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Nonlinear and multiplayer evolutionary games

Mark Broom and Jan Rychtář

Abstract Classical evolutionary game theory has typically considered populations within which randomly selected pairs of individuals play games against each other, and the resulting payoff functions are linear. These simple functions have led to a number of pleasing results for the dynamic theory, the static theory of evolutionarily stable strategies, and the relationship between them. We discuss such games, together with a basic introduction to evolutionary game theory, in Section 1. Realistic populations, however, will generally not have these nice properties, and the payoffs will be nonlinear. In Section 2 we discuss various ways in which nonlinearity can appear in evolutionary games, including pairwise games with strategy-dependent interaction rates, and playing the field games, where payoffs depend upon the entire population composition, and not on individual games. In Section 3 we consider multiplayer games, where payoffs are the result of interactions between groups of size greater than two, which again leads to nonlinearity, and a breakdown of some of the classical results of Section 1. Finally in Section 4 we summarise and discuss the previous sections.

Key words: ESS; payoffs; matrix games; nonlinearity; multi-player games

MSC codes: 91A22; 91A06; 91A80; 92B05

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1 Introduction

In this paper we consider nonlinear and multiplayer evolutionary games. We start in Section 1 with an introduction to evolutionary games for those not familiar with them, focusing on matrix games, which are linear in character, and discussing a number of the key results. We then move on to consider the general idea of nonlinear evolutionary games, including some specific types of such games in Section 2. We believe that these results, and those in the following section, will generally be less familiar to the audience. In Section 3 we consider multiplayer games. The specific type that we consider, and the most commonly used, is multiplayer matrix games, which can be though of as a special type of the nonlinear games in Section 2, although we note that multiplayer games in general do not simply reduce to this type. The text in significant part follows a tutorial talk given by MB at the International Society on Dynamic Games Symposium in Amsterdam in July 2014, which in turn followed aspects of the book Broom and Rychtář (2013).

1.1 What is evolutionary game theory?

Evolutionary game theory as we know it today began in the 1960s, in particular with the consideration of the sex-ratio problem (Hamilton, 1967), although similar reasoning on this problem goes back much earlier to Dusing (see Edwards, 2000) and Fisher (Fisher, 1930). The most influential work on our modern understanding is that of Maynard Smith and collaborators (Maynard Smith and Price, 1973; Maynard Smith, 1982).

In (non-cooperative) game theory, a game is comprised of three key elements, the *players*, the *strategies* available to be employed by the players, and the *payoffs* to the players, which are functions of the strategies chosen. For an evolutionary game we also need a *population*, and a way for our population to evolve through time, an *evolutionary dynamics*.

A *pure strategy* is a choice of what to play in a given interaction. Supposing that the pure strategies comprise the finite set $\{S_1, S_2, \ldots, S_n\}$, then a mixed strategy is defined as a probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, p_i being the probability that the player will play pure strategy S_i. Thus a pure strategy can be written in this way, e.g. S_i is $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the i^{th} place, and a mixed strategy can be written as a convex combination of pure strategies,

$$\mathbf{p} = (p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i S_i. \tag{1}$$

The set of all mixed strategies can be represented by a simplex in $\mathbb{R}^n$ with vertices at $\{S_1, S_2, \ldots, S_n\}$. The *Support of* $\mathbf{p}$, $S(\mathbf{p})$, is defined by $S(\mathbf{p}) = \{i : p_i > 0\}$, so that it is the set of pure strategies which have a positive probability of being played by a $\mathbf{p}$-player. The notion of a mixed strategy is naturally extended even to cases where the set of pure strategies is infinite, as in the "war of attrition" game, for example Bishop and Cannings (1978).

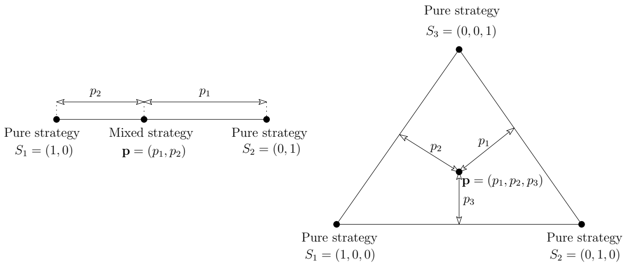

Fig. 1 Visualization of pure and mixed strategies for games with two or three strategies.

Payoffs for a game played by two players with each having a finite number of pure strategies can be represented by two matrices. For example, if player 1 has the strategy set $\mathbf{S} = \{S_1, \ldots, S_n\}$ and player 2 has the strategy set $\mathbf{T} = \{T_1, \ldots, T_m\}$, then the payoffs in this game are written as

$$A = (a_{ij})_{i=1,\ldots,n;j=1,\ldots,m}, B = (b_{ij})_{i=1,\ldots,m;j=1,\ldots,n}, \tag{2}$$

where a_{ij} (b_{ji}) is the reward to players 1 (2) after player 1 (2) chooses pure strategy S_i (T_j). We thus have the payoffs written as a pair of $n \times m$ matrices A and B^T, which is known as a bimatrix representation. This is often written as a single matrix whose entries are ordered pairs of values.

Note that here we write the payoffs from the point of view of the player receiving the reward (i.e. the index of their strategy comes first). It is often the case in other works that the index of player 1 is written first.

Often in evolutionary games, the choice of which player is player 1 is arbitrary, and thus the strategies available to the two players are identical. In this case, $n = m$ and (after a possible renumbering) $S_i = T_i$ for all i. Since the ordering of players is arbitrary, if we swap them their payoffs are unchanged, so that $b_{ij} = a_{ij}$, i.e. $A = B$. This means that all payoffs can be written as a single $n \times n$ matrix

$$A = (a_{ij})_{i,j=1,\ldots,n}, \tag{3}$$

where in this case, a_{ij} is the payoff to a player playing pure strategy S_i when its opponent plays strategy S_j. Such a game is called a *matrix game*.

Consider a game with payoffs given by a matrix A. If player 1 plays $\mathbf{p}$ and player 2 plays $\mathbf{q}$, then the proportion of games involving the first player playing S_i and the second player playing S_j is simply $p_i q_j$. The expected reward to player 1 is thus given by

$$E[\mathbf{p}, \mathbf{q}] = \sum_{i,j} a_{ij} p_i q_j = \mathbf{p} A \mathbf{q}^{\mathbf{T}}. \tag{4}$$

Note that, when comparing payoffs, we can ignore difficult cases involving equalities by assuming our games are *generic* (Samuelson, 1997; Broom and Rychtář, 2013). In most of the following we will make this assumption.

