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Inflation Co-movement across Countries in Multi-maturity Term Structure: An Arbitrage-Free Approach*

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Abstract

Inflation expectation is acknowledged to be an important indicator for policy makers and financial investors. To capture a more accurate real-time estimate of inflation expectation on the basis of financial markets, we propose an arbitrage-free model across different countries in a multi-maturity term structure, where we first estimate inflation expectation by modelling the nominal and inflation-indexed bond yields jointly for each country. The Nelson-Siegel model is popular in fitting the term structure of government bond yields, the arbitrage-free model we proposed is the extension of the arbitrage-free dynamic Nelson-Siegel model proposed by Christensen, Diebold and Rudebusch (2011). We discover that the extracted common trend for inflation expectation is an important driver for each country of interest. Moreover, the model will lead to an improved forecast in a benchmark level of inflation and will provide good implications for financial markets.

Keywords: inflation expectation dynamics, arbitrage free, yield curve modelling, inflation

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risk

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1 Introduction

Today most economists favour a low and steady rate of inflation because it facilitates real wage adjustments in the presence of downward nominal wage rigidity. Hence one of the major objectives of modern monetary policy is to bring inflation expectation under control, which is considered to be the first step in controlling inflation. Meanwhile, hedging the risk around the inflation forecast becomes more attractive in financial markets, as many investors rely on the stability and predictability of future inflation levels. Moreover, price stability is of immense importance to sustain social welfare, job opportunities and economic upturn. The objective of price stability refers to the general level of prices in the economy which implies avoiding both prolonged inflation and deflation. Inflation expectation that is involved in a contemporary macroeconomic framework anticipates future economic trends, will further affect monetary decisions. Since there is large demand on having reasonable estimates of inflation expectation levels, a large amount of literature has focused on analysing the government conventional and inflation-indexed bonds, which can implicitly provide a vast amount of information about the expectations of nominal and real interest rates obtained from the market. Such estimates are known to be an important complement to the estimates provided from the survey data. Despite the fact that inflation indexed bonds have been more frequently and widely issued in recent times, one would still have great difficulties in integrating the market information from multiple countries to get individual level estimates of the inflation expectation. The major problems lie in the relative short period of data availability and the existence of a lot of missing values. While the existing literature's focus is mainly on specific country, we would like to consider an estimation framework that allows us to analyse the co-movement of inflation expectation for multiple countries, and also provide the country specific estimates

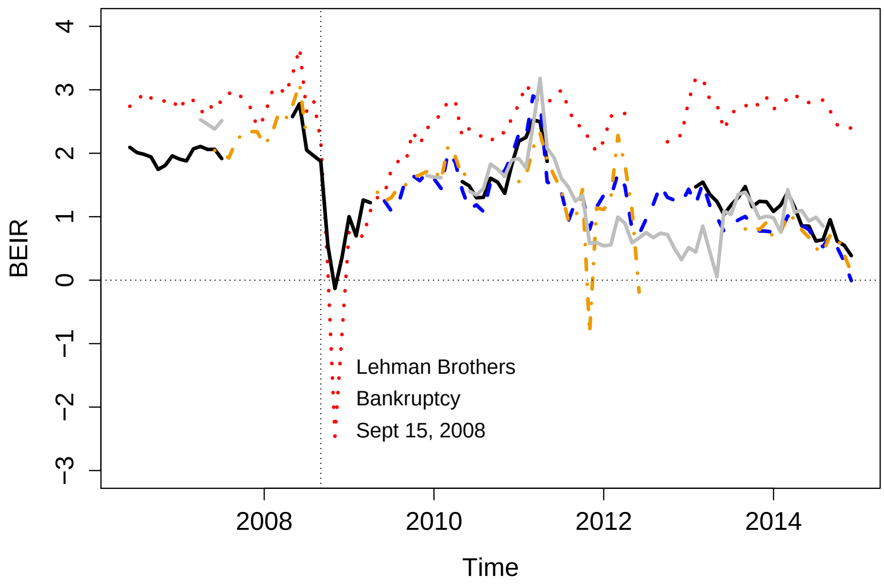

Figure 1: BEIR for five industrialized European countries - U.K.(red dotted line), Germany(blue dashed line), France(black line), Italy(orange dot-dashed line) and Sweden(grey line).

MTS_BEIR

of inflation expectation(IE) and the inflation risk premium (IRP).

The starting point of our research is to analyse the break-even inflation rate (BEIR), which is known to be the difference between the yield on a nominal fixed-rate bond and the real yield on an inflation-linked bond of similar maturity and credit quality. The BEIR can generally indicate how the inflation expectations are priced into the market. However they are not a perfect measure for IEs, as they may also encompass inflation risk premium, liquidity premium and "technical" market factors. We show the BEIR for five European countries - U.K., Germany, France, Italy and Sweden in Figure 1, which exhibits some degree of co-movement. This facilitates the following study in a multiple country framework. It is known that the euro-zone annual inflation rate was recorded at -0.2 percent in December of 2014 which matches, but are slightly higher than the overall BEIR shown in Figure 1. A fall in consumer prices first only appears since September 2009 due to a drop in energy costs. This motivates us to extract a joint time-varying structure of IEs estimated from individual (country-specific) BEIR.

The modelling of BEIR requires a model for the joint dynamics of the nominal and the real yields. For instance, Härdle and Majer (2014) investigated the yield curves using a Dynamic Semiparametric Factor Model (DSFM). To adopt a real time approach to help access the term structure of nominal and inflation-linked yields, in this study we consider a three-factor term structure model originally from Nelson and Siegel (1987). The attractiveness of factor models of the Nelson-Siegel type is due to its convenient linear functions and good empirical performance. Diebold and Li (2006) extend the original Nelson-Siegel model to a dynamic environment. Theoretically, the Nelson-Siegel (NS) model does not ensure the absence of arbitrage opportunities, as shown by Bjork and Christensen (1999). Christensen et al. (2011) further develop the NS model to an AFNS model by imposing the arbitrage-free hypothesis, which reflects most of the real activities of financial markets. With arbitrage-free pricing, financial institutions apply arbitrage conditions to prices that are observable in financial markets in order to determine other prices that are not. The standard approaches for pricing forwards, swaps are all derived from such arbitrage arguments for both complete and incomplete markets. In our paper, we will use an AFNS model for the dynamics of the nominal and the real yield respectively, and combine the two models later on.

Based on the joint dynamics of the nominal and the real yields, a sizable amount of literature has analysed how to isolate IE and IRP from BEIR. Earlier work mainly focuses on U.K. data because the U.K. was one of the first developed economies to issue inflation-indexed bonds for institutional investors. Since the 1981 launch of the original U.K. index-linked gilts, various developments have occured in the international markets. Barr and Campbell (1997) estimated market expectations of real interest rates and inflation from observed prices of U.K. government nominal and inflation-linked bonds. Joyce, Lildholdt and Sorensen (2010) developed an affine term structure model to decompose forward rates to obtain IRP. Notably, Christensen, Lopez and Rudebusch (2010) used an affine arbitrage-free model of the term structure to decompose BEIR that captures the pricing of both nominal and inflation-indexed securities. A four-factor joint AFNS model was achieved by combining the AFNS models for nominal and inflation-linked yields, which proved efficient for fitting and forecasting analysis. Unlike Christensen et al. (2010), we align the four factor models over different maturities to make the factors consistent over maturities.

With the AFNS model for the joint dynamics on hand, we proceed with our European country analysis. Most of the existing literature mentions little about the story of multiple countries. Diebold, Li and Yue (2008) are the first to consider a global multiple country model for nominal yield curves. There are a few European central bank reports, which focus on household and expert inflation expectation and the anchoring of inflation expectations in the two currency areas before and during the 2008 crisis.

Here we would like to look into five industrialised European countries by constructing a joint model of country-specific IEs. We construct an AFNS model in multi-maturity term structure for modelling nominal and inflation-indexed bonds simultaneously, we also propose a joint model of IE dynamics over European countries, which discovers the extracted common trend for IE is an important driver for each country of interest. Then we conduct an analysis to explore the estimated common factor by decomposing the variation into parts driven by common effect variation and macroeconomic effect variation.

The rest of the paper proceeds as follows. Section 2 estimates the joint AFNS model in multi-maturity term structure for estimating yields on nominal and inflation-linked bonds and also covers the decomposition method of BEIR. In section 3, we discuss the econometric methodology used in the joint modelling of IE dynamics. The technical details are in the Appendix. The empirical results are shown in Section 4. Finally section 5 concludes.

2 Preliminary Analysis

In this section, we introduce the methodology to obtain the model-implied BEIR. Sub-section 2.1 briefly introduces the Nelson-Siegel model, and sub-section 2.2 constructs the joint AFNS structure for modelling nominal and inflation-indexed bonds. Sub-section 2.3

introduces the joint AFNS model across countries in a multi-maturity term structure. In the last sub-section 2.4, we describe the decomposition method of BEIR.

2.1 A factor model representation

The classic Nelson-Siegel (NS) yield curve model for fitting to static yield curves with simple functional form,

$$y(\tau) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \tag{1}$$

where $y(\tau)$ is a zero-coupon yield with τ months to maturity, and $\beta_0, \beta_1, \beta_2$ and λ are parameters. This model is popular because it is simple and tractable. For a fixed value of parameter λ the remaining three βs can be estimated by the OLS method. Maturity τ determines the decay speed of parameters.

The aforementioned dynamic version of Nelson-Siegel (DNS) model enables institutional investors and policy makers to understand the evolution of the bond market over time, the DNS model can be written as,

$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \tag{2}$$

where $y_t(\tau)$ denotes the continuously zero-coupon yield of maturity τ at time t. the time-varying parameters are defined as level L_t, slope S_t and curvature C_t. Such choice of the latent factors is motivated by principal component analysis, which gives us three principal components corresponding to the latent factors. For instance, the most variation of yields is accounted for by the first principal component - level factor L_t.

