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From integrability to conductance, impurity systems

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Abstract

We compute the DC conductance with two different methods, which both exploit the integrability of the theories under consideration. On one hand we determine the conductance through a defect by means of the thermodynamic Bethe ansatz and standard relativistic potential scattering theory based on a Landauer transport theory picture. On the other hand we propose a Kubo formula for a defect system and evaluate the current-current two-point correlation function it involves with the help of a form factor expansion. For a variety of defects in a fermionic system we find excellent agreement between the two different theoretical descriptions.
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1 Introduction

Conductance (conductivity) measurements belong to the easiest and most direct experiments which can be carried out. They attract a lot of attention, due to the fact that in general they can be performed without perturbing very much the behaviour of the system, e.g. a rigid-lattice bulk metal, such that the uncertainty of experimental artefacts is reduced to a minimum. There exist various well-known theoretical descriptions, such as semi-classical transport theories (Landauer [1] and Boltzmann-Drude [2]), dynamical linear-response theory [3, 4] and also Green function linear-response theory [5]. To carry out the latter, in particular at finite temperature, is still poorly understood in generality [6], even in 1+1 space-time dimensions [7]. Since recent experimental progress allows conductance measurements also in 1+1 space-time dimensions [8], one can on the theoretical side fully exploit the special features of low dimensionality.

It is in particular very suggestive to exploit the full scope of non-perturbative techniques which have been developed in the context of integrable quantum field theories in 1+1 space-time dimensions, such as the thermodynamic Bethe ansatz (TBA) [9, 10] and the form factor bootstrap approach [11, 12]. Generalizing the Landauer transport picture a proposal for the conductance through a quantum wire with a defect (impurity) has been made in [13, 14]

$$G^{\boldsymbol{\alpha}}(T)=\sum_{i}\lim_{(\mu_i^l-\mu_i^r)\to 0}\frac{q_i}{2}\int_{-\infty}^{\infty}d\theta\left[\rho_i^r(\theta,T,\mu_i^l)|T_i^{\boldsymbol{\alpha}}(\theta)|^2-\rho_i^r(\theta,T,\mu_i^r)|\tilde{T}_i^{\boldsymbol{\alpha}}(\theta)|^2\right], \tag{1.1}$$

which we only modify to accommodate parity breaking, known to occur in integrable lattice models, see e.g. [15]. This means in particular we allow the transmission amplitudes to be different for a particle of type i with charge q_i passing with rapidity θ through a defect of type $\boldsymbol{\alpha}$ from the left $T_i^{\boldsymbol{\alpha}}(\theta)$ and right $\tilde{T}_i^{\boldsymbol{\alpha}}(\theta)$. The density distribution function $\rho_i^r(\theta,T,\mu_i)$, being a function the temperature T, and the potential at the left μ_i^l and right μ_i^r constriction of the wire, can be determined by means of the TBA. We have already restricted (1.1) to the abelian (diagonal) situation. It is clear that the effect resulting from the defect is most interesting when $|T_i^{\boldsymbol{\alpha}}(\theta)|\neq 1$, which requires the occurrence of simultaneous transmission and reflection (see (2.6), (2.17)). In this paper we will therefore be mainly interested in that situation. One may adapt (1.1) also to the case of pure reflection, which physically describes the influence of the constriction to the conducting process. From the previous statement it is clear that such boundary theories are only interesting in this physical context when they are non-abelian.

The other prominent way of determining the conductance is a result from linear response theory, which yields an expression for the conductance in form of the Fourier transform of the current-current two-point correlation function. This Kubo formula has been adopted to the situation with a boundary [16]. As we mentioned, this will only capture effects coming from the constriction of the wire, we propose here a generalization to the analogous situation as described in (1.1), i.e. when a defect is present

$$G^{\boldsymbol{\alpha}}(T)=-\lim_{\omega\to 0}\frac{1}{2\omega\pi^2}\int_{-\infty}^{\infty}dt\; e^{i\omega t}\;\;\langle J(t)Z_{\boldsymbol{\alpha}}\,J(0)\rangle_{T,m}\,. \tag{1.2}$$

Here the defect operator $Z_{\boldsymbol{\alpha}}$ enters in-between the two currents J within the temperature and mass m dependent correlation function. The Matsubara frequency is denoted by ω.