In the above, we have considered a single game between two individuals. However, evolutionary games consist of populations, and individuals are not (usually) involved in only a single contest. They may play many different contest, against many different opponents, with each contributing a relatively small contribution to the total reward.

We consider a function $\mathscr{E}[\sigma;\Pi]$, the fitness of an individual using a strategy σ in a population represented by Π. The term $\delta_{\mathbf{p}}$ is used to represent a population where the probability of a randomly selected player being a $\mathbf{p}$-player is 1. The term δ_i similarly denotes a population consisting only of individuals playing pure strategy S_i (with probability 1). The term $\sum_i p_i\delta_i$ thus means a population where the proportion of S_i-playing individuals is p_i.

1.2 Two approaches to game analysis

1.2.1 Dynamic analysis

In all that follows we assume a very large (effectively infinite) population, with overlapping generations and asexual reproduction, where offspring are direct copies of their parent. The evolution of a population can be modelled using evolutionary dynamics, where the proportion of individuals playing a given strategy changes according to their fitness.

In the following we shall assume a population consisting only of pure strategists. Thus we consider a population represented by $\mathbf{p^T} = \sum_i p_i\delta_i$, i.e. where the frequency of S_i-playing individuals is p_i. We denote the fitness of individuals playing S_i in this population to be $f_i(\mathbf{p})$. The birth rate of individuals in the population is proportional to their fitness.

We assume that the composition of the population changes according to the differential equation

$$\frac{d}{dt}p_i = p_i\Big(f_i\big(\mathbf{p}(t)\big) - \bar{f}\big(\mathbf{p}(t)\big)\Big). \tag{5}$$

This is the *continuous replicator dynamics*, the most commonly used evolutionary dynamics, originating in Taylor and Jonker (1978) (see also Hofbauer and Sigmund, 1998). For a derivation see Broom and Rychtář (2013). We also note the existence of the discrete replicator dynamics, the equivalent dynamics for non-overlapping generations (see Bishop and Cannings, 1978).

For matrix games the continuous replicator dynamics (5) becomes

$$\frac{d}{dt}p_i = p_i\bigg(\Big(A\big(\mathbf{p}(t)\big)^{\mathbf{T}}\Big)_i - \mathbf{p}(t)A\big(\mathbf{p}(t)\big)^{\mathbf{T}}\bigg). \tag{6}$$

1.2.2 Static analysis

An alternative methodology is to use a static analysis, which does not consider how the population reached a particular point in the strategy space, but assuming that the population is at that point, asks whether other strategies can do better within the population?

Consider a population where the vast majority of individuals play strategy S, while a very small proportion $\varepsilon > 0$ of "mutants" play strategy M. The strategies S and M thus compete within the population $(1-\varepsilon)\delta_S + \varepsilon\delta_M$. A strategy S is *e*volutionarily stable against strategy M if there is $\varepsilon_M > 0$ such that

$$\mathscr{E}[S;(1-\varepsilon)\delta_S+\varepsilon\delta_M] > \mathscr{E}[M;(1-\varepsilon)\delta_S+\varepsilon\delta_M] \tag{7}$$

for all $\varepsilon < \varepsilon_M$. S is an *evolutionarily stable strategy (ESS)* if it is *e*volutionarily stable against M for every other strategy $M \neq S$ (Maynard Smith and Price, 1973; Maynard Smith, 1982).

For matrix games, the linearity of the payoffs gives

$$\mathscr{E}[\mathbf{p};(1-\varepsilon)\delta_{\mathbf{p}}+\varepsilon\delta_{\mathbf{q}}] = E[\mathbf{p},(1-\varepsilon)\mathbf{p}+\varepsilon\mathbf{q}] = \tag{8}$$

$$\mathbf{p}A((1-\varepsilon)\mathbf{p}+\varepsilon\mathbf{q})^{\mathbf{T}} = (1-\varepsilon)\mathbf{p}A\mathbf{p}^{\mathbf{T}} + \varepsilon\mathbf{p}A\mathbf{q}^{\mathbf{T}}. \tag{9}$$

It is easy to show that this means a strategy $\mathbf{p}$ is an *Evolutionarily Stable Strategy (ESS)* for a matrix game, if and only if for any mixed strategy $\mathbf{q} \neq \mathbf{p}$

$$E[\mathbf{p},\mathbf{p}] \geq E[\mathbf{q},\mathbf{p}], \tag{10}$$

$$\text{If } E[\mathbf{p},\mathbf{p}] = E[\mathbf{q},\mathbf{p}], \text{ then } E[\mathbf{p},\mathbf{q}] > E[\mathbf{q},\mathbf{q}], \tag{11}$$

(see e.g. Broom and Rychtář, 2013).

We note that inequality (10) is the Nash equilibrium condition, but that, while necessary, it is not sufficient for stability. If (11) does not hold, then $\mathbf{p}$ may be invaded by a mutant that does equally well against the majority of individuals in the population (that play $\mathbf{p}$) but gets a (tiny) advantage against them by outperforming them in the (rare) contests with other mutants (playing $\mathbf{q}$).

Alternatively there is the possibility that the mutant and the residents do equally well against the mutants too. In this latter case invasion can occur by "drift"; both types do equally well, so in the absence of selective advantage random chance decides whether the frequency of mutants increases or decreases.

We define $T(\mathbf{p})$ as the set of pure strategies with equal payoffs against $\mathbf{p}$, i.e.

$$T(\mathbf{p}) = \{i : E[S_i,\mathbf{p}] = E[\mathbf{p},\mathbf{p}]\}. \tag{12}$$

Theorem 1 (Bishop Cannings Theorem). *If* $\mathbf{p}$ *is an ESS of the matrix game A and* $\mathbf{q} \neq \mathbf{p}$ *is such that* $S(\mathbf{q}) \subseteq T(\mathbf{p})$*, then* $\mathbf{q}$ *is not an ESS of matrix game A.*

For a proof, see Bishop and Cannings (1976).

