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CORRIDOR OPTIONS AND ARC-SINE LAW

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We study a generalization of the arc-sine law. In particular we provide new results about the distribution of the time spent by a BM with drift inside a band, giving the Laplace transform of the characteristic function. If one of the extremes of the band goes to infinity, our formula agrees with the results given in Akahori and Takàcs. We apply these results to the pricing of exotic option contracts known as corridor derivatives. We then discuss the inversion problem comparing different numerical methods.

1. Introduction. In this paper we obtain new results on a generalization of the Lèvy arc-sine law. Lèvy studied the density of the time spent by a standard Brownian motion (SBM) below a given level. We will provide results about the case of the Brownian motion with drift below a given level and inside a given band. This problem has been solved for the case of Brownian motion with drift below a given level [3], [9], [26], [11]. In this paper we derive the same expression as in [26] and simpler than that given in [3] and [9]. So the main results are related to the case of the time spent by the Brownian motion (without drift and with drift) inside a band.

The results also have financial applications to the pricing of corridor and hurdle options, as we illustrate in Section 2. Other applications are, as suggested in [27], pages 66 and 67, to the management of a portfolio for the computation of the expected amount of time a trader is expected to spend in the red. A similar problem for the standard Brownian excursion can be found in chemistry as well and in particular in the theory of ring polymers, as explained in [18].

Table 1 illustrates the random variables for which we are interested in deriving the density law and the characteristic function.

In the next section we will illustrate the problem of corridor derivative pricing. In order to price such a contract we need the knowledge of the distribution function of occupation time. In Section 3 we will give the expression for the Laplace transform of the characteristic function of the occupation time of the Brownian motion of the interval $(-\infty; l]$ and $[l; u]$. In Section 4, we discuss different inversion techniques (univariate and multidimensional) and we provide numerical examples with a comparison with the Monte Carlo simulation method.

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TABLE 1

	Bm no drift	**Bm with drift**
1 barrier	$\int_0^t 1_{(W_s<u)}\,ds$	$\int_0^t 1_{(W_s^m<u)}\,ds$
2 barrier	$\int_0^t 1_{(l<W_s<u)}\,ds$	$\int_0^t 1_{(l<W_s^m<u)}\,ds$

2. The corridor derivative. Corridor derivatives are exotic options paying at expiry an amount that depends on the time spent by a reference index, usually an exchange rate or an interest rate, below a given level or inside a band. The structure of the payoff is common to FX range floaters and boost structures as described in [12], [17], [23], [28] and [29]. These kinds of products are suitable for investors believing in stable markets, because with the actual low interest rate level they allow a higher performance than investing in bonds or directly in stocks.

In order to price the contract, we apply the well-known result that in a arbitrage-free market, according to the well-known Harrison–Kreps theorem of asset pricing [16], the price of any contract is just the expected value, under the risk-neutral measure, of the discounted payoff of the contract. So our aim will be to find the distribution function of the occupation time.

Let us suppose that the price of the underlying asset is described by a stochastic differential equation,

$$\begin{aligned} dP_t &= rP_t\,dt + \sigma P_t\,dW_t, \\ P_0 &= p, \end{aligned}$$

where r is the instantaneous risk-free interest rate, so the dynamics of the asset price under the martingale measure is described by a geometric Brownian process and $\ln P(t)$ has normal distribution with mean $\ln p + (r - \sigma^2/2)t$ and variance $\sigma^2 t$.

If we define the random variable

$$\tau(t, p; L, U) = \int_0^t \mathbf{1}_{(L<P(s)<U)}\,ds$$

then a corridor option (hurdle option if $L = 0$) at maturity has payoff given by $\max[\tau - K; 0] := (\tau - K)^+$, and the price at time 0 of the contract having a residual life equal to t and a strike $K < t$, is given by

$$\begin{aligned} &e^{-rt}\mathbf{E}_{0,p}(\max[\tau(t, p; L, U) - K; 0]) \\ &\quad = e^{-rt}\biggl(\int_0^t (s-K)^+ f_\tau(s, t, p; L, U)\,ds + (t-K)^+ \\ &\qquad \times \Pr_{0,p}[\tau(t, p; L, U) = t]\biggr) \end{aligned} \tag{2.1}$$

where $f_\tau(s, t, p; L, U)$ is the density function of the r.v. $\tau(t, p; L, U)$ calculated at s, when $0 < s < t$. Note that in order to calculate the price we need to take into account the fact that the index can always stay inside the band or below the level U (when $L = 0$). This explains the presence of the term $(t - K)^+ \Pr[\tau(t, p; L, U) = t]$. In the following we write $f_\tau(s, t, p)$ to mean $f_\tau(s, t, p; L, U)$. We observe that

$$\mathbf{1}_{(L<P(s)<U)} = \mathbf{1}_{(L<p\ \exp((r-\sigma^2/2)s+\sigma W(s))<U)}$$

$$= \mathbf{1}_{((1/\sigma)\ln(L/p)<1/\sigma(r-\sigma^2/2)s+W(s)<(1/\sigma)\ln(U/p))}.$$

So we can calculate the density function of the r.v. $\tau(t, p; L, U)$ using the density of the occupation time of the SBM $W(t)$, if $(r - \sigma^2/2)/\sigma = 0$, and of the BM with drift $W^{(m)}(t) := mt + W(t)$, where $m = (r - \sigma^2/2)/\sigma \neq 0$. In both cases the barriers are fixed at the levels $l = (\ln L)/\sigma$ and $u = (\ln U)/\sigma$ and with starting value $x = (\ln p)/\sigma$. We study this problem in the following sections.

It is natural to see that we have the same problem to solve if the barriers increase exponentially at rate δ: $a(t) = Le^{\delta t}$ and $b(t) = Ue^{\delta t}$. Indeed,

$$\mathbf{1}_{(Le^{\delta t}<P(t)<Ue^{\delta t})} = \mathbf{1}_{(l<(r-\delta-\sigma^2/2)(1/\sigma)t+W(t)<u)}$$

so we can consider the occupation time for a Brownian motion with an adjusted drift equal to $(r - \delta - \sigma^2/2)/\sigma$ and fixed barriers as above.

We observe that if the strike price is set to zero, we have a corridor bond (hurdle bond if $L = 0$) and the price can be obtained discounting the following expression:

$$\mathbf{E}_{0,x}(\tau(t, p; L, U)) = \mathbf{E}_{0,x}\left[\int_0^t \mathbf{1}_{(l<ms+W(s)<u)}\, ds\right]. \tag{2.2}$$

Applying Fubini's theorem, we obtain

$$= \int_0^t \mathbf{E}_{0,x}(\mathbf{1}_{(l<ms+W(s)<u)})\, ds$$

$$= \int_0^t \Pr_{0,x}(l < ms + W(s) < u)\, ds$$

$$= \int_0^t (\Phi(h(x, u, s)) - \Phi(h(x, l, s)))\, ds,$$

where $\Phi(x) = \int_{-\infty}^x (\exp(-w^2/2)/\sqrt{2\pi})\, dw$ is the cumulative normal distribution and $h(x, l, t) = (1/\sqrt{t})(l - x - mt)$.

3. The characteristic function of the occupation time of the interval [l; u]. In order to price corridor derivatives, we are interested in the evaluation of the distribution of the r.v.

$$\tau(t,x) := \tau(t,x;u,l,m) = \int_0^t \mathbf{1}_{(l<W(s)+ms<u)}\,ds,$$

$$W(0) = x,$$

representing the amount of time spent inside the interval $[l;u]$ up to time t by a Brownian motion with drift m and starting at x.

If we define the characteristic function of the r.v. $\tau(t,x)$,

$$\begin{aligned} v(t,x) &:= v(t,x;u,l,m) = \mathbf{E}_{0,x}[e^{i\mu\tau(t,x)}] \\ &= \int_0^t e^{i\mu\tau} f_\tau(\tau,t,x;u,l,m)\,d\tau + 1\times \mathrm{Pr}_{0,x}[\tau(t,x;u,l)=0] \\ &\quad + e^{i\mu t}\times \mathrm{Pr}_{0,x}[\tau(t,x;u,l)=t] \end{aligned} \tag{3.1}$$

using the Feynman–Kac formula ([19], page 366), it can be shown that $v(t,x)$ satisfies the following partial differential equation (pde):

$$-\frac{\partial v(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 v(t,x)}{\partial^2 x} + m\frac{\partial v(t,x)}{\partial x} + i\mu\mathbf{1}_{(l<x<u)}v(t,x) = 0 \tag{3.2}$$

with initial condition

$$v(0,x) = 1 \qquad \forall\ x\in(-\infty;+\infty) \tag{3.3}$$

and boundary conditions

$$v(t,\pm\infty) = 1 \qquad \forall\ t>0. \tag{3.4}$$

Given a function of time t, we denote with $\mathscr{L}[\cdot; t\to\gamma]$ its Laplace transform with respect to the variable time t, and with $\mathscr{L}^{-1}[\cdot;\gamma\to t]$ the inverse Laplace transform. We have the following result.

THEOREM 1. *The characteristic function of the r.v. $\tau(t,x;u,l,m)$ admits the following representation*:

$$\begin{aligned} v(t,x;l,u,m) &= \Omega(t,\mu,x;l,u,m) \\ &\quad + \begin{cases} 1\times \mathrm{Pr}_{0,x\in[u,+\infty)}\left(\inf\limits_{0\le s\le t} ms+W(s)>u\right), \\ e^{i\mu t}\times \mathrm{Pr}_{0,x\in(u,l)}\left(\sup\limits_{0\le s\le t} ms+W(s)<u; \inf\limits_{0\le s\le t} ms+W(s)>l\right), \\ 1\times \mathrm{Pr}_{0,x\in(-\infty,l]}\left(\sup\limits_{0\le s\le t} ms+W(s)<l\right), \end{cases} \end{aligned} \tag{3.5}$$

where

$$\Omega(t,\mu,x;l,u,m) := \int_0^t e^{i\mu\tau} f_\tau(\tau,t,x;u,l,m)\,d\tau$$
$$= \exp\left(-mx - \frac{m^2}{2}t\right)\mathscr{L}^{-1}[\omega(\gamma,\mu,x;l,u,m);\gamma\to t]$$

and

$$\omega(\gamma,\mu,x;l,u,m)$$
$$= \begin{cases} \mathbf{1}_{(x\ge u)}\ \exp(-\sqrt{2}(x-u)\sqrt{\gamma})\mathscr{L}[y(t,1);\gamma], \\ \mathbf{1}_{(l<x<u)}\dfrac{1}{\sinh(a\pi)}\Big[\mathscr{L}[y(t,0);\gamma]\ \sinh\left(a\pi\left(\dfrac{u-x}{u-l}\right)\right) \\ \qquad\qquad + \mathscr{L}[y(t,1);\gamma]\ \sinh\left(a\pi\left(\dfrac{x-l}{u-l}\right)\right)\Big], \\ \mathbf{1}_{(x\le l)}\ \exp(-\sqrt{2}(l-x)\sqrt{\gamma})\mathscr{L}[y(t,0);\gamma] \end{cases}$$

