of competing procedures, quite inaccurate. Numerical results are plotted in Figure 3.

Finally, Figure 4 shows that our approximate methods can be used to compute option Greeks. We consider the computation of Deltas for a basket spread option with strike price $K = 100$ and model parameters as in Table 11. We compare our approximate sensitivities with those computed via Monte Carlo simulation and a finite difference scheme. Both approximate methods provide accurate estimates of the Delta. With respect to the Monte Carlo benchmark, the relative error is smaller than 2%.

6. Conclusions

Most methodologies in the basket option pricing literature are either restricted to a simple lognormal setting or to a model-dependent framework. In this paper we introduced novel methods which allow for a fast and reliable approximation of basket options, via lower and upper bounds. Such bounds rely on a rather weak assumption, i.e. the characteristic function of the vector of log-prices is known. This assumption is very general as most models which are commonly found in the literature allow for an explicit expression of this quantity either by means of the Lévy-Khintchine formula for Lévy processes or systems of (generalized) Riccati equations for affine processes. However, our approach is not restricted to those classes of models, provided the multivariate characteristic function is known. We study the case of strictly positive basket weights as well as the negative one, i.e. the so-called basket spread option. We test our methodologies on different models: a Gaussian model, a jump diffusion model, a mean reverting jump diffusion model, a multi-factor stochastic volatility model with common factors among the underlyings and finally a stochastic correlation model. In particular we study two kinds of price approximations: an accurate lower bound based on an approximating set and a fast bounded approximation based on the arithmetic-geometric mean inequality. Both approximations are particularly appealing for higher dimensional problems, versus most existing methods in the literature that cannot be applied when the basket dimension is large, due to the significant computational cost. Moreover, by using one of our bounds as a control variate, we can largely improve the accuracy of the Monte Carlo estimate.

We believe that our solutions opens up the possibility of performing further investigations, for example in the study of the relationship between the implied volatility of a basket and its constituents, such as the S&P 500 index, and in portfolio selection applications.

References

Alexander, C. and Venkatramanan, A., Analytic approximations for multi-asset option pricing. *Mathematical Finance*, 2012, **22**, 667–689.

Beisser, J., Topics in finance—A conditional expectation approach to value Asian, basket and spread options. PhD thesis, Johannes Gutenberg University Mainz, 2001.

Bertsimas, D. and Popescu, I., On the relation between option and stock prices: A convex optimization approach. *Operations research*, 2002, **50**, 358–374.

Bjerksund, P. and Stensland, G., Closed form spread option valuation. *Quantitative Finance*, 2014, **14**, 1785–1794.

Black, F. and Scholes, M., The pricing of option and corporate liabilities. *Journal of Political Economy*, 1973, **81**, 637–654.

Borovkova, S., Permana, F.J. and Weide, H., A closed-form approach to the valuation and hedging of basket and spread options. *Journal of Derivatives*, 2007, **14**, 8–24.

Caldana, R. and Fusai, G., A general closed-form spread option pricing formula. *Journal of Banking & Finance*, 2013, **37**, 4893—4906.

Carmona, R. and Durrelman, V., Pricing and Hedging Spread Options. *SIAM Review*, 2003, **45**.

Carr, P. and Madan, D., Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2000, **2**, 61–73.

Cuchiero, C., Filipović, D., Mayerhofer, E. and Teichmann, J., Affine Processes on Positive Semidefinite Matrices. *Annals of Applied Probability*, 2011, **21**, 397–463.

Curran, M., Valuing Asian and portfolio options by conditioning on the geometric mean price. *Managment Science*, 1994, **40**, 1705–1711.

Da Fonseca, J. and Grasselli, M., Riding on the smiles. *Quantitative Finance*, 2011, **11**, 1609–1632.

Da Fonseca, J., Grasselli, M. and Tebaldi, C., Option pricing when correlations are stochastic: an analytical framework. *Review of Derivatives Research*, 2007, **10**, 151–180.

D'Aspremont, A. and El Ghaoui, L., Static arbitrage bounds on basket option prices. *Mathematical Programming*, 2006, **106**, 467–489.

De Col, A., Gnoatto, A. and Grasselli, M., Smiles all around: FX joint calibration in a multi-Heston model.

Journal of Banking & Finance, 2013, **37**, 3799–3818.

Deelstra, G., Diallo, I. and Vanmaele, M., Bounds for Asian basket options. *Journal of Computational and Applied Mathematics*, 2008, **218**, 215–228.

Deelstra, G., Liinev, J. and Vanmaele, M., Pricing of arithmetic basket options by conditioning. *Insurance: Mathematics and Economics*, 2004, **31**, 55–77.

Deelstra, G., Petkovic, A. and M.Vanmaele, Pricing and hedging Asian basket spread options. *Journal of Computational and Applied Mathematics*, 2010, **234**, 2814–2830.

Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D., The concept of comonotonicity in actuarial science and finance: applications.. *Insurance: Mathematics and Economics*, 2002a, **31**, 133–161.

Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D., The concept of comonotonicity in actuarial science and finance: Theory.. *Insurance: Mathematics and Economics*, 2002b, **31**, 3–33.

Escobar, M., Arian, H. and Seco, L., CreditGrades Framework within Stochastic Covariance Models. *Journal of Mathematical Finance*, 2012, **4**, 303—314.

Flamouris, D. and Giamouridis, D., Approximate basket option valuation for a simplified jump process. *Journal of Futures Markets*, 2007, **27**, 819– 837.

Gentle, D., Basket weaving. *Risk*, 1993, **6**, 51–52.

Glasserman, P., *Monte Carlo Methods in Financial Engineering*, Stochastic Modelling and Applied Probability Vol. 53, , 2003, Springer.

Grasselli, M. and Tebaldi, C., Solvable Affine Term Structure Models. *Mathematical Finance*, 2008, **18**, 135–153.

Hambly, B., Howison, S. and Kluge, T., Modelling spikes and pricing swing options in electricity markets. *Quantitative Finance*, 2009, **9**, 937–949.

Hobson, D., Laurence, P. and Wang, T., Static-arbitrage optimal sub-replicating strategies for basket options. *Insurance: Mathematics and Economics*, 2005a, **37**, 553–572.

Hobson, D., Laurence, P. and Wang, T., Static-arbitrage upper bounds for the price of basket options. *Quantitative Finance*, 2005b, **5**.

Huang, Z. and Kou, S., First passage times and analytical solutions for options on two assets with jump risk [online]. , 2006. (accessed ????).