By incorporating the theoretical restriction of arbitrage-free, the AFNS model bridges the best of the Nelson-Siegel model and the AF model. Thus, the AFNS model consists of two equations by taking the structure of the DNS model and the real-world dynamics (under P-measure) equation derived from the AF model respectively,

$$
\begin{aligned}
y_t(\tau) &= X_t^1 + X_t^2\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + X_t^3\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) - \frac{A(\tau)}{\tau} \\
dX_t &= K^P(\theta^P - X_t)dt + \Sigma dW_t^P
\end{aligned}
\tag{3}
$$

where $X_t^\top = (X_t^1, X_t^2, X_t^3)$ is a vector of latent factors, $\dfrac{A(\tau)}{\tau}$ is an unavoidable yield-adjustment term and only depends on maturity. K^P and θ^P correspond to drifts and dynamics terms, and both are allowed to vary freely. Σ is identified as a diagonal volatility matrix.

2.2 A joint factor model

The AFNS structure is a useful representation for term structure research. Christensen et al. (2010) employed and conducted a separate AFNS model estimation of nominal and inflation-linked Treasury bonds respectively. Here we construct an extended AFNS structure for modelling nominal and inflation-indexed bonds simultaneously without exploring the estimated correlation of separate AFNS models.

The separate AFNS model of nominal and inflation-indexed type for a specific country i can be written as,

$$
\begin{aligned}
y_{it}^N(\tau) &= L_{it}^N + S_{it}^N\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_{it}^N\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) - \frac{A_i^N(\tau)}{\tau} \\
y_{it}^R(\tau) &= L_{it}^R + S_{it}^R\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_{it}^R\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) - \frac{A_i^R(\tau)}{\tau}
\end{aligned}
$$

To explore the relationship between nominal and inflation-indexed bond yields within a country, we need combine two types and model them jointly.

To work with a simplified version of the yield curve, we assume the correlation between the latent factors of nominal and inflation-indexed bonds as follows,

$$
\begin{aligned}
S_{it}^{R} &= \alpha_i^S S_{it}^{N} \\
C_{it}^{R} &= \alpha_i^C C_{it}^{N}
\end{aligned}
\tag{4}
$$

The assumption will be justified by the performance of the joint model illustrated in sub-section 4.2. The yield curve of the joint AFNS model is:

$$
\begin{aligned}
\begin{pmatrix} y_{it}^{N}(\tau) \\ y_{it}^{R}(\tau) \end{pmatrix} &= \begin{pmatrix} 1 & \dfrac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} & \dfrac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} - e^{-\lambda_i\tau} & 0 \\ 0 & \alpha_i^S \dfrac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} & \alpha_i^C (\dfrac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} - e^{-\lambda_i\tau}) & 1 \end{pmatrix} \begin{pmatrix} L_{it}^{N} \\ S_{it}^{N} \\ C_{it}^{N} \\ L_{it}^{R} \end{pmatrix} \\
&+ \begin{pmatrix} \varepsilon_{it}^{N}(\tau) \\ \varepsilon_{it}^{R}(\tau) \end{pmatrix} - \begin{pmatrix} \dfrac{A_i^N(\tau)}{\tau} \\ \dfrac{A_i^R(\tau)}{\tau} \end{pmatrix}
\end{aligned}
\tag{5}
$$

where y_{it}^N and y_{it}^R represent the nominal and inflation-linked yields for country i at time t. The real-world dynamics (under P-measure) equation takes the form of,

$$
dX_t = K^P(\theta^P - X_t)dt + \Sigma dW_t^P
$$

where the state variable $X_{it}^{\top} = \left(L_{it}^N, S_{it}^N, C_{it}^N, L_{it}^R\right)$ evolves dynamically.

2.3 Multiple Yield Curve Modelling

Diebold et al. (2008) extend the DNS model to a global version by modelling a potentially large set of country yield curves in a framework that allows for both global and country-specific factors. The model proposed here employs the joint AFNS model introduced in sub-section 2.2 and we further extend it to a multiple-maturity case.

For a specific country i, we first assume the state variable $X_{it}^{\top}$ introduced in sub-section 2.2 is a common state variable for the yield curves across different maturities. The multiple yield curve model may very well lead to efficient estimation. As in the following analysis examined, the small size of model residual represented in sub-section 4.4 accounts for the overall good fit of the model. More specifically, the joint AFNS yield curve in multi-maturity term structure is,

$$\begin{pmatrix} y_{it}^{N}(\tau_1) \\ y_{it}^{R}(\tau_1) \\ y_{it}^{N}(\tau_2) \\ y_{it}^{R}(\tau_2) \\ \vdots \\ y_{it}^{N}(\tau_n) \\ y_{it}^{R}(\tau_n) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda_i\tau_1}}{\lambda_i\tau_1} & \frac{1-e^{-\lambda_i\tau_1}}{\lambda_i\tau_1} - e^{-\lambda_i\tau_1} & 0 \\ 0 & \alpha_i^S\frac{1-e^{-\lambda_i\tau_1}}{\lambda_i\tau_1} & \alpha_i^C(\frac{1-e^{-\lambda_i\tau_1}}{\lambda_i\tau_1} - e^{-\lambda_i\tau_1}) & 1 \\ 1 & \frac{1-e^{-\lambda_i\tau_2}}{\lambda_i\tau_2} & \frac{1-e^{-\lambda_i\tau_2}}{\lambda_i\tau_2} - e^{-\lambda_i\tau_2} & 0 \\ 0 & \alpha_i^S\frac{1-e^{-\lambda_i\tau_2}}{\lambda_i\tau_2} & \alpha_i^C(\frac{1-e^{-\lambda_i\tau_2}}{\lambda_i\tau_2} - e^{-\lambda_i\tau_2}) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda_i\tau_n}}{\lambda_i\tau_n} & \frac{1-e^{-\lambda_i\tau_n}}{\lambda_i\tau_n} - e^{-\lambda_i\tau_n} & 0 \\ 0 & \alpha_i^S\frac{1-e^{-\lambda_i\tau_n}}{\lambda_i\tau_n} & \alpha_i^C(\frac{1-e^{-\lambda_i\tau_n}}{\lambda_i\tau_n} - e^{-\lambda_i\tau_n}) & 1 \end{pmatrix} \begin{pmatrix} L_{it}^{N} \\ S_{it}^{N} \\ C_{it}^{N} \\ L_{it}^{R} \end{pmatrix}$$

$$+ \begin{pmatrix} \varepsilon_{it}^{N}(\tau_1) \\ \varepsilon_{it}^{R}(\tau_1) \\ \varepsilon_{it}^{N}(\tau_2) \\ \varepsilon_{it}^{R}(\tau_2) \\ \vdots \\ \varepsilon_{it}^{N}(\tau_n) \\ \varepsilon_{it}^{R}(\tau_n) \end{pmatrix} - \begin{pmatrix} \frac{A_i^N(\tau_1)}{\tau_1} \\ \frac{A_i^R(\tau_1)}{\tau_1} \\ \frac{A_i^N(\tau_2)}{\tau_2} \\ \frac{A_i^R(\tau_2)}{\tau_2} \\ \vdots \\ \frac{A_i^N(\tau_n)}{\tau_n} \\ \frac{A_i^R(\tau_n)}{\tau_n} \end{pmatrix} \tag{6}$$

where $y_{it}^{N}(\tau_n)$ and $y_{it}^{R}(\tau_n)$ represent the nominal and inflation-linked yields for country i at time t with maturity τ_n. The real-world dynamics equation is in the same form as before,

$$dX_t^P = K^P(\theta^P - X_t)dt + \Sigma dW_t^P$$

where state variables $X_{it}^{\top} = \left(L_{it}^N, S_{it}^N, C_{it}^N, L_{it}^R\right)$ evolves dynamically.

The methodology used to obtain the estimates of yield curves is the Kalman filtering technique. The technical details are in Appendix A.

2.4 BEIR decomposition

In order to find a more appropriate measure of expected inflation, it is necessary to understand the components of the bond yields intuitively and economically, for both nominal and inflation-linked types. A sizable amount of literature has adopted a parameterized approach for modelling the term structure of interest rates to estimate the IE and risk premia using data from both nominal and indexed bonds. Adrian and Wu (2009), Campbell and Viceira (2009), Pflueger and Viceira (2011) decomposed the yield of an inflation-linked bond into current expectation of a future real interest rate and a real interest rate premium. The yield on a nominal bond can be decomposed into parts of the yield on a real bond, expectations of future inflation and IRP. Therefore the spread between both yields, the BEIR, reflects the level of IE and IRP.

In the environment of an arbitrage-free model, there are no opportunities for investors to make risk-free profits, the bonds can be priced by basic pricing equations according to Cochrane (2005),

$$P_t = \mathsf{E}_t \left\{ \beta \frac{u^{'}(c_{t+1})}{u^{'}(c_t)} x_{t+1} \right\} \tag{7}$$

where the price is denoted by P_t, the value c_t at time t has a payoff x_{t+1}, β is the discount factor. We break up the basic consumption-based pricing equation and get the stochastic discount factor (SDF) M_{t+1} at time $t+1$,

$$M_{t+1} = \beta \frac{u^{'}(c_{t+1})}{u^{'}(c_t)} \tag{8}$$

Then the prices of the zero-coupon bonds that pay one unit measured by the consumption basket at time t with maturity τ are formed as follows,

$$\begin{aligned} P_t^N(\tau) &= \mathsf{E}_t \left(M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N \right) \\ P_t^R(\tau) &= \mathsf{E}_t \left(M_{t+1}^R M_{t+2}^R \cdots M_{t+\tau}^R \right) \end{aligned} \tag{9}$$

where the nominal and the real (for inflation-linked bond) SDFs at time t are denoted by M_t^N and M_t^R. P_t^N and P_t^R represent the prices of nominal and real bonds respectively.