and

$$\mathscr{L}[y(t,1);t\to\gamma] = \frac{e^{mu}}{\sqrt{\gamma}(\sqrt{\gamma}-m/\sqrt{2})} - \frac{c}{2\sqrt{\gamma}}(s(\gamma)+d(\gamma)),$$
$$\mathscr{L}[y(t,0);t\to\gamma] = \frac{e^{ml}}{\sqrt{\gamma}(\sqrt{\gamma}+m/\sqrt{2})} + \frac{c}{2\sqrt{\gamma}}(s(\gamma)-d(\gamma))$$

with

$$\frac{c}{\sqrt{\gamma}}\,d(\gamma) = \frac{\sqrt{\gamma-i\mu}\ \sinh(a\pi)}{(\sqrt{\gamma-i\mu}\ \sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1))}$$
$$\times\left(\frac{e^{mu}}{\sqrt{\gamma}(\sqrt{\gamma}-m/\sqrt{2})} + \frac{e^{ml}}{\sqrt{\gamma}(\sqrt{\gamma}+m/\sqrt{2})}\right.$$
$$+ \frac{1}{\sqrt{\gamma-i\mu}\ \sinh(a\pi)(\gamma-i\mu-m^2/2)}$$
$$\times\left[\left(\frac{-m}{\sqrt{2}}(e^{ml}-e^{mu})(\cosh(a\pi)+1)\right.\right.$$
$$\left.\left.\left. -\sqrt{\gamma-i\mu}(e^{mu}+e^{ml})\ \sinh(a\pi)\right)\right]\right),$$

(3.6)

$$\frac{c}{\sqrt{\gamma}}s(\gamma) = \frac{\sqrt{\gamma-i\mu}\ \sinh(a\pi)}{\sqrt{\gamma-i\mu}\ \sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)-1)}$$

$$
\begin{aligned}
&\times\Bigg(\frac{e^{mu}}{\sqrt{\gamma}(\sqrt{\gamma}-m/\sqrt{2})}-\frac{e^{ml}}{\sqrt{\gamma}(\sqrt{\gamma}+m/\sqrt{2})}\\
&\qquad-\frac{1}{\sqrt{\gamma-i\mu}\,\sinh(a\pi)(\gamma-i\mu-m^2/2)}\\
&\qquad\quad\times\Bigg[\Bigg(\frac{-m}{\sqrt{2}}(e^{mu}+e^{ml})(\cosh(a\pi)-1)\\
&\qquad\qquad+\sqrt{\gamma-i\mu}(e^{mu}-e^{ml})\sinh(a\pi)\Bigg)\Bigg]\Bigg),
\end{aligned}
$$

$$
a\pi=\sqrt{\frac{\gamma-i\mu}{c^2}};\qquad \alpha=-m;\qquad \beta=-\frac{m^2}{2};\qquad c^2=\frac{1}{2(u-l)^2}. \tag{3.7}
$$

Moreover, we can express the density function $f(\tau,t,x;u,l,m)$ *of the occupation time for a generic starting point* x *and for* $0<\tau<t$ *in terms of the density function of the occupation time when* $x=u$ *and* $x=l$ *in the following way*:

$$
\begin{aligned}
&f_\tau(\tau,t,x;u,l,m)\\
&=\begin{cases}
1_{(x\geq u)}e^{-m(x-u)}\displaystyle\int_\tau^t\frac{x-u}{\sqrt{2\pi(t-\eta)^3}}\\
\qquad\times\exp\left(-\dfrac{(x-u)^2}{2(t-\eta)}-\dfrac{m^2}{2}(t-\eta)\right)f_\tau(\tau,\eta,u)\,d\eta,\\
1_{(l<x<u)}2\pi c^2\displaystyle\sum_{n=1}^{\infty}n\,\sin\left(n\pi\left(\frac{x-l}{u-l}\right)\right)\int_0^\tau\\
\qquad\times\exp\left(-(\dfrac{m^2}{2}+\lambda_n)\xi\right)(e^{-m(x-l)}f_\tau(\tau-\xi,t-\xi,l)\\
\qquad\qquad-(-1)^n e^{m(u-x)}f_\tau(\tau-\xi,t-\xi,u))\,d\xi,\\
1_{(x\leq l)}\exp(m(l-x))\displaystyle\int_\tau^t\frac{(l-x)}{\sqrt{2\pi(t-\eta)^3}}\\
\qquad\times\exp\left(-\dfrac{(l-x)^2}{2(t-\eta)}-\dfrac{m^2}{2}(t-\eta)\right)f_\tau(\tau,\eta,l)\,d\eta,
\end{cases}
\end{aligned}
\tag{3.8}
$$

3.1. *Remarks.* In the Appendix, we solve the pde and prove the theorem. We can now make some remarks.

REMARK 1. A natural way of solving the pde (3.2) could be to take the Laplace transform with respect to t and then obtain three second-order differential equations. The continuity and differentiability of the solution at the barriers and its boundedness at $\pm\infty$ require then the determination of four constants, generalizing the example in [19], page 273. However, using this approach, we can incur two problems.

(a) The final expression of the solution will be the Laplace transform of the characteristic function $v(t, x)$ and then it will include the Laplace transform of the mass of probabilities concentrated at $\tau = 0$ and $\tau = t$. This fact as explained in [1] can create problems in the numerical inversion and it is advisable to remove the atoms of probability before the inversion. Attacking directly the pde and using the Laplace transform only in a successive step, we avoid this problem. Indeed, we are able to identify these probabilities in the expression of the c.f. and so we can give the Laplace transform of the function $\Omega(t, \mu, x; l, u, m)$ and not directly of the c.f.

(b) If we want to use two univariate numerical inversions for limiting the programming effort and to use well-tested numerical inversion routines, as discussed in the next section, we should provide the Laplace transform of the real part and of the imaginary part of the function $\Omega(t, \mu, x; l, u, m)$. Our approach consists of solving (A.8), separating the real and the imaginary part of the functions $D(t)$ and $S(t)$ and then, taking the Laplace transform, we obtain two linear systems of just two equation each. (The Laplace transforms of the real and imaginary part are available on request.) Instead, if we take directly the Laplace transform of (3.1), we should solve three systems of two differential equations each and then the continuity and differentiability and boundedness of the solution will require the determination of eight constants in a linear system with eight equations.

REMARK 2. The characteristic function is continuous and differentiable at $x = l$ and $x = u$, because, as we will show later, the pde (3.2) has been solved, requiring continuity and differentiability of the solution at these points. This property will be transmitted to the price of the corridor option.

REMARK 3. Comparing expressions (3.2) and (3.5), we obtain the natural results,

$$\Pr_{0,x}[\tau(t, x; u, l) = 0] = \begin{cases} \mathbf{1}_{(x>u)} \Pr_{0,x}\left(\inf_{0\le s\le t} ms + W(s) > u\right), \\ 0; l \le x \le u, \\ \mathbf{1}_{(x<l)} \Pr_{0,x}\left(\sup_{0\le s\le t} ms + W(s) < l\right), \end{cases}$$

and

$$\Pr_{0,x}[\tau(t, x; u, l) = t] = \begin{cases} \mathbf{1}_{(l<x<u)} \Pr_{0,x}\left(\sup_{0\le s\le t} ms + W(s) < u; \inf_{0\le s\le t} ms + W(s) > l\right) \\ 0; x \le l \text{ or } u \le x. \end{cases}$$

The expressions for these quantitities can be found in (A.17), (A.18) and (A.23) in the Appendix and can be compared with the same expressions in [4].

REMARK 4. We can show that the above expressions allow us to recover known results. In particular, now we discuss the following cases: (a) $m = 0$, that is, the case of the occupation time of the SBM of the interval $[l, u]$, (b)$m = 0$ and $l = -\infty$, that is, the time spent below the level u by the SBM and we obtain the Lévy arc-sine law, (c) $l = -\infty$, that is, the time spent by a BM with drift below the upper barrier u, the case studied by [3], [9] and [26].

(a) If $m = 0$, we are considering the occupation time of the SBM of the interval $[l; u]$. The expression for the functions $d(\gamma)$ and $s(\gamma)$, $\mathscr{L}[y(t, 0); t \to \gamma]$ and $\mathscr{L}[y(t, 1); t \to \gamma]$ simplify to

$$\begin{aligned} d(\gamma) &= \frac{1}{c} \frac{2\sqrt{\gamma}\sqrt{\gamma - i\mu} \sinh(a\pi)}{(\sqrt{\gamma - i\mu} \sinh(a\pi) + \sqrt{\gamma}(\cosh(a\pi) + 1))} \left(\frac{1}{\gamma} - \frac{1}{\gamma - i\mu} \right) \\ s(\gamma) &= 0 \end{aligned}$$

$$\begin{aligned} \mathscr{L}[y(t, 0); t \to \gamma] &= \mathscr{L}[y(t, 1); t \to \gamma] \\ &= \frac{\sqrt{\gamma} \sinh(a\pi) + \sqrt{\gamma - i\mu}(\cosh(a\pi) + 1)}{(\sqrt{\gamma - i\mu} \sinh(a\pi) + \sqrt{\gamma}(\cosh(a\pi) + 1))} \frac{1}{\sqrt{\gamma}\sqrt{\gamma - i\mu}} \end{aligned}$$

and

$$\begin{aligned} &\omega(\gamma, \mu, x; l, u, m) \\ &= \begin{cases} \mathbf{1}_{(x \geq u)} \dfrac{\exp(-\sqrt{2}(x - u)\sqrt{\gamma})}{\sqrt{\gamma}} \\ \qquad \times \dfrac{\sqrt{\gamma} \sinh(a\pi) + \sqrt{\gamma - i\mu}(\cosh(a\pi) + 1)}{\sqrt{\gamma - i\mu}(\sqrt{\gamma - i\mu} \sinh(a\pi) + \sqrt{\gamma}(\cosh(a\pi) + 1))}, \\ \mathbf{1}_{(l < x < u)} \dfrac{\sinh(a\pi((u - x)/(u - l))) + \sinh(a\pi((x - l)/(u - l)))}{\sinh(a\pi)} \\ \qquad \times \mathscr{L}[y(t, 1); \gamma], \\ \mathbf{1}_{(x \leq l)} \dfrac{\exp(-\sqrt{2}(l - x)\sqrt{\gamma})}{\sqrt{\gamma}} \\ \qquad \times \dfrac{\sqrt{\gamma} \sinh(a\pi) + \sqrt{\gamma - i\mu}(\cosh(a\pi) + 1)}{\sqrt{\gamma - i\mu}(\sqrt{\gamma - i\mu} \sinh(a\pi) + \sqrt{\gamma}(\cosh(a\pi) + 1))}. \end{cases} \end{aligned}$$

This expression does not seem to admit a simple analytical inverse, although in [4], formula 1.7.4, pages 140 and 141, a very complicated expression is given. We remark that $\mathscr{L}[v(t,0); t \to \gamma] = \mathscr{L}[v(t,1); t \to \gamma]$, a consequence of the reflection principle.

