City Research Online

City, University of London Institutional Repository

This is the accepted version of the paper.

This version of the publication may differ from the final published version.

On a systematic approach to defects in classical integrable field theories

V. Caudrelier[1]

City University
Centre for Mathematical Science
Nothampton Square
LONDON EC1V 0HB
UK

Abstract

We present an inverse scattering approach to defects in classical integrable field theories. Integrability is proved systematically by constructing the generating function of the infinite set of modified integrals of motion. The contribution of the defect to all orders is explicitly identified in terms of a defect matrix. The underlying geometric picture is that those defects correspond to Bäcklund transformations localized at a given point. A classification of defect matrices as well as the corresponding defect conditions is performed. The method is applied to a collection of well-known integrable models and previous results are recovered (and extended) directly as special cases. Finally, a brief discussion of the classical r-matrix approach in this context shows the relation to inhomogeneous lattice models and the need to resort to lattice regularizations of integrable field theories with defects.

PACS numbers: 02.30.Ik, 02.30.Jr, 02.30.Zz, 11.10.Kk, 11.10.Lm

---

[1]E-mail: v.caudrelier@city.ac.uk

Introduction

The topic of defects, or impurities, in integrable systems has quite a rich literature, especially for quantum aspects [1, 2, 3, 4, 6, 7, 8, 9], even if quite a lot remains to be done. Strangely enough, the problem of integrable defects in classical field theories had received less attention. The pioneering paper [10] is worth mentioning for the introduction of a so-called "spin impurity" in the nonlinear Schrödinger equation as a first step to tackle the problem on the half-line with integrable boundary conditions. This topic has been revived recently in the series of papers by P. Bowcock, E. Corrigan and C. Zambon [11, 12]. The lagrangian formalism is used in all these papers to described integrable field theories with internal boundary conditions interpreted as the presence of a defect. The defect conditions emerge from a local lagrangian density concentrated at some fixed point and are obtained from a variational argument. The question that is addressed is then: how to select conditions which leave the full theory integrable? The common underlying philosophy is to impose that a modified momentum, taking into account the presence of the defect, should be a conserved quantity while the breaking of translation invariance obviously entails that the bulk momentum will not be conserved. It turns out that this idea allows to pick up certain classes of defect lagrangian. Then, a general argument for integrability is based on the construction of modified Lax pair involving a limiting procedure. It is checked explicitly for a few conserved charges of certain models. One must note the nice observation made in each case: this procedure yields frozen Bäcklund transformations as the defect conditions for the fields.

The object of this paper is to unify the results obtained by this case by case approach. We take advantage of the common features that have been observed. To do so, we use the efficient inverse scattering method formalism (instead of the lagrangian formalism) and implement defect conditions corresponding to frozen Bäcklund transformations. It is important to note that the role of Bäcklund transformations as a means to generate integrable boundary-initial value systems solvable by inverse scattering method has been discovered and used in [13, 14, 15]. The idea was to fold two copies of the original integrable system related by Bäcklund transformations by using compatible reductions on the fields (for example $u(x) = u(-x)$). Here, we do not fold and use the fact that Bäcklund transformations have a very nice formulation in the inverse scattering method. They can be encoded in matrices, representing gauge transformations of the underlying auxiliary problem and, in the present context, giving rise to defect matrices. Thanks to this formulation, we are able to prove systematically the existence of an infinite set of modified conservation laws, ensuring integrability. The main result of this paper is the explicit identification of the generating function of the defect contributions at all orders, *i.e.* for any conserved charge, for any integrable evolution equation of the AKNS [16] or KN [17] schemes of the inverse scattering method. This provides an efficient algorithm to compute the modified conserved quantities, given the defect matrices. One of the advantage of the method is that the proof of integrability does not require any modification of the usual Lax pair formulation for integrable field theories. Another is

that there is no guess work for finding the defect contributions. They are obtained from a classification of defect matrices.

The paper is organized as follows. In Section 1, the general auxiliary problem formalism we use is presented. We establish our main results about the infinite set of conservation laws in the presence of a defect. The generalization to several defects is also explained. In Section 2, the defect matrices are classified within a certain class of gauge transformations of the auxiliary problem. In section 3, we illustrate our systematic method on several well-known examples of integrable nonlinear equations. They correspond to all the classical field theories that have been explored in the lagrangian formalism (with the exception of the affine Toda field theories). For these models, all the previous results are recovered (and even generalized) and are extended to higher orders. Section 4 is devoted to the extension of the method to another inverse scattering method scheme, the Kaup-Newell scheme [17], which describes other classes of integrable nonlinear equations including e.g. the derivative nonlinear Schrödinger equation. In section 5, we discuss in more detail the question of integrability of such models with defects. It is argued that our approach allows to make a connection between the lagrangian approach and the classical r-matrix formalism. This requires the use of lattice regularizations. Our conclusions and perspectives for future investigations are gathered in the last section.

1   General settings and results

1.1   Lax pair formulation

In the AKNS scheme [16], an integrable evolution equation on the line can be formulated as a compatibility condition, or zero curvature condition, of a linear differential problem for an auxiliary wavefunction $\Psi(x,t,\lambda)$ involving two 2×2 matrix-valued functions $U(x,t,\lambda)$ and $V(x,t,\lambda)$ such that

$$\begin{cases} \Psi_x = U\Psi\,, \\ \Psi_t = V\Psi\,, \end{cases} \tag{1.1}$$

where the subscripts x and t denote differentiation with respect to these variables. In the rest of the paper, we will drop the arguments whenever this is not misleading. The parameter λ is called spectral parameter. Then, for appropriate choices of U and V, the integrable system at hand is equivalent to the compatibility condition $\Psi_{xt} = \Psi_{tx}$ giving rise to the so-called *zero curvature condition*

$$\forall\,\lambda \quad,\quad U_t - V_x + [U,V] = 0\,. \tag{1.2}$$

Large classes of integrable nonlinear evolution equations can be described this way among which some of the most famous are the cubic nonlinear Schrödinger (NLS), sine/sinh-Gordon (sG), Liouville, Korteweg-de Vries (KdV) or its modified version (mKdV). It

is known that this presentation allows one to construct generically the infinite set of conservation laws associated to the integrable equation. For self-containedness, we recall the main ideas. Let us fix U and V to be 2×2 traceless matrices as follows

$$U = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \equiv -i\lambda\sigma_3 + W \quad , \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} , \tag{1.3}$$

where $q(x,t)$ and $r(x,t)$ are the fields satisfying the evolution equation. In this paper, we will fix the class of solutions to be that of sufficiently smooth [18] decaying fields[1] as $|x| \to \infty$ and the following behaviour is assumed[2]

$$A(x,t,\lambda) \to \omega(\lambda) \quad , \quad B(x,t,\lambda), C(x,t,\lambda) \to 0 \text{ as } |x| \to \infty . \tag{1.4}$$

The vector-valued function Ψ is split as

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} . \tag{1.5}$$

Let $\Gamma = \Psi_2 \Psi_1^{-1}$, then the identification of the infinite set of conservation laws follows from a conservation equation

$$(q\Gamma)_t = (B\Gamma + A)_x , \tag{1.6}$$

and a Ricatti equation for Γ

$$\Gamma_x = 2i\lambda\Gamma + r - q\Gamma^2 . \tag{1.7}$$

This equation follows directly from the x part of (1.1). The conservation equation is obtained from the t part to get Γ_t, from the (12) element of (1.2) to get q_t and combining with (1.7) and with the (11) element of (1.2) giving A_x.

Thus, expanding Γ as $\lambda \to \infty$

$$\Gamma = \sum_{n=1}^{\infty} \frac{\Gamma_n}{(2i\lambda)^n} , \tag{1.8}$$

the conserved quantities read

$$I_n = \int_{-\infty}^{\infty} q\Gamma_n dx \quad , \quad n \geq 1 , \tag{1.9}$$

where

$$\Gamma_1 = -r \quad , \quad \Gamma_{n+1} = \Gamma_{nx} + q \sum_{k=1}^{n-1} \Gamma_k \Gamma_{n-k} \quad , \quad n \geq 1 . \tag{1.10}$$

---

[1]The decaying properties are chosen so as to ensure certain analytic properties of the scattering data for (1.1), see e.g. [16]. Typically, a polynomial decay is sufficient.
[2]The only exception is the Liouville equation for which no specific boundary condition is assumed.

1.2 Implementing defect boundary conditions

Generally speaking, a defect in $(1+1)$-dimensional integrable field theories can be viewed as internal boundary conditions on the field and its time and space derivatives at a given point on the line. In other words, one wants to glu together two solutions of the evolution equation in a specific way and at a particular point. To this end, let us consider another copy of the auxiliary problem. We introduce another Lax pair $\widetilde{U}$, $\widetilde{V}$ defined as in (1.3) and (1.4) with q, r replaced by $\tilde{q}$, $\tilde{r}$. We consider the analog of (1.1) for

$$\widetilde{\Psi}(x,t,\lambda) = L(x,t,\lambda)\Psi(x,t,\lambda)\,. \tag{1.11}$$

The matrix valued function $L(x,t,\lambda)$ satisfies the following partial differential equations for any x and t,

$$L_x = \widetilde{U}L - LU\,, \tag{1.12}$$

$$L_t = \widetilde{V}L - LV\,. \tag{1.13}$$

In this paper, we want to think of this matrix as generating the defect conditions at a specific point, x_0 say. Following the terminology of [11, 12], L is called the *defect matrix*. We present a classification of the simplest nontrivial such matrices in the next section.