The price of the consumption basket, which is known as the overall price level Q_t has the following link with SDFs given the assumption of no arbitrage,

$$\frac{M_t^N}{M_t^R} = \frac{Q_{t-1}}{Q_t} \tag{10}$$

Converting the price into the yield by the equation of $y_t(\tau) = -\frac{1}{\tau}\log P_t(\tau)$,

$$\begin{aligned}
y_t^N(\tau) &= -\frac{1}{\tau}\log \mathsf{E}_t\left(M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right)\\
&= -\frac{1}{\tau}\mathsf{E}_t\left(\log M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right)\\
y_t^R(\tau) &= -\frac{1}{\tau}\log \mathsf{E}_t\left(M_{t+1}^R M_{t+2}^R \cdots M_{t+\tau}^R\right)\\
&= -\frac{1}{\tau}\mathsf{E}_t\left(\log M_{t+1}^R M_{t+2}^R \cdots M_{t+\tau}^R\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log M_{t+1}^R M_{t+2}^R \cdots M_{t+\tau}^R\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_t^N(\tau) - y_t^R(\tau) &= -\frac{1}{\tau}\mathsf{E}_t\left(\log \frac{M_{t+1}^N}{M_{t+1}^R}\frac{M_{t+2}^N}{M_{t+2}^R}\cdots\frac{M_{t+\tau}^N}{M_{t+\tau}^R}\right) + \frac{1}{2\tau}\mathsf{Var}_t\left(\log \frac{M_{t+1}^N}{M_{t+1}^R}\frac{M_{t+2}^N}{M_{t+2}^R}\cdots\frac{M_{t+\tau}^N}{M_{t+\tau}^R}\right)\\
&- \frac{1}{\tau}\mathsf{Cov}_t\left(\log \frac{M_{t+1}^N}{M_{t+1}^R}\frac{M_{t+2}^N}{M_{t+2}^R}\cdots\frac{M_{t+\tau}^N}{M_{t+\tau}^R}, \log M_{t+1}^R M_{t+2}^R \cdots M_{t+\tau}^R\right)
\end{aligned}$$

Given the log inflation is $\pi_{t+1} = \log\dfrac{Q_{t+1}}{Q_t}$ and the relationship between SDFs according to equation (10), the BEIR can be defined as,

$$\begin{aligned}
y_t^N(\tau) - y_t^R(\tau) &= \frac{1}{\tau}\mathsf{E}_t\left(\log \pi_{t+1}\pi_{t+2}\cdots\pi_{t+\tau}\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log \pi_{t+1}\pi_{t+2}\cdots\pi_{t+\tau}\right)\\
&+ \frac{1}{\tau}\mathsf{Cov}_t\left(\log \pi_{t+1}\pi_{t+2}\cdots\pi_{t+\tau}, \log M_{t+1}^R M_{t+2}^R \cdots M_{t+\tau}^R\right)
\end{aligned} \tag{11}$$

that is,

$$BEIR_t(\tau) = y_t^N(\tau) - y_t^R(\tau) = \pi_t(\tau) + \eta_t(\tau) + \phi_t(\tau) \tag{12}$$

where $\pi_t(\tau)$ is the IE, $\eta_t(\tau)$ is the corresponding convexity effect and $\phi_t(\tau)$ is IRP. To link the $BEIR_t(\tau)$ with the estimated state variable mentioned in sub-section 2.3, we assume that the P-dynamics equations of the SDFs are,

$$\begin{aligned}\frac{dM_t^N}{M_t^N} &= -(r_t^N - r_{t-1}^N)dt - (\Gamma_t^N - \Gamma_{t-1}^N)dW_t^P \\ \frac{dM_t^R}{M_t^R} &= -(r_t^R - r_{t-1}^R)dt - (\Gamma_t^R - \Gamma_{t-1}^R)dW_t^P\end{aligned} \tag{13}$$

where r_t is the stochastic risk-free rate and Γ_t represents the corresponding risk premium; their dynamics can be connected to the underling state variable $X_t^\top$ in equation (6), details are given in Appendix B. Hence the dynamics of the overall price level is,

$$\begin{aligned}&d\log\left(\frac{Q_{t-1}}{Q_t}\right) = -(r_t^N - r_t^R)dt + (r_{t-1}^N - r_{t-1}^R)dt \\ &d\log(Q_t) = (r_t^N - r_t^R)dt\end{aligned} \tag{14}$$

The IE is given by,

$$\pi_t(\tau) = -\frac{1}{\tau}\log \mathsf{E}_t^P\left[\exp\left\{-\int_t^{t+\tau}(r_s^N - r_s^R)ds\right\}\right] \tag{15}$$

which can be solved by a system of ODEs with a Runge-Kutta method, see Appendix B. The convexity effect can be written as,

$$\eta_t(\tau) = -\frac{1}{\tau}\mathsf{E}_t^P\left[\log\exp\left\{-\int_t^{t+\tau}(r_s^N - r_s^R)ds\right\}\right] \tag{16}$$

Then the IRP can be easily calculated out by equation (12).

3 Econometric Modelling of Inflation Expectation

Diebold et al. (2008) extended the dynamic Nelson-Siegel (DNS) model proposed by Diebold and Li (2006) to a global version by modelling a potentially large set of country yield curves in a framework that allows for both global and country-specific factors. As far as we have obtained the country-specific estimates of IE, we can tell a story of multiple countries in this section. We aim to investigate the country-specific idiosyncratic factors to load on a common time-varying factor and country-specific factors. The dynamics of an extracted common trend is also evaluated.

The model without a macroeconomic factor is structured as follows, the idiosyncratic factors $\hat{\pi}^e_{it}$ for each country i at time t, to load on a common time-varying latent factor Π_t,

$$\hat{\pi}^e_{it} = m_i + n_i \Pi_t + \mu_{it} \tag{17}$$

The dynamics of common factor,

$$\Pi_t = p + q\Pi_{t-1} + \nu_t \tag{18}$$

where m, n, p and q are unknown parameters. The errors μ_{it} and ν_{it} are assumed to be i.i.d white noise.

Since there is a dynamic interaction between macro-economy and the yield curve as evidenced by Diebold, Rudebusch and Aruoba (2006), and in Figure 1 we can observe that the decrease of the BEIR appears around 2012 due to the European sovereign debt crisis. A straightforward extension of the joint modelling equation (16) is adding a proxy of the macroeconomic factor - default risk factor. The model with a macroeconomic factor is,

$$\hat{\pi}^e_{it} = m_i + n_i \Pi_t + l_i d_{it} + \mu_{it} \tag{19}$$

where d_{it} is the default risk factor varying over time and m, n, p, l and q are unknown parameters. The noises μ_{it} and ν_{it} are assumed to be i.i.d. The dynamics of the common factor is the same form as in (18).

4 Empirical Results

We now turn to the analysis of the results obtained using the model proposed. Subsection 4.1 describes the data, we then discuss the fitting performance of the multiple yield curve modelling, setting up a brief discussion on the estimation results of the preliminary analysis in sub-section 4.2. Sub-section 4.3 establishes the estimated IE for each country, and sub-section 4.4 discusses the common trend extracted from the joint modelling of country-specific IEs. The results of the cross-sectional forecast is shown in sub-section 4.5.

4.1 Data

Monthly nominal and inflation-linked yield data of zero-coupon government bonds used for model estimation are taken from Bloomberg and Datastream. The research databases are supported by the Research Data Center (RDC) from the Collaborative Research Center 649, Humboldt Universität zu Berlin. The time series for a specific country are estimated using data from the same source. We consider five samples from the industrialized European countries - United Kingdom (U.K.), France, Germany, Italy and Sweden, all member states of European Union (EU). It should be noted that, even though two of the selected five European countries are outside the euro-zone - U.K. and Sweden, they have their own currencies therefore independent central banks and monetary policy, the inflation co-movement can be observed across the selected countries in sub-section 4.4. This also motivates our analysis to extract a joint time-varying structure of country-specific IEs.

The lack of short-maturity inflation-linked bonds of the sample countries indicates that inflation-linked yield at short-maturity tends to be less reliable and accessible, so the shortest maturities we could get access for each country are limited. We therefore selected three maturities for each country to ensure that enough observations are available. The sample period involves the subprime crisis in 2008 and is sightly different for each country due to the integrity of the data. The surfaces of the yield data are plotted in Figure 2. The blank areas in the Figure are for missing values and not zeros. Summary statistics are depicted in Table 1.