(b) If we let $l \to -\infty$ in the above expression, we are considering the time spent below the level u by the SBM, so we should recover the Lévy arc-sine law. We get

$$\lim_{l \to -\infty} \mathscr{L}[y(t,1); t \to \gamma] = \frac{1}{\sqrt{\gamma}\sqrt{\gamma - i\mu}}$$

and then for $x > u$,

$$\omega(\gamma, \mu, x; l, u, m) = \mathbf{1}_{(x \geq u)} \frac{\exp(-\sqrt{2}(x-u)\sqrt{\gamma})}{\sqrt{\gamma}\sqrt{\gamma - i\mu}}$$

and using the inversion formula in [2], we obtain

$$\lim_{l \to -\infty} f_{\tau}(\tau, t, x; u, l, 0) = \begin{cases} 1_{(x>u)} \dfrac{1}{\pi} \dfrac{\exp\left(-\frac{1}{2}(x-u)^2/(t-\tau)\right)}{\sqrt{\tau(t-\tau)}}, \\ 1_{(x=u)} \dfrac{1}{\pi} \dfrac{1}{\sqrt{\tau(t-\tau)}}, \\ 1_{(x<u)} \dfrac{\exp\left(-\frac{1}{2}((u-x)^2/\tau)\right)}{\pi\sqrt{\tau(t-\tau)}}, \end{cases} \tag{3.9}$$

where the expression for the case $x < u$ has been found exploiting the symmetry property $\tau(t, x; u, -\infty, m) = t - \tau(t, -x; -u, -\infty, -m)$; compare [26]. Expression (3.9) is the well-known arc-sine law, [22] and [4], formula 1.4.4, page 129, where the time spent above the level u is given.

(c) If we let $l \to -\infty$, we are considering the time spent by BM with positive drift below the level u. This case has been studied by [3] and [9]. [26] provides an expression simpler than [3].

From (3.6), we obtain

$$\begin{aligned} \lim_{l \to -\infty} \frac{c}{\sqrt{\gamma}} d(\gamma) &= e^{-\alpha u} \frac{\sqrt{\gamma - i\mu}}{\sqrt{\gamma - i\mu} + \sqrt{\gamma}} \left(\frac{1}{\sqrt{\gamma}(\sqrt{\gamma} + \alpha/\sqrt{2})} \right. \\ &\qquad \left. - \frac{\sqrt{\gamma - i\mu} + \alpha/\sqrt{2}}{\sqrt{\gamma - i\mu}(\gamma - i\mu - \alpha^2/2)} \right) \\ \lim_{l \to -\infty} \frac{c}{\sqrt{\gamma}} s(\gamma) &= e^{-\alpha u} \frac{\sqrt{\gamma - i\mu}}{\sqrt{\gamma - i\mu} + \sqrt{\gamma}} \left(\frac{1}{\sqrt{\gamma}(\sqrt{\gamma} + \alpha/\sqrt{2})} \right. \\ &\qquad \left. - \frac{\sqrt{\gamma - i\mu} + \alpha/\sqrt{2}}{\sqrt{\gamma - i\mu}(\gamma - i\mu - \alpha^2/2)} \right) \end{aligned} \tag{3.10}$$

and then

$$\begin{aligned}\lim_{l\to-\infty}\mathscr{L}[y(t,1);t\to\gamma] &= \frac{e^{-\alpha u}}{(\sqrt{\gamma}+\alpha/\sqrt{2})(\sqrt{\gamma-i\mu}-\alpha/\sqrt{2})}\\ &= \frac{e^{-\alpha u}(\sqrt{\gamma}+\alpha/\sqrt{2}-\alpha/\sqrt{2})}{\sqrt{\gamma}(\sqrt{\gamma}+\alpha/\sqrt{2})(\sqrt{\gamma-i\mu}-\alpha/\sqrt{2})}\\ &= \frac{e^{-\alpha u}}{(\sqrt{\gamma-i\mu}-\alpha/\sqrt{2})}\left(\frac{1}{\sqrt{\gamma}}-\frac{\alpha/\sqrt{2}}{\sqrt{\gamma}(\sqrt{\gamma}+\alpha/\sqrt{2})}\right)\end{aligned}$$

so when $x > u$,

$$\begin{aligned}&\Omega(t,\mu,x;l,u,m)\\ &\quad= \exp\left(-m(x-u)-\frac{m^2}{2}t\right)\mathscr{L}^{-1}\\ &\qquad\times\left[\frac{e^{mu}}{(\sqrt{\gamma-i\mu}+m/\sqrt{2})}\left(\frac{\exp(-\sqrt{2}(x-u)\sqrt{\gamma})}{\sqrt{\gamma}}\right.\right.\\ &\qquad\qquad\left.\left.+\frac{m}{\sqrt{2}}\frac{\exp(-\sqrt{2}(x-u)\sqrt{\gamma})}{\sqrt{\gamma}(\sqrt{\gamma}-m/\sqrt{2})}\right);\gamma\to t\right]\\ &\quad= 1_{(x>u)}\exp\left(-m(x-u)-\frac{m^2}{2}t\right)\\ &\qquad\times\left(\int_0^t e^{i\mu\theta}\left(\frac{1}{\sqrt{\pi\theta}}-\frac{m}{\sqrt{2}}\exp\left(\frac{m^2}{2}\theta\right)\operatorname{Erfc}\left(\frac{m\sqrt{\theta}}{\sqrt{2}}\right)\right)\right)\\ &\qquad\times\left(\frac{\exp(-(x-u)^2/2(t-\theta))}{\sqrt{\pi(t-\theta)}}+\frac{m}{\sqrt{2}}\exp\left(\frac{m^2}{2}(t-\theta)-m(x-u)\right)\right.\\ &\qquad\qquad\left.\times\operatorname{Erfc}\left(\frac{-m\sqrt{t-\theta}}{\sqrt{2}}+\frac{(x-u)}{\sqrt{2(t-\theta)}}\right)\right)d\theta,\end{aligned}$$

where we have used the inversion formulas in [2] and the convolution property of the Laplace transform. The limits can appear to depend on the value of m, but we can suppose $m > 0$ without loss of generality. Indeed, the key in the determination of the density is the symmetry property in [26] that allows us to find an expression for the density when $x < u$ in terms of the density when $x > u$. So if the drift is negative, we can suppose that $x < l$ and we let $u \to \infty$ so we consider the time spent above the level l. Then using the symmetry property, we find the expression for $x > l$ as well. So the result does not depend on the sign of m.

For $x < u$ and $l \to -\infty$, we can again use the symmetry argument in [26], so we then obtain that the density function of the occupation time with only one barrier is given by

$$
\begin{aligned}
&\lim_{l\to-\infty} f_\tau(\tau, t, x; u, l, m) \\
&= \begin{cases}
1_{(x\geq u)}\left(\dfrac{\exp(-(m^2/2\tau))}{\sqrt{\pi\tau}} - \dfrac{m}{\sqrt{2}}\mathrm{Erfc}\left(\dfrac{m\sqrt{\tau}}{\sqrt{2}}\right)\right) \\
\qquad \times \left(\dfrac{\exp\left(-\frac{1}{2}(((x-u+m(t-\tau)))^2/(t-\tau))\right)}{\sqrt{\pi(t-\tau)}}\right. \\
\qquad\qquad \left. + \dfrac{m}{\sqrt{2}}e^{-2m(x-u)}\mathrm{Erfc}\left(\dfrac{(x-u)-m(t-\tau)}{\sqrt{2(t-\tau)}}\right)\right) \\
1_{(x<u)}\left(\dfrac{\exp(-(m^2/2)(t-\tau))}{\sqrt{\pi(t-\tau)}} + \dfrac{m}{\sqrt{2}}\mathrm{Erfc}\left(-\dfrac{m\sqrt{t-\tau}}{\sqrt{2}}\right)\right) \\
\qquad \times \left(\dfrac{\exp\left(-\frac{1}{2}(((u-x-m\tau))^2/\tau)\right)}{\sqrt{\pi\tau}} - \dfrac{m}{\sqrt{2}}e^{+2m(u-x)}\right. \\
\qquad\qquad \left. \times\ \mathrm{Erfc}\left(\dfrac{(u-x)+m\tau}{\sqrt{2\tau}}\right)\right)
\end{cases}
\end{aligned}
\tag{3.11}
$$

In order to make comparable the expression above with equation (12) in Takacs [26], where the average time spent below the level $x < u$ is considered, it is necessary to set in [26] $\alpha = (u-x)/\sqrt{t}$ and substitute m with $-m\sqrt{t}$. We need to use as well the relationship $\mathrm{Erfc}(x) = 2\Phi(-\sqrt{2}x)$.

Moreover, if we have $m = 0$, the above density function reduces again to (3.9).

4. The numerical inversion. In this section we discuss the problem of the numerical inversion. From a computational point of view it is convenient to distinguish the problem of finding the density function from the problem of pricing the corridor option. Indeed, in the first case we like to use two single univariate inversions, while in the second it is better to use a multivariate inversion.

This choice is also because, up to now, relatively little attention has been given to inversion of multidimensional transforms, so in order to limit the programming effort a possibility is to obtain the density function using well-tested univariate inversion routines as described in the following two steps:

1. Numerically find the inverse Laplace transform of the function $\Omega(t, \mu, x; l, u, m)$. We have solved this problem using the Crump's [8] method implemented in the IMSL-library subroutine FLINV. This method is ranked among the most accurate available numerical inversion techniques in the Davies and Martin [10] comparison.

2. Using the numerically computed $\Omega(t, \mu, x; l, u, m)$, find the function $f_\tau(\tau, t, x; U, L, m)$ through a further numerical Fourier transform inversion. We have computed this inverse using the fast Fourier transform and this

TABLE 2

Price of the corridor bond using the analytical solution and integrating the density function obtained by the two step inversion. Parameters: $r = 0.05$, $\sigma = 0.2$, $L = 100$, $U = 100$, $t = 1$. n denotes the number of chosen points (a power of 2) in the FFT

p	**Analytical sol.**	***n*=2048**	***n*=1024**	***n*=512**	***n*=256**
80	0.04609	0.04607	0.04611	0.04620	0.04642
85	0.08149	0.08149	0.08155	0.08170	0.08204
90	0.13134	0.13137	0.13146	0.13169	0.13218
95	0.19606	0.19613	0.19627	0.19657	0.19723
100	0.27463	0.27471	0.27489	0.27528	0.27610
105	0.30959	0.30983	0.31006	0.31052	0.31145
110	0.25770	0.25772	0.25788	0.25824	0.25901
115	0.18058	0.18061	0.18073	0.18100	0.18159
120	0.12478	0.12478	0.12486	0.12506	0.12551
125	0.08509	0.08508	0.08514	0.08529	0.08562
Average time	$< 1''$	$170''$	$104''$	$56''$	$32''$

allows us to reduce greatly the computational time. The advantage of using the FFT routine is that we obtain simultaneously the entire density function whilst using a different procedure (like a double Laplace inversion described later on) we need to repeat the inversion as many times as the number of points at which we desire the density function. Moreover we can exploit the high efficiency of the FFT.