Hurd, T.R. and Zhou, Z., A Fourier transform method for spread option pricing. *SIAM Journal of Financial Mathematics*, 2010, **1**, 142–157.

Huynh, C.B., Back to Baskets. *Risk*, 1993, **7**, 59–61.

Jackson, K.R., Jaimungal, S. and Surkov, V., Fourier space time-stepping for option pricing with Lévy models. *The Journal of Computational Finance*, 2008, **12**, 1–29.

Jaimungal, S. and Surkov, V., Lévy Based Cross-Commodity Models and Derivative Valuation. *SIAM Journal of Financial Mathematics*, 2011, **2**, 464–487.

Johnson, N.L., Systems of Frequency Curves Generated by Methods of Translation. *Biometrika*, 1949, **36**, 149–176.

Ju, E., Pricing Asian and basket options via Taylor expansion. *Journal of Computational Finance*, 2002, **5**, 79–103.

Krekel, M., Kock, J.D., Korn, R. and Man, T., An Analysis of Pricing Methods for Baskets Options. *Wilmott*, 2004, **3**, 82–89.

Laurence, P. and Wang, T., What's a basket worth?. *Risk*, 2004, pp. 73–77.

Laurence, P. and Wang, T., Sharp upper and lower bounds for basket options. *Applied Mathematical Finance*, 2005, **12**, 253–282.

Laurence, P. and Wang, T., Distribution-free upper bounds for spread options and market-implied antimonotonicity gap. *The European Journal of Finance*, 2008, **14**, 717–734.

Lee, R., Option pricing by transform methods: extensions, unification and error control. *Journal of Computational Finance*, 2004, **7**, 51—86.

Leentvaar, C.C.W. and Oosterlee, C.W., Multi-asset option pricing using a parallel Fourier-based technique. *The Journal of Computational Finance*, 2008, **12**, 1–26.

Levy, E., Pricing European average rate currency options.. *Journal of International Money and Finance*, 1992, **11**, 474–491.

Lewis, A., *Option valuation under stochastic volatility*, 2000, Finance Press.

Li, M., Zhou, J. and Deng, S.J., Multi-asset Spread Option Pricing and Hedging. *Quantitative Finance*, 2010, **10**, 305–324.

Lord, R., Partially exact and bounded approximations for arithmetic Asian options. *Journal of Computational Finance*, 2006, **10**, 1–52.

Lord, R., Fang, F., Bervoets, F. and Oosterlee, C.W., A fast and accurate FFT-based method for pricing early-exercise options under Lévy processes. *SIAM J. Scientific Computing*, 2008, **30**, 1678–1705.

Milevsky, M.A. and Posner, S.E., A Closed-Form Approximation for Valuing Basket Options. *Journal of Derivatives*, 1998a, **5**, 54–61.

Milevsky, M.A. and Posner, S.E., Valuing exotic options by approximating the SPD with higher moments. *Journal of Financial Engineering*, 1998b, **7**, 109–125.

Nielsen, J.A. and Sandmann, K., Pricing bounds on Asian options. *Journal of Financial and Quantitative Analysis*, 2003, **38**, 449–473.

Rogers, L.C.G. and Shi, Z., The value of an Asian option. *Journal of Applied Probability*, 1995, **32**, 1077–1088.

Takayama, A., *Mathematical Economics*, 1974, The Dryden Press.

Thompson, G.W.P., Topics in mathematical finance. PhD thesis, University of Cambridge, 1999.

Vanmaele, M., Deelstra, G. and Liinev, G., Approximation of stop-loss premiums involving sums of lognormals by conditioning on two variables. *Insurance: Mathematics and Economics*, 2004, **35**, 343–367.

Vorst, T., Prices and Hedge Ratios of Average Exchange Rate Options. *International Review of Financial Analysis*, 1992, **1**, 179–193.

Vyncke, D., Goovaerts, M.J. and Dhaene, J., An accurate analytical approximation for the price of a European-style arithmetic Asian option. *Finance*, 2004, **25**, 121–139.

Xu, G. and Zheng, H., Approximate basket options valuation for a jump-diffusion model. *Insurance: Mathematics and Economics*, 2009, **45**, 188–194.

Xu, G. and Zheng, H., Basket options valuation for a local volatility jump-diffusion model with the asymptotic expansion method. *Insurance: Mathematics and Economics*, 2010, **47**, 415–422.

Xu, G. and Zheng, H., Lower Bound Approximation to Basket Option Values for Local Volatility Jump-Diffusion Models. *International Journal of Theoretical and Applied Finance*, 2014, **17**.

Zhou, J. and Wang, X., Accurate closed-form approximation for pricing Asian and basket options. *Applied Stochastic Models in Business and Industry*, 2008, **24**, 343–358.

Appendix A: Proof of Proposition 1.1

We denote by $f(X_k, Y_n)$ the joint bivariate probability density of $X_k(T)$ and the log-geometric average $Y_n(T)$. We consider the lower bound to the basket option payoff in T, as in formula (4):

$$\mathbb{E}_t\left[(A_n(T)-K)\,I(\mathcal{G})\right],$$

where $\mathcal{G}=\{\omega : Y_n(T)>\varkappa\}$. We introduce the damping factor $\exp(\delta\varkappa)$ according to Carr and Madan (2000) and compute the Fourier transform with respect to $\varkappa$. We obtain

$$\begin{aligned}
\Psi_T(\gamma;\delta) &= \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\mathbb{E}_t\left[(A_n(T)-K)\,I(\mathcal{G})\right]d\varkappa \\
&= \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\left[\mathbb{E}_t\left[\left(\sum_{k=1}^n w_k S_k(T)-K\right)I(\mathcal{G})\right]\right]d\varkappa \\
&= \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\left[\sum_{k=1}^n w_k\mathbb{E}_t\left[S_k(T)I(\mathcal{G})\right]\right]d\varkappa - \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\mathbb{E}_t\left[KI(\mathcal{G})\right]d\varkappa \\
&= \Psi_T^1(\gamma;\delta)-\Psi_T^2(\gamma;\delta).
\end{aligned}$$