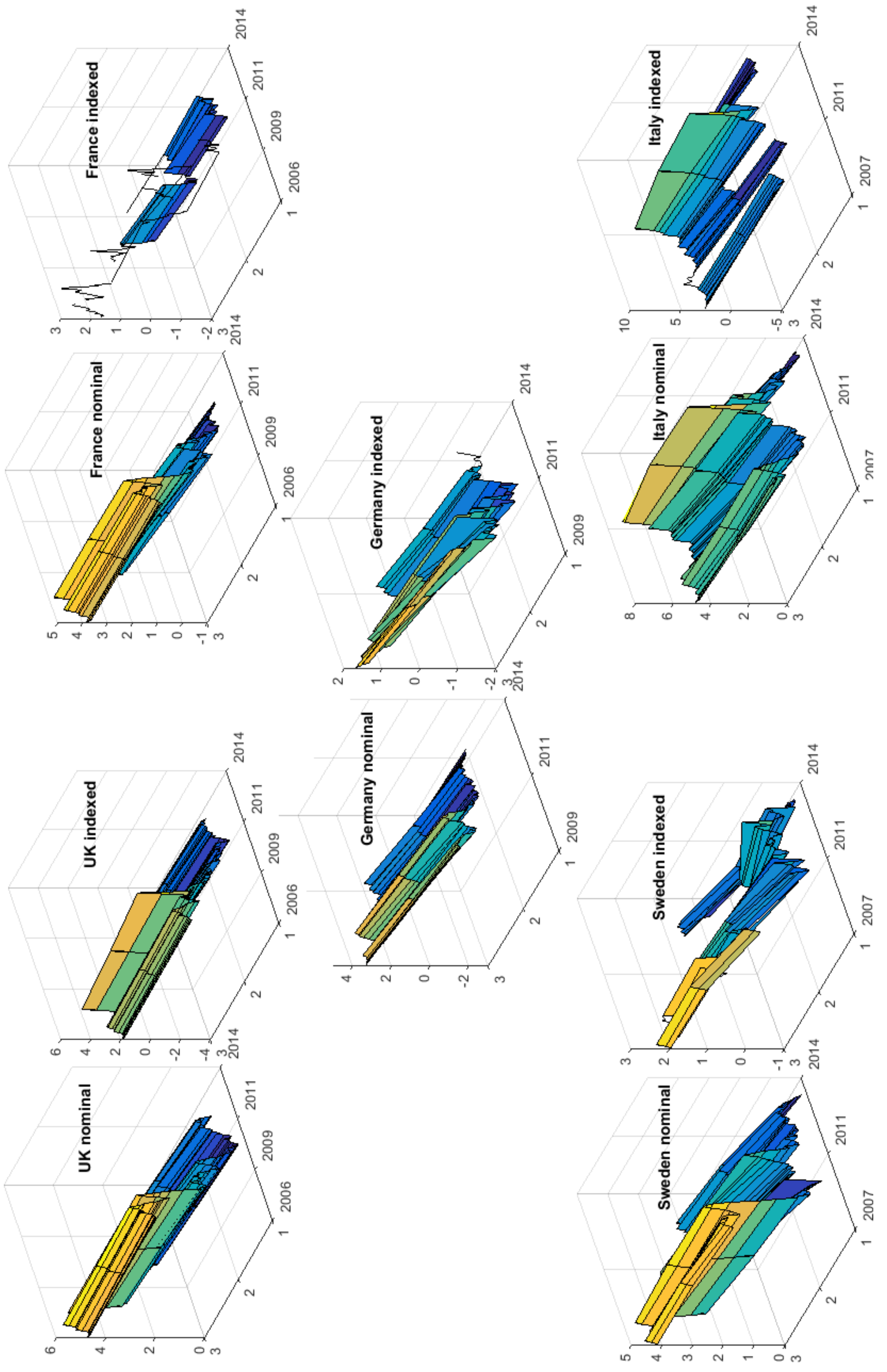

Figure 2: Term structures of nominal and inflation-linked bond yields across five European countries.

	Period	Maturity	Type	Min	Mean	Max	SD
UK	30.06.2006	3	nominal	0.16	2.20	5.69	1.73
			inflation-indexed	-2.87	-0.14	5.35	1.81
	—	4	nominal	0.35	2.44	5.62	1.59
			inflation-indexed	-2.62	-0.04	4.74	1.64
	31.12.2014	5	nominal	0.57	2.66	5.56	1.48
			inflation-indexed	-2.37	0.11	4.27	1.49
Sweden	30.04.2007	3	nominal	0.18	1.80	4.67	1.27
			inflation-indexed	-0.71	0.15	1.83	0.54
	—	5	nominal	0.58	2.33	4.71	1.11
			inflation-indexed	-0.84	0.51	2.33	0.79
	29.08.2014	10	nominal	0.88	2.59	4.61	1.03
			inflation-indexed	-0.30	0.98	2 29	0.64
France	30.06.2006	3	nominal	-0.02	1.86	4.74	1.46
			inflation-indexed	-1.19	0.43	2.75	1.29
	—	5	nominal	0.06	2.10	4.80	1.37
			inflation-indexed	-1.29	-0.40	1.06	0.60
	31.12.2014	10	nominal	0.18	2.34	4.80	1.28
			inflation-indexed	-1.09	1.03	2.66	1.03
Germany	30.06.2009	5	nominal	-0.07	0.90	2.38	0.77
			inflation-indexed	-1.39	-0.35	1.00	0.54
	—	7	nominal	0.05	1.37	2.85	0.86
			inflation-indexed	-1.16	0.05	1.36	0.65
	31.12.2014	10	nominal	0.39	1.94	3.29	0.84
			inflation-indexed	-0.53	0.33	1.67	0.67
Italy	30.06.2007	3	nominal	0.55	2.94	7.37	1.33
			inflation-indexed	-0.34	1.51	8.21	1.46
	—	5	nominal	0.95	3.53	7.54	1.20
			inflation-indexed	0.20	2.00	7.84	1.29
	31.12.2014	10	nominal	1.89	4.45	7.11	0.93
			inflation-indexed	1.02	2.77	6.72	1.06

Table 1: Descriptive statistics of the monthly bond yields data. SD is standard deviation.

4.2 Estimates of Yield Curve Modelling

We assess the performance of the previously discussed multiple yield curve model by conducting Kalman filtering, whose recursion is a set of equations allowing for an estimator to be updated once a new observation y_t is available; more technical details can be found in Appendix A. The objective is to see if the yields of nominal and inflation-linked bonds are suited to the model proposed in sub-section 2.3. Figure 3 shows the model residuals over different maturities for all five European countries.

While the model residuals have jumps for short periods, it is not entirely surprising as the jumps can be identified as the occurrence of extreme events. The outliers observed in the sub-figure of Italy happened to be the financial default crisis of Italy in 2012. We can also observe a jump around September 2008 for the U.K. and Sweden due to the well-known subprime crisis. Basically the overall small size of the model residuals indicates the good fit of the joint multiple yield curve model. The summary statistics of the model fit is represented in Table 2. Again, the value of the mean and RMSE of the model residuals are smaller due to the outliers in Figure 3.

The country-specific state variables are plotted in Figure 4, with four underlying latent factors $L_{it}^N, S_{it}^N, C_{it}^N, L_{it}^R$ presented. We observe that the level factors L_{it}^N, L_{it}^R are significantly positive, which in turn verifies the previously discussed choice of latent factors in sub-section 2.1. That is, the choice of the latent factors is motivated by principal component analysis, which gives us three principal components corresponding to the latent factors L_{it}, S_{it}, C_{it}. For instance, the most variation of yields is accounted for by the first principal component from principal component analysis - level factor L_t.

4.3 IE

We started by fitting the multiple yield curve modelling, where the model residuals are used to indicate the efficiency of the four-factor AFNS model over different maturities. We then conducted the decomposition of BEIR, as already described in sub-section 2.4, into parts of IE, the convexity effect and IRP to facilitate the following analysis. Figure 5

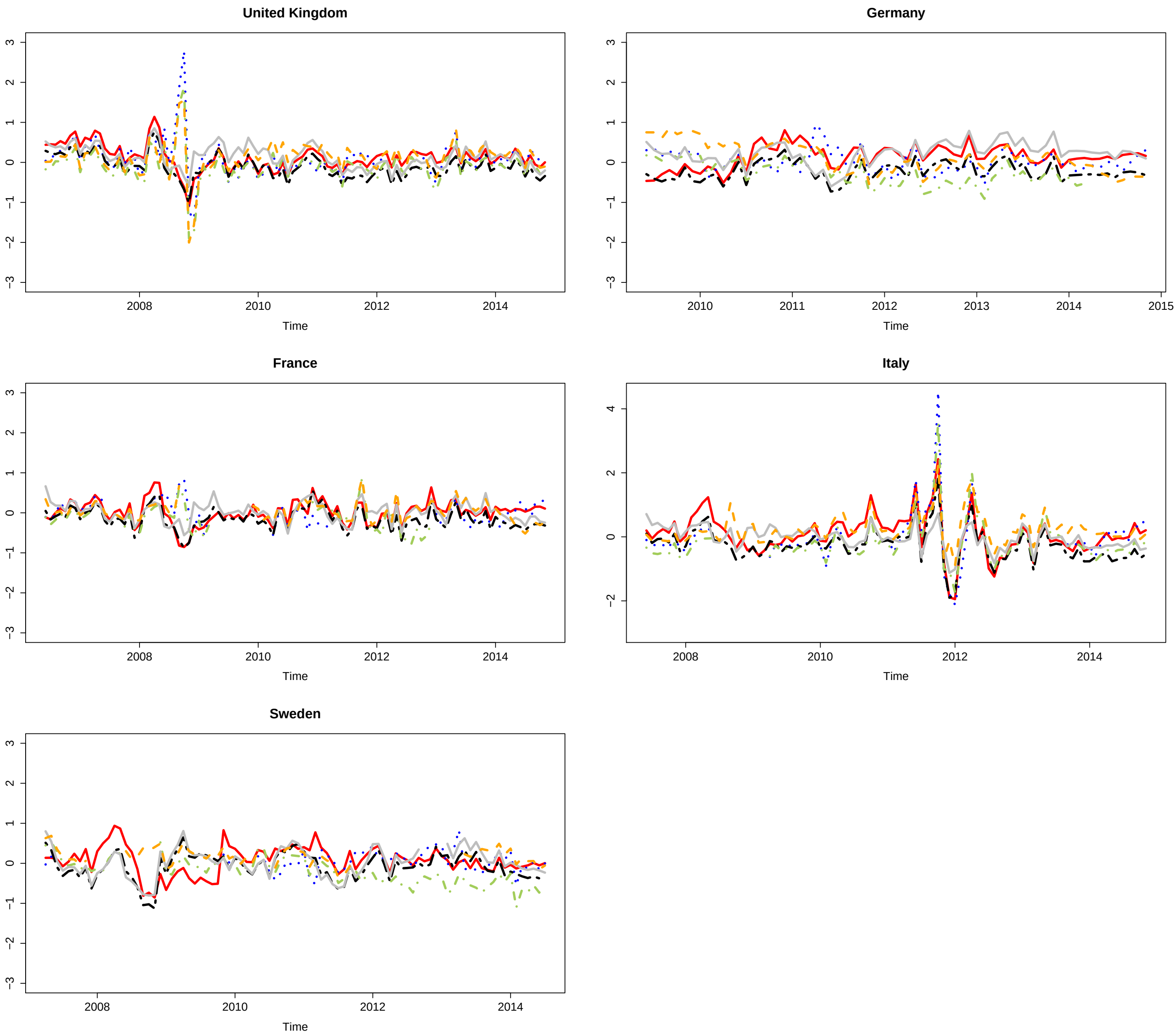

Figure 3: The model residuals of multiple yield curve modelling over different maturities ($\tau_1 < \tau_2 < \tau_3$). The nominal type with τ_1 is the red line and the real type is the blue dotted line. The nominal and real types with τ_2 are the black long-dashed and green dot-dashed lines. For maturity of τ_3, the nominal type is grey and real type is an orange dashed line.