The main purpose of this manuscript is to compare the two alternative descriptions (1.1) and (1.2) for massive bulk theories with a defect which allows for simultaneous reflection and transmission. There exist various investigations, e.g., [17, 13, 14, 18] for conformal (massless) theories with defect, which exploit the original folding idea of Wong and Affleck [17]. The idea is that a conformal field theory with a *purely* transmitting or reflecting defect can be mapped into a boundary theory, i.e. a theory living in half space, which has the advantage that the full restriction of modular invariance can be exploited in the construction of boundary states as pioneered by Cardy [19]. Since this folding idea relies on the vanishing of either the reflection or transmission, our considerations do in general not reduce to that set up, even in the conformal limit. As was already pointed out in [17], and as can be seen directly from (1.1) and (1.2), in that case the conductance is less interesting because it is either zero or perfect for abelian theories.

In section 2 we outline the procedure of how the defect scattering matrices may be determined, since they are needed as input in both approaches. In section 3 we newly formulate the defect TBA equations and use them to determine the density distribution functions. We evaluate numerically the Landauer formula (1.1) for various defects and provide some analytical approximations in certain regimes. In section 4 we propose a Kubo formula (1.2) for a configuration in which an impurity is present and compute the current-current two-point correlation functions occurring in there by means of a form factor expansion. We find very good agreement between (1.1) and (1.2) for the complex free Fermion theory with various types of defects. Our final conclusions and an outlook into open problems is provided in section 5.

2 Determining the defect scattering matrices

An essential input required in both non-perturbative methods which are exploited to compute the conductance (1.1) and (1.2), that is the TBA and the form factor bootstrap approach, respectively, is the knowledge of the exact (defect) scattering matrix. It is one of the most intriguing facts of two dimensional quantum field theories that these matrices can be determined exactly to all orders in perturbation theory. In the following section we will recall how much (little) of this approach can be carried over to the situation when defects are present and compute explicitly the transmission and reflection amplitudes for a variety of concrete defects.

2.1 Defect Yang-Baxter equations

A cornerstone in the context of integrable models in 1+1 space-time dimensions are the Yang-Baxter equations [20]. They can be derived most easily simply by exploiting the associativity of the so-called Zamolodchikov-Faddeev (ZF) algebra [21] and its extended version which includes an additional generator representing a boundary [22, 23, 24] or a defect [25, 26]. Indicating particle types by Latin and degrees of freedom of the impurity by Greek letters, the "braiding" (exchange) relations of annihilation operators $Z_i(\theta)$ of a particle of type i moving with rapidity θ and defect operators Z_α in the state α can be written as

$$Z_i(\theta_1)Z_j(\theta_2) = S_{ij}^{kl}(\theta_1-\theta_2)Z_k(\theta_2)Z_l(\theta_1), \tag{2.1}$$

$$Z_i(\theta_1)Z_j^\dagger(\theta_2) = S_{ij}^{kl}(\theta_1-\theta_2)Z_k^\dagger(\theta_2)Z_l(\theta_1)+2\pi\delta(\theta_1-\theta_2), \tag{2.2}$$

$$
\begin{aligned}
Z_i(\theta) Z_\alpha &= R_{i\alpha}^{j\beta}(\theta) Z_j(-\theta) Z_\beta + T_{i\alpha}^{j\beta}(\theta) Z_\beta Z_j(\theta)\,, && (2.3)\\
Z_\alpha Z_i(\theta) &= \tilde{R}_{i\alpha}^{j\beta}(-\theta) Z_\beta Z_j(-\theta) + \tilde{T}_{i\alpha}^{j\beta}(-\theta) Z_j(\theta) Z_\beta . && (2.4)
\end{aligned}
$$

The bulk scattering matrix is indicated by S, and the left/right reflection and transmission amplitudes through the defect are denoted by $R/\tilde{R}$ and $T/\tilde{T}$, respectively. We employed Einstein's sum convention, that is we assume sums over doubly occurring indices. We suppress the explicit mentioning of the dependence of Z_α on the position in space and assume for the time being that it is included in α. For the treatment of a single defect this is not relevant, but it will become important when we consider multiple defects. The same relations hold when we replace the annihilation operators by the creation operators $Z_i^\dagger(\theta)$ with $R/\tilde{R}$, $T/\tilde{T}$ and S replaced by their complex conjugates. The algebra (2.3)-(2.4) can be used to derive various relations amongst the scattering amplitudes. Using extended ZF-algebra twice leads to the constraints

$$
\begin{aligned}
S_{ij}^{kl}(\theta) S_{kl}^{mn}(-\theta) &= \delta_i^m \delta_j^n, && (2.5)\\
R_{i\alpha}^{j\beta}(\theta) R_{j\beta}^{k\gamma}(-\theta) + T_{i\alpha}^{j\beta}(\theta) \tilde{T}_{j\beta}^{k\gamma}(-\theta) &= \delta_i^k \delta_\alpha^\gamma, && (2.6)\\
R_{i\alpha}^{j\beta}(\theta) T_{j\beta}^{k\gamma}(-\theta) + T_{i\alpha}^{j\beta}(\theta) \tilde{R}_{j\beta}^{k\gamma}(-\theta) &= 0\,. && (2.7)
\end{aligned}
$$

