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Exact analytical solutions for time-dependent Hermitian Hamiltonian systems from static unobservable non-Hermitian Hamiltonians

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We propose a procedure to obtain exact analytical solutions to the time-dependent Schrödinger equations involving explicit time-dependent Hermitian Hamitonians from solutions to time-independent non-Hermitian Hamiltonian systems and the time-dependent Dyson relation together with the time-dependent quasi-Hermiticity relation. We illustrate the working of this method for a simple Hermitian Rabi-type model by relating it to a non-Hermitian time-independent system corresponding to the one-site lattice Yang-Lee model.

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I. INTRODUCTION

The time-dependent Schrödinger equations (TDSE) is central in the description of almost all quantum mechanical phenomena. When it involves explicitly time-depedent Hamiltonians it is usually extremely difficult to solve, so that only few exact analytical solutions have been found so far. In most cases one resorts to approximative approaches. For instance, for systems with quantum potentials and classical electromagnetic fields one relies almost entirely on perturbative methods either in the weak or strong field regime [1, 2] with only a few available approximative methods that go beyond [3]. Schemes that allow to construct exact analytical solutions, such as for instance the method of invariants proposed by Lewis and Riesenfeld [4], are extremely rare and only very few exactly solvable models are known. Thus almost any workable alternative procedure will constitute an advance of the subject area.

Here we propose a new method that allows in principle to find exact analytical solutions, but it may also be adapted to a perturbative setting. Our approach exploits some special solutions of the time-dependent Dyson and time-dependent quasi-Hermiticity relations [5–7] in which we take the non-Hermitian Hamiltonian to be time-independent. The problem of solving the TDSE for a time-dependent Hermitian Hamiltonian $h(t) = h^\dagger(t)$ is replaced by the much easier one to solve the TDSE for a time-independent non-Hermitian Hamiltonian $H \neq H^\dagger$ and the time-dependent Dyson relation for the Dyson map or time-dependent quasi-Hermiticity relation for the metric operator.

Hence our starting point are the two TDSEs

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \qquad H\Psi(t) = i\hbar\partial_t\Psi(t), \tag{1}$$

for which the two wave functions $\phi(t)$ and $\Psi(t)$ are assumed to be related by a time-dependent invertible operator $\eta(t)$, the time-dependent Dyson map, as

$$\phi(t) = \eta(t)\Psi(t). \tag{2}$$

It then follows by direct substitution of (2) into (1) that the two Hamiltonians are related to each other by the time-dependent Dyson relation

$$h(t) = \eta(t)H\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t). \tag{3}$$

Thus computing $\phi(t)$ from the first equation in (2) becomes equivalent to computing $\Psi(t)$ from the second equation in (1) and $\eta(t)$ from (3). Alternatively we may also compute $\eta(t)$ by solving the quasi-Hermiticity relation

$$H^\dagger\rho(t) - \rho(t)H = i\hbar\partial_t\rho(t). \tag{4}$$

for the metric operator $\rho(t)$ and subsequently use $\rho(t) := \eta^\dagger(t)\eta(t)$. So despite the fact that the Hamiltonian H is static we will associate it to a time-dependent metric. We also note, as previously argued in [7], that because of the presence of the gauge-like term in (3) the non-Hermitian Hamiltonian H is not quasi-Hermitian and therefore not observable. Instead the operator