In Figure 1, we represent the density function of the occupation time obtained using the two step inversion. In Table 2 we compare the price of the corridor bond obtained using the density coming from the double inversion described above and the analytical formula for the corridor bond as from Section 2. The expected value has been calculated through a simple trapezoidal rule. In the numerical inversion, we have to choose the Fourier transform maximum frequency, so from the table we can see that by increasing it we obtain great accuracy, but at the cost of increasing the computational time. (All the calculations have been performed on a Pentium 133 machine.)

Using the results in Theorem 1, we can obtain the double Laplace transform of the price of the corridor option. Indeed, we observe that we can write the undiscounted price of the corridor option with strike K and residual live t as

$$C(t, K; x) = \int_K^t (\tau - K) f_\tau(\tau, t, x; u, l, m)\, d\tau + (t - K)^+ \times \Pr_{0, x\in(l,u)}[\tau(t, x; u, l) = t]$$

and with some simple passage, we obtain

$$C(t, K; x) = E_{0,x}[\tau(t, x; u, l)] - K(1 - \Pr_{0,x\in(l,u)}[\tau(t, x; u, l) = 0]) + \int_0^K (K - \tau) f_\tau(\tau, t, x; u, l, m)\, d\tau,$$

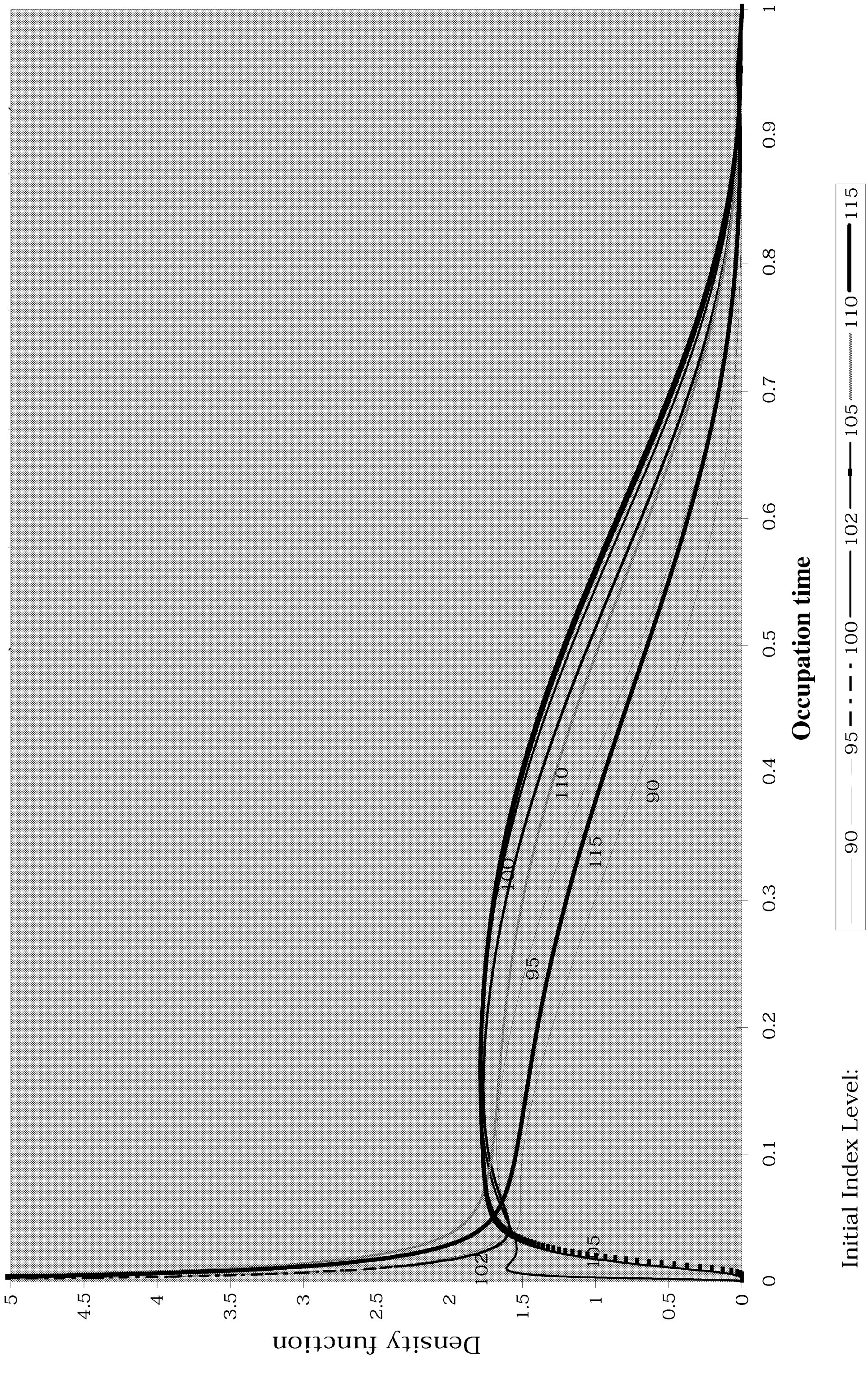
Density function
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Occupation time
102
105
95
100
115
110
90
Initial Index Level:
90
95
100
102
105
110
115

FIG. 1.

where $E_{0,x}[\tau(t, x; u, l)]$ is the expected value of the r.v. $\tau(t, x; u, l)$ and is given by (2.2). If we consider now the Laplace transform with respect to K of the third term, and using the convolution property, we obtain

$$\mathscr{L}\left[\int_0^K (K-\tau) f_\tau(\tau, t, x; U, L, m)\, d\tau; K \to \mu\right] = \frac{\Omega(t, \mu, x; l, u, m)}{\mu^2}$$

and then $C(t, K; x)$ is given by

$$\begin{aligned} &= E_{0,x}[\tau(t, x; u, l)] - K(1 - \Pr_{0, x\in(l,u)}[\tau = 0]) \\ &\quad + \mathscr{L}^{-1}\left[\frac{\Omega(t, \mu, x; l, u, m)}{\mu^2}; \mu \to K\right] \\ &= E_{0,x}[\tau(t, x; u, l)] - K\left(1 - \Pr_{0, x\in(l,u)}[\tau = 0]\right) \\ &\quad + \mathscr{L}^{-1}\left[\frac{\omega(\gamma, \mu, x; l, u, m)}{\mu^2}; \mu \to K, \gamma \to t\right]. \end{aligned}$$

So the calculation of the price of the corridor option requires the determination of the double Laplace inverse of the quantity $\omega(\gamma, \mu, x; l, u, m)/\mu^2$. We can use here the expressions given in Theorem 1, once we have substituted in all expressions the quantity $\gamma - i\mu$ with the quantity $\gamma + \mu$. This is because here we are using a double Laplace transform, while in Theorem 1 we have used a Fourier transform and a Laplace transform. Moreover, for numerical purposes, it is convenient to divide in (3.6) the numerator and the denominator by $\sinh(a\pi)$ and to use the fact that $tgh(a\pi/2) = (\cosh(a\pi) - 1)/\sinh(a\pi) = \sinh(a\pi)/(\cosh(a\pi) + 1)$.

If $w(\gamma, \mu) := \omega(\gamma, \mu, x; l, u, m)/\mu^2$ is the double Laplace transform, the inverse Laplace transform $W(t, K)$ is obtained applying the inversion formula in two variables,

$$W(t, K) = \left(\frac{1}{2\pi j}\right)^2 \int_{c_1 - j\infty}^{c_1 + j\infty} \int_{c_2 - j\infty}^{c_2 + j\infty} e^{\gamma t} e^{\mu K} w(\gamma, \mu)\, d\gamma\, d\mu.$$

where c_1 and c_2 are arbitrary but greater than the real part of all singularities of $\omega(\gamma, \mu)$, that is analytic when $\mathrm{Re}(\gamma) > m^2/2$ and $\mathrm{Re}(\gamma + \mu) > m^2/2$.

We have considered two methods for numerically computing this quantity: (a) the Fourier series method first introduced for muldimensional transform inversion by [7]; (b) the Padé approximation as suggested in [25].

The Fourier series inversion procedure, formula (2.11) in [7], consists essentially of an enhancement of the Euler algorithm in [1] based on truncating the inversion integral and applying the trapezoidal rule. The idea consists of damping the function to be inverted, multiplying it by a two-dimensional decaying exponential function and then approximating the damped function by a periodic function constructed by aliasing. The inversion formula is then the two-dimensional Fourier series of the periodic function. The computation of the series can be greatly accelerated by the use of the Euler algorithm for

TABLE 3

Price of the corridor option using the different inversion routines and the Monte Carlo simulation (50,000 *simulations with* 1200 *steps) with antithetic variate. The parameters are* $r = 0.05, \sigma = 0.2, L = 100, U = 110, t = 1$. *In the Crump + FFT we have used* 2048 *sampling points; in both cases for the Fourier method we have set* $A_1 = A_2 = 20, l_1 = l_2 = 2$ *and the Euler algorithm has been used with a total of* $m+n+1$ *terms. In the Padé approximation the degree of the denominator has been set to* 18 *and the degree of the numerator has been set to* 4. *The poles and the residues have been calculated in Mathematica* 3.0 *with the functions Pade[ ], NResidue[ ] and NSolve[ ]. In the Monte Carlo column appears also the* 1000× *standard error*