1.2.3 Dynamic versus static analysis

Dynamic and static analyses are mainly complementary, however the relationship between the two is not straightforward, and there is some apparent inconsistency between the theories. Comparing the static ESS analysis and replicator dynamics, we see that the information required for each type of analysis is different. To determine whether $\mathbf{p}$ is an ESS, we need the minimum of a function

$$\mathbf{q} \to \mathscr{E}[\mathbf{p}; (1-\varepsilon)\delta_{\mathbf{p}} + \varepsilon\delta_{\mathbf{q}}] - \mathscr{E}[\mathbf{q}; (1-\varepsilon)\delta_{\mathbf{p}} + \varepsilon\delta_{\mathbf{q}}] \tag{13}$$

to be attained for $\mathbf{q} = \mathbf{p}$ for all sufficiently small $\varepsilon > 0$.

To understand the replicator dynamics, however, we need $\mathscr{E}[S_i; \mathbf{p}^{\mathbf{T}}]$ for all i and all $\mathbf{p}$. Thus a major difference between the two methods is that the static analysis considers monomorphic populations $\delta_{\mathbf{p}}$ while the dynamic analysis considers mixed populations $\mathbf{p}^{\mathbf{T}} = \sum_i p_i \delta_i$.

The analyses can thus produce the same (or at least similar) results only if there is an identification between $\delta_{\mathbf{p}}$ and $\mathbf{p}^{\mathbf{T}}$, as in the case of matrix games, and we note that most of the comparative analysis between the methods has assumed matrix games.

Theorem 2 (Folk theorem of evolutionary game theory, Hofbauer and Sigmund (2003)). *For a matrix game with payoffs given by matrix A, we have:*

1) *If* $\mathbf{p}$ *is a Nash equilibrium, and so an ESS, of a matrix game, then* $\mathbf{p}^{\mathbf{T}}$ *is a rest point of the dynamics , i.e. the population does not evolve further from the state* $\mathbf{p}^{\mathbf{T}} = \sum_i p_i \delta_i$.
2) *If* $\mathbf{p}$ *is a strict Nash equilibrium, then* $\mathbf{p}$ *is locally asymptotically stable.*
3) *If the rest point* $\mathbf{p}*$ *of the dynamics is also the limit of an interior orbit, then it is a Nash equilibrium.*
4) *If the rest point* $\mathbf{p}$ *is Lyapunov stable, then* $\mathbf{p}$ *is a Nash equilibrium.*

An ESS is an attractor of the replicator dynamics, and the population converges to the ESS for every strategy sufficiently close to it. If $\mathbf{p}$ is an internal ESS, then global convergence to $\mathbf{p}$ is assured (Zeeman, 1980).

It is also true that if the replicator dynamics has a unique internal rest point $\mathbf{p}*$, under certain conditions (satisfied for matrix games)

$$\lim_{t\to\infty} \frac{1}{T} \int_0^T p_i(t)dt = p_i^*, \tag{14}$$

so that the long-term average strategy is given by this rest point, even if there is considerable variation at any given time.

Thus for matrix games, identifying ESSs and Nash equilibria of a game gives a lot of important information about the dynamics. For example, if $\mathbf{p}$ is an internal ESS, then global convergence to $\mathbf{p}$ is assured.

However, there are cases when an ESS analysis does not provide such a complete picture. In particular, there are attractors of the replicator dynamics that are not ESSs. To see this, consider the matrix

$$\begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \\ 2 & -1 & 0 \end{pmatrix}. \tag{15}$$

(see Hofbauer and Sigmund, 1998). The replicator dynamics for this game has a unique internal attractor, but this attractor is not an ESS. This happens because we can find an invading mixture for **p** where the dynamics effectively forces the mixture into a combination that no longer invades. Thus if the invading group is comprised of mixed strategists it can invade, whereas if it is comprised of a mixture of pure strategists it cannot. Note that for the discrete dynamics the situation is even more complex, since then it is not guaranteed that an ESS is an attractor (Cannings, 1990).

1.3 Two classic matrix games

Two well-known examples of matrix games are the Hawk-Dove game (Maynard Smith and Price, 1973) and the prisoner's dilemma (Tucker, 1980). These both represent important biological/ social scenarios.

1.3.1 The Hawk Dove game

For the Hawk-Dove game, individual compete against other randomly chosen individuals for a reward (e.g. a territory) of value $V > 0$. Each of the contestants has two pure strategies available, Hawk (H) and Dove (D). Hawks fight, whereas Doves merely display. Doves divide the reward, a Hawk always beats a Dove, whereas two Hawks fight, with the loser incurring a cost C. This gives the payoff matrix as

$$\begin{array}{c} \\ Hawk \\ Dove \end{array} \begin{array}{c} \begin{array}{cc} Hawk & Dove \end{array} \\ \begin{pmatrix} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{pmatrix} \end{array}. \tag{16}$$

Denoting a mixed strategy $\mathbf{p} = (p, 1-p)$ to mean to play Hawk with probability p and to play Dove otherwise, it is easy to show that pure Dove is never an ESS, pure Hawk is an ESS if $V \geq C$. For $V < C$, $\mathbf{p} = (V/C, 1-V/C)$ is the unique ESS (see e.g. Broom and Rychtář, 2013).

1.3.2 The prisoner's dilemma

In the Prisoner's dilemma, a pair of individuals can either cooperate (play C) or try to obtain an advantage by defecting and exploiting the other (play D). The payoffs are given by the payoff matrix

$$\begin{array}{c|cc} & Cooperate & Defect \\ \hline Cooperate & R & S \\ Defect & T & P \end{array}. \tag{17}$$

Whilst the individual numbers are not important, for the classical dilemma we need $T > R > P > S$. We also need the additional condition $2R > S+T$ which is necessary for the evolution of cooperation. In this game Defect is the unique ESS, although if both players cooperated they would do better. The game is widely used to consider the issue of (especially human cooperation), and of how it can be established against cheating. Many variants of the above game, usually using multiple interactions of some kind, have been developed to this end (see e.g. Axelrod, 1981; Nowak, 2006).

2 Nonlinear games

2.1 Overview and general theory

In the previous section we considered matrix games, where

$$\mathscr{E}[\mathbf{p};\mathbf{q}^T] = \mathbf{p}A\mathbf{q}^{\mathbf{T}}. \tag{18}$$

The above payoffs can alternatively be written in the form $\sum_i p_i(A\mathbf{q}^{\mathbf{T}})_i$ or $\sum_j(\mathbf{p}A)_j q_j$, and so payoffs are linear in both the strategy of the focal individual and the strategy of the population and, as we have seen, this has nice static and dynamic properties.