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H + i\hbar\eta^{-1}(t)\partial_t\eta(t), \tag{5}$$

is quasi-Hermitian and interpreted as the physical operator that plays the role of the energy in the non-Hermitian system. It does, however, not satisfy the relevant TDSE. One may of course define for $\tilde{H}(t)$ a new time-dependent Schrödinger equation $\tilde{H}(t)\tilde{\Psi}(t) = i\hbar\partial_t\tilde{\Psi}(t)$, but that would be a new system with different Hilbert space and therefore with different physical content. Having solved (2), we can subsequently construct an exact form for the unitary time-evolution operator

$$u(t,t') = T\exp\left[-i\int_{t'}^{t} ds\, h(s)\right], \tag{6}$$

that evolves a state $\phi(t) = u(t,t')\phi(t')$ from a time t' to t satisfying $h(t)u(t,t') = i\hbar\partial_t u(t,t')$, $u(t,t')u(t',t'') = u(t,t'')$, $u(t,t) = \mathbb{I}$ and preserves by definition the inner product $\left\langle u(t,t')\phi(t') \middle| u(t,t')\tilde{\phi}(t')\right\rangle = \left\langle \phi(t) \middle| \tilde{\phi}(t)\right\rangle$. The time-evolution operator $U(t,t') = \eta^{-1}(t)u(t,t')\eta(t')$ that evolves states $\Psi(t) = U(t,t')\Psi(t')$ in the non-Hermitian system from a time t' to t is not expected to be unitary

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in a standard matrix representation, but it preserves the modified inner product

$$\left\langle U(t,t')\Psi(t') \left| U(t,t')\tilde{\Psi}(t') \right\rangle_\rho = \left\langle \Psi(t) \left| \tilde{\Psi}(t) \right\rangle_\rho \right. \right. \tag{7}$$
$$= \left\langle \Psi(t) \left| \rho(t)\tilde{\Psi}(t) \right\rangle \right. .$$

where $\langle \cdot | \cdot \rangle$ denoted the standard inner product. In that sense it guarantees unitary time-evolution.

A priori it is not clear whether the equations above actually admit nontrivial and meaningful solutions in the way described above. In fact, it was doubted that they make sense at all and were interpreted as a no-go theorem for the possibility to have consistent non-Hermitian systems with time-dependent metric [6]. This view was already challenged in [8] and in [7, 9, 10] it was demonstrated that nontrivial solutions exist. Besides mathematical arguments questioning the solvability of these equations the main physical objection was based on the fact that the Hamiltonian operator H, or $H(t)$, that governs the TDSE is no longer observable. In [7] it was argued that this is an unnecessary requirement. It is already well accepted that in the non-Hermitian setting many operators, such as for instance the standard position or momentum operator, become mere auxiliary operators that do not correspond to observable quantities. In the time-dependent setting one simply need to add $H(t)$ to that list and interpret $\tilde{H}(t)$ as the observable quantity.

Here we will elaborate on the special type of solutions for which the non-Hermitian Hamiltonian is kept independent of time and study a simple (2×2)-matrix Hamiltonian with a periodic time-dependent potential

$$h(t) = -\frac{1}{2}\left[\omega\mathbb{I} + \frac{2\phi^2}{2+\gamma^2\sin(t\phi)-\gamma^2}\sigma_z\right]. \tag{8}$$

where $\sigma_{x,y,z}$ denote the standard Pauli matrices and $\omega, \gamma, \phi \in \mathbb{R}$ are constants constrained as $\phi = \sqrt{1-\gamma^2}$, $\gamma \leq 1$. This Hamiltonian is similar in type to the Hermitian Rabi model solved in [11] using perturbation theory [12]. A perturbative treatment of the non-Hermitian $\mathcal{PT}$-symmetric version of this model was recently considered in [13]. Here we are providing a non-perturbative analytical solution following the procedure outlined above.