	Crump+FFT	**Fourier**	**Fourier**	**Padé**	**MC+AV**
$K = 0.2$		$n = 20, m = 20$	$n = 100, m = 20$		
$x = 90$	0.0462793	0.0463038	0.0463038	0.0463039	0.046606; 0.265
$x = 95$	0.0792503	0.0792444	0.0792444	0.0792445	0.078995; 0.308
$x = 100$	0.1247528	0.1247227	0.1247228	0.1247226	0.124951; 0.518
$x = 105$	0.1470881	0.1469239	0.1469239	0.1469239	0.147282; 0.665
$x = 110$	0.1161007	0.1161262	0.1161262	0.1161261	0.115834; 0.358
$x = 115$	0.0735259	0.0735554	0.0735554	0.0735555	0.073820; 0.311
$x = 120$	0.0456788	0.0457253	0.0457253	0.0457254	0.046075; 0.271
Average CPU	$170''$	$8''$	$39''$	$< 1''$	$337''$
$K = 0.4$					
$x = 90$	0.0100981	0.0101457	0.0101457	0.0101459	0.010328; 0.123
$x = 95$	0.0213042	0.0213357	0.0213358	0.0213361	0.021146; 0.175
$x = 100$	0.0400273	0.0400375	0.0400376	0.0400375	0.040250; 0.27
$x = 105$	0.0504331	0.0503482	0.0503483	0.0503485	0.050574; 0.376
$x = 110$	0.0372272	0.0372753	0.0372754	0.0372753	0.037048; 0.227
$x = 115$	0.0202404	0.0202948	0.0202948	0.0202951	0.020376; 0.176
$x = 120$	0.0107088	0.0107697	0.0107697	0.0107699	0.010868; 0.285
Average CPU	$170''$	$8''$	$38''$	$< 1''$	$337''$
$K = 0.6$					
$x = 90$	0.0008576	0.0009014	0.0009014	0.0009019	0.000927; 0.030
$x = 95$	0.0026490	0.0026893	0.0026893	0.0026899	0.002592; 0.053
$x = 100$	0.0067578	0.0067873	0.0067874	0.0067831	0.006865; 0.092
$x = 105$	0.0094893	0.0094617	0.0094618	0.0094619	0.009545; 0.131
$x = 110$	0.0062664	0.0063189	0.0063191	0.0063188	0.006258; 0.087
$x = 115$	0.0026123	0.0026664	0.0026664	0.0026670	0.002687; 0.056
$x = 120$	0.0010299	0.0010822	0.0010822	0.0010826	0.001093; 0.034
Average CPU	$170''$	$8''$	$37''$	$< 1''$	$337''$

alternating series. The implementation of the algorithm requires the selection of two parameters, m and n, on which depends the accuracy of the inversion. In our case we have seen that a good choice consists in setting $m = 20$ and $n = 20$. The proposed algorithm allows a simultaneous control of the aliasing error and the round-off error coming from multiplying large numbers by small ones. [7] shows indeed that both errors can be controlled choosing in a appropriate way four different constants A_1, A_2, l_1 and l_2. The aliasing error is bounded by $C(e^{-A_1} + e^{-A_2})$, where C is a constant that can be found fixing an upper bound to the price of the corridor option. This upper bound in our case is given by the price of the corridor bound. The round-off error depends on the quantity $\exp(A_1/2l_1 + A_2/2l_2)/(4l_1l_2tK)$ that is decreasing when l_1 and l_2 increase. There is, however, a trade-off between error control and computation time, since this increases proportionally to the product of $l_1$and l_2, and increasing A_1 and A_2 requires an increase in l_1 and l_2. In [7] it is suggested to use $A_1 = A_2 = 20$ and $l_1 = l_2 = 2$, and we have verified that this choice is suitable also in our case. Table 3 compares the two possibilities, so particular care has to been taken in choosing the parameters. We remark that, as explained in [1], pages 75 and 76, this inversion method is not accurate when we have a not-bounded density or with very high peaks as in our case (compare Figure 1). For this reason it is more appropriate to apply this method to the calculation of the corridor option price and avoid its use in the computation of density function.

The method proposed by [25] is very accurate if the pricing function is smooth as in our case. The idea consists of approximating the functions e^z appearing in the Laplace inversion formula by a Padé rational function. Then, if $w(\gamma, \mu)$ is the double Laplace transform, the inverse Laplace transform $W(t, K)$ can be computed as

$$W(t, K) = \frac{1}{tK} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} r_{1i} r_{2j} w\left(\frac{z_{1i}}{t}, \frac{z_{2j}}{K}\right), \tag{4.1}$$

where z_{1i} and z_{2j} are the poles of the Padé approximations to $e^{\gamma t}$ and $e^{\mu K}$, and r_{1i} and r_{2j} are the corresponding residues. M_1 and M_2 are the degree of the denominator in the Padé approximations. This inverse requires choosing the degree of the numerator and denominator of the Padé approximant. From some numerical experiments, we have observed that in order to obtain a good accuracy and avoid round-off errors the degree of the denominator has to be chosen greater than 13 and not larger than 18. The degree of the numerator has to be chosen not greater than 4 or 5. This choice gives us an agreement to the seventh digit between the numerical computed inverse and the analytical formula for the hurdle option. Moreover this inversion can be performed very quickly (less than one second) and with a limited programming effort, once we have computed the poles and the residues of the Padé approximants with programs like Mathematica or Maple and we have stored them. An easy check for avoiding round-off errors is that the sum of the residues has to be equal to 0. However, in respect to the previous method, particular care has to be used, because there is no way to tell how accurate the Padé approximant is. “It is a powerful technique, but in the end still mysterious technique” ([24], page 202).

In Table 3 we compare the three methods and the Monte Carlo simulation method, reporting as well the computational time and the standard error of the Monte Carlo estimate. We remark that the Monte Carlo method suffers from the intrinsic discreteness of the simulation so we do not know if the process has crossed the barriers or not and we cannot compute exactly the time spent inside the barriers during each step of the simulation. The Monte Carlo simulation has been performed using 50,000 simulation and using the antithetic variate technique [5]. We have tried also to obtain a greater reduction in the standard error using as control variate the price of the corridor bond, but this technique performed poorly as K increases and we do not report the results here.

We can remark how the different methods agree in general to the third digit and sometimes more. In this respect, the Padé and the Euler-Fourier inversion algorithm with $m = 20$ and $n = 20$ appear the most preferable methods, mainly for their high accuracy (they agree up to the seventh digit). The Padé inversion is impressive because it is very simple to implement and requires a very low computational time. The Euler-Fourier is slightly slower but allows a clear control of the different type of errors that incur in the inversion. The Monte Carlo method appears very time consuming.

We observe that in order to calculate the Greeks of the contract we can simply calculate the derivatives of the double Laplace transform and invert them. So the numerical routines used for finding the price can be adapted in a simple way to the calculations of the sensitivities, while in the Monte Carlo method, the calculation of the Greeks is usually reputed to be inaccurate. In Figure 2 we present the delta of the corridor option, varying strike and underlying index.

More details can be found in [13], while a comparason between continuous and discrete time monitoring has been analyzed in [14].

APPENDIX

A. Solution of the pde. In order to solve the pde (3.2) we consider the following transformation:

$$v(t, x) = e^{\alpha x+\beta t} h(t, x)$$

and setting $\alpha = -m$ and $\beta = -m^2/2$, we get the following pde for the function $h(t, x)$:

$$-\frac{\partial h(t, x)}{\partial t} + \frac{1}{2}\frac{\partial^2 h(t, x)}{\partial^2 x} + i\mu \mathbf{1}_{(l<x<u)} h(t, x) = 0 \tag{A.1}$$

with initial condition $h(0, x) = e^{-\alpha x}$. We can make a second transformation defining $z = (x - l)/(u - l)$ and, introducing a new function,

$$y(t, z) = h(t, (u - l)z + l)$$

we get the following pde for the function $y(t, z)$:

$$-\frac{\partial y(t, z)}{\partial t} + c^2\frac{\partial^2 y(t, z)}{\partial^2 z} + i\mu \mathbf{1}_{(0<z<1)} y(t, z) = 0, \tag{A.2}$$

where $c^2 = 1/2(u - l)^2$, and $y(0, z) = \exp(-\alpha((u - l)z + l))$.

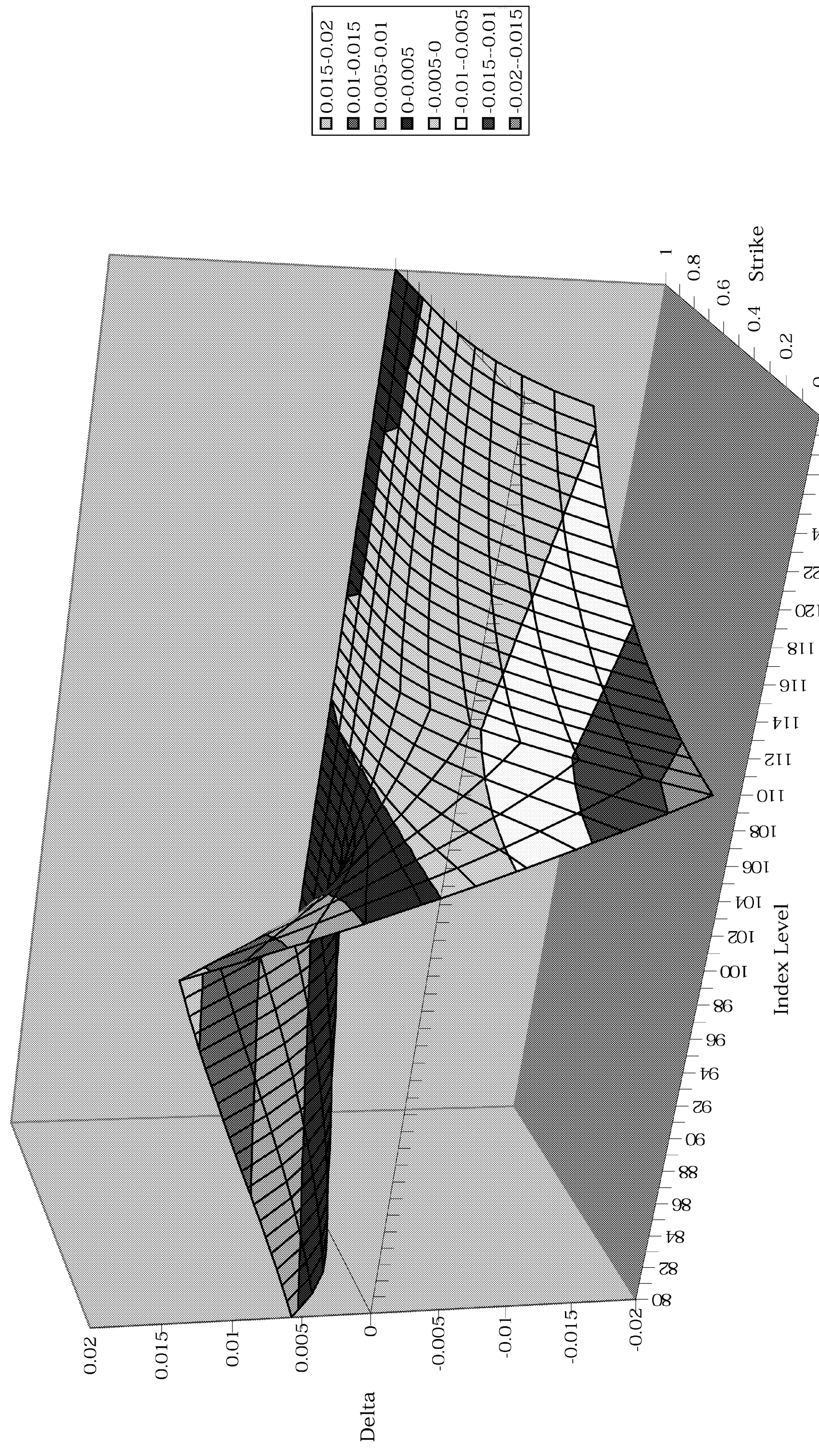
0.015-0.02
0.01-0.015
0.005-0.01
0-0.005
-0.005-0
-0.01--0.005
-0.015--0.01
-0.02--0.015
Delta
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
Index Level
80
82
84
86
88
90
92
94
96
98
100
102
104
106
108
110
112
114
116
118
120
122
124
126
128
130
Strike
0
0.2
0.4
0.6
0.8
1

FIG. 2.