More generally, we say that $\mathscr{E}$ is *linear on the left* if it is linear in the strategy of the focal player, i.e.

$$\mathscr{E}\left[\sum_i \alpha_i \mathbf{p}_i;\Pi\right] = \sum_i \alpha_i \mathscr{E}[\mathbf{p}_i;\Pi] \tag{19}$$

for every population Π, every $m-$tuple of individual strategies $\mathbf{p}_1,\ldots,\mathbf{p}_m$ and every collection of constants $\alpha_i \geq 0$ such that $\sum_i \alpha_i = 1$ (Broom and Rychtář, 2013).

We say that $\mathscr{E}$ is *linear on the right* if it is linear in the strategy of the population, i.e.

$$\mathscr{E}\left[\mathbf{p};\sum_i \alpha_i \delta_{\mathbf{q}_i}\right] = \sum_i \alpha_i \mathscr{E}[\mathbf{p};\delta_{\mathbf{q}_i}] \tag{20}$$

for every individual strategy $\mathbf{p}$, every $m-$tuple $\mathbf{q}_1,\ldots,\mathbf{q}_m$ and every collection of α_i's from $[0,1]$ such that $\sum_i \alpha_i = 1$ (Broom and Rychtář, 2013).

Recall that for matrix games, the payoff to an individual is the same whether it faces opponents playing a polymorphic mixture of pure strategies or a monomorphic population. We say that a game has *polymorphic-monomorphic equivalence* if for every strategy $\mathbf{p}$, any finite collection of strategies $\{\mathbf{q}_i\}_{i=1}^m$ and any corresponding collection of m constants $\alpha_i \geq 0$ such that $\sum_i^m \alpha_i = 1$ we have

$$\mathscr{E}\left[\mathbf{p};\sum_i \alpha_i \delta_{\mathbf{q}_i}\right] = \mathscr{E}\left[\mathbf{p};\delta_{\sum_i \alpha_i \mathbf{q}_i}\right], \tag{21}$$

(Broom and Rychtář, 2013). Note that he concept of polymorphic-monomorphic equivalence holds only in respect of static analyses, and there is no such concept in terms of dynamics.

The payoff is linear on the left for many evolutionary games because $\mathscr{E}[\mathbf{p};\Pi]$ is often defined to be the average of the payoffs to players of pure strategy S_i, weighted by the selection probability p_i, for all i. It is common, however, that the payoff is nonlinear on the right, which occurs whenever the game does not involve pairwise contests against randomly selected opponents.

The payoff function can be nonlinear on the left, if a strategy is a pure strategy drawn from a continuum, but that the payoff is nonlinear as a function of this pure strategy, such as in the tree height game from Kokko (2007) that we consider in Section 2.4. Nearly all real situations feature nonlinearity of some type, and when models of real behaviours are developed, the payoffs involved are indeed generally nonlinear in some way.

Some results for linear games can be generalized and reformulated for nonlinear games. The conditions (10) and (11) can be generalized as follows:

Theorem 3. *For games with generic payoffs, if the incentive function*

$$h_{\mathbf{p},\mathbf{q},u} = \mathscr{E}[\mathbf{p};(1-u)\delta_{\mathbf{p}} + u\delta_{\mathbf{q}}] - \mathscr{E}[\mathbf{q};(1-u)\delta_{\mathbf{p}} + u\delta_{\mathbf{q}}] \tag{22}$$

is differentiable (from the right) at $u = 0$ for every $\mathbf{p}$ and $\mathbf{q}$, then $\mathbf{p}$ is an ESS if and only if for every $\mathbf{q} \neq \mathbf{p}$;

1. *$\mathscr{E}[\mathbf{p};\delta_{\mathbf{p}}] \geq \mathscr{E}[\mathbf{q};\delta_{\mathbf{p}}]$ and*
2. *if $\mathscr{E}[\mathbf{p};\delta_{\mathbf{p}}] = \mathscr{E}[\mathbf{q};\delta_{\mathbf{p}}]$, then $\frac{\partial}{\partial u}\Big|_{(u=0)} h_{\mathbf{p},\mathbf{q},u} > 0$.*

For a proof, see (Broom and Rychtář, 2013).

Theorem 4. *Let $\mathscr{E}$ be linear in the focal player strategy, i.e. (19) holds, and let the function $h_{\mathbf{p},\mathbf{q},u}$ be differentiable w.r.t u at $u = 0$. Let $\mathbf{p} = (p_i)$ be an ESS. Then $\mathscr{E}[\mathbf{p};\delta_{\mathbf{p}}] = \mathscr{E}[S_i;\delta_{\mathbf{p}}]$ for any pure strategy S_i such that $i \in S(\mathbf{p}) = \{j; p_j > 0\}$.*

For a proof see Broom and Rychtář (2013). We note that it is enough to assume $h_{\mathbf{p},\mathbf{q},u}$ to be continuous.

If the payoff is not linear but strictly convex so that, for all $\mathbf{q}$ and all $\mathbf{p}$ with at least two elements in $S(\mathbf{p})$,

$$\sum_i p_i \mathscr{E}[S_i;\delta_{\mathbf{q}}] > \mathscr{E}[\mathbf{p};\delta_{\mathbf{q}}], \tag{23}$$

then any ESS must be a pure strategy.

Lemma 1 below shows that the payoffs of games that are linear in the focal player strategy and satisfy polymorphic monomorphic equivalence (21) must be of a special form. These games are called *population games*, or *playing the field games*.

Lemma 1. *If the payoffs of the game are linear in the focal player strategy (i.e. satisfy (19)) and satisfy polymorphic monomorphic equivalence (21), then for every* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *and every* $\varepsilon \in [0,1]$

$$\mathscr{E}[\mathbf{x}; (1-\varepsilon)\delta_{\mathbf{y}} + \varepsilon\delta_{\mathbf{z}}] = \sum_i x_i f_i\big((1-\varepsilon)\mathbf{y} + \varepsilon\mathbf{z}\big) \tag{24}$$

where $f_i(\mathbf{q}) = \mathscr{E}[S_i; \delta_{\mathbf{q}}]$.