Now we turn to the general contruction of the generating function of the infinite set of modified conservation laws due to the presence of a defect. This is the main result of this paper and establishes integrability for any nonlinear integrable equation of the AKNS scheme with a defect realizing a frozen Bäcklund transformation. For particular models (NLS, sG, Liouville, KdV and mKdV), it proves to all orders the results of [11, 12] about the defect contribution and gives an explicit form for it. In addition, this is done without resorting to a modified Lax pair formalism involving a complicated limiting procedure to construct the conserved charges. To illustrate this, we will discuss those particular examples in Section 3.

To fix ideas, we choose a point $x_0 \in \mathbb{R}$ and we suppose that the auxiliary problem (1.1) exists for $x > x_0$ while the one for $\widetilde{U}$ and $\widetilde{V}$ exists for $x < x_0$. We also assume that the two systems are connected by the relations (1.12) and (1.13) at $x = x_0$. Then, the following holds

Proposition 1.1 *The generating function for the integral of motions reads*

$$I(\lambda) = I_{bulk}^{left}(\lambda) + I_{bulk}^{right}(\lambda) + I_{defect}(\lambda)\,, \tag{1.14}$$

where

$$I_{bulk}^{left}(\lambda) = \int_{-\infty}^{x_0} \tilde{q}\widetilde{\Gamma}dx\,, \tag{1.15}$$

$$I_{bulk}^{right}(\lambda) = \int_{x_0}^{\infty} q\Gamma dx\,, \tag{1.16}$$

$$I_{defect}(\lambda) = -\ln(L_{11} + L_{12}\Gamma)|_{x=x_0}\,, \tag{1.17}$$

and L_{ij}'s are the entries of the defect matrix L.

Proof: From the general result (1.6), we get

$$(q\Gamma)_t = (B\Gamma + A)_x \quad, \quad \forall x > x_0\,, \tag{1.18}$$

$$\left(\tilde{q}\widetilde{\Gamma}\right)_t = \left(\widetilde{B}\widetilde{\Gamma} + \widetilde{A}\right)_x \quad, \quad \forall x < x_0\,, \tag{1.19}$$

where $\widetilde{\Gamma}$ is defined from $\widetilde{\Psi}$ as in the previous section, $\widetilde{\Gamma} = \widetilde{\Psi}_2\widetilde{\Psi}_1^{-1}$. From this and the rapid decay of the fields, we get

$$\partial_t \int_{x_0}^{\infty} q\Gamma dx + \partial_t \int_{-\infty}^{x_0} \tilde{q}\widetilde{\Gamma}dx = \left(\widetilde{B}\widetilde{\Gamma} + \widetilde{A} - (B\Gamma + A)\right)|_{x=x_0} \tag{1.20}$$

The crucial point now is that the right-hand-side is a total time derivative of a quantity evaluated at $x = x_0$: it is the contribution of the defect to the conserved quantities as we now show. From (1.11) we get $\widetilde{\Gamma} = (L_{21} + L_{22}\Gamma)(L_{11} + L_{12}\Gamma)^{-1}$. Then, using (1.13) at $x = x_0$ to eliminate $\widetilde{A}$ and $\widetilde{B}$, one gets

$$\left(\widetilde{B}\widetilde{\Gamma} + \widetilde{A} - (B\Gamma + A)\right)|_{x=x_0} \tag{1.21}$$

$$= \left\{\partial_t L_{11} + \partial_t L_{12}\Gamma + L_{12}(C - 2A\Gamma - B\Gamma^2)\right\}(L_{11} + L_{12}\Gamma)^{-1}|_{x=x_0}\,. \tag{1.22}$$

The final step consists in noting that the t part of (1.1) implies another Ricatti equation

$$\Gamma_t = C - 2A\Gamma - B\Gamma^2\,, \tag{1.23}$$

so that

$$\partial_t \int_{x_0}^{\infty} q\Gamma dx + \partial_t \int_{-\infty}^{x_0} \tilde{q}\widetilde{\Gamma}dx = \frac{(L_{11} + L_{12}\Gamma)_t}{(L_{11} + L_{12}\Gamma)}|_{x=x_0}\,, \tag{1.24}$$

Therefore,

$$\partial_t I(\lambda) = 0\,. \tag{1.25}$$

■

Note that this holds in all generality for any integrable evolution equation in the AKNS scheme with decaying fields. The latter can be relaxed for some models, and we discuss in detail below the Liouville equation for which this does not hold.

1.3 Several defects

It is quite straightforward to repeat the general argument for the construction of conservations laws in the case of several defects. Suppose we have $N + 1$ auxiliary problems connected two by two at points $x_1, \ldots, x_N$ by matrices $L^1, \ldots, L^N$. Then, it is easy to see that the contribution of these N defects is the sum of the contributions from each defect.

Indeed, using $x_0 = -\infty$, $x_{N+1} = +\infty$ and otherwise obvious notations, the generating function for the integral of motion reads

$$
I(\lambda) \;=\; \sum_{j=1}^{N+1} I_{bulk}^j(\lambda) + \sum_{j=1}^{N} I_{defect}^j(\lambda)\,, \tag{1.26}
$$

$$
I_{bulk}^j(\lambda) \;=\; \int_{x_{j-1}}^{x_j} q_j \Gamma_j dx \;\;,\;\; j = 1,\dots,N+1\,, \tag{1.27}
$$

$$
I_{defect}^j(\lambda) \;=\; \ln((L_j)_{11} + (L_j)_{12}\Gamma_j)|_{x=x_0} \;\;,\;\; j = 1,\dots,N\,. \tag{1.28}
$$

2 Defect matrices

In this section, we derive a large class of defect matrices satisfying (1.12) and (1.13) together with the associated conditions they entail on the fields: the Bäcklund transformations. The latter will become the defect conditions when imposed at $x = x_0$.

2.1 Generalities

The matrix L preserves the zero curvature condition as is easily seen by writing $L_{xt} = L_{tx}$,

$$
(\forall\,\lambda\,,\; U_t - V_x + [U,V] = 0) \Leftrightarrow (\forall\,\lambda\,,\; \widetilde{U}_t - \widetilde{V}_x + [\widetilde{U},\widetilde{V}] = 0)\,. \tag{2.1}
$$

In other words, if q, r are solutions of the evolution equation described by (U,V) then $\tilde{q}$, $\tilde{r}$ are solutions of the evolution equation described by $(\widetilde{U},\widetilde{V})$ under the transformation induced by L and vice versa. This is just the usual definition of a Bäcklund transformation and this shows the connection with the idea of frozen Bäcklund transformations discussed above. In other words, we look for defect matrices in the class of matrices realizing Bäcklund transformations between two nonlinear integrable evolution equations. Note that the evolution equations need not be the same in general. If they are, the terminology auto-Bäcklund transformation is usually used. Such matrices are sometimes referred to as Darboux matrices (see e.g. [19]). Even if a lot is known on these matrices, we proceed with their derivation in the form needed for this paper. We adopt a pedestrian method which does not require any previous knowledge of their theory. In particular, no reference to the wavefunction of the auxiliary problem or to a special Riemann problem is needed (which are usually the methods encountered in the literature).

Let us establish some general facts about L. First, there is some freedom in its normalization coming from the invariance of the zero-curvature condition under the transformation $(U,V) \rightarrow (M^{-1}UM, M^{-1}VM)$ for any invertible matrix M independent of x and t. This also obviously preserves the tracelessness property. In particular, left multiplication of L by M^{-1} amounts to apply this transformation to $(\widetilde{U},\widetilde{V})$ while right multiplication by M applies it to (U,V). Then, we have the

Proposition 2.1 *The determinant of L is independent of x and t*

$$\det L(x, t, \lambda) = f(\lambda) \,. \tag{2.2}$$

Proof: The result follows from the Jacobi formula

$$
\begin{aligned}
(\det L(x, t, \lambda))_x &= \det L(x, t, \lambda) \mathrm{Tr}(\widetilde{U} - U) \,, & (2.3) \\
(\det L(x, t, \lambda))_t &= \det L(x, t, \lambda) \mathrm{Tr}(\widetilde{V} - V) \,, & (2.4)
\end{aligned}
$$

and the tracelessness of $U, \widetilde{U}, V, \widetilde{V}$. ∎

At this stage, it is hard to go further without specifying $U, \widetilde{U}, V, \widetilde{V}$ a bit more. Let us simply note that given $U, \widetilde{U}, V, \widetilde{V}$ and initial-boundary values for the fields, the integration of (1.12) and (1.13) gives for instance (the path of integration being irrelevant)

$$L(x, t, \lambda) = L(x_0, t_0, \lambda) + \int_{x_0}^{x} (\widetilde{U}L - LU)|_{\tau=t} dy + \int_{t_0}^{t} (\widetilde{V}L - LV)|_{y=x_0} d\tau \,. \tag{2.5}$$

The formal iteration of the previous equation suggests that, in general, L has a complicated Laurent series structure as a function of λ. In the following, we will assume that L has only a finite number of terms and, recalling that it is defined up to a scalar function in λ, we will look for a solution of the form

$$L(x, t, \lambda) = \sum_{n=0}^{N} L^{(n)}(x, t) \lambda^{-n} \,. \tag{2.6}$$

Actually, we shall consider the case $N = 1$ which we study in detail. For convenience, we also restrict our attention to auto-Bäcklund matrices. We comment later on on the fact that this is not necessary in our approach, one of the crucial ingredient being simply that the evolution equations have the same dispersion relation (see (2.18) below).