In order to solve (A.2) we can distinguish three cases, $z < 0, 0 < z < 1$ and $z > 1$ and require a continuous and differentiable solution at these boundary points. As a consequence, we can guarantee that the characteristic function as well is continuous and differentiable at $x = l$ and at $x = u$. So we can solve the pde (A.2) imposing Neumann boundary conditions at $z = 0$ and at $z = 1$,

$$\begin{aligned} \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=0^-} &= L(t) = \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=0^+}, \\ \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=1^-} &= U(t) = \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=1^+}, \end{aligned} \tag{A.3}$$

where $L(t)$ and $U(t)$ are unknown functions to be chosen in order to have continuity of the solution at $z = 0$ and at $z = 1$, that is,

$$\begin{aligned} y(t,z)\big|_{z=0^-} &= y(t,z)\big|_{z=0^+}, \\ y(t,z)\big|_{z=1^-} &= y(t,z)\big|_{z=1^+}. \end{aligned} \tag{A.4}$$

In the case $z < 0$ and $z > 1$, to solve (A.2) with boundary conditions (A.3) amounts to solving the heat equation in a semiinfinite region with Neumann boundary condition. The solution can be found in [21], page 268, equation 5.121. In the case $0 < z < 1$, considering the new transformation $g(t,z) = e^{-i\mu t}y(t,z)$, we get that $g(t,z)$ satisfies the heat equation in a finite strip with Neumann boundary $e^{-i\mu t}L(t)$ and $e^{-i\mu t}U(t)$ at $z = 0$ and $z = 1$, respectively. The solution in this case can be found in [21], page 62, equation 2-73a. In conclusion, the function $y(t,z)$ can be expressed in terms of the unknown functions $L(t)$ and $U(t)$ in the following way:

$$\begin{aligned} &y(t,z) \\ &= \begin{cases} 1_{(z>1)} \left(\begin{array}{l} e^{-\alpha u} \displaystyle\int_0^{+\infty} [G(z-1-\xi,t) + G(z-1+\xi,t)] e^{-\alpha(u-l)\xi}\, d\xi \\ -2c^2 \displaystyle\int_0^t G(z-1,t-\theta) U(\theta)\, d\theta \end{array} \right) \\ 1_{(0<z<1)} \left(\begin{array}{l} e^{i\mu t} \dfrac{e^{-\alpha l} - e^{-\alpha u}}{\alpha(u-l)} + c^2 \displaystyle\int_0^t e^{i\mu(t-\theta)} (U(\theta) - L(\theta))\, d\theta \\ + 2c^2 \displaystyle\sum_{n=1}^{+\infty} y_n(t) \cos n\pi z + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t} \phi_n \cos n\pi z \end{array} \right) \\ 1_{(z<0)} \left(\begin{array}{l} e^{-\alpha l} \displaystyle\int_0^{+\infty} [G(-z-\xi,t) + G(-z+\xi,t)] e^{\alpha(u-l)\xi}\, d\xi \\ +2c^2 \displaystyle\int_0^t G(-z,t-\theta) L(\theta)\, d\theta \end{array} \right), \end{cases} \end{aligned} \tag{A.5}$$

where

$$G(x,t) = \frac{\exp(-x^2/4c^2t)}{\sqrt{4\pi c^2 t}},$$

$$\phi_n = 2\int_0^1 \exp(-\alpha((u-l)\xi + l))\cos(n\pi\xi)\,d\xi,$$

$$y_n(t) = \int_0^t \exp((i\mu - \lambda_n)(t-\theta))((-1)^n U(\theta) - L(\theta)),$$

$$\lambda_n = (n\pi c)^2.$$

In order to find the unknown functions $U(t)$ and $L(t)$ we now require the continuity of the function $y(t,z)$ at $z=0$ and $z=1$; that is, we impose conditions (A.4). However, it is more convenient to transform the continuity conditions in the following way:

$$\lim_{z\to 1^+} y(t,z) + \lim_{z\to 0^-} y(t,z) = \lim_{z\to 1^-} y(t,z) + \lim_{z\to 0^+} y(t,z),$$

$$\lim_{z\to 1^+} y(t,z) - \lim_{z\to 0^-} y(t,z) = \lim_{z\to 1^-} y(t,z) - \lim_{z\to 0^+} y(t,z)$$

and introducing the functions $D(t) := U(t) - L(t)$ and $S(t) := U(t) + L(t)$, we obtain

$$\begin{aligned}
&\lim_{z\to 1^+} y(t,z) + \lim_{z\to 0^-} y(t,z) \\
&\quad = \exp\left(-\alpha u + \frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(\alpha\frac{\sqrt{t}}{\sqrt{2}}\right) + \exp\left(-\alpha l + \frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) \\
&\qquad - 2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}} D(\theta)\,d\theta
\end{aligned}$$

$$\begin{aligned}
&\lim_{z\to 1^+} y(t,z) - \lim_{z\to 0^-} y(t,z) \\
&\quad = \exp\left(-\alpha u + \frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(\alpha\frac{\sqrt{t}}{\sqrt{2}}\right) - \exp\left(-\alpha l + \frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) \\
&\qquad - 2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}} S(\theta)\,d\theta
\end{aligned}$$

$$\begin{aligned}
&\lim_{z\to 1^-} y(t,z) + \lim_{z\to 0^+} y(t,z) \\
&\quad = 2\,\exp(i\mu t e^{-\alpha l} - e^{-\alpha u}/\alpha(u-l)) + 2c^2\int_0^t e^{i\mu(t-\theta)} D(\theta)\,d\theta
\end{aligned}$$

$$+\sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n(1+(-1)^n)$$

$$+2c^2\sum_{n=1}^{+\infty}(1+(-1)^n)\int_0^t \exp((i\mu-\lambda_n)(t-\theta))D(\theta)\,d\theta$$

$$\lim_{z\to 1^-} y(t,z)-\lim_{z\to 0^+} y(t,z)$$

$$=2c^2\sum_{n=1}^{+\infty}(1-(-1)^n)\int_0^t \exp((i\mu-\lambda_n)(t-\theta))S(\theta)\,d\theta$$

$$+\sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n((-1)^n-1)$$

The determination of the functions $D(t)$ and $S(t)$ requires solving in respect to them the following integral equations:

$$\begin{aligned}
&\exp\left(-\alpha u+\frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)+\exp\left(-\alpha l+\frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)\\
&\quad-2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}}D(\theta)\,d\theta\\
&=2e^{i\mu t}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)}+2c^2\int_0^t e^{i\mu(t-\theta)}D(\theta)d\theta\\
&\quad+\sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n(1+(-1)^n)\\
&\quad+2c^2\sum_{n=1}^{+\infty}(1+(-1)^n)\int_0^t \exp((i\mu-\lambda_n)(t-\theta))D(\theta)\,d\theta
\end{aligned}\tag{A.6}$$

$$\begin{aligned}
&\exp\left(-\alpha u+\frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)-\exp\left(-\alpha l+\frac{1}{2}\alpha^2 t\right)\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)\\
&\quad-2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}}S(\theta)d\theta\\
&=2c^2\sum_{n=1}^{+\infty}(1-(-1)^n)\int_0^t \exp((i\mu-\lambda_n)(t-\theta))S(\theta)\,d\theta\\
&\quad+\sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n((-1)^n-1).
\end{aligned}\tag{A.7}$$

We remark that these integral equations involve separately the functions $S(t)$ and $D(t)$. We solve them using the Laplace transform. We call $s(\gamma)$ and $d(\gamma)$ the Laplace transforms, with respect to the time variable t, of the functions $S(t)$ and $D(t)$,

$$s(\gamma) := \mathscr{L}[S(t); t \to \gamma] = \int_0^t e^{-\gamma t} S(t)\, dt,$$

$$d(\gamma) := \mathscr{L}[D(t); t \to \gamma] = \int_0^t e^{-\gamma t} D(t)\, dt.$$

Laplace transforming (A.6) and (A.7), we obtain two linear equations, one for $s(\gamma)$ and the other for $d(\gamma)$,

$$\begin{aligned} &\frac{e^{-\alpha u}}{\sqrt{\gamma}(\sqrt{\gamma}+\alpha/\sqrt{2})} + \frac{e^{-\alpha l}}{\sqrt{\gamma}(\sqrt{\gamma}-\alpha/\sqrt{2})} - \frac{2c^2}{\sqrt{4c^2\gamma}} d(\gamma) \\ &\quad = 2\frac{1}{\gamma - i\mu}\frac{e^{-\alpha l} - e^{-\alpha u}}{\alpha(u-l)} + 2c^2 \frac{1}{\gamma - i\mu} d(\gamma) + \frac{1}{\pi^2 c^2} \sum_{n=1}^{+\infty} \frac{(1+(-1)^n)}{n^2+a^2}\phi_n \\ &\qquad + d(\gamma)\frac{2c^2}{\pi^2 c^2}\sum_{n=1}^{+\infty}\frac{(1+(-1)^n)}{n^2+a^2}, \end{aligned} \tag{A.8}$$

$$\begin{aligned} &\frac{e^{-\alpha u}}{\sqrt{\gamma}(\sqrt{\gamma}+\alpha/\sqrt{2})} - \frac{e^{-\alpha l}}{\sqrt{\gamma}(\sqrt{\gamma}-\alpha/\sqrt{2})} - \frac{2c^2}{\sqrt{4c^2\gamma}} s(\gamma) \\ &\quad = +s(\gamma)\frac{2c^2}{\pi^2 c^2}\sum_{n=1}^{+\infty}\frac{(1-(-1)^n)}{n^2+a^2} + \frac{1}{\pi^2 c^2}\sum_{n=1}^{+\infty}\frac{((-1)^n - 1)}{n^2+a^2}. \end{aligned}$$