MTS_multi_modelres

	Maturity	Type	Mean	RMSE
UK	3	nominal	0.12	0.10
		inflation-indexed	0.10	0.22
	4	nominal	-0.10	0.08
		inflation-indexed	-0.09	0.18
	5	nominal	0.13	0.08
		inflation-indexed	0.12	0.17
Sweden	3	nominal	0.06	0.12
		inflation-indexed	0.06	0.04
	5	nominal	-0.07	0.11
		inflation-indexed	-0.20	0.51
	10	nominal	0.02	0.12
		inflation-indexed	0.18	0.03
France	3	nominal	0.01	0.08
		inflation-indexed	0.01	0.05
	5	nominal	-0.15	0.07
		inflation-indexed	-0.12	-0.06
	10	nominal	0.02	0.06
		inflation-indexed	0.04	0.05
Germany	5	nominal	0.14	0.08
		inflation-indexed	0.02	0.08
	7	nominal	-0.22	0.05
		inflation-indexed	-0.25	0.09
	10	nominal	0.23	0.08
		inflation-indexed	0.12	0.14
Italy	3	nominal	0.07	0.40
		inflation-indexed	0.01	0.50
	5	nominal	-0.25	0.27
		inflation-indexed	-0.18	0.40
	10	nominal	-0.02	0.13
		inflation-indexed	0.24	0.22

Table 2: Summary statistics of the model fit using the multiple yield curve model. RMSE is a root mean square error.

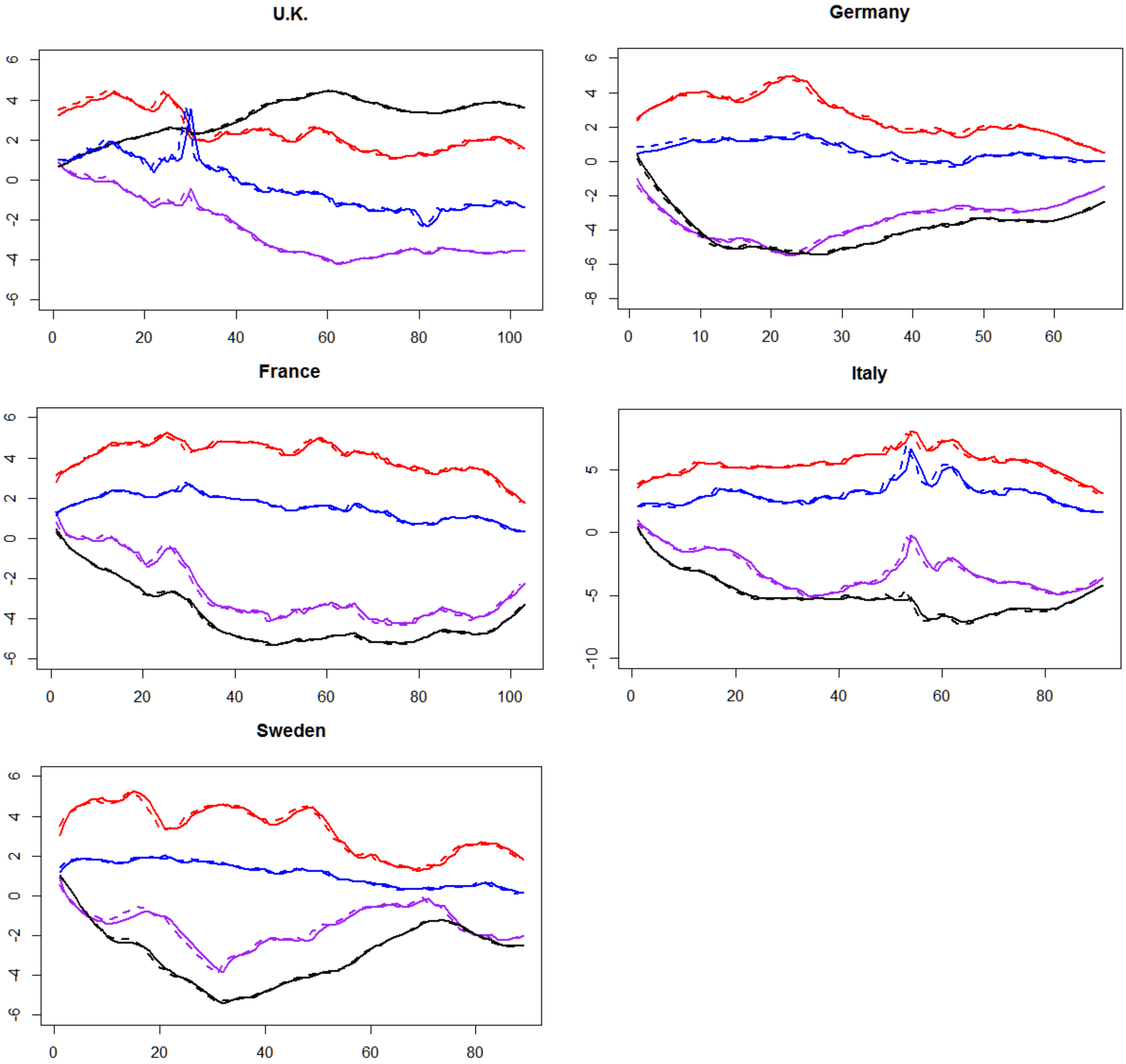

Figure 4: The estimated four latent factors of state variable $X_t = (L_{it}^N, S_{it}^N, C_{it}^N, L_{it}^R)$ for each European country - the nominal level factor L_{it}^N (red), the real level factor L_{it}^R (blue), the nominal slope factor S_{it}^N (purple) and the nominal curvature factor C_{it}^N (black). The preidicted state variables are presented as line type and the filtered state variables are dashed.

MTS_afns_uk, MTS_afns_de, MTS_afns_fr, MTS_afns_it, MTS_afns_sw compares the estimated three-year and five-year forecasts of IE for each European country.

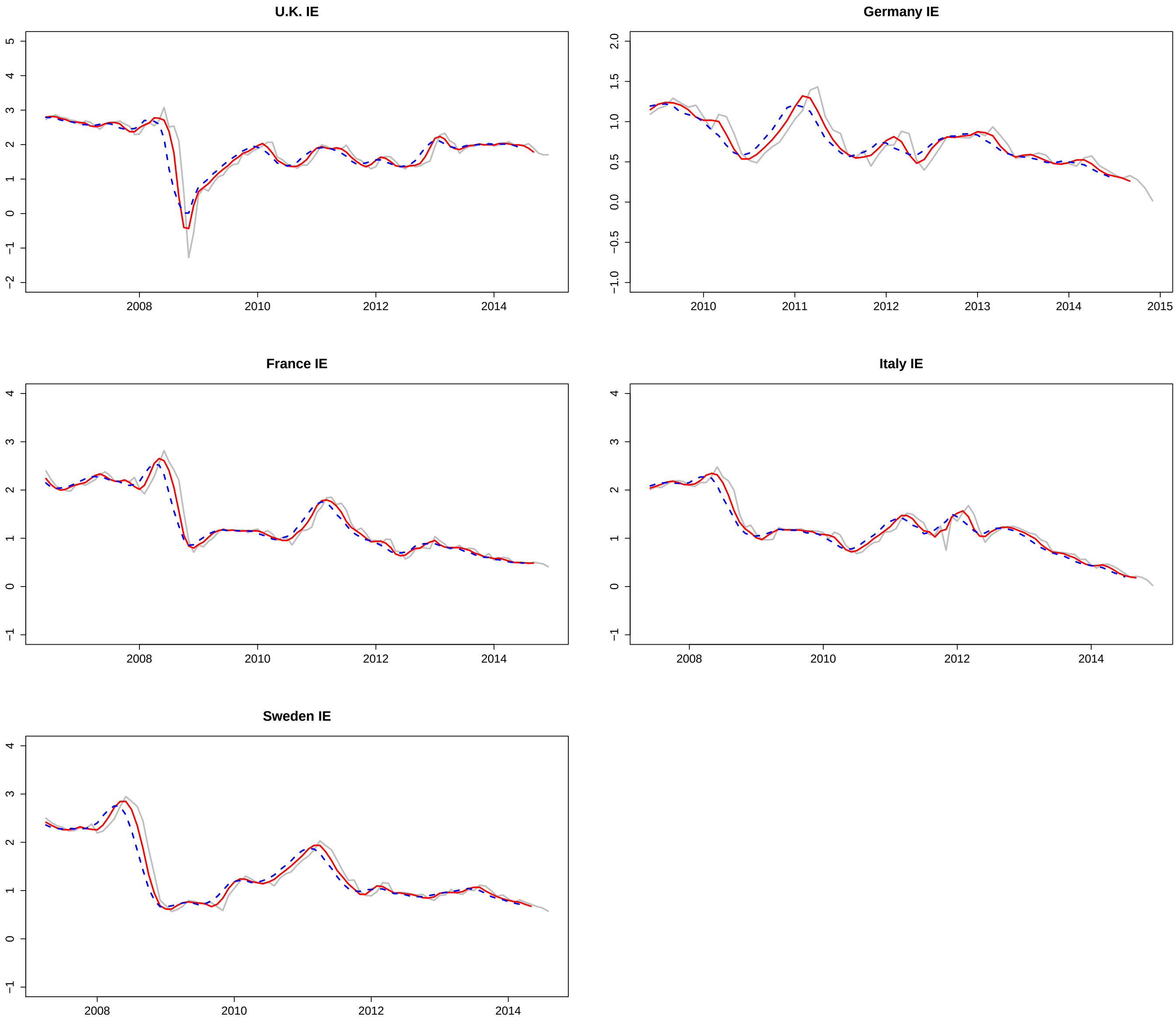

Figure 5: The model-implied IE for each European country. The 3-year IE is the red line and the 5-year IE is dashed blue.