So the first part becomes:

$$
\begin{aligned}
\Psi_T^1(\gamma;\delta) &= \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\left[\sum_{k=1}^{n} w_k \mathbb{E}_t\left[S_k(T)I(\mathcal{G})\right]\right] d\varkappa \\
&= \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\left[\sum_{k=1}^{n} w_k \int_{\mathbb{R}}\int_{\varkappa}^{+\infty} S_k(t)\, e^{X_k(T)} f(X_k, Y_n)\, dX_k dY_n\right] d\varkappa \\
&= \sum_{k=1}^{n} w_k \int_{\mathbb{R}}\int_{\mathbb{R}}\left[\int_{-\infty}^{Y_n} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa} d\varkappa\right] S_k(t)\, e^{X_k(T)} f(X_k, Y_n)\, dX_k dY_n \\
&= \frac{1}{\mathrm{i}\gamma+\delta}\sum_{k=1}^{n} w_k \int_{\mathbb{R}}\int_{\mathbb{R}} e^{\mathrm{i}(\gamma-\mathrm{i}\delta)Y_n(T)} S_k(t)\, e^{X_k(T)} f(X_k, Y_n)\, dX_k dY_n \\
&= \frac{1}{\mathrm{i}\gamma+\delta}\sum_{k=1}^{n} w_k S_k(t)\, \mathbb{E}_t\left[e^{\mathrm{i}(\gamma-\mathrm{i}\delta)Y_n(T)+X_k(T)}\right] \\
&= \frac{1}{\mathrm{i}\gamma+\delta}\sum_{k=1}^{n} w_k S_k(t)\, \Phi_T\left(\gamma-\mathrm{i}\delta, -\mathbf{i}\mathbf{e}_k, \mathbf{w}, Y_n(t)\right).
\end{aligned}
$$

The second part becomes:

$$
\begin{aligned}
\Psi_T^2(\gamma;\delta) &= \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\mathbb{E}_t\left[KI(\mathcal{G})\right] d\varkappa \\
&= \int_{\mathbb{R}} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa}\left[\int_{\mathbb{R}}\int_{\varkappa}^{+\infty} Kf(X_k, Y_n)\, dX_k dY_n\right] d\varkappa \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\left[\int_{-\infty}^{Y_n} e^{\mathrm{i}\gamma\varkappa+\delta\varkappa} d\varkappa\right] Kf(X_k, Y_n)\, dX_k dY_n \\
&= \frac{1}{\mathrm{i}\gamma+\delta}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\mathrm{i}(\gamma-\mathrm{i}\delta)Y_n(T)} Kf(X_k, Y_n)\, dX_k dY_n \\
&= \frac{1}{\mathrm{i}\gamma+\delta} K\mathbb{E}_t\left[e^{\mathrm{i}(\gamma-i\delta)Y_n(T)}\right] \\
&= \frac{1}{\mathrm{i}\gamma+\delta} K\Phi_T\left(\gamma-\mathrm{i}\delta, \mathbf{0}, \mathbf{w}, Y_n(t)\right).
\end{aligned}
$$

Remembering the damping factor, we read the Fourier inversion as

$$
\frac{e^{-\delta\varkappa}}{\pi}\int_0^{+\infty} e^{-\mathrm{i}\gamma\varkappa}\Psi(\gamma;\delta)d\gamma.
$$

Formula (8) is obtained by discounting, taking the positive part and maximizing with respect to $\varkappa$. The moment condition $\{\mathbf{e}_k, \delta\mathbf{w}+\mathbf{e_k}\} \in \mathcal{A}_{\mathbf{X}},\ \forall k = 1, \cdots, n$, can be easily deduced from (Lee 2004, Theorem 5.1)

Appendix B: Proof of Proposition 2.1

Due to the arithmetic-geometric mean inequality, $A_n^{pos}(T) \geq G_n^{pos}(T)$, and $A_n^{neg}(T) \geq G_n^{neg}(T)$. Through the put-call parity we have

$$
\begin{gathered}
(c^{pos}A_n^{pos}(T) - c^{neg}A_n^{neg}(T) - K)^+ \geq (c^{pos}G_n^{pos}(T) - c^{neg}A_n^{neg}(T) - K)^+ = \\
(c^{neg}A_n^{neg}(T) + K - c^{pos}G_n^{pos}(T))^+ + c^{pos}G_n^{pos}(T) - c^{neg}A_n^{neg}(T) - K \geq \\
(c^{neg}G_n^{neg}(T) + K - c^{pos}G_n^{pos}(T))^+ + c^{pos}G_n^{pos}(T) - c^{neg}A_n^{neg}(T) - K = \\
(c^{pos}G_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K)^+ + c^{neg}G_n^{neg}(T) - c^{neg}A_n^{neg}(T),
\end{gathered}
$$

and

$$
\begin{gathered}
(c^{pos}A_n^{pos}(T) - c^{neg}A_n^{neg}(T) - K)^+ \leq (c^{pos}A_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K)^+ = \\
(c^{neg}G_n^{neg}(T) + K - c^{pos}A_n^{pos}(T))^+ + c^{pos}A_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K \leq \\
(c^{neg}G_n^{neg}(T) + K - c^{pos}G_n^{pos}(T))^+ + c^{pos}A_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K = \\
(c^{pos}G_n^{pos}(T) - c^{neg}G_n^{neg}(T) - K)^+ + c^{pos}A_n^{pos}(T) - c^{pos}G_n^{pos}(T).
\end{gathered}
$$

The proof ends taking the expectation of above inequalities and discounting.

Appendix C: Spread option approximation

We recall here the spread option pricing formula proposed in Caldana and Fusai (2013). Let $S_1(t)$ and $S_2(t)$ be two stock price processes. The time 0 no-arbitrage fair price of a spread option is

$$
Spread_K(0) = e^{-rT}\mathbb{E}\left[(S_1(T) - S_2(T) - K)^+\right], \tag{C1}
$$

Let $\mathbf{u} = (u_1, u_2)^\intercal \in \mathbb{R}^2$ and $\mathbf{X}(t) = (\ln S_1(t), \ln S_2(t))^\intercal$ and consider the joint characteristic function

$$
\Phi_T(\mathbf{u}) = \Phi_T(u_1, u_2) = \mathbb{E}\left[e^{iu_1 \ln S_1(T) + iu_2 \ln S_2(T)}\right] = \mathbb{E}\left[e^{i\mathbf{u}^\intercal \mathbf{X}(T)}\right].
$$