II. A GENERAL NON HERMITIAN SU(2)-HAMILTONIAN

The non-Hermitian counterpart to $h(t)$ in (8) falls into the general class of Hamiltonians built from generators of an SU(2)-Lie algebra represented here by standard Pauli matrices

$$H = \frac{1}{2}\left[\kappa_0 + i\lambda_0\right]\mathbb{I} + \frac{1}{2}\sum\nolimits_{j=x,y,z}\left[\kappa_j + i\lambda_j\right]\sigma_j, \tag{9}$$

with $\kappa_0, \lambda_0, \kappa_j, \lambda_j \in \mathbb{R}$. In what follows we will drop the explicit sum and use the standard sum convention over repeated indices. Trying to solve the time-dependent quasi-Hermiticity relation (4) we make the generic Ansatz

$$\rho(t) = \alpha(t)\mathbb{I} + \beta_j(t)\sigma_j, \qquad \alpha(t), \beta_j(t) \in \mathbb{R}, \tag{10}$$

for the metric operator. Substituting (10) into (4) and reading off the coefficients of the generators then leads to the constraining first order differental equations

$$\alpha_t = -\alpha\lambda_0 - \vec{\beta}\cdot\vec{\lambda}, \tag{11}$$
$$\vec{\beta}_t = \vec{\kappa}\times\vec{\beta} - \lambda_0\vec{\beta} - \alpha\vec{\lambda}, \tag{12}$$

for the as yet unknown functions $\alpha(t), \beta_j(t)$. Demanding the metric operator to be positive definite imposes the additional constraint

$$\det\rho = \alpha^2 - \vec{\beta}\cdot\vec{\beta} > 0. \tag{13}$$

Next we will solve the constraints (11)-(13).

A. Time-independent Hamiltonian and time-independent metric

At first we consider the simplest scenario that is obtained when we just reduce the equations to the standard time-independent scenario, see [14, 15] for reviews. In this case (11)-(12) simplify to

$$\vec{\beta}\cdot\vec{\lambda} = -\alpha\lambda_0, \quad \text{and} \quad \vec{\kappa}\times\vec{\beta} = \lambda_0\vec{\beta} + \alpha\vec{\lambda}, \tag{14}$$

which when taking $\lambda_0 = 0$ is easily solved by

$$\vec{\beta} = \frac{\alpha}{|\vec{\kappa}|^2}\vec{\lambda}\times\vec{\kappa} + \nu\vec{\kappa}, \qquad \nu \in \mathbb{R}. \tag{15}$$

This solution was also reported in [9] and only serves here as a benchmark when taking the limit to the time-independent case. Using the parameterization (10) then reproduces solutions previously obtained for the Hamiltonians falling into the class reported in (9) for the time-independent scenario, see for instance [16] for an example.

B. Time-independent Hamiltonian and time-dependent metric

Next we allow $\partial_t\rho$ to be non-vanishing so that H is no longer quasi-Hermitian because (4) has a nonvanishing right hand side. Guided by the solution in the previous section we keep $\lambda_0 = 0$ and substitute the Ansatz

$$\vec{\beta}(t) = \zeta_1(t)\vec{\kappa} + \zeta_2(t)\vec{\lambda} + \zeta_3(t)\vec{\kappa}\times\vec{\lambda} \tag{16}$$

into (10) and (4) and thus obtaining a set of simple first order coupled differential equations

$$\partial_t\zeta_1 = \zeta_3\vec{\kappa}\cdot\vec{\lambda}, \quad \partial_t\zeta_2 = -\alpha - \zeta_3|\vec{\kappa}|^2, \quad \partial_t\zeta_3 = \zeta_2, \tag{17}$$

as constraints. Assuming that $\vec{\kappa}\cdot\vec{\lambda}=0$ the general solutions to these equations are easily obtained as

$$\zeta_1(t) = c_4, \tag{18}$$

$$\zeta_2(t) = c_1 \sin(\phi t) + c_2 \cos(\phi t), \tag{19}$$

$$\zeta_3(t) = -\frac{c_1}{\phi}\cos(\phi t) + \frac{c_2}{\phi}\sin(\phi t) + c_3, \tag{20}$$

$$\alpha(t) = \left(\frac{c_1}{\phi}\left|\vec{\kappa}\right|^2 - c_1\phi\right)\cos(\phi t) \tag{21}$$

$$\quad + \left(c_2\phi - \frac{c_2}{\phi}\left|\vec{\kappa}\right|^2\right)\sin(\phi t) - c_3\left|\vec{\kappa}\right|^2 \tag{22}$$

with $\phi=\sqrt{|\vec{\kappa}|^2-|\vec{\lambda}|^2}$ and $c_1,\ldots,c_4\in\mathbb{R}$ being arbitrary constants.