MTS_expinf

Figure 5 observes a decrease of the expected inflation for the U.K., which is also seemingly present in the other countries. To illustrates the similar trend among the five European countries, we present the country-specific three-year IE in Figure 6 to facilitate the following study of the similarity and difference among these five countries. Because the model-implied inflation expectation is on the three-year basis, the difference existing between the realized inflation level is understandable. We still find that the estimated IEs using the multiple AFNS model show similar trends as the realized levels. For instance, the realized inflation level of Sweden has two fluctuations in magnitude around the second half of the years 2008 and 2011, which is consistent with our finding.

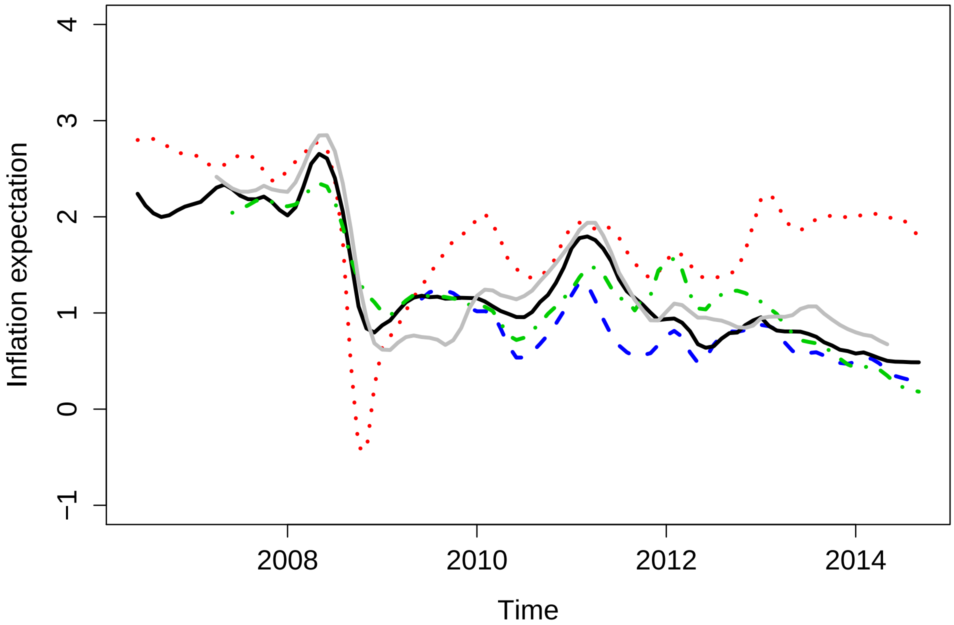

Figure 6: Model-implied inflation expectation for different countries - U.K.(red dotted line), Germany(blue dashed line), France(black line), Italy(green dot-dashed line) and Sweden(grey line)

MTS_expinf

4.4 Common Inflation Factor

Based on the methodology proposed in section 3, we can built the relationship among idiosyncratic countries to find out the similarities and differences for the model-implied IEs. The common inflation factor is extracted from the joint time-varying structure of IE dynamics and depicted in Figure 7. To be more specific, the estimated parameters for the joint modelling of IE dynamics are presented in the Table 3.

We conduct a variance decomposition to split the variation in model-implied IE into parts driven by the estimated common factor and the corresponding idiosyncratic factor. The

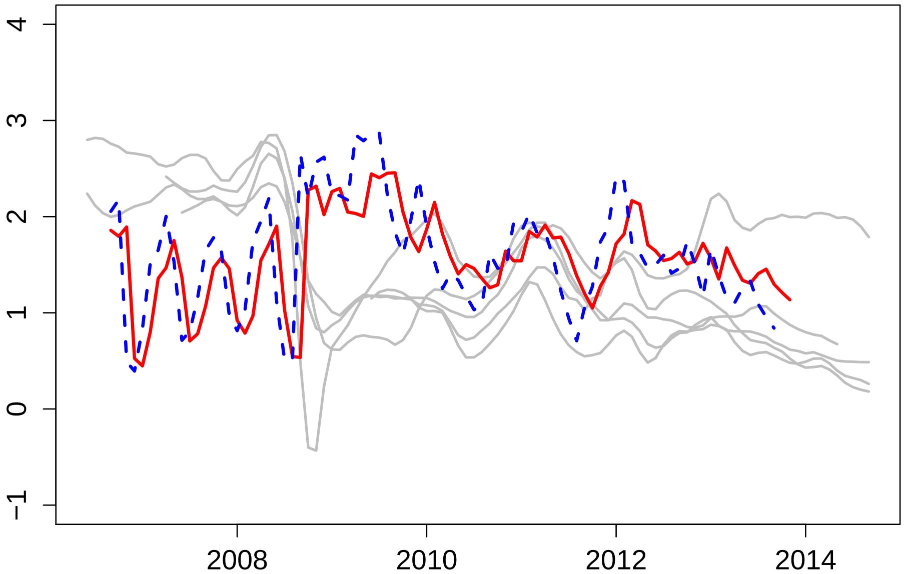

Figure 7: Common inflation factor in the red. The grey lines are the country-specific IEs. The predicted Π_t is the red line and the filtered Π_t is the blue dashed line.

MTS_comexpinf

Country-specific equations		
UK	$\pi^e_{1t}(\tau) =$	$0.166 + 0.576\Pi_t$
France	$\pi^e_{2t}(\tau) =$	$-0.022 + 0.665\Pi_t$
Italy	$\pi^e_{3t}(\tau) =$	$-0.347 + 0.822\Pi_t$
Sweden	$\pi^e_{4t}(\tau) =$	$-0.057 + 0.665\Pi_t$
Germany	$\pi^e_{5t}(\tau) =$	$0.008 + 0.644\Pi_t$
Common Effect equation		
	$\Pi_t =$	$0.588 + 0.651\Pi_{t-1}$

Table 3: Estimates for the dynamics of IE.

	U.K.	France	Italy	Sweden	Germany
Common effect	24.91	30.66	40.32	30.65	29.32
Country-specific effect	69.34	50.69	69.35	58.50	70.68

Table 4: Variations explained in percentage

variance equation is listed,

$$\mathsf{Var}(\pi_{it}^{e}) = \beta_i^2 \mathsf{Var}\left(\Pi_t\right) + \mathsf{Var}\left(\mu_{it}\right) \tag{20}$$

The variations explained by the common inflation factor are shown in the Table 4. The common inflation factor explains the least variation in the U.K., which can be explained by the U.K. being outside the euro-zone and having its own currency, therefore being independent of European central bank and monetary policy. Even though Sweden is also outside the euro-zone, we find out it has closer relationships with other European countries compared with the U.K.. The international interaction among countries can also be observed through the estimation results.

To illustrate the efficiency of joint modelling of country-specific IEs, the model residuals are reported in Figure 8. The small size of the model residuals represents the overall good fit of the joint modelling of IE dynamics. However the model residuals are relatively high, around 2012, due to the European sovereign debt crisis including Italy's default. To eradicate this, we try to incorporate one more macroeconomic factor- default risk proxy to improve the model performance. By applying the method using the equations (18) and (19) proposed in section 3, we assess the joint model of IE dynamics by checking the model residuals in Figure 10 and the estimation results listed in Table 6.

The data we implement is the three-year CDS of Italy, the extracted common inflation factor derived from the joint model of IE dynamics with default proxy is presented in Figure 9, and successfully captures the decrease of IE caused by the subprime crisis. The estimated parameters for the joint modelling of inflation dynamics with macroeconomic factors - default proxy are presented in Table 5.

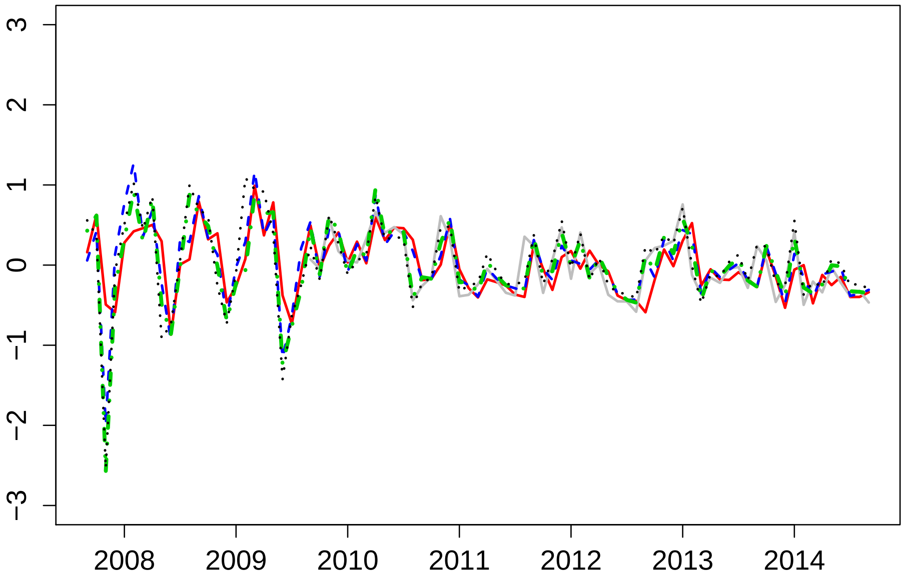

Figure 8: Model residual for modelling of inflation expectation dynamics over different countries - U.K.(red line), Germany(grey line), France(blue dashed line), Italy(black dotted) and Sweden(green dot-dashed).