Below we write payoffs in the form $\mathscr{E}[\mathbf{p}; \delta_{\mathbf{q}}] = \sum_i p_i f_i(\mathbf{q})$ for some functions f_i, and this indicates that payoffs are linear in the focal player strategy and also satisfy polymorphic monomorphic equivalence.

Theorem 5. *Let the payoffs be such that* $\mathscr{E}[\mathbf{p}; \delta_{\mathbf{q}}] = \sum_i p_i f_i(\mathbf{q})$ *for some continuous functions* f_i. *Then the strategy* $\mathbf{p}$ *is an ESS if and only if it is locally superior, i.e. there is* $U(\mathbf{p})$ *a neighbourhood of* $\mathbf{p}$ *such that*

$$\mathscr{E}[\mathbf{p}; \delta_{\mathbf{q}}] > \mathscr{E}[\mathbf{q}; \delta_{\mathbf{q}}], \textit{ for all } \mathbf{q}(\neq \mathbf{p}) \in U(\mathbf{p}). \tag{25}$$

For a proof, see Palm (1984).

2.2 Playing the field

In this section we consider payoff functions of the form

$$\mathscr{E}[\mathbf{p}; \Pi] = \sum p_i f_i(\Pi) \tag{26}$$

where the f_i's are (in general nonlinear) functions of the population strategy Π. Such playing the field games are the most natural way of incorporating nonlinearity into a game model, since the fitness function automatically includes the population frequencies of the different strategies.

An example is the sex ratio game, one of the classical models of evolutionary game theory (Hamilton, 1967). The model considers the question of why the sex ratio in most animals is close to a half? At first sight there needs to be far less males than females, since often the only male contribution is in mating; in many species most offspring are fathered by a small number of males and the rest make no contribution.

Assume that the strategy of an individual female is its choice of the proportion of male offspring. Let p be the strategy of a small invading group in a population that plays strategy m. Every individual has the same number of offspring, so fitness is given proportional to the number of grandchildren. Given that every individual has one mother and one father, if generation sizes remain constant it is easy to show that the fitness of an individual with strategy p is given by

$$\mathscr{E}[p;\delta_m] = \frac{p}{m} + \frac{1-p}{1-m} \tag{27}$$

so that in the notation of equation (26) we have

$$f_1(m) = \frac{1}{m}, f_2(m) = \frac{1}{1-m}. \tag{28}$$

The unique ESS of this game is $m = 1/2$, i.e. an equal sex ratio. The sex ratio game is in fact effectively just a special case of the following foraging problem (with $N = 2$ and $r_1 = r_2$).

Consider a population of animals foraging on N food patches, with resources $r_i > 0$ per unit time for $i = 1,\dots,N$, equally shared by all individuals on the patch (Parker, 1978).

The game has N pure strategies for this game, each corresponding to foraging on a given patch, and a mixed strategy $\mathbf{x} = (x_i)$ means to forage at patch i with probability x_i. The payoff to an individual using strategy $\mathbf{x} = (x_i)$ against a population playing $\mathbf{y} = (y_i)$ is

$$\mathscr{E}[\mathbf{x};\delta_{\mathbf{y}}] = \begin{cases} \infty, & \text{if } x_i > 0 \text{ for some } i \text{ such that } y_i = 0, \\ \sum_{i;x_i>0}^{N} x_i \frac{r_i}{y_i} & \text{otherwise.} \end{cases} \tag{29}$$

It is clear from (29) that any ESS $\mathbf{p}$ must have $p_i > 0$ for all $i = 1,\dots,N$. Thus any potential problems with infinite payoffs do not need to be considered. In particular Theorem 3 holds despite the discontinuities in the fitness functions, since they are continuous in the vicinity of any potential ESS.

The unique ESS $\mathbf{p} = (p_i)$ is given by $p_i = r_i/\sum_{i=1}^{N} r_i$. This solution can alternatively be written as

$$\frac{p_i}{p_j} = \frac{r_i}{r_j}. \tag{30}$$

This is called *Parker's matching principle*.

We can show this as follows. It is clear that $\mathscr{E}[\mathbf{q};\delta_{\mathbf{p}}] = \mathscr{E}[\mathbf{p};\delta_{\mathbf{p}}]$ for all $\mathbf{q}$. Moreover, since this game satisfies polymorphic monomorphic equivalence (21) then

$$\mathscr{E}[\mathbf{x};(1-u)\delta_{\mathbf{y}} + u\delta_{\mathbf{z}}] = \mathscr{E}[\mathbf{x};\delta_{(1-u)\mathbf{y}+u\mathbf{z}}] \tag{31}$$

and so

$$h_{\mathbf{p},\mathbf{q},u} = \mathscr{E}[\mathbf{p};(1-u)\delta_{\mathbf{p}} + u\delta_{\mathbf{q}}] - \mathscr{E}[\mathbf{q};(1-u)\delta_{\mathbf{p}} + u\delta_{\mathbf{q}}] = \tag{32}$$

$$\sum_{i=1}^{N} (p_i - q_i) \frac{r_i}{p_i + u(q_i - p_i)} = \tag{33}$$

$$\sum_{i=1}^{N} \frac{p_i - q_i}{p_i} r_i \left(1 - u\frac{q_i - p_i}{p_i} + \dots\right). \tag{34}$$

This implies that

$$\frac{\partial}{\partial u}\bigg|_{u=0} h_{\mathbf{p},\mathbf{q},u} = \sum_{i=1}^{N} r_i \left(\frac{p_i - q_i}{p_i}\right)^2 > 0. \tag{35}$$

So from Theorem 3, $\mathbf{p}$ is an ESS.

2.3 Nonlinearity due to non-constant interaction rates

Another way for nonlinear games to occur is where the strategies employed by the players affect the frequency of their interactions. The pairwise interactions may be simple, but if the strategy affects the interaction rate, then the overall payoff function can be complicated.

The simplest non-trivial scenario is a two player contest with two pure strategies S_1 and S_2, with payoffs given by the usual payoff matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{36}$$

but where the three types of interaction happen with probabilities not proportional to their frequencies.