Proposition C.1 *The approximate spread option value $C_K^{k,\alpha}(0)$ is given in terms of a Fourier inversion formula as*

$$
Spread_K^{k,\alpha}(0) = \left(\frac{e^{-\delta k - rT}}{\pi}\int_0^{+\infty} e^{-i\gamma k}\Psi_T(\gamma;\delta,\alpha)d\gamma\right)^+, \tag{C2}
$$

where

$$
\begin{aligned}
\Psi_T(\gamma;\delta,\alpha) = {} & \frac{e^{i(\gamma - i\delta)\ln(\Phi_T(0,-i\alpha))}}{i(\gamma - i\delta)}\left[\Phi_T((\gamma - i\delta) - i, -\alpha(\gamma - i\delta)) - \right. \\
& \left. \Phi_T(\gamma - i\delta, -\alpha(\gamma - i\delta) - i) - K\Phi_T(\gamma - i\delta, -\alpha(\gamma - i\delta))\right]
\end{aligned} \tag{C3}
$$

and

$$\alpha = \frac{F_2(0,T)}{F_2(0,T)+K}, \tag{C4}$$

$$k = \ln\big(F_2(0,T)+K\big). \tag{C5}$$

The quantity $F_2(0,T) = \mathbb{E}[S_2(T)]$ in formulas (C4) and (C5) is the forward price of the second asset at time 0 for delivery at future date T. Using the characteristic function properties, we can write $F_2(0,T) = \Phi_T(0,-i)$. The parameter δ tunes an exponentially decaying term introduced to allow the integration in the Fourier space.

Appendix D: Proofs for the geometric Brownian motion case

Let us introduce the notation

$$\ln(\mathbf{S}(t)) = \begin{pmatrix} \log S_1(t) \\ \vdots \\ \log S_n(t) \end{pmatrix}.$$

We consider the set

$$\begin{aligned} \mathcal{G} &= \{\omega : Y_n(T) > \varkappa\} \\ &= \left\{\omega : \mathbf{w}^\intercal \left(\ln(\mathbf{S}(t)) + \left(r\mathbf{1} - \mathbf{q} - \frac{1}{2}Vec(\Sigma_{kk})\right)(T-t) + \sqrt{\Sigma}\mathbf{W}(T-t)\right) > \varkappa\right\}. \end{aligned}$$

We see that $\mathbf{w}^\intercal\sqrt{\Sigma}\mathbf{W}(T-t)$ has the same distribution as a univariate Brownian motion $\sigma^* W^*(T-t)$, where $\sigma^* = \sqrt{\mathbf{w}^\intercal\Sigma\mathbf{w}}$. Considering $\mathbf{m}$ as in formula (25), we can write the set $\mathcal{G}$ as

$$\mathcal{G} = \left\{\omega : Z > d = \frac{\varkappa - \mathbf{w}^\intercal\left(\ln(\mathbf{S}(t)) + \mathbf{m}(T-t)\right)}{\sigma^*\sqrt{T-t}}\right\},$$

where Z is a standard Normal random variable. We can write the expectation in (4) as

$$\begin{aligned} \mathbb{E}_t\left[(A_n(T) - K)I(\mathcal{G})\right]^+ &= \mathbb{E}_t\left[\mathbb{E}_t\left[A_n(T) - K|\mathcal{G}\right]I(\mathcal{G})\right]^+ \\ &= \mathbb{E}_t\left[\mathbb{E}_t\left[A_n(T) - K|Z\right]I(Z > d)\right]^+. \end{aligned}$$

Conditionally to the random variable Z, the vector $\sqrt{\Sigma}\mathbf{W}(T-t)$ is distributed like a multivariate Normal with mean $\boldsymbol{\mu}$ and variance V, with their elements defined for $k, j = 1, \ldots, n$ as

$$\mu_k = Za_k\sqrt{T-t}, \quad V_{kj} = (T-t)(\Sigma_{kj} - a_k a_j), \quad a_k = \frac{\sum_{j=1}^n w_j\Sigma_{kj}}{\sigma^*},$$

and we indicate with Σ_{kj} the element of Σ in position (k,j). Due to this fact, $\mathbf{S}(T)|Z$ follows a multivariate lognormal $\mathcal{MLN}(\hat{\boldsymbol{\mu}}, \hat{V})$, where, for $k, j = 1, \ldots, n$,

$$\hat{\mu}_k = \ln S_k(t) + (r - q_k - \Sigma_{kk}/2)(T-t) + Za_k\sqrt{T-t},$$

$$\hat{V}_{kj} = V_{kj}.$$

We can now compute the inner expectation of the payoff, using the lognormal distribution properties

$$
\begin{aligned}
&\mathbb{E}_t\left[\mathbb{E}\left[A_n\left(T\right)-K|Z\right]1_{(Z\geq d)}\right]^+ \\
&=\mathbb{E}_t\left[\left(\sum_{k=1}^{n} w_k e^{\ln S_k(t)+(r-q_k-a_k^2/2)(T-t)+Za_k\sqrt{T-t}}-K\right)I\left(Z\geq d\right)\right]^+.
\end{aligned}
$$

We solve the above expectation by using the partial expectation property of the lognormal distribution. Discounting and maximizing with respect to $\varkappa$, we obtain the lower bound

$$
C_K^{\mathcal{G}}(t)=\max_{\varkappa} e^{-r(T-t)}\left(\sum_{k=1}^{n} w_k S_k(t)e^{(r-q_k)(T-t)}\mathcal{N}(a_k\sqrt{T-t}-d)-K\mathcal{N}(-d)\right)^+. \tag{D1}
$$

We indicate with $\mathcal{N}(\cdot)$ the standard Normal distribution function. The formula above still depends on maximization with respect to the parameter $\varkappa$, involved in the definition of d. Maximization must be carried out by a numerical search, equating to zero the first derivative with respect to $\varkappa$. We need to solve the equation

$$
\sum_{k=1}^{n} w_k S_k(t)e^{(r-q_k)(T-t)}\phi(a_k\sqrt{T-t}-d)-K\phi(-d)=0, \tag{D2}
$$

where we indicate with $\phi(\cdot)$ the standard Normal density function. Using a linearization argument, we can provide the starting point $\varkappa^{start}$ of the numerical search. We approximate the term $\phi(a_k\sqrt{T-t}-d)$ in formula (D2) with a first-order Taylor expansion centered at $-d$,

$$
\phi(a_k\sqrt{T-t}-d)\approx\phi(-d)+a_k\sqrt{T-t}\phi'(-d)=\phi(-d)+da_k\sqrt{T-t}\phi(-d),
$$

obtaining

$$
\sum_{k=1}^{n} w_k S_k(t)e^{(r-q_k)(T-t)}\left(1+da_k\sqrt{T-t}\right)-K=0.
$$

Substituting the definition of d and rearranging terms, it is easy to obtain the following approximation for the value of $\varkappa$ in which the option price is maximum:

$$
\varkappa^{start}=\sigma^*\frac{K-\sum_{k=1}^{n} w_k S_k(t)e^{(r-q_k)(T-t)}}{\sum_{k=1}^{n} w_k a_k S_k(t)e^{(r-q_k)(T-t)}}+\sum_{k=1}^{n} w_k\left(\ln S_k(t)+\left(r-q_k-\frac{\Sigma_{kk}}{2}\right)(T-t)\right).
$$