MTS_comexpinf

Country-specific equations		
UK	$\pi^e_{1t}(\tau) =$	$-0.358 d_t + 0.798 \Pi_t$
France	$\pi^e_{2t}(\tau) =$	$0.085 d_t + 0.714 \Pi_t$
Italy	$\pi^e_{3t}(\tau) =$	$1.078 d_t + 0.531 \Pi_t$
Sweden	$\pi^e_{4t}(\tau) =$	$-0.621 d_t + 0.805 \Pi_t$
Germany	$\pi^e_{5t}(\tau) =$	$0.045 d_t + 0.700 \Pi_t$
Common Effect equation		
	$\Pi_t =$	$0.382 + 0.976 \Pi_{t-1}$

Table 5: Estimates for the dynamics of IE.

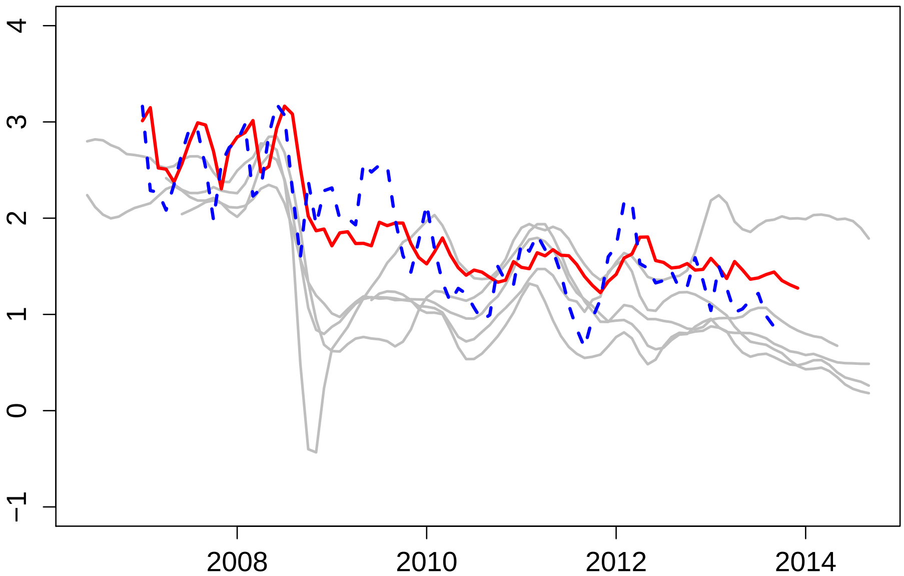

Figure 9: Common inflation factor with default proxy. The predicted estimation of common factor Π_t is the red line and the filtered Π_t is the blue dashed line.

MTS_comexpinf_cds

	U.K.	France	Italy	Sweden	Germany
Common effect	36.08	33.59	11.54	31.87	32.84
Country-specific effect	56.66	65.88	40.92	49.17	67.02
Default risk effect	7.26	0.53	47.55	18.96	0.14

Table 6: Variations explained in percentage

We also conduct a variance decomposition to split the variation in model-implied IE into parts driven by the estimated common factor Π_t and the default proxy d_t. The variations explained by the common inflation factor and default factor are reported in Table 6. The default factor explains the most variation in Italy and least variation in Germany, which can be explained by the stability of the economy because Germany is generally considered to be the benchmark in the European financial system. It is generally known that the European sovereign debt crisis including Italy's default happened around 2012, the 47.55% variation explained by the default factor for Italy is understandable.

The model residuals are presented in Figure 10 to illustrate the efficiency of the joint model of IE dynamics with a default factor. Even though the model residuals with a default factor remain unchanged compared with Figure 8, we discover that the extracted common trend for IE is an important driver for each country of interest. Table 6 reports that the variation explained by the common inflation factor accounts for more than 30% of almost all the sample countries.

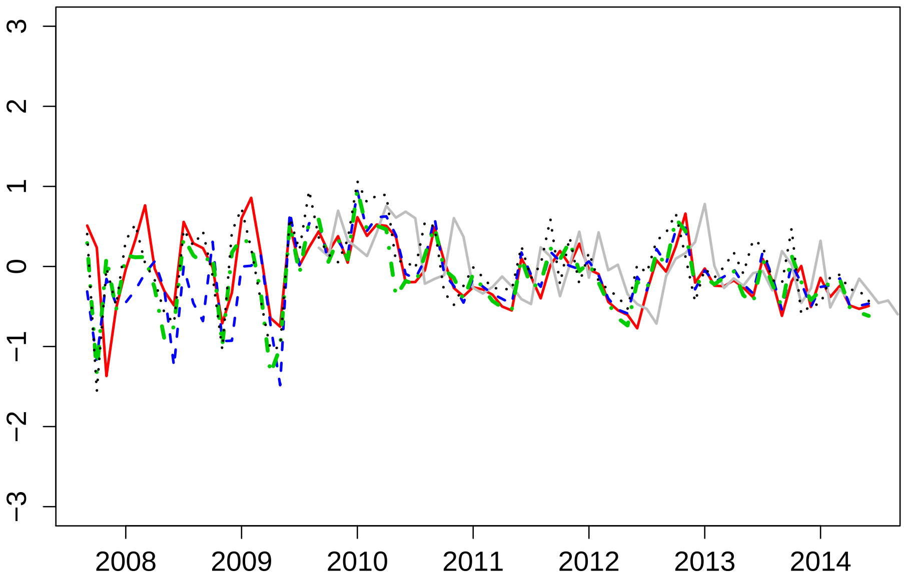

Figure 10: Model residual for modelling of inflation expectation dynamics with a default proxy factor over different countries - U.K.(red line), Germany(grey line), France(blue dashed line), Italy(black dotted) and Sweden(green dot-dashed).

MTS_comexpinf_cds

4.5 Forecast

Having obtained the estimation from the joint model of IE dynamics, we continue in this section by forecasting the common inflation factor. Figure 11 displays a forecast (in blue) containing 30 observations, that is, a two and a half year prediction. The predicted linear model we use involves trend and seasonality components. The confidence intervals are presented graphically and are shown at confidence levels of 80% and 95%.

Figure 12 clearly shows the difference among different measures of inflation. The real-time approach to measure IE proposed previously performs better than the other measures.

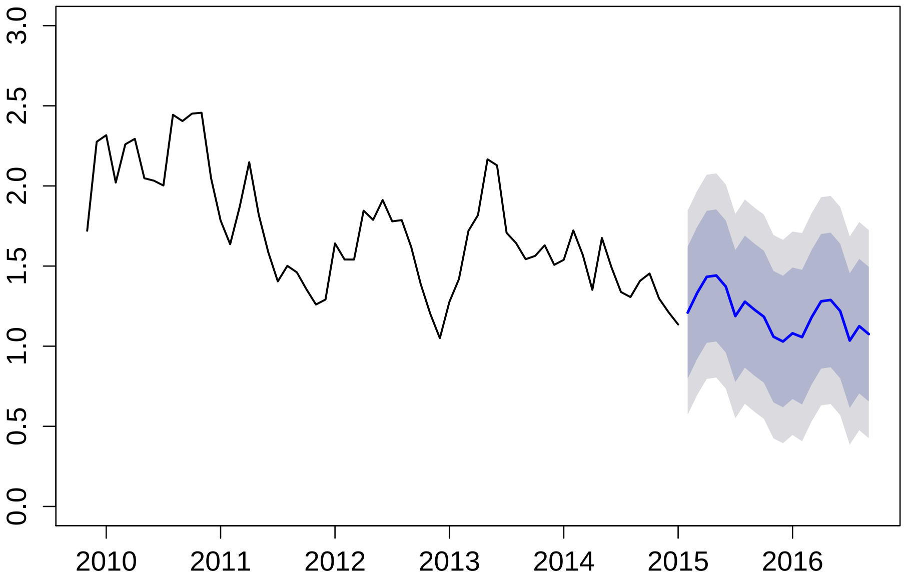

Figure 11: The forecast of common inflation factor derived from the joint model of IE dynamics with default factor. The 80% and 95% confidence intervals are marked in the shaded area.

A similar co-movement is seemingly present between the realized inflation level and the three-year IE estimated derived from our model. The 1 year and 2 year SPF (Survey Professional Forecast) data plotted in Figure 12 vary slightly over time therefore contains limited information of financial markets.