2.2   Construction for N=1

The defect matrix is of the form[3]

$$L(x, t, \lambda) = L^{(0)}(x, t) + L^{(1)}(x, t) \lambda^{-1} \,. \tag{2.7}$$

The defining relation (1.12) is equivalent to

$$
\begin{aligned}
0 &= \left[L^{(0)}, \sigma_3 \right] \,, & (2.8) \\
L_x^{(0)} &= i \left[L^{(1)}, \sigma_3 \right] + \widetilde{W} L^{(0)} - L^{(0)} W \,, & (2.9) \\
L_x^{(1)} &= \widetilde{W} L^{(1)} - L^{(1)} W \,. & (2.10)
\end{aligned}
$$

---

[3]Note that it is implicitly assumed that both $L^{(0)}$ and $L^{(1)}$ are not trivial since otherwise, the Bäcklund transformation is essentially the trivial one $\widetilde{W} = W$.

If $V, \widetilde{V}$ are polynomials in λ with coefficients $V^{(j)}(x,t)$, $\widetilde{V}^{(j)}(x,t)$, $j = 0, \cdots, N$, equation (1.13) is equivalent to

$$
\begin{align}
L_t^{(1)} &= \widetilde{V}^{(0)} L^{(1)} - L^{(1)} V^{(0)}, & (2.11)\\
L_t^{(0)} &= \widetilde{V}^{(1)} L^{(1)} - L^{(1)} V^{(1)} + \widetilde{V}^{(0)} L^{(0)} - L^{(0)} V^{(0)}, & (2.12)\\
0 &= \widetilde{V}^{(2)} L^{(1)} - L^{(1)} V^{(2)} + \widetilde{V}^{(1)} L^{(0)} - L^{(0)} V^{(1)}, & (2.13)\\
&\vdots & (2.14)\\
0 &= \widetilde{V}^{(N)} L^{(0)} - L^{(0)} V^{(N)}. & (2.15)
\end{align}
$$

If $V, \widetilde{V}$ are polynomials in λ^{-1}, the equations are the same under the exchange $L^{(0)} \leftrightarrow L^{(1)}$.

Let us make a few remarks. First, when we have found the matrix L, equations (2.10) and (2.11) will give the x and t parts of the corresponding Bäcklund transformations for the fields. Then, in traditional approaches, equation (2.9) is used to construct new soliton solutions, $\widetilde{W}$, from given solutions W and the knowledge of the Bäcklund transformation. We will not discuss this last step in this paper and refer the reader to the vast literature on the subject (see e.g. [20] and references therein).

We now proceed with the statement of the general results of this section.

Proposition 2.2 Defect matrix
The defect matrix L has the following general form

$$
L = \mathbb{1}_2 + \lambda^{-1} \begin{pmatrix} \frac{1}{2}\left\{\alpha_+ \pm \left[\alpha_-^2 - 4a_2 a_3\right]^{1/2}\right\} & a_2 \\ a_3 & \frac{1}{2}\left\{\alpha_+ \mp \left[\alpha_-^2 - 4a_2 a_3\right]^{1/2}\right\} \end{pmatrix}, \quad (2.16)
$$

where

$$
a_2 = -\frac{i}{2}(\tilde{q} - q) \quad , \quad a_3 = \frac{i}{2}(\tilde{r} - r), \quad (2.17)
$$

and $\alpha_\pm \in \mathbb{C}$ are the (x, t-independent) parameters of the defect.

<u>Proof:</u> Equation (2.8) implies that $L^{(0)}$ in (2.7) is diagonal and then, equation (2.9) shows that the diagonal elements do not depend on x. Therefore, we can consider equation (2.12) as $|x| \to \infty$. Recall that we have

$$
V, \widetilde{V} \to \omega(\lambda)\, \sigma_3 \quad , \quad |x| \to \infty. \quad (2.18)
$$

Writing $\omega(\lambda) = \sum_{n=0}^{N} \omega^{(n)} \lambda^n$ and denoting $L_\infty^{(1)} = \lim_{|x| \to \infty} L^{(1)}(x,t)$, we get

$$
\begin{align}
L_t^{(0)} &= \omega^{(1)}\left[\sigma_3, L_\infty^{(1)}\right] + \omega^{(0)}\left[\sigma_3, L^{(0)}\right] & (2.19)\\
&= -i\omega^{(1)} \lim_{|x| \to \infty}\left(\widetilde{W} L^{(0)} - L^{(0)} W\right) & (2.20)\\
&= 0, & (2.21)
\end{align}
$$

8

where we have used equation (2.8) and (2.9) in the second equality and the fact that we consider decaying fields in the last equality. The proof is similar if V, $\widetilde{V}$ are polynomials in λ^{-1}. So, as explained above, we can left multiply with $\left(L^{(0)}\right)^{-1}$ and work with $L' = \left(L^{(0)}\right)^{-1} L$. We drop the $'$ in the following but remember that $L^{(0)}$ should now be $\mathbb{1}_2$ in all the equations (2.8-2.15). Next, denote

$$L^{(1)} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \tag{2.22}$$

and α_1, α_2 its eigenvalues. Then, equation (2.9) gives immediately $a_2 = -\frac{i}{2}(\tilde{q} - q)$, $a_3 = \frac{i}{2}(\tilde{r} - r)$. Now, the elements a_1 and a_4 are easily computed from

$$a_1 a_4 - a_2 a_3 = \alpha_1 \alpha_2 \quad \text{and} \quad a_1 + a_4 = \alpha_1 + \alpha_2, \tag{2.23}$$

and introducing $\alpha_\pm = \alpha_1 \pm \alpha_2$. Finally, we need to show that $\alpha_\pm$ is independent of x and t. Let ℓ_1 and ℓ_2 be the eigenvalues of L. It is enough to prove that ℓ_1 and ℓ_2 are independent of x and t. In turn, it is sufficient to prove that $\ell_1 \ell_2$ and $\ell_1 + \ell_2$ are independent of x and t. From proposition 2.1, we already know that $\ell_1 \ell_2 = f(\lambda)$. Next, we prove $\ell_1 + \ell_2 = g(\lambda)$. Suppose $\ell_1 + \ell_2 = g(x, t, \lambda)$, then, using (1.12) and (2.9)

$$g_x = \text{Tr} \left[L(\widetilde{W} - W) \right] \tag{2.24}$$

$$= \lambda^{-1} \text{Tr} \left[L^{(1)}(\widetilde{W} - W) \right] \tag{2.25}$$

$$= \frac{i}{\lambda} \text{Tr} \left[L^{(1)} \left[\sigma_3, L^{(1)} \right] \right] \tag{2.26}$$

$$= 0. \tag{2.27}$$

Now, $g_t = \text{Tr} \left[L(\widetilde{V} - V) \right]$ can be evaluated as $x \to \infty$ for which we know that $\widetilde{V} - V \to 0$. So $g_t = 0$. ∎

The Bäcklund transformations associated with the matrix L read:

- For the x part,

$$a_{1x} = \tilde{q} a_3 - r a_2, \tag{2.28}$$

$$a_{2x} = \tilde{q} a_4 - q a_1, \tag{2.29}$$

$$a_{3x} = \tilde{r} a_1 - r a_4, \tag{2.30}$$

$$a_{4x} = \tilde{r} a_2 - q a_3, \tag{2.31}$$

- For the t part if V, $\widetilde{V}$ are polynomials in λ,

$$a_{1t} = (\widetilde{A}^{(0)} - A^{(0)}) a_1 + \widetilde{B}^{(0)} a_3 - C^{(0)} a_2, \tag{2.32}$$

$$a_{2t} = (\widetilde{A}^{(0)} + A^{(0)}) a_2 + \widetilde{B}^{(0)} a_4 - B^{(0)} a_1, \tag{2.33}$$

$$a_{3t} = -(\widetilde{A}^{(0)} + A^{(0)}) a_3 + \widetilde{C}^{(0)} a_1 - C^{(0)} a_4, \tag{2.34}$$

$$a_{4t} = -(\widetilde{A}^{(0)} - A^{(0)}) a_4 + \widetilde{C}^{(0)} a_2 - B^{(0)} a_3, \tag{2.35}$$

9

- For the t part if V, $\widetilde{V}$ are polynomials in λ^{-1},

$$a_{1t} = (\widetilde{A}^{(1)} - A^{(1)})d_1 + B^{(0)}a_3 - C^{(0)}a_2 \,, \tag{2.36}$$

$$a_{2t} = 2A^{(0)}a_2 + B^{(0)}(a_4 - a_1) + d_2\widetilde{B}^{(1)} - d_1 B^{(1)} \,, \tag{2.37}$$

$$a_{3t} = -2A^{(0)}a_3 + C^{(0)}(a_1 - a_4) + d_1\widetilde{C}^{(1)} - d_2 C^{(1)} \,, \tag{2.38}$$

$$a_{4t} = -(\widetilde{A}^{(1)} - A^{(1)})a_4 + C^{(0)}a_2 - B^{(0)}a_3 \,, \tag{2.39}$$

where a_1, a_2, a_3, a_4 are as in Proposition 2.2. At this stage, it seems that there is an overdetermination since there are four equations for each part whereas only two of each type are needed (the x and t transforms relating $\tilde{q}$ and q and those relating $\tilde{r}$ and r). It turns out that half of them are indeed redundant.