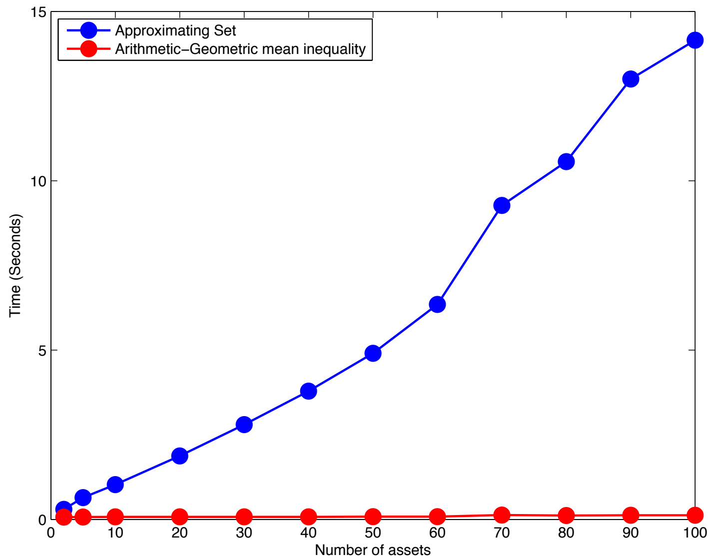

Figure 3. The CPU time (seconds) for the jump diffusion model of Section 4.1 as a function of the basket dimension. Numerical values are given in Table 12.

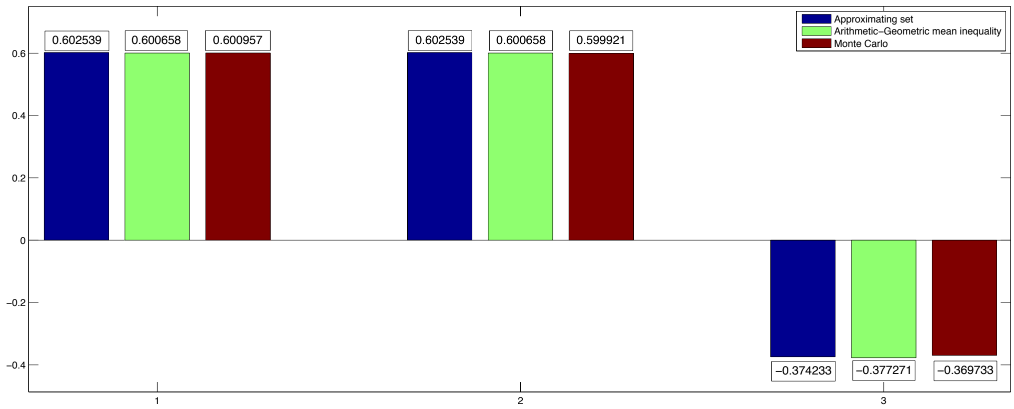

Figure 4. Deltas for a three-asset basket spread option with strike price $K = 100$ and model parameters as in Table 11. We consider sensitivities computed via the approximating set lower bound of Section 1 and via the price approximation of Section 2. A benchmark is obtained through Monte Carlo simulation and a finite difference scheme.

Table 1. The control variate (MC) and the crude (MC^{cr}) Monte Carlo are compared for each model. Simulation settings and model parameters are set as in Tables 2–6. The option strike price K is chosen to be close to at the money. The confidence interval length of the Monte Carlo simulation is also provided.

Model	K	Parameters	MC	C.I. length	MC^{cr}	C.I. length^{cr}
GBM	100	Table 2	28.0070	3.3707×10^{-3}	28.0447	8.2416×10^{-2}
JD	100	Table 3	18.3813	1.0106×10^{-2}	18.2822	2.0322×10^{-1}
MRJD	25	Table 4	10.0060	6.8898×10^{-3}	9.9641	2.0639×10^{-1}
FX-SV	100	Table 5	7.4990	8.8989×10^{-4}	7.4717	1.2348×10^{-1}
WASC	5000	Table 6	1024.27	2.4257×10^{-1}	1032.2709	9.6196×10^{1}

Table 2. Prices of basket options computed for different strikes K in the geometric Brownian motion model of Section 3. The basket weights are $\mathbf{w} = [0.25; 0.25; 0.25; 0.25]$. The parameter values are $T - t = 5$, $r = 0$, $S_k(t) = 100$, $q_k = 0$, $\sigma_k = 40\%$ and $\rho_{kj} = 0.5$, for $k = 1, \ldots, 4$ and $k \neq j$. Column $C_K^{\mathcal{G}}(t)$ contains the lower bound in formula (4). Column $L_K^{AG}(t)$, $C_K^{AG}(t)$ and $U_K^{AG}(t)$ contains the lower bound, the upper bound and the approximation based on the arithmetic-geometric mean inequality as in formulae (17), (19) and (18), respectively. Columns MC and C.I. length contain the Monte Carlo prices and confidence intervals for 10^7 random trials respectively.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
50	54.1580	41.7569	51.9919	55.6861	54.3108	1.4961×10^{-3}
60	47.2699	35.6507	44.4340	49.5799	47.4805	1.9255×10^{-3}
70	41.2575	30.4718	37.9328	44.4010	41.5219	2.3448×10^{-3}
80	36.0411	26.0978	32.4038	40.0270	36.3522	2.7257×10^{-3}
90	31.5296	22.4084	27.7284	36.3376	31.8765	3.0672×10^{-3}
100	27.6326	19.2949	23.7836	33.2241	28.0070	3.3707×10^{-3}
110	24.2664	16.6634	20.4559	30.5926	24.6578	3.6160×10^{-3}
120	21.3562	14.4344	17.6453	28.3636	21.7626	3.8746×10^{-3}
130	18.8368	12.5412	15.2667	26.4704	19.2490	4.0523×10^{-3}
140	16.6519	10.9286	13.2487	24.8578	17.0660	4.2393×10^{-3}
150	14.7532	9.5511	11.5319	23.4803	15.1638	4.3800×10^{-3}
Time	0.0451	0.0085			9.4026	