5 Conclusion

This study attempts to provide an additional measure of IE on the basis of financial markets. We firstly construct an AFNS model in multi-maturity term structure of modelling nominal and inflation-indexed bonds simultaneously. The performance of this multiple yield curve modelling was assessed by conducting Kalman filtering, whose recursion was a set of equations allowing for an estimator to be updated once a new observation y_t is available. We then conducted the decomposition of model-implied BEIR into parts of IE, convexity effect and IRP to facilitate the modelling of joint structure of IE dynamics. The

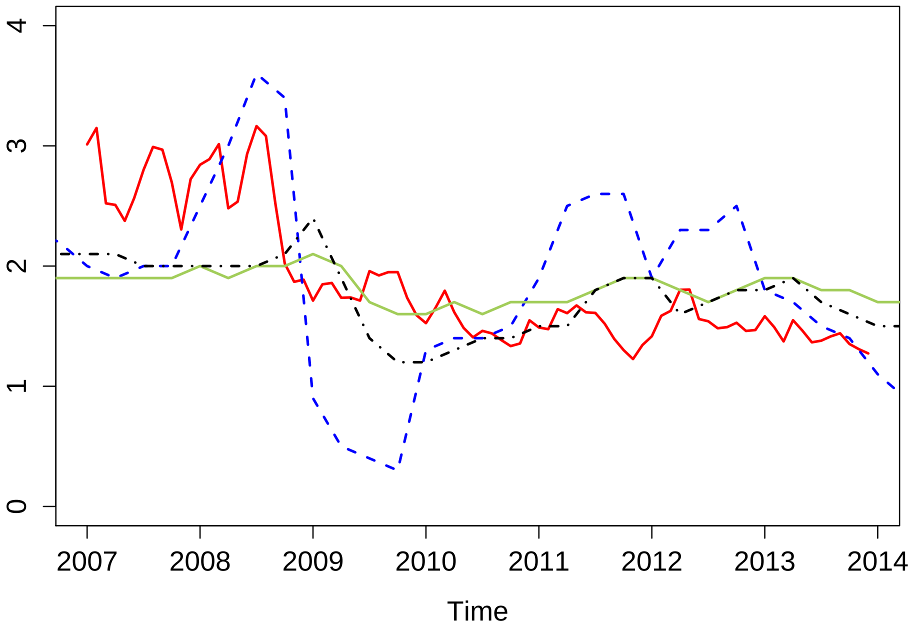

Figure 12: The comparison of different measures of inflation - the model-implied common inflation level (in red line), the observed inflation level (blue dashed line), the 1 year SPF forecast level of inflation (black dot-dashed) and the 2 year SPF forecast (in green).

joint models of IE dynamics with, and without, macroeconomic factors indicated the extracted common inflation factor and was an important driver for each country of interest. Moreover, the model should lead to a better forecast in benchmark levels of inflation and give good implications for financial markets.

6 Appendix

A Estimation of multiple yield curve modelling

The analysis starts by introducing the yield-adjustment term proposed in the original AFNS model. Derived in an analytical form, the yield-adjustment term $\dfrac{A(\tau)}{\tau}$ with τ

months to maturity can be written as,

$$
\begin{aligned}
\frac{A(\tau)}{\tau} &= \bar{A}\frac{\tau^2}{6} + \bar{B}\left\{\frac{1}{2\lambda^2} - \frac{1}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau} + \frac{1}{4\lambda^3}\frac{1-\exp(-2\lambda\tau)}{\tau}\right\} \\
&+ \bar{C}\left\{\frac{1}{2\lambda^2} + \frac{1}{\lambda^2}\exp(-\lambda\tau) - \frac{1}{4\lambda}\tau\exp(-2\lambda\tau) - \frac{3}{4\lambda^2}\exp(-2\lambda\tau)\right\} \\
&+ \bar{C}\left\{-\frac{2}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau} + \frac{5}{8\lambda^3}\frac{1-\exp(-2\lambda\tau)}{\tau}\right\} \\
&+ \bar{D}\left\{\frac{1}{2\lambda}\tau + \frac{1}{\lambda^2}\exp(-\lambda\tau) - \frac{1}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau}\right\} \\
&+ \bar{E}\left\{\frac{3}{\lambda^2}\exp(-\lambda\tau) + \frac{1}{2\lambda}\tau + \frac{1}{\lambda}\exp(-\lambda\tau) - \frac{3}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau}\right\} \\
&+ \bar{F}\left\{\frac{1}{\lambda^2} + \frac{1}{\lambda^2}\exp(-\lambda\tau) - \frac{1}{2\lambda^2}\exp(-2\lambda\tau) - \frac{3}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau} + \frac{3}{4\lambda^3}\frac{1-\exp(-2\lambda\tau)}{\tau}\right\}
\end{aligned} \tag{A.1}
$$

where the six terms $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$ and $\bar{F}$ can be identified by the volatility matrix Σ defined in the dynamics equation under P-measure. The value of the adjustment term is constant in time t, but depends on time to maturity τ, coefficient λ that governs the mean reversion rate of slope and curvature factors, and the volatility parameters $\bar{A}$, $\bar{D}$ and $\bar{F}$.

The four latent factors defined in the state variable $X_{it}^{\top} = \left(L_{it}^N, S_{it}^N, C_{it}^N, L_{it}^R\right)$ evolve dynamically and hence we can identify their shocks accordingly,

$$
\begin{pmatrix} dL_{it}^N \\ dS_{it}^N \\ dC_{it}^N \\ dL_{it}^R \end{pmatrix} = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{34} \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & \kappa_{44} \end{pmatrix} \begin{pmatrix} L_{it}^N \\ S_{it}^N \\ C_{it}^N \\ L_{it}^R \end{pmatrix} dt + \Sigma \begin{pmatrix} dW_t^{L^N} \\ dW_t^{S^N} \\ dW_t^{C^N} \\ dW_t^{L^R} \end{pmatrix} \tag{A.2}
$$

where $W_t^{L^N}$, $W_t^{S^N}$, $W_t^{C^N}$ and $W_t^{L^R}$ are independent Brownian motions.

We estimate the parameters in (A.2) using the Kalman filter technique. The Kalman filter recursion is a set of equations which allow for an estimator to be updated once a new observation y_t becomes available. It first forms an optimal predictor of the unobserved state variable vector given its previously estimated value. This prediction is obtained using the distribution of unobserved state variables, conditional on the previous estimated values. These estimates for unobserved state variables are then updated using the information provided by the observed variables.

By rewriting the yield equation (6) of the joint AFNS model in multi-maturity term structure proposed in sub-section 2.3, we obtain the measurement equation as,

$$\begin{pmatrix} y_{it}^{N}(\tau_1) \\ y_{it}^{R}(\tau_1) \\ \vdots \\ y_{it}^{R}(\tau_n) \end{pmatrix} = AX_{it} + \begin{pmatrix} \varepsilon_{it}^{N}(\tau_1) \\ \varepsilon_{it}^{R}(\tau_1) \\ \vdots \\ \varepsilon_{it}^{R}(\tau_n) \end{pmatrix} - \begin{pmatrix} \frac{A_i^N(\tau_1)}{\tau_1} \\ \frac{A_i^R(\tau_1)}{\tau_1} \\ \vdots \\ \frac{A_i^R(\tau_n)}{\tau_n} \end{pmatrix} \tag{A.3}$$

The transition equation derived from Christensen et al. (2011) takes the form of,

$$X_{i,t} = \left[I - expm\left(-K^P \Delta t\right)\right] \theta^P + expm\left(-K^p \Delta t\right) X_{i,t-1} + \eta_t \tag{A.4}$$

where *expm* is a matrix exponential. The measurement and transition equations are assumed to have the error structure as,

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} = N\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right\}$$

where Q has a special structure,

$$Q = \int_0^{\Delta t} e^{-K^P s} \Sigma \Sigma^\top e^{-(K^P)^\top s} ds$$

For estimation, the transition and measurement errors are assumed orthogonal to the initial state. The initial value of the filter is given by the unconditional mean and variance of the state variable $X_t^\top$ under the P measure,

$$\begin{aligned} X_0 &= \theta^P \\ \Sigma_0 &= \int_0^{\infty} e^{-K^P s} \Sigma \Sigma^\top e^{-(K^P)^\top s} ds \end{aligned}$$

which can be calculated using the analytical solution provided in Fisher and Gilles (1996).

B BEIR Decomposition

In the environment of an AF model, there are no opportunities to make risk-free profits. Based on the pricing equation from Cochrane (2005), the bond can be priced by the equation,

$$P_t = \mathsf{E}_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right\} \tag{B.1}$$

where the value of the bond c_t has a payoff x_{t+1}, β is the discount factor. We break up the basic consumption-based pricing equation (B.1) and get the stochastic discount factor (SDF) M_{t+1} at time $t+1$,

$$M_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \tag{B.2}$$

To estimate the expected value of inflation using the stochastic discount factor (SDF) M_t. Firstly we use the Taylor series to approximate the moments of the logarithm. Assuming that M_t, in a sense, significant from 0, so the yield for a nominal bond can be extended as follows,

$$\log\left(M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) = \log\left\{\left(\mu_M + M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N - \mu_M\right)\right\} \tag{B.3}$$

where

$$\mu_M = \mathsf{E}_t\left(M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) \tag{B.4}$$

Expand equation (B.3) using Taylor series and take the expectation on both sides,

$$\mathsf{E}_t\left(\log M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) = \log \mu_M - \mathsf{Var}_t\left(\log M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) \tag{B.5}$$

Therefore,

$$\begin{aligned} y_t^N(\tau) &= -\frac{1}{\tau} \log \mathsf{E}_t\left(M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) \\ &= -\frac{1}{\tau}\mathsf{E}_t\left(\log M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log M_{t+1}^N M_{t+2}^N \cdots M_{t+\tau}^N\right) \end{aligned} \tag{B.6}$$

Similar solution could be obtained for the inflation-indexed bonds by the same logic.

To facilitate the calculation of equation (12), the instantaneous risk-free rate r_t and the risk premium Γ_t are given, more details can be found in Christensen et al. (2011) and Christensen et al. (2010),

$$\begin{aligned} r_t &= \rho_0(t) + \rho_1(t) X_t && \text{(B.7)} \\ \Gamma_t &= \gamma_0 + \gamma_1 Y_t && \text{(B.8)} \end{aligned}$$

where $\rho_0(t)$, $\rho_1(t)$, γ_0 and γ_1 are bounded, continuous functions. X_t is the state variable and Y_t is the realized observations.

The estimation of the inflation expectation can be calculated by

$$\pi_t(\tau) = -\frac{1}{\tau} \log \mathsf{E}_t^P \left[\exp\left\{ -\int_t^{t+\tau} (r_s^N - r_s^R) ds \right\} \right] \tag{B.9}$$

which are the solutions to a system of ordinary differential equations using the fourth-order Runge Kutta method.

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