The same equations also hold after performing a parity transformation, that is for $R \leftrightarrow \tilde{R}$ and $T \leftrightarrow \tilde{T}$ in (2.6)-(2.7). From the associativity of the extended ZF-algebra one derives the equations [22, 23, 24, 25, 26]

$$
\begin{aligned}
&S(\theta_{12})[\mathbb{I} \otimes R_\alpha^\beta(\theta_1)] S(\hat{\theta}_{12})[\mathbb{I} \otimes R_\beta^\gamma(\theta_2)] = [\mathbb{I} \otimes R_\alpha^\beta(\theta_2)] S(\hat{\theta}_{12})[\mathbb{I} \otimes R_\beta^\gamma(\theta_1)] S(\theta_{12}), && (2.8)\\
&S(\theta_{12})[\mathbb{I} \otimes R_\alpha^\beta(\theta_1)] S(\hat{\theta}_{12})[\mathbb{I} \otimes T_\beta^\gamma(\theta_2)] = R_\beta^\gamma(\theta_1) \otimes T_\alpha^\beta(\theta_2), && (2.9)\\
&S(\theta_{12})[T_\alpha^\beta(\theta_2) \otimes T_\beta^\gamma(\theta_1)] = [T_\alpha^\beta(\theta_1) \otimes T_\beta^\gamma(\theta_2)] S(\theta_{12}), && (2.10)
\end{aligned}
$$

where we employed the convention $(A \otimes B)_{ij}^{kl} = A_i^k B_j^l$ for the tensor product and abbreviated the rapidity sum $\hat{\theta}_{12} = \theta_1 + \theta_2$ and difference $\theta_{12} = \theta_1 - \theta_2$. Once again the same equations also hold for $R \leftrightarrow \tilde{R}$ and $T \leftrightarrow \tilde{T}$. Starting with another initial asymptotic state one derives [26]

$$
\begin{aligned}
R_\alpha^\beta(\theta_1) \otimes \tilde{R}_\beta^\gamma(\theta_2) &= R_\beta^\gamma(\theta_1) \otimes \tilde{R}_\alpha^\beta(\theta_2), && (2.11)\\
[T_\alpha^\beta(\theta_2) \otimes \mathbb{I}] S(\hat{\theta}_{12}) [\tilde{R}_\beta^\gamma(\theta_1) \otimes \mathbb{I}] S(\theta_{12}) &= T_\beta^\gamma(\theta_2) \otimes \tilde{R}_\alpha^\beta(\theta_1), && (2.12)\\
[\mathbb{I} \otimes \tilde{T}_\alpha^\beta(\theta_2)] S(\hat{\theta}_{12}) [\mathbb{I} \otimes R_\beta^\gamma(\theta_1)] S(\theta_{12}) &= R_\alpha^\beta(\theta_1) \otimes \tilde{T}_\beta^\gamma(\theta_2), && (2.13)\\
[T_\alpha^\beta(\theta_1) \otimes \mathbb{I}] S(\hat{\theta}_{12}) [\tilde{T}_\beta^\gamma(\theta_2) \otimes \mathbb{I}] &= [\mathbb{I} \otimes \tilde{T}_\alpha^\beta(\theta_2)] S(\hat{\theta}_{12}) [\mathbb{I} \otimes T_\beta^\gamma(\theta_1)]. && (2.14)
\end{aligned}
$$