Assume that each pair of S_1 individuals meet at rate r_{11}, each pair of S_1 and S_2 individuals meet at rate r_{12} and each pair of S_2 individuals meet at rate r_{22} (see Taylor and Nowak, 2006). This yields the following nonlinear payoff function

$$\mathscr{E}[S_1;\mathbf{p}^{\mathbf{T}}] = \frac{ar_{11}p_1 + br_{12}p_2}{r_{11}p_1 + r_{12}p_2}, \tag{37}$$

$$\mathscr{E}[S_2;\mathbf{p}^{\mathbf{T}}] = \frac{cr_{12}p_1 + dr_{22}p_2}{r_{12}p_1 + r_{22}p_2}. \tag{38}$$

This reduces to the standard payoffs for a matrix game for the case $r_{11} = r_{12} = r_{22}$.

In the standard game with uniform interaction rates, if $a < c$ and $b > d$ there is a mixed ESS, and this is also true for non-uniform interaction rates, although the ESS proportions change. If $a > c$ and $b < d$ then there are two ESSs in the uniform case, and this is also true for non-uniform interactions, although we note that the location of the unstable equilibrium between the pure strategies changes.

Otherwise for the uniform case there is a unique ESS. For non-uniform interaction rates, there is always a single pure ESS, but sometimes there is a mixed ESS too. For $c > a > d > b$, and setting $r_{12} = 1$, this occurs if

$$r_{11}r_{22} > \left(\frac{\sqrt{(a-b)(c-d)} + \sqrt{(a-c)(b-d)}}{d-a}\right)^2. \tag{39}$$

The Prisoner's Dilemma is an example where $c > a > d > b$. Setting $r_{11} = r_{22} = r$ and letting $r \to \infty$ the proportion of cooperators in the mixture tends to 1 and the basin of attraction of the proportion of cooperators p in the replicator dynamics increases, tends to $p \in (0, 1]$. Thus in extreme cases, the eventual outcome of the game can be effectively the opposite to that implied by the game with uniform interaction rates.

2.4 Nonlinearity in the strategy of the focal player

Here we consider a third case, involving games where the strategy of an individual is described by a single number (or a vector) that does not represent the probability of playing a given pure strategy, but rather represents a unique behaviour such as the intensity of a signal. We note that this is also the scenario generally considered in Adaptive Dynamics (see e.g. Metz et al, 1992; Metz, 2008), though in practice stronger assumptions are generally made than we use here.

Consider the following game-theoretical model of tree growth Koch et al (2004); Kokko (2007). We assume that a tree has to grow large enough in order to get sunlight and not get overshadowed by neighbours; yet the more the tree grows the more of its energy has to be devoted to "standing" rather than photosynthesis.

Let $h \in [0, 1]$ be the normalized height of the tree, so that 1 is the maximum possible height of a tree. In Kokko (2007), the fitness of a tree of height h in a forest where all other trees are of height H was given by

$$\mathscr{E}[h; \delta_H] = (1 - h^3) \cdot \big(1 + \exp(H - h)\big)^{-1}, \tag{40}$$

where $f(h) = 1 - h^3$ represents the proportion of leaf tissue of a tree of height h and $g(h - H) = \big(1 + \exp(H - h)\big)^{-1}$ represents the advantage or disadvantage of being taller/ shorter than neighbouring trees.

What are the ESSs for the tree, i.e. the evolutionarily stable heights? Differentiating (40) with respect to h obtains the unique maximum for h, i.e. the best response to a given H in the population. Any ESS must be a best response to itself, and so setting $h = H$ after the above differentiation yields

$$\frac{1}{4}\big(-6H^2 + (1 - H^3)\big) = 0. \tag{41}$$

Equation (41) has only one root in (0,1) and the crossing of the x axis happens with negative derivative, so that the root is the unique ESS.

3 Multi-player games

In the previous sections we have considered games with two individuals only, or games played against "the population". We shall now consider situations with contests involving groups of individuals which are of size three or larger, selected randomly from a large population. We shall only consider multi-player matrix games (Broom et al, 1997) here. Note that another important example of a multi-player game is the multi-player war of attrition (Haigh and Cannings, 1989). For an extensive review of multiplayer evolutionary games, see Gokhale and Traulsen (2014).

3.1 Introduction to multi-player matrix games

Consider an infinite population, from which groups of *m* players are selected at random to play a game. The expected payoff to an individual is obtained by simply averaging over the rewards for all possible cases, weighted by their probabilities, as for matrix games.

In general where the ordering of individuals matter, extending the bimatrix game case to *m* players, the payoff to each individual in position *k* is governed by an *m*-dimensional payoff matrix. However, as in matrix games, as opposed to bimatrix games, we assume that there is no significance to the ordering of the players. Thus an individual's payoff depends only upon its strategy and the combination of its opponents' strategies. We will call such games *symmetric*, and we have the following symmetry conditions:

$$a_{i_1\ldots i_m} = a_{i_1\sigma(i_2)\ldots\sigma(i_m)} \tag{42}$$

for any permutation σ of the indices $i_2,\ldots,i_m$. For the three player case, these are simply

$$a_{pqr} = a_{prq}, \text{ for all } p,q,r = 1,2,\ldots,n. \tag{43}$$

The payoff to an individual playing $\mathbf{p}$ in a contest with individuals playing $\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{m-1}$ respectively is written as $E[\mathbf{p};\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{m-1}]$. As the ordering is irrelevant, for convenience when some strategies are identical we use a power notation, for example $E[\mathbf{p};\mathbf{p}_1,\mathbf{p}_2,{\mathbf{p}_3}^{m-3}]$.

The payoffs function is given as follows

$$E[\mathbf{p};\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{m-1}] = \sum_{i=1}^{n} p_i \sum_{i_1=1}^{n} \cdots \sum_{i_{m-1}=1}^{n} a_{ii_1i_2\ldots i_{(m-1)}} \prod_{j=1}^{k-1} p_{j,i_j}, \tag{44}$$

where $\mathbf{p}_j = (p_{j,1},p_{j,2},\ldots,p_{j,n})$.

We note that, as pointed out by Gokhale and Traulsen (2010), as long as groups are selected from the population completely at random, as is usually assumed, then symmetric and non-symmetric games will have identical payoff functions. For example in the case of 3-player games, every individual is equally likely to occupy any of the ordered positions. In particular the term a_{ijk} has identical weighting to a_{ikj} in the payoff to an i-player, so that the sum of these two can be replaced by twice their average.