Proposition 2.3 Bäcklund transformations
The Bäcklund transformations corresponding to L are given by the equations for a_2 and a_3 in (2.29, 2.30) and (2.33, 2.34) or (2.37, 2.38). The remaining equations for a_1 and a_4 can be deduced from them.

<u>Proof:</u> We know that the eigenvalues of $L^{(1)}$ are constant so

$$(a_1 a_4)_x = (a_2 a_3)_x \quad , \quad a_{1x} + a_{4x} = 0 \,. \tag{2.40}$$

From this and the equations $(2.29, 2.30)$ for a_2 and a_3, we deduce

$$a_{1x}(a_4 - a_1) = a_4(\tilde{q}a_3 - ra_2) + a_1(\tilde{r}a_2 - qa_3) \,. \tag{2.41}$$

Now, using $(\tilde{q} - q)a_3 + (\tilde{r} - r)a_2 = 0$,

$$(a_{1x} - \tilde{q}a_3 + ra_2)(a_4 - a_1) = 0 \,. \tag{2.42}$$

The possibility $a_4 = a_1$ must be rejected in general since together with $a_{1x} + a_{4x} = 0$ it would imply that a_1 and a_4 are independent of x. Thus, we obtain the equation for a_1 and hence for a_4.

The proof for the t part is similar. Useful identities in getting the result are obtained from

$$\mathrm{Tr}\, L_t = \mathrm{Tr}\left[L(\widetilde{V} - V)\right] = 0 \,, \tag{2.43}$$

and expanding in powers of λ or λ^{-1}. ∎

We finish this general discussion by making a connection with Darboux matrices (see e.g. [19]). Suppose that $\alpha_1 \neq \alpha_2$ then we can define

$$P = \frac{1}{\alpha_2 - \alpha_1}(L^{(1)} - \alpha_1 \mathbb{1}_2) \,, \tag{2.44}$$

10

and multiply $L(\lambda)$ by $\frac{\lambda}{\lambda+\alpha_1}$ (since it is defined up to a function of λ) to get

$$L(\lambda) = \mathbb{1}_2 + \frac{\alpha_2 - \alpha_1}{\lambda + \alpha_1} P \,. \tag{2.45}$$

The important fact is that P is a projector ($P^2 = P$ can be checked directly from the definition (2.44)). The form (2.45) for L is usually encountered where the so-called Bäcklund-Darboux transformations are used to generate multi-soliton solutions from a given one. This form is also useful to exhibit the inverse of the Bäcklund matrix

$$L^{-1}(\lambda) = \mathbb{1}_2 - \frac{\alpha_2 - \alpha_1}{\lambda + \alpha_2} P \,. \tag{2.46}$$

3   Examples

In this section, our results are applied on a variety of examples. We are able to reproduce the results of [11, 12] in a very simple way. For the NLS equation, we obtain a more general result corresponding to a Bäcklund transformation with two real parameters (as it should, see e.g. [21]) instead of one. For the Korteweg-de Vries (KdV) and modified KdV equations, we obtain the defect contribution directly in terms of the original fields and reproduce the lagrangian approach expressions in terms of "potential" fields. For each model, the defect conditions are given and consistently reproduces the associated well-known Backlünd transformations, but taken at $x = x_0$ here, as expected by construction. We gather the examples in three classes according to certain symmetry considerations yielding information on the defect parameters $\alpha_\pm$.

3.1   Class I: $q = u$, $r = \epsilon u^*$, $\epsilon = \pm 1$, u complex scalar field

Let us introduce

$$K = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \,. \tag{3.1}$$

We have the following symmetries (we drop x and t)

$$U^*(\lambda^*) = KU(\lambda)K^{-1} \quad , \quad \widetilde{U}^*(\lambda^*) = K\widetilde{U}(\lambda)K^{-1} \,, \tag{3.2}$$

and we assume that V and $\widetilde{V}$ have the same properties. Therefore, we can look for the Bäcklund matrix L such that $L^*(\lambda^*) = KL(\lambda)K^{-1}$. This implies $\alpha_2 = \alpha_1^*$ so $\alpha_+ \in \mathbb{R}$ and $\alpha_- \in i\mathbb{R}$. Then, it can be shown that the remaining conditions imply that the nontrivial Bäcklund matrix reads

$$L(\lambda) = \mathbb{1}_2 + \lambda^{-1} \begin{pmatrix} \frac{1}{2}\left\{\alpha_+ \pm i\sqrt{\beta^2 + \epsilon|\tilde{u} - u|^2}\right\} & -\frac{i}{2}(\tilde{u} - u) \\ \frac{i}{2}\epsilon(\tilde{u}^* - u^*) & \frac{1}{2}\left\{\alpha_+ \mp i\sqrt{\beta^2 + \epsilon|\tilde{u} - u|^2}\right\} \end{pmatrix} \,, \tag{3.3}$$

11

where $\beta = i\alpha_- \in \mathbb{R}$. For this class, the defect matrix, and hence the defect conditions are parametrized by two arbitrary real numbers α_+ and β. In the case $\epsilon = -1$, we see that the transformation is such that $|\tilde{u} - u|^2 \leq \beta^2$. We must take into account the fact that $r = \epsilon q^*$ in the discussion of the integrals of motion. In particular, it turns out that to generate real integral of the motion (real classical observables), one has to consider the following combination

$$I^{sym}(\lambda) = i(I(\lambda) - I^*(\lambda^*)). \tag{3.4}$$

So in practice, we will compute the contribution of the defect as

$$I^{sym}_{defect}(\lambda) = -i\left(\ln(L_{11} + L_{12}\Gamma) - \ln(L_{11} + L_{12}\Gamma)^*\right)|_{x=x_0}. \tag{3.5}$$

Now we expand (3.4) in powers of λ^{-1} up to order 3 to illustrate the method. For convenience, we define $\Omega_\epsilon = \sqrt{\beta^2 + \epsilon|\tilde{u} - u|^2}$. At order λ^{-1}, we find that the modified conserved density reads

$$\int_{-\infty}^{x_0} |\tilde{u}|^2 dx + \int_{x_0}^\infty |u|^2 dx \mp \epsilon\,\Omega_\epsilon|_{x=x_0}, \tag{3.6}$$

where the last term is the explicit defect contribution. Similarly, at order λ^{-2}, the modified conserved momentum is

$$\int_{-\infty}^{x_0} i(\tilde{u}\tilde{u}_x^* - \tilde{u}^*\tilde{u}_x)dx + \int_{x_0}^\infty i(uu_x^* - u^*u_x)dx - i\left[(u^*\tilde{u} - u\tilde{u}^*) \mp 2\epsilon\beta\Omega_\epsilon\right]|_{x=x_0}. \tag{3.7}$$

Finally, the modified conserved energy is found to be

$$\int_{-\infty}^{x_0} (|\tilde{u}_x|^2 + \epsilon|\tilde{u}|^4)dx + \int_{x_0}^\infty (|u_x|^2 + \epsilon|u|^4)dx$$
$$+ \left[\mp\Omega_\epsilon(|\tilde{u}|^2 + |u|^2) \mp \frac{\epsilon}{3}\Omega_\epsilon(3\beta^2 - \Omega_\epsilon^2) - i\beta(\tilde{u}u^* - u\tilde{u}^*)\right]|_{x=x_0}. \tag{3.8}$$

These results hold for any member of class I so in particular they hold for the cubic focusing ($\epsilon = -1$) or defocusing ($\epsilon = 1$) nonlinear Schrödinger equation for the complex scalar field u

$$iu_t + u_{xx} = \epsilon|u|^2 u, \tag{3.9}$$

and similarly for $\tilde{u}$. Indeed, this equation is obtained in the AKNS scheme by taking

$$A(\lambda) = -2i\lambda^2 + i|u|^2, \quad B(\lambda) = \epsilon C^*(\lambda^*) = 2\lambda + iu_x, \tag{3.10}$$

and similarly for $\tilde{u}$.

Now the corresponding defect conditions can be derived from the general Bäcklund transformations given in the previous section. From the symmetry $a_3 = \epsilon a_2^*$, we need only

consider (2.29) and (2.33). Note that the x part of the defect conditions is the same for all the models in class I. Here, for NLS, we have at $x = x_0$

$$
\begin{aligned}
(\tilde{u} - u)_x &= i\alpha_+(\tilde{u} - u) \pm (\tilde{u} + u)\sqrt{\beta^2 + \epsilon|\tilde{u} - u|^2}\,, & (3.11) \\
(\tilde{u} - u)_t &= -\alpha_+(\tilde{u} - u)_x \pm i(\tilde{u} + u)_x\sqrt{\beta^2 + \epsilon|\tilde{u} - u|^2} + i(\tilde{u} - u)(|u|^2 + |\tilde{u}|^2)\, & (3.12)
\end{aligned}
$$

Setting $\epsilon = -1$ and choosing the $-$ sign in (3.6), (3.7), (3.8), we note that we have to impose further $\beta = 0$ to recover the results of [12] (where the notation $\Omega = \sqrt{\alpha^2 - |\tilde{u} - u|^2}$ is used). This is due to the fact that the lagrangian the authors took for the defect (the $\mathcal{B}$ functional in their notations) is not the most general one. It corresponds to particular Bäcklund transformations with $\beta = 0$.