On the basis of the equations (2.8)-(2.10), it was shown in [25], for the abelian case without defect degrees of freedom, that one can not have reflection and transmission simultaneously. In [26] this result was extended to the non-abelian parity breaking case and it was proven that for the simultaneous occurrence of reflection and transmission the scattering matrix has to be rapidity independent and of the form

$$
S(\theta) = \mathbb{P}\sigma\,, \qquad (2.15)
$$

with $\mathbb{P}$ being a permutation operator and σ a constant matrix. When assuming in addition that σ is a diagonal matrix with the property $\sigma_{ij}\sigma_{ji} = 1$, the free Fermion ($\sigma_{ij} = \sigma_{ji} = -1$), free Boson ($\sigma_{ij} = \sigma_{ji} = 1$) and also the Federbush model [27] and the generalized coupled Federbush models [28] are solutions to (2.15).
As a further set of consistency equations, which serve for the determination of the defect scattering matrix, we report the crossing relations, which are as usual less obvious to justify. In analogy to the relations which have to hold for the bulk scattering matrix $S_{ij}(\theta) = S_{i\bar{j}}(i\pi-\theta) = S^*_{ji}(-\theta)$, ($\bar{j}$ is the anti-particle of j and $*$ denotes the complex conjugation) we deduce from (2.3)-(2.4) the crossing-hermiticity relations

$$R^{\alpha}_{\bar{j}}(\theta) = \tilde{R}^{\alpha}_{\bar{j}}(-\theta)^* = S_{j\bar{j}}(2\theta)\tilde{R}^{\alpha}_{j}(i\pi-\theta)\,, \tag{2.16}$$

$$T^{\alpha}_{\bar{j}}(\theta) = \tilde{T}^{\alpha}_{\bar{j}}(-\theta)^* = \tilde{T}^{\alpha}_{j}(i\pi-\theta)\,. \tag{2.17}$$

The first equalities follow when taking $Z_i^{\dagger}(\theta)^* = Z_i(\theta)$ and $Z_\alpha = Z_\alpha^{\dagger}$. The latter relations in (2.17) simply result by considering the relations for S while letting one of the particles freeze, i.e., setting its rapidity to zero, and viewing it as a defect. Relations (2.16) are obtainable in a similar fashion as the interpretation put forward in [29, 24]. Our equations (2.16) and (2.17) disagree slightly from the crossing relations in [29, 24, 25, 30], which is due to the fact that when parity is broken real analyticity is replaced by Hermitian analyticity [31]. Later on in our example, this will also be reflected in the representation of the free Fermion field (2.35), being Dirac rather than Majorana. There is of course no consequence of this choice of conventions on the physics, since the ambiguity just exploits the fact that only the moduli of these amplitudes are observable.

Similar as for the bulk scattering matrices an additional powerful constraint results from the singularity structure of the defect scattering amplitudes. In [26] it was shown that the defect does not admit any excited state once one demands a simultaneous occurrence of reflection and transmission. Supposing that the defect scattering matrices have a pole on the imaginary axis at $i\theta_0 \in i\mathbb{R}$, the corresponding residues are therefore constraint as

$$\operatorname*{Res}_{\theta\to i\theta_0} R^{\alpha}_{j}(\theta) = \operatorname*{Res}_{\theta\to i\theta_0} \tilde{R}^{\alpha}_{j}(\theta) = \operatorname*{Res}_{\theta\to i\theta_0} T^{\alpha}_{j}(\theta) = \operatorname*{Res}_{\theta\to i\theta_0} \tilde{T}^{\alpha}_{j}(\theta) \left\{ \begin{array}{ll} <0 & \text{for} \quad \theta_0 \in (0,\pi) \\ >0 & \text{for} \quad \theta_0 \notin (0,\pi) \end{array} \right. . \tag{2.18}$$

The intervals $(0,\pi)$ are understood to be mod 2π. Hence, there is no pole with positive residue in the physical sheet.

2.2 Multiple defects

Assuming that we have determined the defect scattering matrices $R/\tilde{R}$ and $T/\tilde{T}$ for a single defect, for instance by solving the consistency equations in the previous subsection, it is straightforward to use them in order to compute the related quantities for several defects. This type of situation is of interest since, unlike for a single defect, it leads to the occurrence of resonance phenomena and when the number of defects tends to infinity even to band structures. Let us therefore commence by exploiting the extended ZF-algebra (2.3)-(2.4) for a double defect. For the reasons mentioned in the introduction we are interested in the situation when $R/\tilde{R}$ and $T/\tilde{T}$ are simultaneously non-vanishing, and in the light of the result (2.15), we shall therefore focus on the diagonal case from now onwards. We compute

$$Z_i(\theta)Z_\alpha Z_\beta = R_i^{\alpha\beta}(\theta)Z_i(-\theta)Z_\alpha Z_\beta + T_i^{\alpha\beta}(\theta)Z_\alpha Z_\beta Z_i(\theta)\,, \tag{2.19}$$