Having obtained $\rho(t)$ we may easily compute $\eta(t)$, but in order to carry out the second step in our procedure, that is solving the TDSE, we need to be more specific. Let us therefore study a concrete model that falls into the general class of Hamiltonians treated in this section.

III. THE ONE-SITE LATTICE YANG-LEE MODEL

We consider an Ising quantum spin chain in the presence of a magnetic field in the z-direction and a longitudinal imaginary field in the x-direction [17] that has been identified as the discretised lattice version of the Yang-Lee model [18], described by the non-Hermitian Hamiltonian

$$H_N = -\frac{1}{2}\sum\nolimits_{j=1}^{N}(\sigma_j^z + \lambda\sigma_j^x\sigma_{j+1}^x + i\kappa\sigma_j^x), \tag{23}$$

with $\lambda,\kappa\in\mathbb{C}$. In [7] it was demonstrated that when taking $N=1$ the time-dependent quasi-Hermiticity relation admits nontrivial solutions. In this case the non-Hermitian Hamiltonian just reduces to a particular example of (9)

$$H_1 = -\frac{1}{2}\left[\lambda\mathbb{I} + \sigma_z + i\kappa\sigma_x\right]. \tag{24}$$

We will now solve the time-dependent Dyson relation (3) together with the time-dependent quasi-Hermiticity relation (4) and the TDSE for H_1 in more detail.

A. Time-independent Hamiltonian and time-independent metric

Specifying the quantities in section II A as $\lambda=\omega\equiv$ const and $\kappa=\gamma\equiv$ const we identify $\kappa_0=-\omega$, $\lambda_0=0$, $\vec{\kappa}=(0,0,-1)$, $\vec{\lambda}=(-\gamma,0,0)$, so that

$$\vec{\beta} = -\alpha\gamma\vec{e}_y - \nu\vec{e}_z, \quad \rho = \alpha\mathbb{I} - \alpha\gamma\sigma_y - \nu\sigma_z, \tag{25}$$

with $\det\rho=\alpha^2(1-\gamma^2)-\nu^2>0$ and $\vec{e}_i$ denoting unit vector in the direction $i=x,y,z$. As expected, the metric ceases to be positive definite when the eigenvalues

$$E_\pm = \frac{1}{2}\left(-\omega\pm\phi\right) \tag{26}$$

of H become complex conjugate, that is when $\gamma>1$. Taking now $\nu=0$ for simplicity, we compute the Dyson map η as the square root of the metric ρ as

$$\eta = \sqrt{\rho} = \frac{\sqrt{\alpha}}{2}\left[\left(\phi_- + \phi_+\right)\mathbb{I} + \left(\phi_- - \phi_+\right)\sigma_y\right]. \tag{27}$$

with $\phi_\pm=\sqrt{1\pm\gamma}$. Assuming η to be Hermitian, it is computed from ρ simply by taking the square root in the standard way by diagonalizing it first as $\rho=UDU^{-1}$ and subsequently computing $\sqrt{\rho}=UD^{1/2}U^{-1}$. We select here the plus sign for the square root without loss of generality, as a minus sign will cancel out in all relevant computations. Using this expression in (3) leads to the isospectral Hermitian counterpart

$$h = -\frac{1}{2}\left(\omega\mathbb{I} + \phi\sigma_z\right). \tag{28}$$

It is of course well known how to obtain these type of relations in the time-independent case, but the expressions obtained here serve as benchmarks for the time-dependent case.