3.2 Class II: $q = u$, $r = \epsilon u$, , $\epsilon = \pm 1$, u real scalar field

This class is a subclass of class I with $u^* = u$ (and $\tilde{u}^* = \tilde{u}$). this immediately implies

$$
\alpha_+ = 0\,. \tag{3.13}
$$

The defect matrix for this class reads

$$
L(\lambda) = \mathbb{1}_2 + \lambda^{-1}\begin{pmatrix} \pm\frac{i}{2}\sqrt{\alpha^2 + \epsilon(\tilde{u} - u)^2} & -\frac{i}{2}(\tilde{u} - u) \\ \frac{i}{2}\epsilon(\tilde{u} - u) & \mp\frac{i}{2}\sqrt{\alpha^2 + \epsilon(\tilde{u} - u)^2} \end{pmatrix}\,. \tag{3.14}
$$

So we can simply use the result of the previous class, setting $u^* = u$ and $\tilde{u}^* = \tilde{u}$ (and thus, without symetrizing). We exhibit the first orders for a few examples, taking advantage of specific forms of the defect matrix in each case.

3.2.1 Modified Korteweg-de Vries equation

The modified Korteweg-de Vries equation equation

$$
u_t - 6\epsilon u^2 u_x + u_{xxx} = 0\,, \tag{3.15}
$$

is obtained in the AKNS scheme by taking

$$
A(\lambda) = -4i\lambda^3 - 2i\lambda\epsilon u^2 \quad,\quad B(\lambda) = \epsilon C^*(\lambda^*) = 4\lambda^2 u + 2i\lambda u_x - u_{xx} + 2\epsilon u^3\,, \tag{3.16}
$$

and similarly for $\tilde{u}$. The modified conserved density reads

$$
\int_{-\infty}^{x_0} \tilde{u}^2 dx + \int_{x_0}^{\infty} u^2 dx \mp \epsilon\sqrt{\alpha^2 + \epsilon(\tilde{u} - u)^2}|_{x=x_0}\,. \tag{3.17}
$$

The modified conserved momentum is

$$
\int_{-\infty}^{x_0} \tilde{u}\tilde{u}_x dx + \int_{x_0}^{\infty} uu_x dx - \frac{1}{2}(\tilde{u}^2 - u^2 + \alpha^2)|_{x=x_0}\,, \tag{3.18}
$$

13

as can be directly checked by integration by parts (the constant α^2 being irrelevant). Finally, the next order yields

$$\int_{-\infty}^{x_0} (\tilde{u}_x^2 + \epsilon \tilde{u}^4) dx + \int_{x_0}^{\infty} (u_x^2 + \epsilon u^4) dx \mp \Omega_\epsilon \left[(\tilde{u}^2 + u^2) - \frac{\epsilon}{3} \Omega_\epsilon^2 \right] |_{x=x_0}, \qquad (3.19)$$

where here $\Omega_\epsilon = \sqrt{\alpha^2 + \epsilon(\tilde{u} - u)^2}$. The corresponding defect conditions read

$$(\tilde{u} - u)_x = \pm(\tilde{u} + u)\sqrt{\alpha^2 + \epsilon(\tilde{u} - u)^2}, \qquad (3.20)$$

$$(\tilde{u} - u)_t = \pm \left\{ 2\epsilon(\tilde{u}^3 + u^3) - (\tilde{u} + u)_{xx} \right\} \sqrt{\alpha^2 + \epsilon(\tilde{u} - u)^2}. \qquad (3.21)$$

It is worth noting that everything is expressed directly in terms of the fields u and $\tilde{u}$. This should be compared with the Lagrangian approach of [12] where this was not possible. The use of "potential" fields p and q such that $\tilde{u} = p_x$ and $u = -q_x$ is required in this formulation. Under this substitution, an alternative form of the defect matrix can be derived

$$L(\lambda) = \mathbb{1}_2 \pm \lambda^{-1} \frac{i\alpha}{2} \left(\begin{array}{cc} \cos(\tilde{v} - v) & -\sin(\tilde{v} - v) \\ -\sin(\tilde{v} - v) & -\cos(\tilde{v} - v) \end{array} \right). \qquad (3.22)$$

and we consistently recover their result for the defect contribution to the first conserved quantity (setting $\epsilon = -1$), that is $\epsilon \sqrt{\alpha^2 + \epsilon(\tilde{u} - u)^2}|_{x=x_0}$ becomes

$$\alpha(\cos(p - q) - 1)|_{x=x_0}, \qquad (3.23)$$

(a constant can always be added).

3.2.2 Sine/sinh-Gordon equation

This example illustrates the case $V, \widetilde{V}$ polynomials in λ^{-1}. The sine-Gordon equation in light-cone coordinates

$$v_{xt} = \sin v, \qquad (3.24)$$

is obtained by setting $u = -\frac{v_x}{2}$, $\epsilon = -1$ and taking

$$A(\lambda) = \frac{i \cos v}{4\lambda}, \quad B(\lambda) = \frac{i \sin v}{4\lambda}, \qquad (3.25)$$

and similarly for $\tilde{v}$. For this model, the defect matrix takes the nice following form

$$L(\lambda) = \mathbb{1}_2 \pm \frac{i\alpha}{2\lambda} \left(\begin{array}{cc} \cos \frac{\tilde{v}+v}{2} & -\sin \frac{\tilde{v}+v}{2} \\ -\sin \frac{\tilde{v}+v}{2} & -\cos \frac{\tilde{v}+v}{2} \end{array} \right), \qquad (3.26)$$

where α is a nonzero real parameter. Therefore, the modified conserved momentum reads

$$\frac{1}{4} \int_{-\infty}^{x_0} \tilde{v}_x^2 dx + \frac{1}{4} \int_{x_0}^{\infty} v_x^2 dx \pm \alpha \cos \frac{\tilde{v} + v}{2}|_{x=x_0}. \qquad (3.27)$$

This is just the result in [30] for the momentum and energy but expressed here in light-cone coordinates. At the next order, as can be anticipated from (3.18), the bulk contribution can be integrated by parts and combines nicely with the defect contribution, leaving the constant $-\frac{\alpha^2}{2}$ as the conserved quantity. Finally, the third order conserved quantity is

$$\frac{1}{2}\int_{-\infty}^{x_0}(-\tilde{v}_{xx}^2 + \frac{\tilde{v}_x^4}{4})dx + \frac{1}{2}\int_{x_0}^{\infty}(-v_{xx}^2 + \frac{v_x^4}{4})dx \pm \frac{\alpha}{3}\cos\frac{\tilde{v}+v}{2}(\tilde{v}_x^2 + \tilde{v}_x v_x + v_x^2 + 2\alpha^2)|_{x=x_0} \quad (3.28)$$

The defect conditions at $x = x_0$ are given by

$$(\tilde{v} - v)_x = \pm 2\alpha \sin\frac{\tilde{v}+v}{2}, \quad (3.29)$$

$$(\tilde{v} + v)_t = \pm\frac{2}{\alpha}\sin\frac{\tilde{v}-v}{2}. \quad (3.30)$$

3.2.3 Liouville equation

This is another example of evolution equation obtained from V and $\tilde{V}$ polynomials in λ^{-1}. We need to discuss this equation in detail since, as mentioned above, the condition of vanishing fields at infinity and the boundary conditions (1.4) are not applicable. However, it is straightforward to prove that Proposition 2.2 is still valid. Also, it is still possible to prove Proposition 1.1 with a slight modification, as we now show. The Liouville equation in light-cone coordinates for the field v

$$v_{xt} = 2e^v, \quad (3.31)$$

is obtained by taking

$$u = \frac{v_x}{2}, \quad \epsilon = 1, \quad A(\lambda) = \frac{ie^v}{2\lambda} = -B(\lambda), \quad (3.32)$$

and similarly for $\tilde{v}$. The conservation laws (1.18) and (1.19) still hold since they do not depend on the boundary conditions. But now, (1.20) becomes

$$\partial_t\int_{x_0}^{\infty}q\Gamma dx + \partial_t\int_{-\infty}^{x_0}\tilde{q}\tilde{\Gamma}dx = \lim_{x\to\infty}(B\Gamma + A) - \lim_{x\to-\infty}(\tilde{B}\tilde{\Gamma} + \tilde{A}) \quad (3.33)$$

$$+ \left(\tilde{B}\tilde{\Gamma} + \tilde{A} - (B\Gamma + A)\right)|_{x=x_0}. \quad (3.34)$$

The last term in the right-hand-side is treated as before. The point is to recast the other two terms as time derivatives. For this equation, one has $A = C = -B$ so $B\Gamma + A = A(1 - \Gamma)$ and the Ricatti equation (1.23) becomes

$$\Gamma_t = A(1 - \Gamma)^2. \quad (3.35)$$

Therefore, since $\Gamma \neq 1$ (this can be seen from (1.7))

$$B\Gamma + A = -\partial_t\ln(1 - \Gamma). \quad (3.36)$$

15

The same result holds for $\widetilde{\Gamma}$. Therefore, Proposition 1.1 is modified to the following.