Table 3. Prices of basket options computed for different strikes K in the jump diffusion model of Section 4.1. The basket weights are $\mathbf{w} = \frac{1}{20}\mathbf{1}_{20}$. The parameter values are $T - t = 1$, $r = 1\%$, $S_k(t) = 100$, $\sigma_k = 40\%$, $\xi_k = 0.5$, $\xi_{kk} = 0.3$, $\alpha_k = \alpha_{kk} = -0.05$, $\lambda = 1$, $\lambda_k = 0.5$, $\rho_{kj} = 0.5$ and $\rho_{kj}^Y = 0.5$ for $k = 1, \ldots, 20$ and $k \neq j$. The Monte Carlo price is obtained with 10^6 simulations. Column labels are as in Table 2.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
50	51.3843	40.4134	51.0643	52.1230	51.4620	3.6600×10^{-3}
60	42.6134	32.2563	41.9998	43.9659	42.7501	5.4294×10^{-3}
70	34.7380	25.2013	33.6724	36.9109	34.9339	6.9961×10^{-3}
80	27.9956	19.4182	26.3980	31.1277	28.2421	8.2449×10^{-3}
90	22.4645	14.8891	20.3784	26.5987	22.7487	9.1949×10^{-3}
100	18.0726	11.4574	15.6289	23.1670	18.3813	1.0106×10^{-2}
110	14.6576	8.9072	12.0121	20.6168	14.9775	1.0812×10^{-2}
120	12.0282	7.0255	9.3174	18.7351	12.3448	1.1089×10^{-2}
130	10.0046	5.6332	7.3281	17.3428	10.3153	1.1819×10^{-2}
140	8.4370	4.5925	5.8579	16.3021	8.7358	1.2298×10^{-2}
150	7.2090	3.8030	4.7614	15.5126	7.4913	1.2342×10^{-2}
Time	2.0381	0.0430			10.5049	

Table 4. Prices of basket options computed for different strikes K in the mean reverting jump diffusion model of Section 4.2. The basket weights are $\mathbf{w} = [0.25; 0.25; 0.25; 0.25]$. The parameter values are $T - t = 1$, $r = 0$, $f_k(T) = \log(25)$, $\boldsymbol{\alpha} = [0.1; 0.2; 0.1; 0.3]$, $f_k(t) = X_k(t) = Y_k(t) = 0$ for $k = 1, \dots, 4$. Jump parameters are $\boldsymbol{\lambda}^+ = \boldsymbol{\lambda}^- = [0.1; 0.2; 0.3; 0.2]$, $\boldsymbol{\mu}^+ = \boldsymbol{\mu}^- = [0.1; 0.1; 0.3; 0.3]$. The covariance matrix of the Brownian noise is $[0.5, 0.35, 0.35, 0.25; 0.35, 0.5, 0.475, 0.15; 0.35, 0.475, 0.5, 0.15; 0.25, 0.15, 0.15, 0.5]$. The Monte Carlo price is obtained with 10^5 random trials and 100 time steps. Column labels are as in Table 2.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
5	26.5940	24.0587	26.5936	26.5947	26.5942	2.0449×10^{-4}
10	21.6486	19.1468	21.6104	21.6829	21.6546	1.4028×10^{-3}
15	17.0447	14.6763	16.8630	17.2123	17.0666	3.2606×10^{-3}
20	13.1066	10.9662	12.7435	13.5023	13.1485	5.0113×10^{-3}
25	9.9462	8.0804	9.4451	10.6164	10.0060	6.8898×10^{-3}
30	7.5059	5.9190	6.9338	8.4551	7.5768	8.0097×10^{-3}
35	5.6608	4.3327	5.0748	6.8688	5.7404	1.0593×10^{-2}
40	4.2798	3.1793	3.7181	5.7154	4.3615	1.1832×10^{-2}
45	3.2497	2.3431	2.7337	4.8792	3.3314	1.2047×10^{-2}
50	2.4809	1.7364	2.0200	4.2724	2.5603	1.2114×10^{-2}
55	1.9054	1.2945	1.5013	3.8306	1.9791	1.2693×10^{-2}
Time	0.6525		0.0461		6.9586	

Table 5. Prices of basket options computed for different strikes K in the stochastic volatility model of Section 4.3. The basket weights are $\mathbf{w} = [0.4; 0.6]$. The parameter values are $T - t = 1$, $S^{JPY,USD} = 86.90$, $S^{JPY,EUR} = 112.29$, $V_1 = 0.0137$, $V_2 = 0.0391$, $a_1^{USD} = 0.6650$, $a_2^{USD} = 1.0985$, $a_1^{EUR} = 1.6177$, $a_2^{EUR} = 1.3588$, $a_1^{JPY} = 0.2995$, $a_2^{JPY} = 1.6214$, $\varkappa_1 = 0.9418$, $\varkappa_2 = 1.7909$, $\theta_1 = 0.0370$, $\theta_2 = 0.0909$, $\xi_1 = 0.4912$, $\xi_2 = 1$, $\rho_1 = 0.5231$ and $\rho_2 = -0.3980$. The Monte Carlo price is obtained with 10^5 random trials and 100 time steps. Column labels are as in Table 2.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
50	52.2555	51.1796	52.2871	52.2988	52.2560	3.3805×10^{-4}
60	42.3929	41.3311	42.4279	42.4503	42.3939	4.6861×10^{-4}
70	32.6765	31.6344	32.7076	32.7536	32.6782	6.5341×10^{-4}
80	23.2842	22.2763	23.2967	23.3955	23.2873	7.7204×10^{-4}
90	14.6123	13.6795	14.5812	14.7986	14.6172	1.0181×10^{-3}
100	7.4935	6.7392	7.3923	7.8584	7.4990	8.8989×10^{-4}
110	3.0988	2.6440	2.9503	3.7631	3.1043	9.6854×10^{-4}
120	1.2843	1.0498	1.1551	2.1690	1.2883	8.6991×10^{-4}
130	0.6279	0.4926	0.5318	1.6117	0.6305	7.0428×10^{-4}
140	0.3561	0.2672	0.2845	1.3863	0.3581	7.4411×10^{-4}
150	0.2237	0.1601	0.1689	1.2793	0.2250	6.7663×10^{-4}
Time	0.7684		0.0260		2.9647	