Using the following summation series formula in [15], page 40, as formula 1.445.1,

$$\sum_{n=1}^{\infty} \frac{n\,\sin(nx)}{n^2+a^2} = \frac{\pi}{2}\frac{\sinh[a(\pi - x)]}{\sinh[a\pi]}; \qquad 0 < x < 2\pi, \tag{A.9}$$

formula 1.445.2,

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+a^2} = \frac{\pi}{2a}\frac{\cosh[a(\pi - x)]}{\sinh[a\pi]} - \frac{1}{2a^2}; \qquad 0 < x \le 2\pi, \tag{A.10}$$

formula 1.445.3,

$$\sum_{n=1}^{\infty} \frac{(-1)^n\cos(nx)}{n^2+a^2} = \frac{\pi}{2a}\frac{\cosh[ax]}{\sinh[a\pi]} - \frac{1}{2a^2}; \qquad -\pi \le x \le \pi \tag{A.11}$$

and formula 1.445.4,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n \sin(nx)}{n^2+a^2} = -\frac{\pi}{2}\frac{\sinh[ax]}{\sinh[a\pi]}; \qquad -\pi < x < \pi, \tag{A.12}$$

where $a\pi = \sqrt{(\gamma - i\mu)/c^2}$, we obtain

$$\begin{aligned} &\frac{e^{-\alpha u}}{\sqrt{\gamma}(\sqrt{\gamma}+\alpha/\sqrt{2})} + \frac{e^{-\alpha l}}{\sqrt{\gamma}(\sqrt{\gamma}-\alpha/\sqrt{2})} \\ &\qquad - \frac{1}{c}\int_0^1 \frac{\cosh(a\pi(1-x)) + \cosh(a\pi x)}{\sqrt{\gamma - i\mu}\,\sinh(a\pi)} \exp(-\alpha((u-l)x+l))\,dx \\ &\qquad = c\left(\frac{1}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma-i\mu}}\frac{\cosh(a\pi)+1}{\sinh(a\pi)}\right) d(\gamma), \\ &\frac{e^{-\alpha u}}{\sqrt{\gamma}(\sqrt{\gamma})+\alpha/\sqrt{2})} - \frac{e^{-\alpha l}}{\sqrt{\gamma}(\sqrt{\gamma}-\alpha/\sqrt{2})} \\ &\qquad - \frac{1}{c}\int_0^1 \frac{\cosh(a\pi x) - \cosh(a\pi(1-x))}{\sqrt{\gamma - i\mu}\,\sinh(a\pi)} \exp(-\alpha((u-l)x+l))\,dx \\ &\qquad = +c\left(\frac{1}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma-i\mu}}\frac{\cosh(a\pi)-1}{\sinh(a\pi)}\right) s(\gamma). \end{aligned} \tag{A.13}$$

We observe

$$\begin{aligned} &\frac{1}{c}\int_0^1 \frac{\cosh(a\pi(1-x)) + \cosh(a\pi x)}{\sqrt{\gamma - i\mu}\,\sinh(a\pi)} \exp(-\alpha((u-l)x+l))\,dx \\ &\quad = \frac{(\sqrt{\gamma - i\mu}(e^{-\alpha u}+e^{-\alpha l})\sinh(a\pi) + (\alpha/\sqrt{2})(e^{-\alpha u}-e^{-\alpha l})(\cosh(a\pi)+1))}{\sqrt{\gamma-i\mu}\,\sinh(a\pi)(\gamma - i\mu - \alpha^2/2)}, \end{aligned}$$

$$\begin{aligned} &\frac{1}{c}\int_0^1 \frac{\cosh(a\pi x) - \cosh(a\pi(1-x))}{\sqrt{\gamma - i\mu}\,\sinh(a\pi)} \exp(-\alpha((u-l)x+l))\,dx \\ &\quad = \frac{(\sqrt{\gamma - i\mu}(e^{-\alpha u}-e^{-\alpha l})\sinh(a\pi) + (\alpha/\sqrt{2})(e^{-\alpha u}+e^{-\alpha l})(\cosh(a\pi)-1))}{\sqrt{\gamma-i\mu}\,\sinh(a\pi)(\gamma - i\mu - \alpha^2/2)}, \end{aligned}$$

so substituting these expressions in (A.13) and solving in respect to the quantities $cd(\gamma)/\sqrt{\gamma}$ and $cs(\gamma)/\sqrt{\gamma}$, we obtain (3.6) in Theorem 1.

From (A.5), we get as well the Laplace transform of the function $y(t,z)$ when $z=0$ and when $z=1$. Then the expressions for the characteristic function when $x=l$ and when $x=u$, are

$$\begin{aligned} v(t,u) &= \exp\left(-mu - \frac{m^2}{2}t\right) y(t,1), \\ v(t,l) &= \exp\left(-ml - \frac{m^2}{2}t\right) y(t,0). \end{aligned} \tag{A.14}$$

A.1. *Solution of the pde with Dirichlet boundary condition.* In order to find the expression for the function $v(t,x)$ for a generic value of x, we can now solve the pde (3.2) in three different regions ($x<l, l<x<u$ and $u<x$) using as Dirichlet boundary conditions at $x=l$ and $x=u$ the known values in (A.14). This, after the same transformation as before, amounts to solving

$$-\frac{\partial y(t,z)}{\partial t} + c^2\frac{\partial^2 y(t,z)}{\partial^2 z} + i\mu \mathbf{1}_{(0<z<1)} y(t,z) = 0, \tag{A.15}$$

where $c^2 = 1/2(u-l)^2$, and $y(0,z) = \exp(-\alpha((u-l)z+l))$, in three different regions using Dirichlet boundary conditions at $z=0$ and at $z=1$ the known values of $y(t,0)$ and $y(t,1)$.

A.1.1. *Case $x<l$ and $x>u$.* In this case, we have the heat equation with Dirichlet boundary conditions; the solution can be found in [30], page 265, equation 5.105,

$$y(t,x) = \begin{cases} \mathbf{1}_{(x>u)} \left(\begin{aligned} &\int_0^{+\infty} \left[\frac{\exp(-(x-u-\zeta)^2/2t)}{\sqrt{2\pi t}} \right. \\ &\qquad \left. - \frac{\exp(-(x-u+\zeta)^2/2t)}{\sqrt{2\pi t}} \right] e^{-\alpha(\zeta+u)}\, d\zeta \\ &+ \int_0^t \frac{(x-u)}{\sqrt{2\pi(t-\theta)^3}} \exp\Big(-\frac{(x-u)^2}{2(t-\theta)}\Big) e^{-\alpha u - \beta\theta} q(\theta)\, d\theta \end{aligned} \right) \\ \mathbf{1}_{(x<l)} \left(\begin{aligned} &\int_0^{+\infty} \left[\frac{\exp(-(l-x-\zeta)^2/2t)}{\sqrt{2\pi t}} \right. \\ &\qquad \left. - \frac{\exp(-(l-x+\zeta)^2/2t)}{\sqrt{2\pi t}} \right] e^{-\alpha(l-\zeta)}\, d\zeta \\ &+ \int_0^t \frac{(l-x)\exp(-(l-x)^2/2(t-\theta))}{\sqrt{2\pi(t-\theta)^3}} e^{-\alpha l - \beta\theta} p(\theta)\, d\theta \end{aligned} \right) \end{cases}$$

and then

$$
\begin{aligned}
v(t,x) &= e^{\alpha x+\beta t}y(t,x)\\
&\mathbf{1}_{(x>u)}\left(\begin{aligned}
&e^{\alpha x+\beta t}\int_0^{+\infty}\left[\frac{\exp(-(x-u-\zeta)^2/2t)}{\sqrt{2\pi t}}\right.\\
&\qquad\qquad\left.-\frac{\exp((x-u+\zeta)^2/2t)}{\sqrt{2\pi t}}\right]e^{-\alpha(u+\zeta)}\,d\zeta\\
&+e^{\alpha(x-u)}\int_0^t\frac{(x-u)\exp(-(x-u)^2/2(t-\theta))}{\sqrt{2\pi(t-\theta)^3}}e^{\beta(t-\theta)}q(\theta)\,d\theta
\end{aligned}\right)\\
&\mathbf{1}_{(x<l)}\left(\begin{aligned}
&e^{\alpha x+\beta t}\int_0^{+\infty}\left[\frac{\exp(-(l-x-\zeta)^2/2t)}{\sqrt{2\pi t}}\right.\\
&\qquad\qquad\left.-\frac{\exp(-(l-x+\zeta)^2/2t)}{\sqrt{2\pi t}}\right]e^{-\alpha(l-\zeta)}\,d\zeta\\
&+e^{-\alpha(l-x)}\int_0^t\frac{(l-x)\exp(-(l-x)^2/2(t-\theta))}{\sqrt{2\pi(t-\theta)^3}}e^{\beta(t-\theta)}p(\theta)\,d\theta
\end{aligned}\right),
\end{aligned}
\tag{A.16}
$$

where $q(t)$ and $p(t)$ are the known values of the characteristic function at $x=u$ and $x=l$: $q(t):=v(t,u)$ and $q(t):=v(t,l)$.

Using some algebra and comparing with [4], formula 1.2.4, page 198, we can remark that when $x>u$,

$$
\begin{aligned}
&= e^{\beta t}\int_0^{+\infty}\left(\frac{\exp(-(x-u-\zeta)^2/2t)}{\sqrt{2\pi t}}\right.\\
&\qquad\qquad\left.-\frac{\exp(-(x-u+\zeta)^2/2t)}{\sqrt{2\pi t}}\right)e^{\alpha(x-u-\zeta)}\,d\zeta\\
&= 1-\left[\frac{1}{2}\mathrm{Erfc}\left(\frac{x-u+mt}{\sqrt{2t}}\right)+\frac{e^{-2m(x-u)}}{2}\mathrm{Erfc}\left(\frac{x-u-mt}{\sqrt{2t}}\right)\right]\\
&= \Pr_{0,x\in(u,+\infty)}\left(\inf_{0\le s\le t} ms+W(s)>u\right),
\end{aligned}
\tag{A.17}
$$

where we have used the fact that $\beta=-m^2/2$ and $\alpha=-m$. Similarly, when $x<l$, it can be shown with some algebra and using formula 1.1.4, page 197 in [4], that

$$
\begin{aligned}
&e^{\alpha x+\beta t}\int_0^{+\infty}\left[\frac{\exp(-(l-x-\zeta)^2/2t)}{\sqrt{2\pi t}}-\frac{\exp(-(l-x+\zeta)^2/2t)}{\sqrt{2\pi t}}\right]e^{-\alpha(l-\zeta)}\,d\zeta\\
&\quad= 1-\left[\frac{1}{2}\mathrm{Erfc}\left(\frac{l-x-mt}{\sqrt{2t}}\right)+\frac{e^{2m(l-x)}}{2}\mathrm{Erfc}\left(\frac{l-x+mt}{\sqrt{2t}}\right)\right]\\
&\quad= \Pr_{0,x\in(-\infty,l)}\left(\sup_{0\le s\le t} ms+W(s)<l\right).
\end{aligned}
\tag{A.18}
$$

In order to find the expression for $\Omega(t,\mu,x;u,l,m)$ in Theorem 1, we can use (A.14); substituting it in (A.16), we obtain

$$\Omega(t,\mu,x;u,l,m)=\begin{cases}\mathbf{1}_{(x>u)}e^{\alpha x+\beta t}\displaystyle\int_0^t\frac{(x-u)\exp(-(x-u)^2/2(t-\theta))}{\sqrt{2\pi(t-\theta)^3}}y(\theta,1)d\theta\\[2ex]\mathbf{1}_{(x<l)}e^{\alpha x+\beta t}\displaystyle\int_0^t\frac{(l-x)\exp(-(l-x)^2/2(t-\theta))}{\sqrt{2\pi(t-\theta)^3}}y(\theta,0)d\theta\end{cases}$$

and then if we consider the Laplace transform of the integrals we obtain the expression for $\omega(\gamma,\mu,x;l,u,m)$ in Theorem 1.