The generating function for the integral of motions of the Liouville equation reads

$$I(\lambda) = I_{bulk}^{left}(\lambda) + I_{bulk}^{right}(\lambda) + I_{defect}(\lambda), \tag{3.37}$$

where

$$I_{bulk}^{left}(\lambda) = \frac{1}{2} \int_{-\infty}^{x_0} \tilde{v}_x \widetilde{\Gamma} dx - \lim_{x \to -\infty} \ln(1 - \widetilde{\Gamma}), \tag{3.38}$$

$$I_{bulk}^{right}(\lambda) = \frac{1}{2} \int_{x_0}^{\infty} v_x \Gamma dx + \lim_{x \to \infty} \ln(1 - \Gamma), \tag{3.39}$$

$$I_{defect}(\lambda) = -\ln(L_{11} + L_{12}\Gamma)|_{x=x_0}, \tag{3.40}$$

and L_{ij}'s are the entries of the Bäcklund matrix L. The additional contributions essentially kill terms arising from trivial integration by parts in the integrals of motion.

The defect matrix can be written

$$L(\lambda) = \mathbb{1}_2 + \frac{i\gamma}{4\lambda} e^{\pm \frac{\tilde{v}+v}{2}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \tag{3.41}$$

where γ is a nonzero real constant. At first order, the modified conserved momentum reads

$$\frac{1}{2} \int_{-\infty}^{x_0} \tilde{v}_x^2 dx + \tilde{v}_x|_{-\infty} + \frac{1}{2} \int_{x_0}^{\infty} v_x^2 dx - v_x|_{\infty} - \gamma e^{\pm \frac{\tilde{v}+v}{2}}|_{x=x0}. \tag{3.42}$$

As is now customary, the next order combines nicely to produce

$$(v_x^2 - 2v_{xx})|_{\infty} - (\tilde{v}_x^2 - 2\tilde{v}_{xx})|_{-\infty}, \tag{3.43}$$

which can be checked directly to be a constant. Finally, at the third order, we have

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^{x_0} \left(\tilde{v}_{xx}^2 + \frac{\tilde{v}_x^4}{4} \right) dx - \left(\tilde{v}_{xxx} - \frac{\tilde{v}_x^3}{6} - \tilde{v}_x \tilde{v}_{xx} \right) |_{-\infty} \\ + & \frac{1}{2} \int_{x_0}^{\infty} \left(v_{xx}^2 + \frac{v_x^4}{4} \right) dx + \left(v_{xxx} - \frac{v_x^3}{6} - v_x v_{xx} \right) |_{\infty} \\ - & \frac{\gamma}{6} e^{\pm \frac{\tilde{v}+v}{2}} (\tilde{v}_x^2 + v_x^2 + \tilde{v}_x v_x)|_{x=x0}. \end{aligned} \tag{3.44}$$

The defect conditions are given by

$$(\tilde{v} - v)_x = \gamma e^{\pm \frac{\tilde{v}+v}{2}}, \tag{3.45}$$

$$(\tilde{v} + v)_t = \pm \frac{4}{\gamma} e^{\mp \frac{\tilde{v}+v}{2}} (e^{\tilde{v}} - e^v), \tag{3.46}$$

16

3.3 Class III: $q = u$, $r = \epsilon$, , $\epsilon = \pm 1$, u real scalar field

For this class, there is no special symmetry. The derivation of the defect matrix deserved special attention for this class. Indeed, if we assume that $(\widetilde{U}, \widetilde{V})$ has the same form as (U, V), as we have done so far, then an immediate consequence is $a_3 = 0$ and $a_1 = a_4 = \alpha_1$ are constant. From this, equation (2.28) implies $\tilde{u} = u$ *i.e.* we get the trivial defect conditions. A solution to this problem is to left multiply L by σ_3 or equivalently to consider $(\sigma_3 \widetilde{U} \sigma_3, \sigma_3 \widetilde{V} \sigma_3)$. This amounts to change the sign of the off-diagonal terms. Then, equation (2.30) implies $\alpha_+ = 0$. Finally, taking into account the reality of the field, one gets the defect matrix for this class

$$L(\lambda) = \mathbb{I}_2 + \lambda^{-1} \begin{pmatrix} \pm\frac{i}{2}\sqrt{\beta^2 + 2\epsilon(\tilde{u} + u)} & \frac{i}{2}(\tilde{u} + u) \\ -i\epsilon & \mp\frac{i}{2}\sqrt{\beta^2 + 2\epsilon(\tilde{u} + u)} \end{pmatrix}, \tag{3.47}$$

We apply our method to the Korteweg-de Vries equation

$$u_t - 6\epsilon u u_x + u_{xxx} = 0, \tag{3.48}$$

which is obtained in the AKNS scheme by taking

$$A(\lambda) = -4i\lambda^3 - 2i\epsilon\lambda u + \epsilon u_x, \tag{3.49}$$
$$B(\lambda) = 4\lambda^2 u + 2i\lambda u_x + 2\epsilon u^2 - u_{xx}, \tag{3.50}$$
$$C(\lambda) = 4\epsilon\lambda^2 + 8u, \tag{3.51}$$

and similarly for $\tilde{u}$, with appropriate change of signs. We make direct use of (1.14). The first nontrivial order is λ^{-3} and for the modified conserved density reads

$$\frac{1}{2}\int_{-\infty}^{x_0} \tilde{u}^2 dx + \frac{1}{2}\int_{x_0}^{\infty} u^2 dx \mp \frac{1}{6}\sqrt{\beta^2 + 2\epsilon(\tilde{u} + u)}\left(\epsilon(\tilde{u} + u) - \beta^2\right)|_{x=x_0}. \tag{3.52}$$

The defect conditions are given by

$$(\tilde{u} + u)_x = \pm(\tilde{u} - u)\sqrt{\beta^2 + 2\epsilon(\tilde{u} + u)}, \tag{3.53}$$
$$(\tilde{u} + u)_t = \pm\left(3\epsilon(\tilde{u}^2 - u^2) - (\tilde{u} - u)_{xx}\right)\sqrt{\beta^2 + 2\epsilon(\tilde{u} + u)}. \tag{3.54}$$

Once again, we note that everything is expressed directly in terms of the initial fields u and $\tilde{u}$ while this was not possible in the lagrangian approach. It is easy to make contact with the latter by setting $u = q_x$ and $\tilde{u} = p_x$. Then,

$$(p + q)_x = \frac{\epsilon}{2}\left[(p - q)^2 - \beta^2\right]. \tag{3.55}$$

Taking $\epsilon = 1$ and setting $\beta^2 = -4\alpha$, we recover the result of [12] for the defect contribution to the density (the momentum in their setting)

$$\frac{1}{2}\int_{-\infty}^{x_0} p_x^2 dx + \frac{1}{2}\int_{x_0}^{\infty} q_x^2 dx, \tag{3.56}$$

that is

$$\left(-\alpha(p - q) - \frac{1}{12}(p - q)^3\right)|_{x=x_0}. \tag{3.57}$$

3.4 Remarks

We will not go into the analysis of the higher N case in (2.6). We simply note that a large class of higher N defect matrices is provided by products of $N = 1$ defect matrices. The corresponding defect conditions are then simply compositions of the defect conditions we derived above. In other words, the defect matrices we constructed have a group structure, as can be seen directly from (1.12) and (1.13). Furthermore, Bianchi's theorem of permutability (see e.g. [19]) implies that this is an abelian group. An important question concerns the existence and properties of those higher N defect matrices which do not factorize as products of $N = 1$ ones. Their study would shed new light on possible new defect conditions for the well-known systems we discussed. Also, the question of defect matrices preserving integrability but which do not fall at all in the class discussed here remains entirely open. In this sense, no claim of uniqueness of defect matrices is made and one should remember that the proposed approach here is sufficient to ensure integrability in the presence of a defect. The issue of finding necessary defect conditions for integrability is not answered. It should also be noted that following the linearization argument of [11, 12], it appears that the defect conditions constructed here allow for pure transmission only. This has been checked explicitly for all the examples given above.

4 Extension to another scheme

The Kaup-Newell (KN) [17] scheme goes along the same steps as the AKNS scheme to produce integrable evolution equations with the essential difference that the matrix U involved in the x part of the auxiliary problem (1.1) has the following form

$$U_{KN} = \begin{pmatrix} -i\lambda^2 & \lambda q \\ \lambda r & i\lambda^2 \end{pmatrix} = -i\lambda^2 \sigma_3 + \lambda W. \tag{4.1}$$

One well-known model obtained in this scheme is the derivative nonlinear Schrödinger equation

$$iu_t + u_{xx} = 2\epsilon(|u|^2 u)x, \tag{4.2}$$

where $q = u = \epsilon r$, $\epsilon = \pm 1$ and the matrix V_{KN} should read

$$V_{KN} = \begin{pmatrix} -2i\lambda^4 - i\epsilon\lambda^2 |u|^2 & 2\lambda^3 u + i\lambda u_x + \epsilon\lambda|u|^2 u \\ 2\epsilon\lambda^3 u^* - i\epsilon\lambda u_x^* + \lambda|u|^2 u^* & 2i\lambda^4 + i\epsilon\lambda^2|u|^2 \end{pmatrix}. \tag{4.3}$$

It turns out that our method works for this scheme too with the appropriate modifications. We proceed as before by taking two copies of the auxiliary problem related by a matrix M such that

$$M_x = \widetilde{U}_{KN} M - M U_{KN}, \tag{4.4}$$

$$M_t = \widetilde{V}_{KN} M - M V_{KN}. \tag{4.5}$$

Then we assume that the two copies are related by M at some point $x = x_0$.