A multi-player matrix is super-symmetric if

$$a_{i_1 \dots i_m} = a_{\sigma(i_1) \dots \sigma(i_m)} \tag{45}$$

for any permutation σ of the indices $i_1, \dots, i_m$.

For example, for super-symmetric three-player three strategy games, there are ten distinct payoffs. Without loss of generality we can define the three payoffs $a_{111} = a_{222} = a_{333} = 0$, and this leaves seven distinct payoffs to consider $a_{112}, a_{113}, a_{221}, a_{223}$, a_{331}, a_{332} and a_{123}. Broom et al (1997) considers the replicator dynamics for such games in detail, including every case where the last seven payoffs above take values of either 1 or -1. We will only discuss the simpler two strategy games here.

3.2 ESSs in multi-player matrix games

A strategy $\mathbf{p}$ in an m-player game is called *evolutionarily stable against a strategy* $\mathbf{q}$ if there is an $\varepsilon_{\mathbf{q}} \in (0,1]$ such that for all $\varepsilon \in (0, \varepsilon_{\mathbf{q}}]$

$$\mathscr{E}[\mathbf{p}; (1-\varepsilon)\delta_{\mathbf{p}} + \varepsilon\delta_{\mathbf{q}}] > \mathscr{E}[\mathbf{q}; (1-\varepsilon)\delta_{\mathbf{p}} + \varepsilon\delta_{\mathbf{q}}], \tag{46}$$

where

$$\mathscr{E}[\mathbf{x}; (1-\varepsilon)\delta_{\mathbf{y}} + \varepsilon\delta_{\mathbf{z}}] = \sum_{l=0}^{m-1} \binom{m-1}{l} (1-\varepsilon)^l \varepsilon^{m-1-l} E[\mathbf{x}; \mathbf{y}^l, \mathbf{z}^{m-1-l}]. \tag{47}$$

$\mathbf{p}$ is an *ESS* for the game if for every $\mathbf{q} \neq \mathbf{p}$, there is $\varepsilon_{\mathbf{q}} > 0$ such that (46) is satisfied for all $\varepsilon \in (0, \varepsilon_{\mathbf{q}}]$ (Broom et al, 1997).

Similarly as in inequalities (10) and (11), we have the following:

Theorem 6. *For an m-player matrix game, the mixed strategy* $\mathbf{p}$ *is evolutionarily stable against* $\mathbf{q}$ *if and only if there is a* $j \in \{0, 1, \dots, m-1\}$ *such that*

$$E[\mathbf{p}; \mathbf{p}^{m-1-j}, \mathbf{q}^j] > E[\mathbf{q}; \mathbf{p}^{m-1-j}, \mathbf{q}^j], \tag{48}$$

$$E[\mathbf{p}; \mathbf{p}^{m-1-i}, \mathbf{q}^i] = E[\mathbf{q}; \mathbf{p}^{m-1-j}, \mathbf{q}^i] \textit{ for all } i < j. \tag{49}$$

For a proof see Broom et al (1997) or Bukowski and Miękisz (2004).

A strategy $\mathbf{p}$ is an *ESS at level J* if, for every $\mathbf{q} \neq \mathbf{p}$, the conditions (48-49) of Theorem 6 are satisfied for some $j \leq J$ and there is at least one $\mathbf{q} \neq \mathbf{p}$ for which the conditions are met for $j = J$ precisely.

If $\mathbf{p}$ is an ESS, then by Theorem 6, for all q,

$$E[\mathbf{p}; \mathbf{p}^{m-1}] \geq E[\mathbf{q}; \mathbf{p}^{m-1}]. \tag{50}$$

The payoffs are linear on the left so that

$$E[\mathbf{p};\mathbf{p}^{m-1}] = E[\mathbf{q};\mathbf{p}^{m-1}], \text{ for all } \mathbf{q} \text{ with } S(\mathbf{q}) \subseteq S(\mathbf{p}). \tag{51}$$

We note that in the generic case, any pure ESS is of level 0. A mixed ESS cannot be of level 0, but in the generic case, any mixed ESS must be of level 1.

Analogues of the strong restrictions on possible combinations of ESSs for matrix games do not hold for multi-player games. In particular, the Bishop-Cannings Theorem fails already for $m = 3$. For $m > 3$, there can be more than one ESS with the same support as we shall see in Section 3.3. On the other hand, we still have the following for $m = 3$.

Theorem 7. *It is not possible to have two ESSs with the same support in a three player matrix game.*

For a proof, see Broom et al (1997) or Broom and Rychtář (2013).

3.3 Two-strategy multi-player games

We shall now consider games with only two pure strategies. The possible situations for a given individual are thus all combinations of that individual playing pure strategy $i = 1,2$ against $m-1$ players, j of which play strategy S_1 (and the other $m-1-j$ play strategy S_2), for any $0 \leq j \leq m-1$. We shall denote these payoffs by α_{ij}.

We consider an individual playing strategy $\mathbf{x}$ in a population playing $\mathbf{y}$. A group of $m-1$ opponents is chosen and each one of them chooses to play strategy S_1 with probability y_1 (and so strategy S_2 with probability $y_2 = 1-y_1$). We obtain

$$\mathscr{E}[\mathbf{x};\delta_{\mathbf{y}}] = \sum_{l=0}^{m-1} \binom{m-1}{l} y_1^l y_2^{m-1-l} E[\mathbf{x};S_1^l S_2^{m-1-l}], \tag{52}$$

where

$$E[\mathbf{x};S_1^l,S_2^{m-1-l}] = \sum_{i=1}^{2} x_i \alpha_{il}. \tag{53}$$

Note that it does not matter whether the population is polymorphic or monomorphic and playing the mean strategy; thus multi-player matrix games have the polymorphic-monomorphic equivalence property.