Table 6. Prices of basket options computed for different strikes K in the WASC model of Section 4.4. The basket weights are $\mathbf{w} = [0.5; 0.5]$. The parameter values are $T - t = 1$, $\mathbf{S}(t) = [5371.80; 3295.28]$, $Q = [0.3296, 0.2866; 0.3446, 0.3524]$, $M = [-0.9886, -0.3631; -0.4464, -0.7599]$, $\rho = [-0.2675; -0.5496]$, $\beta = 10.8247$ and $r = 0$. The Monte Carlo price is obtained with 10^4 random trials and 100 time steps. Column labels are as in Table 2.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
4000	1389.46	1291.48	1348.57	1428.48	1389.78	1.9805×10^{-1}
4500	1192.76	1102.68	1151.49	1239.68	1193.17	2.5758×10^{-1}
5000	1023.87	941.50	983.14	1078.50	1024.27	2.4257×10^{-1}
5500	879.12	804.14	839.61	941.14	879.52	2.8705×10^{-1}
6000	755.22	687.22	717.40	824.22	755.60	2.9166×10^{-1}
6500	649.24	587.76	613.43	724.76	649.55	2.4128×10^{-1}
7000	558.60	503.16	525.00	640.16	558.91	2.5190×10^{-1}
7500	481.09	431.20	449.78	568.20	481.42	3.0524×10^{-1}
8000	414.79	369.97	385.78	506.97	415.18	3.0420×10^{-1}
8500	358.04	317.83	331.30	454.83	358.50	3.7031×10^{-1}
9000	309.43	273.40	284.88	410.41	309.69	2.1837×10^{-1}
Time	3.6146		0.3652		85.9311	

Table 7. Prices of basket spread options computed for different strikes K in the geometric Brownian motion model of Section 3. The basket weights are $\mathbf{w} = [1; -1; -1]$. The parameter values are $T - t = 1$, $r = 5\%$ $\mathbf{S}(t) = [100; 63; 12]$, $\mathbf{q} = [0; 0; 0]$, $\boldsymbol{\sigma} = [0.21; 0.34; 0.63]$, $\rho_{12} = 0.87$, $\rho_{13} = 0.3$, $\rho_{23} = 0.43$. Column $C_K^{\mathcal{G}}(t)$ contains the lower bound in formula (4). Column $L_K^{AG}(t)$, $C_K^{AG}(t)$ and $U_K^{AG}(t)$ contains the lower bound, the upper bound and the approximation based on the arithmetic-geometric mean inequality as in formulae (13), (15) and (14), respectively. Columns MC and C.I. length contain the Monte Carlo prices and confidence intervals for 10^7 random trials.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
5	25.6962	23.6509	24.6434	27.8943	26.7762	4.6745×10^{-3}
10	21.3408	20.0257	21.3284	24.2691	23.0466	6.1873×10^{-3}
15	17.2434	16.7488	18.3674	20.9922	19.6852	7.7895×10^{-3}
20	13.4984	13.8257	15.7524	18.0691	16.7042	9.4315×10^{-3}
25	10.1955	11.2470	13.4645	15.4904	14.1003	1.1021×10^{-2}
30	7.4021	8.9929	11.4781	13.2363	11.8478	1.2448×10^{-2}
35	5.1488	7.0373	9.7641	11.2807	9.9260	1.3713×10^{-2}
40	3.4223	5.3509	8.2927	9.5943	8.2952	1.4684×10^{-2}
45	2.1695	3.9039	7.0345	8.1473	6.9192	1.5344×10^{-2}
50	1.3100	2.6671	5.9622	6.9105	5.7612	1.5659×10^{-2}
55	0.7528	1.6132	5.0507	5.8567	4.8008	1.5656×10^{-2}
Time	0.1748		0.0295		2.8915	

Table 8. Prices of basket spread options computed for different strikes K in the jump diffusion model of Section 4.1. The basket weights are $\mathbf{w} = [1; 1; 1; 1; 1; 1; 1; 1; 1; 1; -0.9; -0.9; -0.9; -0.9; -0.9; -0.9; -0.9; -0.9; -0.9; -0.9]$. The parameter values are $T - t = 1$, $r = 1\%$, $S_k(t) = 100$, $\sigma_k = 20\%$, $\xi_k = 0.25$, $\xi_{kk} = 0.15$, $\alpha_k = \alpha_{kk} = -0.05$, $\lambda = 1$, $\lambda_k = 0.1$, $\rho_{kj} = 0.75$ and $\rho^Y_{kj} = 0.75$ for $k = 1, \ldots, 20$ and $k \neq j$. The Monte Carlo price is obtained with 10^6 simulations. Column labels are as in Table 7.

K	$C^{\mathcal{G}}_K(t)$	$L^{AG}_K(t)$	$C^{AG}_K(t)$	$U^{AG}_K(t)$	MC	C.I. length
50	59.7381	47.9137	59.9191	71.2563	60.2348	1.7135×10^{-2}
60	52.4154	40.5551	52.4967	63.8976	52.8422	1.5334×10^{-2}
70	45.6244	33.7432	45.6139	57.0857	45.9951	1.4100×10^{-2}
80	39.4076	27.5264	39.3213	50.8690	39.7330	1.2740×10^{-2}
90	33.7918	21.9352	33.6515	45.2777	34.0936	1.1653×10^{-2}
100	28.7861	16.9788	28.6166	40.3214	29.0839	1.1070×10^{-2}
110	24.3817	12.6459	24.2075	35.9884	24.6909	1.0848×10^{-2}
120	20.5538	8.9062	20.3964	32.2487	20.8955	1.1240×10^{-2}
130	17.2644	5.7151	17.1400	29.0577	17.6438	1.1814×10^{-2}
140	14.4663	3.0185	14.3852	26.3611	14.8924	1.2630×10^{-2}
150	12.1067	0.7576	12.0735	24.1002	12.5764	1.3666×10^{-2}
Time	6.3236		0.2214		12.3182	

Table 9. Prices of basket spread options computed for different strikes K in the mean reverting jump diffusion model of Section 4.2. The basket weights are $\mathbf{w} = [2; 1; -1; -1]$. The parameter values are $T - t = 1$, $r = 0$, $f_k(T) = \log(25)$, $\boldsymbol{\alpha} = [0.1; 0.2; 0.1; 0.3]$, $f_k(t) = X_k(t) = Y_k(t) = 0$ for $k = 1, \ldots, 4$. Jump parameters are $\boldsymbol{\lambda}^+ = \boldsymbol{\lambda}^- = [0.1; 0.2; 0.3; 0.2]$, $\boldsymbol{\mu}^+ = \boldsymbol{\mu}^- = [0.1; 0.1; 0.3; 0.3]$. The covariance matrix of the Brownian noise is $[0.5, 0.35, 0.35, 0.25; 0.35, 0.5, 0.475, 0.15; 0.35, 0.475, 0.5, 0.15; 0.25, 0.15, 0.15, 0.5]$. The Monte Carlo price is obtained with 10^5 random trials and 100 time steps. Column labels are as in Table 7.