We now show how to find the density function of the occupation time given in (3.8) in Theorem 1. Using in (A.16) the fact that

$$\text{(A.19)}\qquad \begin{aligned}q(t)&=v(t,u;l,u)=\int_0^t e^{i\mu\theta}f_\tau(\theta,t,u)\,d\theta,\\ p(t)&=v(t,l;l,u)=\int_0^t e^{i\mu\theta}f_\tau(\theta,t,l)\,d\theta,\end{aligned}$$

we can observe that, for $x>u$, we have

$$\begin{aligned}&e^{\alpha(x-u)}\int_0^t\frac{(x-u)\exp(-(x-u)^2/2(t-\tau))}{\sqrt{2\pi(t-\tau)^3}}e^{\beta(t-\tau)}q(\tau)\,d\tau\\
&\quad=e^{\alpha(x-u)}\int_0^t\frac{(x-u)\exp(-(x-u)^2/2(t-\tau))}{\sqrt{2\pi(t-\tau)^3}}e^{\beta(t-\tau)}\\
&\qquad\qquad\times\left(\int_0^\tau e^{i\mu\theta}f_\tau(\theta,\tau,u)\,d\theta\right)d\tau\\
&\quad=e^{\alpha(x-u)}\int_0^t e^{i\mu\theta}\left(\int_\theta^t\frac{(x-u)\exp(-(x-u)^2/2(t-\tau))}{\sqrt{2\pi(t-\tau)^3}}\right.\\
&\qquad\qquad\qquad\left.\times e^{\beta(t-\tau)}f_\tau(\theta,\tau,u)d\tau\right)d\theta\end{aligned}$$

and then

$$\begin{aligned}v(t,x;l,u)&=1\times\Pr_{0,x\in(u,+\infty)}\left(\inf_{0\le s\le t}ms+W(s)>u\right)\\
&\quad+e^{\alpha(x-u)}\int_0^t e^{i\mu\theta}\left(\int_\theta^t\frac{(x-u)\exp(-(x-u)^2/2(t-\tau))}{\sqrt{2\pi(t-\tau)^3}}\right.\\
&\qquad\qquad\left.\times e^{\beta(t-\tau)}f_\tau(\theta,\tau,u)\,d\tau\right)d\theta\end{aligned}$$

and so comparing with (3.2) the density function of the occupation when $x>u$ can be expressed in terms of the density function when $x=u$. Similarly, for

$x < l$, we obtain

$$v(t, x; l, u) = 1 \times \mathrm{Pr}_{0, x\in(-\infty, l)}\left(\sup_{0\le s\le t} ms + W(s) < l\right)$$
$$+ e^{-\alpha(l-x)} \int_0^t e^{i\mu\theta}\left(\int_\theta^t \frac{(l-x)\exp(-(l-x)^2/2(t-\tau))}{\sqrt{2\pi(t-\tau)^3}}\right.$$
$$\left. \times e^{\beta(t-\tau)} f_\tau(\theta, \tau, l)\, d\tau\right) d\theta.$$

Comparing these expressions with (3.2), the density function of the occupation when $x < l$ can be expressed in terms of the density function when $x = l$.

A.1.2. *Case* $l < x < u$. In this case the pde (3.2) becomes

$$-\frac{\partial v(t, x)}{\partial t} + \frac{1}{2}\frac{\partial^2 v(t, x)}{\partial^2 x} + m\frac{\partial v(t, x)}{\partial x} + i\mu v(t, x) = 0 \tag{A.20}$$

and has to be solved in the finite region $l < x < u$, with boundary conditions

$$v(t, u) = q(t),$$
$$v(t, l) = p(t)$$

and initial condition

$$v(0, x) = 1.$$

If we consider the transformation $y(t, z) = \exp(-\alpha((u-l)z+l) - kt)v(t, (u-l)z + l)$, we get for the function $y(t, z)$ the heat equation in a finite region $0 < z < 1$,

$$-\frac{\partial y(t, z)}{\partial t} + c^2\frac{\partial^2 y(t, z)}{\partial^2 z} = 0 \tag{A.21}$$

with boundary conditions

$$y(t, 1) = e^{-\alpha u - kt} q(t); \qquad y(t, 0) = e^{-\alpha l - kt} p(t)$$

and initial condition $y(0, z) = \exp(-\alpha((u-l)z+l))$. The solution can be found in [21], page 62, equation 2-73a. Then the expression for the characteristic function is given by

$$v(t, x) = e^{i\mu t} 2\exp(\alpha x - \frac{\alpha^2}{2}t)\int_0^1 \left[\sum_{n=1}^{\infty} \exp(-(cn\pi)^2 t)\sin(n\pi z)\sin(n\pi\xi)\right]$$
$$\times \exp(-\alpha(\xi(u-l)+l))\, d\xi + e^{\alpha x + kt}\sum_{n=1}^{\infty} w_n(t)\sin\left(n\pi\left(\frac{x-l}{u-l}\right)\right),$$

where

$$w_n(t) = 2n\pi c^2 \int_0^t e^{-\lambda_n(t-s)}[e^{-\alpha l - ks}p(s) - (-1)^n e^{-\alpha u - ks} q(s)]\, ds. \tag{A.22}$$

For the properties of the theta function, compare [20] at pages 25 and 26 and [6] at page 62, formula 6.3.1, and using the expression in [4], formula 1.15.4, page 211, we obtain

$$\begin{aligned}
&2\,\exp(-mx - \frac{\alpha^2}{2}t)\int_0^1 \left[\sum_{n=1}^{\infty} \exp\left(-(cn\pi)^2 t\right)\,\sin(n\pi z)\,\sin(n\pi\xi)\right] \\
&\qquad \times \exp(m(\xi(u-l)+l))\,d\xi \qquad \text{(A.23)}\\
&\quad = \Pr_{0,x\in(l,u)}\left(\sup_{0\le s\le t} ms + W(s) < u;\ \inf_{0\le s\le t} ms + W(s) > l\right).
\end{aligned}$$

In order to find the expression for $\Omega(t, x; u, l, m)$ when $x \in (l, u)$ in Theorem 1, we can use (A.14) and substituting it in (A.22), we obtain

$$w_n(t) = 2n\pi c^2 \int_0^t \exp(-\lambda_n(t-s) - i\mu s)[y(s,0) - (-1)^n y(s,1)]\,ds$$

and then

$$\begin{aligned}
&\Omega(t, \mu, x; u, l, m)\\
&\quad = e^{\alpha x + \beta t}\sum_{n=1}^{\infty} 2n\pi c^2\,\sin\left(n\pi\left(\frac{x-l}{u-l}\right)\right)\int_0^t\ \exp(-(\lambda_n - i\mu)(t-s))\\
&\qquad\qquad \times(y(s,0) - (-1)^n y(s,1))\,ds.
\end{aligned}$$

If we consider the Laplace transform of the series, we get

$$\begin{aligned}
&\omega(\gamma, \mu, x; l, u, m)\\
&\quad = \frac{1}{\pi^2c^2}\sum_{n=1}^{\infty}\frac{2n\pi c^2}{n^2 + ((\gamma - i\mu)/(\pi^2c^2))}\,\sin\left(n\pi\left(\frac{x-l}{u-l}\right)\right)\\
&\qquad \times(\mathscr{L}[y(t,0); t\to\gamma] - (-1)^n\mathscr{L}[y(t,1); t\to\gamma])\,ds
\end{aligned}$$

and using the summation formulas (A.9) and (A.10) we obtain the expression in Theorem 1.

We now show how to find the density function of the occupation time given in (3.8) in Theorem 1. Substituting in expression (A.22) the functions $q(t)$ and $p(t)$ as given in (A.19), we have

$$\begin{aligned}
&e^{\alpha x + kt} w_n(t)\\
&\quad = -2n\pi c^2\int_0^t \exp(k - (n\pi c)^2)(t-s))\\
&\qquad \times\left[(-1)^n e^{-\alpha(u-x)}\int_0^s e^{i\mu\theta} f_\tau(\theta, s, u) - e^{\alpha(x-l)}\int_0^s e^{i\mu\theta} f_\tau(\theta, s, l)\,d\theta\right] ds.
\end{aligned}$$

With a change of variable, ($\xi = t - s,\ \nu = t + \tau - s = \xi + \tau$), and using the fact that $k = -\alpha^2/2 + i\mu$, we get

$$= -2n\pi c^2\int_0^t \exp\left(-\left(\frac{\alpha^2}{2} + (n\pi c)^2\right)\xi\right)$$

$$
\begin{aligned}
&\times \int_{\xi}^{t} [(-1)^n e^{-\alpha(u-x)} e^{i\mu\theta} f_{\tau}(\theta - \xi, t - \xi, u) \\
&\qquad - e^{\alpha(x-l)} e^{i\mu\theta} f_{\tau}(\theta - \xi, t - \xi, l)]\, d\theta\, d\xi \\
&= 2n\pi c^2 \int_0^t e^{i\mu\theta} \\
&\qquad \times \int_0^{\theta} \exp\left(-\left(\frac{\alpha^2}{2} + (n\pi c)^2\right)\xi\right)(e^{\alpha(x-l)} f_{\tau}(\theta - \xi, t - \xi, l) \\
&\qquad - (-1)^n e^{-\alpha(u-x)} f_{\tau}(\theta - \xi, t - \xi, u))\, d\xi\, d\theta.
\end{aligned}
$$

So for a generic starting point $x \in (l, u)$ we have

$$
\begin{aligned}
&v(t, x; l, u) \\
&= e^{i\mu t} \Pr_{0, x\in(l,u)}\left(\sup_{0\le s\le t} ms + W(s) < u; \inf_{0\le s\le t} ms + W(s) > l\right) \\
&\quad + \int_0^t e^{i\mu\theta}\Bigg[\sum_{n=1}^{\infty} 2n\pi c^2 \sin\left(n\pi\left(\frac{x-l}{u-l}\right)\right) \\
&\qquad\qquad \times \int_0^{\theta} \exp\left(-\left(\frac{\alpha^2}{2} + n^2\pi^2 c^2\right)\xi\right)(e^{\alpha(x-l)} f_{\tau}(\theta - \xi, t - \xi, l) \\
&\qquad\qquad\qquad - (-1)^n e^{-\alpha(u-x)} f_{\tau}(\theta - \xi, t - \xi, u))\, d\xi\Bigg]\, d\theta
\end{aligned}
$$

and so we recognize in the square brackets inside the integral the density function of the occupation when $l < x < u$, as shown in Theorem 1.

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