Recalling that the payoffs of the m-player two strategy matrix game are α_{il} for $i = 1,2$ and $l = 0,1,\ldots,m-1$, we define $\beta_l = \alpha_{1l} - \alpha_{2l}$ and consider the incentive function

$$h(p) = \mathscr{E}[S_1;\delta_{(p,1-p)}] - \mathscr{E}[S_2;\delta_{(p,1-p)}] \tag{54}$$

$$= \textstyle\sum_{l=0}^{m-1} \binom{m-1}{l} \beta_l p^l (1-p)^{m-l-1}. \tag{55}$$

The function h quantifies the benefits of using strategy S_1 over strategy S_2 in a population where all other players use strategy $\mathbf{p} = (p, 1-p)$. Note that h is differentiable, and that the replicator dynamics now becomes

$$\frac{dq}{dt} = q(1-q)h(q). \tag{56}$$

Theorem 8. *In a generic two strategy m-player matrix game*

1. *pure strategy S_1 is an ESS (level 0) if and only if $\beta_{m-1} > 0$,*
2. *pure strategy S_2 is an ESS (level 0) if and only if $\beta_0 < 0$,*
3. *an internal strategy $\mathbf{p} = (p, 1-p)$ is an ESS, if and only if*

1. *$h(p) = 0$, and*
2. *$h'(p) < 0$.*

It is shown in Broom et al (1997) that the possible sets of ESSs are the following:

1. 0 pure ESSs, and l internal ESSs with $l \leq \lfloor \frac{m}{2} \rfloor$;
2. 1 pure ESS, and l internal ESSs with $l \leq \lfloor \frac{m}{2} - 1 \rfloor$;
3. 2 pure ESSs, and l internal ESSs with $l \leq \lfloor \frac{m}{2} - 2 \rfloor$.

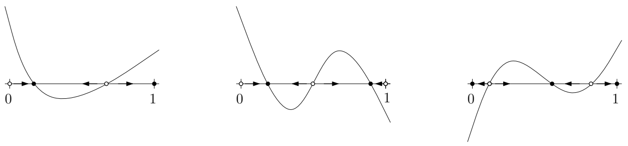

Fig. 2 The incentive function and ESSs in multiplayer games. The full dots show equilibrium points and the arrows show the direction of the evolution under the replicator dynamics.

There can be more than one ESS with the same support in a 4-player game as shown in the example below.

Consider an example with the following payoffs (Bukowski and Miękisz, 2004): with $\alpha_{11} = \alpha_{22} = -\frac{13}{96}, \alpha_{13} = \alpha_{20} = -\frac{3}{32}$ and $\alpha_{10} = \alpha_{12} = \alpha_{21} = \alpha_{23} = 0$. Thus $\beta_0 = 3/32, \beta_1 = -13/96, \beta_2 = 13/96, \beta_3 = -3/32$ giving

$$h(p) = -\frac{3}{32}p^3 + \frac{13}{32}p^2(1-p) - \frac{13}{32}p(1-p)^2 + \frac{3}{32}(1-p)^3 = \tag{57}$$

$$-\left(p - \frac{1}{4}\right)\left(p - \frac{1}{2}\right)\left(p - \frac{3}{4}\right). \tag{58}$$

Using the ESS conditions from Theorem 8, we see that the game has two internal ESSs at $\mathbf{p} = (1/4, 3/4)$ and $\mathbf{p} = (3/4, 1/4)$, and no pure ESSs.

4 Discussion

In this paper we have considered two main recent developments in the theory of evolutionary games. In particular the extension from linear matrix games to nonlinear games, and from two player to multiplayer games.

Nonlinearity within evolutionary games is introduced in its most natural way by considering games played against the population as a whole, so-called playing the field games. These can be generally expressed in the form of equation 26. They often result from situations where individuals do not interact directly, but where their behaviours have a direct effect on the environment, which then affects the payoffs of individuals. Thus in foraging models, the value of food patches depends directly on the intensity of their use by foragers within the population, as we saw from (Parker, 1978). More recent and realistic models of this phenomenon are given in Cressman et al (2004); Křivan et al (2008) for example.

Even when games are pairwise, linearity only occurs because opponents are chosen at random, with equal probability. If some opponents are more likely than others and this is in any way related to the strategy of those involved, either through individuals directly being more likely to interact with those choosing a particular strategy or because evolution has led to different strategy distributions in different geographical locations, then nonlinearity will result, as we saw in Taylor and Nowak (2006). An example of this phenomenon occurs in food-stealing games, see e.g. Broom et al (2004, 2008).

The above games are linear in the strategy of the focal player, as its strategy is a probabilistic weighting of distinct choices. When its strategy is a single trait chosen from a continuum, such as the height of a tree as in Koch et al (2004); Kokko (2007), then there is nonlinearity in the focal player strategy too. Another example is the sperm allocation games of Parker et al (1997); Ball and Parker (2007). We also note that this idea is central to the related concept of adaptive dynamics, where populations evolve by successive small mutations, see Kisdi and Meszéna (1993); Geritz et al (1998).

Multiplayer games have been, and continue to be, common in Economics, for instance see Kim (1996),Wooders et al (2006), Ganzfried and Sandholm (2009). However until recently they have been less common in evolutionary games. An extension of the classical idea of well-mixed populations of pairwise games to consider such populations with multiplayer games was first introduced with the work of Palm (1984) and followed by Haigh and Cannings (1989); Broom et al (1997); Bukowski and Miękisz (2004). More recently Hauert et al (2006), Gokhale and Traulsen (2010), Han et al (2012), Gokhale and Traulsen (2014) have developed the theory further.

As for nonlinear games above, multiplayer games can occur from non-independent pairwise games, for example within the formation of dominance hierarchies, where the results of a contest directly dictate who an individual will face next (if anybody). This was the focus of the games from Broom et al (2000a,b).

Evolutionary game theory has also been extended to finite populations, based upon the original Moran process Moran (1962) where different concepts are needed to deal with the stochastics effects which are not present in infinite populations, and where the single most important concept is that of the fixation probability of a rare mutant (equivalent to a small fraction of mutants within an infinite population, whose establishment within a population is either certain or impossible), important examples include Fogel et al (1998); Nowak et al (2004); Taylor et al (2004); Traulsen et al (2005); Nowak (2006). Within this general theory, there have also been developments based upon multiplayer games, and these are well-reviewed in Gokhale and Traulsen (2014).

Interesting new work on multiplayer games in each of the above areas continues to appear. For example the theory of adaptive dynamics is continually expanding, and the nonlinearity that appeared in the food stealing games of Broom et al (2008), which was due to the effect of time constraints, os being considered more widely, for instance in Cressman et al (2014). The work on finite populations including its multiplayer variants continues to be developed. In particular the modelling of structured populations from evolutionary graph theory Lieberman et al (2005) has been extended to incorporate multiplayer games Broom and Rychtář (2012). This area is at the relatively early stages of development, and there are many possibilities for further research.

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