Proposition 4.1 *The generating function for the integral of motions reads*

$$I(\lambda) \;=\; I_{bulk}^{left}(\lambda) + I_{bulk}^{right}(\lambda) + I_{defect}(\lambda)\,, \tag{4.6}$$

where

$$I_{bulk}^{left}(\lambda) \;=\; \int_{\infty}^{x_0} \lambda \tilde{q}\widetilde{\Gamma} dx\,, \tag{4.7}$$

$$I_{bulk}^{right}(\lambda) \;=\; \int_{x_0}^{\infty} \lambda q\Gamma dx\,, \tag{4.8}$$

$$I_{defect}(\lambda) \;=\; -\ln(M_{11} + M_{12}\Gamma)|_{x=x_0}\,, \tag{4.9}$$

and M_{ij}'s are the entries of the Bäcklund matrix M. For this scheme, Γ and $\widetilde{\Gamma}$ have a different expansion as $\lambda \to \infty$

$$\Gamma = \sum_{n=0}^{\infty} \frac{\Gamma_n}{(2i\lambda)^{2n+1}}\,, \tag{4.10}$$

with

$$\Gamma_0 = -r\;,\quad \Gamma_{n+1} = 2i\Gamma_{nx} + q\sum_{p=0}^{n} \Gamma_p \Gamma_{n-p}\,, \tag{4.11}$$

and similarly for $\widetilde{\Gamma}$.

<u>Proof:</u> The proof is the same as that of Proposition 1.1, the only difference being that the conservation equations now read

$$(q\Gamma)_t \;=\; \frac{1}{\lambda}\left(B\Gamma + A\right)_x\;,\quad \forall x > x_0\,, \tag{4.12}$$

$$\left(\tilde{q}\widetilde{\Gamma}\right)_t \;=\; \frac{1}{\lambda}\left(\widetilde{B}\widetilde{\Gamma} + \widetilde{A}\right)_x\;,\quad \forall x < x_0\,, \tag{4.13}$$

due to the different λ dependence of U_{KN}. The latter is also responsible for the different series expansions of Γ and $\widetilde{\Gamma}$ coming from the following Ricatti equation

$$\Gamma_x = \lambda r + 2i\lambda^2\Gamma - \lambda q\Gamma^2\,, \tag{4.14}$$

and similarly for $\widetilde{\Gamma}$. ∎

To apply this to specific models, we would need some sort of classification of the matrices M along the lines of what is available for L. However, the author is not aware of such results. It is a problem for future investigation.

5 Discussion of integrability

So far, we have shown how the infinite set of conservation laws is modified by the presence of a defect described by a matrix for any evolution equation falling into the AKNS or KN schemes. The role of an infinite set of conserved quantities is well-known in the construction of action-angle variables in the inverse scattering method for evolution equations on the line. In turn, this construction ensure the integrability of the system in the sense that in terms of the new variables, the evolution in time is very easy to solve. Then, using the inverse part of the method (the Gelfan'd-Levitan-Marchenko equations [22, 23]) one can deduce the time evolution of the original fields (see e.g. [16]). So, in this sense, the systems with defect we have considered are integrable[4].

A traditional complementary view is to reformulate the evolution equation as Hamiltonian systems with a Poisson structure. In this formalism, the idea is to show that the conserved quantities previously constructed form a commutative Poisson algebra containing the Hamiltonian which generates the time evolution of the fields. Then, one talks about integrability in the sense of Liouville. One of the advantages of this reformulation is the possibility of quantization and this was the object of the quantum inverse scattering method, see e.g. [24].

The method of the classical r-matrix [25] has proved very useful and fundamental in the discussion of these issues for classical integrable systems on the whole line. The situation on the half-line is also well understood [26]. Our aim in this section is to discuss the situation with a defect.

Let us first recall some facts about the method of the classical r-matrix for ultralocal models. The basic ingredient is the 2×2 transition matrix $T(x, y, \lambda)$, $x < y$, defined as the fundamental solution of the x-part of the auxiliary problem at a given time (not explicitely displayed in T)

$$\partial_y T(x, y, \lambda) = U(y, t, \lambda) T(x, y, \lambda) \quad , \quad T(x, x, \lambda) = \mathbb{1}_2 \,, \tag{5.1}$$

where U is given as in (1.3). The important result reads (see e.g. [27])

$$\{T_1(x, y, \lambda), T_2(x, y, \mu)\} = [r_{12}(\lambda - \mu), T_1(x, y, \lambda) \, T_2(x, y, \mu)] \,, \tag{5.2}$$

where $r_{12}(\lambda)$, the classical r-matrix, is a 4×4 antisymmetric solution of the classical Yang-Baxter equation [28]. For systems on the circle, it allows to show that the coefficients of the series expansion of $Tr T(x, y, \lambda)$ in λ are in involution and the Hamiltonian is one of the them. For systems on the line, the principle is the same but one has to take the infinite volume limit of (5.2), usually with special care.

For the problem with defect, we would like to mimic this procedure. The main difference here is that there is something nontrivial going on at a single point $x = x_0$ and

---

[4]For completeness, one should perform the inverse scattering method and identify the action-angle variables in this context. This is left for future work.

characterized by L (or M[5]). The standard procedure becomes ill-defined in the continuous case as it involves singular Poisson brackets of the type "$\delta(0)$" where $\delta(x)$ is the Dirac distribution.

Indeed, the analog of the transition matrix for $x < x_0 < y$ is

$$\mathcal{T}^{x_0}(x, y, \lambda) = T(x_0, y, \lambda)L^{-1}(x_0, t, \lambda)\widetilde{T}(x, x_0, \lambda)\,, \tag{5.3}$$

and the task of computing

$$\{\mathcal{T}_1^{x_0}(x, y, \lambda), \mathcal{T}_2^{x_0}(x, y, \mu)\}\,, \tag{5.4}$$

involves computing $\{L_1^{-1}(x_0, t, \lambda), L_2^{-1}(x_0, t, \lambda)\}$. There does not seem to be a direct approach starting from the explicit form of L as classified in this paper. However, in the context of finite-dimensional integrable systems, important results were obtained by E. Sklyanin in [29] concerning the canonicity of Bäcklund transformations in the formalism of the r-matrix approach. Transposing the results in the present context and assuming we have the same theories on both sides of the defect, *i.e.* $T(x, y, \lambda)$ and $\widetilde{T}(x', y', \lambda)$ satisfy (5.2), for $x, y > x_0$ and $x', y' < x_0$ respectively, it seems reasonable to postulate that L^{-1} is just another representation of the Poisson algebra (5.2) so that

$$\{L_1^{-1}(x_0, t, \lambda), L_2^{-1}(x_0, t, \lambda)\} = [r_{12}(\lambda - \mu), L_1^{-1}(x_0, t, \lambda)\ L_2^{-1}(x_0, t, \lambda)]\,. \tag{5.5}$$

From this it immediately follows that

$$\{\mathcal{T}_1^{x_0}(x, y, \lambda), \mathcal{T}_2^{x_0}(x, y, \mu)\} = [r_{12}(\lambda, \mu), \mathcal{T}_1^{x_0}(x, y, \lambda)\ \mathcal{T}_2^{x_0}(x, y, \mu)]\,. \tag{5.6}$$

Note that the generalization to N defects can be described in this formalism as well. Using the same notations as Section 1.3, one construct the monodromy matrix

$$\begin{aligned}
\mathcal{T}^N(\lambda) &= T^{N+1}(x_N, x_{N+1}, \lambda)(L^N)^{-1}(x_N, t, \lambda)\ldots(L^2)^{-1}(x_1, t, \lambda)T^1(x_1, x_2, \lambda) \\
&\quad \times (L^1)^{-1}(x_1, t, \lambda)T^0(x_0, x_1, \lambda)\,,
\end{aligned} \tag{5.7}$$

with all the L^j's satisfying (5.5).

This discussion brings us to the connection between the lagrangian approach of [11, 12] to integrable defects and a quite standard procedure to implement inhomogeneities or impurities in discrete integrable systems. Indeed, through the approach of this paper, one can reformulate the lagrangian approach in terms of a transition matrix made of two bulk parts and a localised defect part realising a different representation of the same Poisson algebra. But this is exactly what is usually done to implement so-called impurities or inhomogeneities in discrete integrable systems. Let us consider a discrete ultralocal system of length ℓ with N sites. The transition matrix $T(\lambda)$ is then a product of local matrices $t_j(\lambda)$, $j = 1, \ldots, N$ satisfying

$$\{t_{j0}(\lambda), t_{k0'}\} = \delta_{jk}\left[r_{00'}(\lambda - \mu), t_{j0}(\lambda)t_{k0'}(\mu)\right]\,, \tag{5.8}$$

---

[5]For convenience, in the rest of the paper, we stick to a single notation L for the defect matrix.

in the same representation (the index j represents the space of dynamical variables while 0, $0'$ are auxiliary spaces, $\mathbb{C}^2$ here). One can check then that $T(\lambda) = t_N(\lambda)\dots t_1(\lambda)$ satisfies

$$\{T_0(\lambda), T_{0'}(\mu)\} = [r_{00'}(\lambda - \mu), T_0(\lambda)T_{0'}(\mu)] \,, \tag{5.9}$$

i.e. exactly (5.2). To introduce an inhomogeneity at site j_0 say, one then chooses a different representation $\hat{t}_{j_0}(\lambda)$ of the same algebra. This does not change the properties of $T(\lambda) = t_N(\lambda)\dots\hat{t}_{j_0}(\lambda)\dots t_1(\lambda)$ but influences the physical quantities (e.g. the integrals of motion) that can be computed since the latter depend on the representation at each site. Again, the introduction of several inhomogeneities can be done straightforwardly. This actually provides a solution to the above problem of computing Poisson brackets of defect matrices by considering lattice regularizations of integrable field theories and changing representations appropriately at local sites to generate defects.