B. Time-independent Hamiltonian and time-dependent metric

Switching on the time-dependence we solve first the TDSE. Since H_1 is time-independent this is easily achieved by expanding the solution in terms of the energy eigenstates $E_\pm$ in (26) as $\Psi_\pm(t)=c_\pm e^{-iE_\pm t}$ with some constants $c_\pm$. Substitution of this Ansatz into (1) then yields the solutions to (1) with a suitable normalization

$$\Psi_\pm(t) = \frac{\sqrt{\gamma}}{\sqrt{2\phi}\sqrt{1\pm\phi}}\begin{pmatrix}\gamma\\ i(1\pm\phi)\end{pmatrix}e^{-iE_\pm t}. \tag{29}$$

Next we need to solve (3) and (4), for $\eta(t)$ and $\rho(t)$, respectively. Keeping at first all integration constants generic we obtain from (13)

$$\det[\rho(t)] = c_3^2 - c_4^2 - \gamma^2(c_1^2+c_2^2+c_3^2) > 0. \tag{30}$$

Thus it is vital to maintain $c_3\neq 0$. Considering the solution in section II B, we first notice that $\partial_t\rho=0$ leads to $c_1=c_2=0$, so that we recover the time-independent scenario in this case. A nontrivial convenient choice is for instance $c_1=0$, $c_2=-\phi/\gamma$, $c_3=-1/\gamma$, $c_4=0$, leading to the time-dependent metric

$$\rho(t) = \left[\frac{1}{\gamma}+\gamma\sin(\phi t)\right]\mathbb{I} + \phi\cos(\phi t)\sigma_x - \left[1+\sin(\phi t)\right]\sigma_y. \tag{31}$$

Taking the square root, similarly as in the previous section, then yields the time-dependent Dyson map

$$\eta(t) = \frac{1}{2}\left[p_+(t) + p_-(t)\right]\mathbb{I} \qquad (32)$$
$$+\frac{p_+(t) - p_-(t)}{2\left|p_0(t)\right|}\left[Im\left[p_0(t)\right]\sigma_x - Re\left[p_0(t)\right]\sigma_y\right],$$

where we abbreviated the functions

$$p_\pm(t) = \sqrt{\gamma^{-1} + \gamma\sin(\phi t) \pm |p_0(t)|}, \qquad (33)$$
$$p_0(t) = 1 + \sin(\phi t) + i\phi\cos(\phi t). \qquad (34)$$

Using this expression for $\eta(t)$ in (3) produces the Hermitian time-dependent Hamiltonian $h(t)$ in (8).

We have now obtained explicit analytical solutions for all time-dependent wave functions. Next we verify that they yield meaningful expectation values. Using (2) together with our solutions (29), (31) and (32), we compute

$$\langle\Psi_\pm(t)\left|\rho(t)\Psi_\pm(t)\right\rangle = \langle\phi_\pm(t)\left|\phi_\pm(t)\right\rangle = 1, \qquad (35)$$
$$\langle\Psi_\mp(t)\left|\rho(t)\Psi_\pm(t)\right\rangle = \langle\phi_\mp(t)\left|\phi_\pm(t)\right\rangle = \pm i\gamma. \qquad (36)$$

These states were not expected to be orthonormal, but we can use them to easily find an orthonormal basis. A useful and natural basis is

$$\phi_1(t_0) = \begin{pmatrix}1\\0\end{pmatrix} = c_+\left|\phi_-(t_0)\right\rangle + c_-\left|\phi_+(t_0)\right\rangle \qquad (37)$$

$$\phi_2(t_0) = \begin{pmatrix}0\\1\end{pmatrix} = c_-\left|\phi_-(t_0)\right\rangle - c_+\left|\phi_+(t_0)\right\rangle \qquad (38)$$

with

$$t_0 = -\frac{\pi}{2\phi}, \qquad c_\pm = \frac{1}{\sqrt{2}\phi^2}e^{\frac{i\pi}{4}\left(\frac{\omega}{\phi}\pm 1\right)}\left(\phi_\pm - \gamma\phi_\mp\right). \qquad (39)$$