K	$C^{\mathcal{G}}_K(t)$	$L^{AG}_K(t)$	$C^{AG}_K(t)$	$U^{AG}_K(t)$	MC	C.I. length
5	27.7176	23.9716	27.4510	32.5504	28.6940	1.0589×10^{-1}
10	24.2968	20.2235	23.9533	28.8024	25.0622	9.6171×10^{-2}
15	21.2185	16.8882	20.8709	25.4671	21.8236	8.6890×10^{-2}
20	18.4747	13.9634	18.1838	22.5423	18.9433	6.6397×10^{-2}
25	16.0482	11.4209	15.8554	19.9997	16.4535	6.2356×10^{-2}
30	13.9159	9.2207	13.8434	17.7996	14.2959	5.9968×10^{-2}
35	12.0514	7.3206	12.1065	15.8995	12.4334	4.8932×10^{-2}
40	9.0178	4.2637	9.3106	12.8425	10.8217	4.3863×10^{-2}
45	10.4277	5.6804	10.6068	14.2592	9.4338	4.3187×10^{-2}
50	7.7966	3.0386	8.1888	11.6175	8.2514	4.2944×10^{-2}
55	6.7404	1.9777	7.2163	10.5566	7.2224	4.3764×10^{-2}
Time	0.5670		0.2531		7.5675	

Table 10. Prices of basket spread options computed for different strikes K in the FX stochastic volatility model of Section 4.3. The basket weights are $\mathbf{w} = [2; -1]$. The parameter values are $T - t = 1$, $S^{JPY,USD} = 86.90$, $S^{JPY,EUR} = 112.29$, $V_1 = 0.0137$, $V_2 = 0.0391$, $a_1^{USD} = 0.6650$, $a_2^{USD} = 1.0985$, $a_1^{EUR} = 1.6177$, $a_2^{EUR} = 1.3588$, $a_1^{JPY} = 0.2995$, $a_2^{JPY} = 1.6214$, $\varkappa_1 = 0.9418$, $\varkappa_2 = 1.7909$, $\theta_1 = 0.0370$, $\theta_2 = 0.0909$, $\xi_1 = 0.4912$, $\xi_2 = 1$, $\rho_1 = 0.5231$ and $\rho_2 = -0.3980$. The Monte Carlo price is obtained with 10^5 random trials and 100 time steps. Column labels are as in Table 7.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
5	56.5638	56.7048	56.7048	56.7048	56.7049	6.8517×10^{-5}
10	51.5996	51.7630	51.7630	51.7630	51.7631	1.8136×10^{-4}
15	46.6555	46.8368	46.8368	46.8368	46.8371	2.4466×10^{-4}
20	41.7420	41.9396	41.9396	41.9396	41.9402	4.8875×10^{-4}
30	32.0764	32.3139	32.3139	32.3139	32.3158	7.9035×10^{-4}
40	22.8299	23.0913	23.0913	23.0913	23.0947	1.0828×10^{-3}
50	14.4695	14.7121	14.7121	14.7121	14.7184	1.3303×10^{-3}
60	7.8445	8.0167	8.0167	8.0167	8.0300	2.3239×10^{-3}
70	3.7239	3.8079	3.8079	3.8079	3.8255	3.2946×10^{-3}
75	2.5116	2.5589	2.5589	2.5589	2.5768	3.5316×10^{-3}
80	1.6886	1.7073	1.7073	1.7073	1.7287	4.4332×10^{-3}
Time	0.6483	0.2401			2.4542	

Table 11. Prices of basket spread options computed for different strikes K in the WASC model of Section 4.4. The basket weights are $\mathbf{w} = [1; 1; -1]$. The parameter values are $T - t = 1$, $\mathbf{S}(t) = 100 \cdot \mathbf{1}_3$, $Q = 0.25 \cdot I_3$, $M = -0.5 \cdot I_3$, $\rho = -0.3 \cdot \mathbf{1}_3$, $\beta = 10.8247$ and $r = 0$. The Monte Carlo price is obtained with 10^4 random trials and 100 time steps. Column labels are as in Table 7.

K	$C_K^{\mathcal{G}}(t)$	$L_K^{AG}(t)$	$C_K^{AG}(t)$	$U_K^{AG}(t)$	MC	C.I. length
50	64.3911	55.0700	64.0552	67.6858	65.2016	2.1340×10^{-1}
60	57.4289	48.4728	56.8742	61.0886	58.3278	2.3614×10^{-1}
70	50.9008	42.3712	50.1516	54.9870	51.7403	2.2825×10^{-1}
80	44.8258	36.7837	43.9182	49.3995	45.6697	2.3646×10^{-1}
90	39.2168	31.7180	38.1950	44.3338	40.0216	2.2475×10^{-1}
100	34.0799	27.1708	32.9926	39.7866	34.9146	2.2828×10^{-1}
110	29.4146	23.1290	28.3108	35.7448	30.2865	2.3272×10^{-1}
120	25.2135	19.5703	24.1386	32.1861	26.0794	2.3005×10^{-1}
130	21.4630	16.4654	20.4561	29.0812	22.4785	2.5191×10^{-1}
140	18.1441	13.7799	17.2356	26.3957	19.3800	3.0061×10^{-1}
150	15.2327	11.4761	14.4440	24.0919	16.4885	3.0263×10^{-1}
Time	4.8373	1.1070			98.7995	

Table 12. The CPU time (seconds) for the jump diffusion model of Section 4.1 is given. We consider the two classes of price approximations for an increasing basket dimension n. The basket weights are $\mathbf{w} = \frac{1}{n}\mathbf{1}_n$. The parameter values are $T - t = 1$, $r = 1\%$, $S_k(t) = 100$, $\sigma_k = 40\%$, $\xi_k = 0.5$, $\xi_{kk} = 0.3$, $\alpha_k = \alpha_{kk} = -0.5$, $\lambda = 1$, $\lambda_k = 0.5$, $\rho_{kj} = 0.5$ and $\rho^Y_{kj} = 0.5$ for $k = 1, \ldots, n$ and $k \neq j$. The strike price is $K = 100$.

n	CPU$^{\mathcal{G}}$	CPU^{AG}
2	0.299	0.067
5	0.644	0.071
10	1.029	0.073
20	1.874	0.075
30	2.800	0.075
40	3.788	0.077
50	4.905	0.081
60	6.347	0.081
70	9.273	0.128
80	10.564	0.115
90	13.004	0.122
100	14.152	0.123