From this point of view, one can anticipate the outcome of the quantization of this approach. It is known that (5.9) is the classical ($\hbar \to 0$) limit of the quantum Yang-Baxter algebra

$$R_{00'}(\lambda - \mu)\tau_{j0}(\lambda)\tau_{j0'}(\mu) = \tau_{j0'}(\mu)\tau_{j0}(\lambda)R_{00'}(\lambda - \mu) \,, \tag{5.10}$$

where $R(\lambda)$ is the quantum R matrix associated to r

$$R(\lambda) = \mathbb{1} + i\hbar r(\lambda) + O(\hbar^2) \,, \tag{5.11}$$

and $\tau_j(\lambda) \to t_j(\lambda)$ in the $\hbar \to 0$ limit. So the quantum defect matrix $\mathcal{L}(\lambda)$ encoding the defect conditions will satisfy the quantum Yang-Baxter algebra. This gives some support to the ad hoc quantization procedure adopted in [30] for the sine-Gordon with integrable defect. We would like to stress that the above programme of discretization has been completed for the defect sine-Gordon model in the important paper [31], both at classical and quantum level. The approach is based on the notion of ancestor algebra [32].

Conclusions and outlook

In this paper, we have reformulated the lagrangian approach to the question of integrable defects in the language of the inverse scattering method, taking advantage of the common features that had been observed on a case by case study: frozen Bäcklund transformations as defect conditions ensure integrability. The reformulation allows a systematic proof of this as well as an efficient computation of the modified conserved quantities to all orders in terms of the defect matrix. The latter, and the associated defect conditions, can be classified and we performed these computations for a certain class of matrices. Taking particular examples, we recovered and even generalized all the previous results obtained by the lagrangian method. It should be emphasized that this procedure provides a sufficient

approach to the question of integrable defects in classical field theories and, by no means, represents a complete picture of the story.

Rather, it is a first step for future developments among which further study of the classical r matrix approach and quantization of the method are important. Let us mention also the contruction of other integrable defects allowing if possible reflection as well. If applicable, the quantization should then be related to existing quantum algebraic frameworks like the Reflection-Transmission algebras [6]. Finally, the complete setup of the direct and inverse part of the method for the actual construction of the solutions, especially of soliton type, should shed new light on the results already obtained by the more direct approach of [11, 12].

Acknowledgements

References

[1] G. Delfino, G. Mussardo and P. Simonetti, *Scattering Theory and Correlation Functions in Statistical Models with a Line of Defect*, Nucl. Phys. **B432** (1994) 518 and `hep-th/9409076`; *Statistical Models with a Line of Defect*, Phys. Lett. **B328** (1994) 123 and `hep-th/9403049`.

[2] R. Konik and A. LeClair, *Purely Transmitting Defect Field Theories*, Nucl. Phys. **B538** (1999) 587 and `hep-th/9703085`.

[3] H. Saleur, *Lectures on Non-perturbative field theory and quantum impurity problems*, `cond-mat/9812110`; *Lectures on Non-perturbative field theory and quantum impurity problems. II*, `cond-mat/0007309`.

[4] O. A. Castro-Alvaredo, A. Fring and F. Gohmann, *On the absence of simultaneous reflection and transmission in integrable impurity systems*, `hep-th/0201142`.

[5] O. Castro-Alvaredo, A. Fring, *From integrability to conductance, impurity systems*, Nucl. Phys. **B649** (2003) 449 and `hep-th/0205076`.

[6] M. Mintchev, E. Ragoucy and P. Sorba, *Scattering in the Presence of a Reflecting and Transmitting Impurity*, Phys. Lett. **B547** (2002) 313 and `hep-th/0209052`; *Reflection-Transmission Algebras*, J. Phys. **A36** (2003) 10407 and hep-th/0303187.

[7] V. Caudrelier, M. Mintchev, E. Ragoucy, *Solving the quantum non-linear Schrödinger equation with delta-type impurity*, J.Phys. **A37** (2004) L367 and `hep-th/0404144`.

[8] Z. Bajnok, A. George, *From Defects to Boundaries*, Int. J. Mod. Phys. **A21** (2006) 1063 and `hep-th/0404199` .

[9] V. Caudrelier, M. Mintchev, E. Ragoucy, P. Sorba, *Reflection-Transmission Quantum Yang-Baxter Equations*, J.Phys. **A38** (2005) 3431 and `hep-th/0412159`.

[10] P. N. Bibikov, V. O.Tarasov, *A boundary-value problem for the nonlinear Schrödinger equation*, Theor. Math. Phys. **79** (1989), 334.

[11] P. Bowcock, E. Corrigan, C. Zambon, *Classically integrable field theories with defects*, Int. J. Mod. Phys. **A19S2** (2004) 82 and `hep-th/0305022`; *Affine Toda field theories with defects* , JHEP 0401 (2004) 056 and `hep-th/0401020`;

[12] E. Corrigan, C. Zambon, *Aspects of sine-Gordon solitons, defects and gates*, J. Phys. **A37** (2004) L471 and `hep-th/0407199`; *Jump-defects in the nonlinear Schrödinger model and other non-relativistic field theories*, Nonlinearity 19 (2006) 1447 and `nlin.SI/0512038`.

[13] I.T. Habibullin, *Bäcklund transformation and integrable boundary-initial value problems*, in Nonlinear world", vol. 1 (Kiev, 1989), 130-138, World Sci. Publ., River Edge, NJ, 1990; *Integrable initial-boundary value problems*, Theor. and Math. Phys. **86** (1991), no. 1, 2836.

[14] R.F. Bikbaev, V. O.Tarasov, *Initial-boundary value problem for the nonlinear Schrödinger equation*, J. Phys. **A24** (1991) 2507.

[15] V. O.Tarasov, *The integrable initial-boundary value problem on a semiline: nonlinear Schrödinger and sine-Gordon equations*, Inv. probl. **7** (1991) 435.

[16] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, *The inverse scattering transform - Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53** (1974), 249.

[17] D.J. Kaup, A.C. Newell, *An exact solution for a derivative nonlinear Schrödinger equation,* J. Math. Phys. **19** (1978), 798.

[18] The initial data is typically chosen to be infinitely smooth in the x variable in the bulk but the defect conditions ensure only that derivatives of the fields up to a finite degree are well-behaved as $x \to x_0^{\pm}$. For each higher degree of "smoothness", one has to impose additional conditions at the point of the defect which can be obtained by differentiating for instance (2.28-2.31), using all known relations to get an expression involving only the fields and their derivatives up to the desired degree, and finally set $x = x_0$. This is important to make sense of the higher conserved quantities.

[19] C. Rogers, W.K. Schief, *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, 2002.

[20] P.G. Drazin, R.S. Johnson, *Solitons: an introduction,* Cambridge Texts in Applied Mathematics, CUP 1989.

[21] H.H. Chen, *General Derivation of Bäcklund Transformations from Inverse Scattering Method,* Phys. Rev. Lett. **33** (1974), 925.

[22] I.M. Gel'fand, B.M. Levitan, *On the determination of a differential equation from its spectral function*, Am. Math. Soc. Transl. **1** (1955), 259.

[23] V.A. Marchenko, *On reconstruction of the potential energy from phases of the scattered waves*, Dokl. Acad. Nauk SSSR, **104** (1955), 695.

[24] L.D. Faddeev, L.A. Takhtajan, *Quantum inverse scattering method*, Sov. Sci. Rev. Math. **C**1 (1981), 107.

[25] E.K. Sklyanin, *On complete integrability of the Landau-Lifshitz equation*, preprint Leningrad Branch of the Mathematics Institute, **E**-3-79 (1979).

[26] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A**21 (1988), 2375.

[27] L. Faddeev, *Integrable models in* $(1 + 1)$-*dimensional quantum field theory*, Les Houches lectures, 1982 "Recent advances in Field Theory and Statistical Mechanics", J.6B. Zuber and R. Stora eds, 1984.

[28] A.A. Belavin, V.G. Drinfel'd, *Solutions of the classical Yang-Baxter equation for simple Lie algebras*, Funct. Anal. and Appl. **16** (1982), 159; *Classical Young-Baxter equation for simple Lie algebras*, Funct. Anal. and Appl. **17** (1984), 220; *Triangle equation and simple Lie algebras*, Sov. Sci. Rev. sect. **C**4 (1984), 93.

[29] E. K. Sklyanin, *Canonicity of Bäcklund transformation: r-matrix approach. II*, `solv-int/9903017`.

[30] P. Bowcock, E. Corrigan, C. Zambon, *Some aspects of jump-defects in the quantum sine-Gordon model*, JHEP **0508** (2005), 023.

[31] Ismagil Habibullin, Anjan Kundu, *Quantum and classical integrable sine-Gordon model with defect*, `arXiv:0709.4611`.

[32] A. Kundu, *Yang-Baxter algebra and generation of quantum integrable models*, `nlin.SI/0609001`.