Using these states and t_0 as initial time, the time-evolution operator is easily extracted from the explicit form of $\phi(t)$ as

$$u(t,t_0) = \begin{pmatrix} e^{i\theta(t)} & 0 \\ 0 & e^{\frac{i\pi}{2}\left(\frac{\omega}{\phi}+\frac{2t\omega}{\pi}\right)-i\theta(t)}\end{pmatrix} \qquad (40)$$

with

$$\theta(t) = \frac{\pi}{4} + \frac{\omega}{2}(t-t_0) + \arctan\left[\frac{(1-\phi)^2 + \gamma^2\tan\left(\frac{t\phi}{2}\right)}{\gamma^2 + (1-\phi)^2\tan\left(\frac{t\phi}{2}\right)}\right]. \qquad (41)$$

One may verify that $u(t,t_0)$ indeed satisfies the TDSE. It is now also straightforward to compute the time-evolution operator for the non-Hermitian system $U(t,t_0) = \eta^{-1}(t)u(t,t_0)\eta(t_0)$ using (32) and (40).

In order to sustain our claim that the Hamiltonian $\tilde{H}(t)$ defined in (5) represents the energy in the non-Hermitian system we compute the energy expectation values

$$E_\pm(t) = \langle\Psi_\pm(t)\left|\tilde{H}(t)\rho(t)\Psi_\pm(t)\right\rangle \qquad (42)$$
$$= \langle\phi_\pm(t)\left|h(t)\phi_\pm(t)\right\rangle \qquad (43)$$
$$= \pm\frac{\phi^3}{2+\gamma^2\sin(t\phi)-\gamma^2} - \frac{\omega}{2} \qquad (44)$$

oscillating with Rabi-frequency $\phi = (E_+ - E_-)$ between the values $E_\pm(t_0) = E_\pm$ and $E_\pm(-t_0) = (\pm\phi^3 - \omega)/2$. Thus $h(t)$ the Hermitian side corresponds to $\tilde{H}(t$ on the non-Hermitian side.

IV. CONCLUSIONS

We have demonstrated that the problem of solving the TDSE involving an explicitly time-dependent Hermitian Hamiltonian (1) can be replaced with a two step procedure consisting of first solving the TDSE for a time-independent non-Hermitian Hamiltonian (1) and second solving the time-dependent Dyson relation (3) together with the time-dependent quasi-Hermiticity relation (4). For the simple model presented here it transpires that the equations in our two-step procedure are indeed easier to solve than the original TDSE. For our model the simplicity lies in the actual integrals involved at the cost of some more algebra. The original integral is of the type for which the integrand can be transformed into a rational function by using the substitution $z = \tan(\phi t/2)$, whereas in our proposed method only a simple harmonic oscillator equation need to be integrated. Here we have presented a derivation with H as starting point, but of course all steps are reversible and one may also take $h(t)$ to commence with. For more details on the limitations and alternative solution procedures we refer to [21].

As a by-product we have also obtained further evidence for the solvability of the equations (3) and (4), as already observed in [7, 10], but in addition we also showed here that the solutions obtained constitute meaningful wave-functions and produce physical expectation values.

Clearly our approach can also be adapted to a perturbative treatment. Just as in the time-independent setting, where the Dyson map is often only known perturbatively [14, 15], this limitation is likely to carry over to the time-dependent scenario. So a perturbative series would be in a parameter related to η rather than one occurring in the model itself.

It will be very interesting to investigate the viability of this approach further for more complicated systems of higher rank matrix type, but especially for Hamiltonians related to infinite dimensional Hilbert spaces. More investigations are also desirable for the situation in which the non-Hermitian Hamiltonian in (1) are explicitly time-dependent. For instance, a detailed comparison with adiabatic approaches, e.g. in [19, 20], would be very insightful and valuable.

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