$$Z_\alpha Z_\beta Z_i(\theta) = \tilde{R}_i^{\alpha\beta}(-\theta)Z_\alpha Z_\beta Z_i(-\theta) + \tilde{T}_i^{\alpha\beta}(-\theta)Z_i(\theta)Z_\alpha Z_\beta, \tag{2.20}$$

where we have now introduced overall transmission and reflection amplitudes corresponding to two defects

$$T_i^{\alpha\beta}(\theta) = \frac{T_i^\alpha(\theta)T_i^\beta(\theta)}{1-R_i^\beta(\theta)\tilde{R}_i^\alpha(\theta)}, \qquad R_i^{\alpha\beta}(\theta) = R_i^\alpha(\theta) + \frac{R_i^\beta(\theta)T_i^\alpha(\theta)\tilde{T}_i^\alpha(\theta)}{1-R_i^\beta(\theta)\tilde{R}_i^\alpha(\theta)}, \tag{2.21}$$

$$\tilde{T}_i^{\alpha\beta}(\theta) = \frac{\tilde{T}_i^\alpha(\theta)\tilde{T}_i^\beta(\theta)}{1-R_i^\beta(\theta)\tilde{R}_i^\alpha(\theta)}, \qquad \tilde{R}_i^{\alpha\beta}(\theta) = \tilde{R}_i^\beta(\theta) + \frac{R_i^\alpha(\theta)T_i^\beta(\theta)\tilde{T}_i^\beta(\theta)}{1-R_i^\beta(\theta)\tilde{R}_i^\alpha(\theta)}. \tag{2.22}$$

The term $[1-R_i^\beta(\theta)\tilde{R}_i^\alpha(\theta)]^{-1} = \sum_{n=1}^\infty (R_i^\beta(\theta)\tilde{R}_i^\alpha(\theta))^n$ results from the infinite number of reflections which we have in-between the two defects, well known from Fabry-Perot type devices of classical and quantum optics. For the case $T=\tilde{T}, R=\tilde{R}$ the expressions (2.21) and (2.22) coincide with the formulae proposed in [32]. When absorbing the space dependent phase factor into the defect matrices, the explicit example presented in [25] for the free fermion perturbed with the energy operator agree almost for $T=\tilde{T}, R=\tilde{R}$ with the general formulae (2.21). They disagree in the sense that the equality of $R_i^{\alpha\beta}(\theta)$ and $\tilde{R}_i^{\alpha\beta}(\theta)$ does not hold for generic α,β as stated in [25].

It is now straightforward to extend the expressions to an arbitrary number of defects, say n, in a recursive manner

$$T_i^{\boldsymbol{\alpha}}(\theta) = \frac{T_i^{\alpha_1\ldots\alpha_k}(\theta)T_i^{\alpha_{k+1}\ldots\alpha_n}(\theta)}{1-\tilde{R}_i^{\alpha_1\ldots\alpha_k}(\theta)R_i^{\alpha_{k+1}\ldots\alpha_n}(\theta)}, \qquad 1<k<n\,, \tag{2.23}$$

$$R_i^{\boldsymbol{\alpha}}(\theta) = R_i^{\alpha_1\ldots\alpha_k}(\theta) + \frac{R_i^{\alpha_{k+1}\ldots\alpha_n}(\theta)T_i^{\alpha_1\ldots\alpha_k}(\theta)\tilde{T}_i^{\alpha_1\ldots\alpha_k}(\theta)}{1-\tilde{R}_i^{\alpha_1\ldots\alpha_k}(\theta)R_i^{\alpha_{k+1}\ldots\alpha_n}(\theta)}, \qquad 1<k<n\,. \tag{2.24}$$

For convenience we encoded here the defect degrees of freedom into the vector $\boldsymbol{\alpha}=\{\alpha_1,\cdots,\alpha_n\}$. Similar expressions also hold for $\tilde{T}_i^{\boldsymbol{\alpha}}(\theta)=\tilde{T}_i^{\alpha_1\ldots\alpha_n}(\theta)$ and $\tilde{R}_i^{\boldsymbol{\alpha}}(\theta)=\tilde{R}_i^{\alpha_1\ldots\alpha_n}(\theta)$. It is clear that from the knowledge of the single defect amplitudes we are now in the position to compute the corresponding quantities for multiple defects just by nesting successively the expressions (2.23) and (2.24) for increasing values of n into each other. Nonetheless, in general one does not succeed to provide simple analytical expressions for n-defect amplitudes and a different description is useful.

Alternatively, we can define, in analogy to standard quantum mechanical methods (see e.g. [33, 34]), a transmission matrix which takes the particle from one side of the defect to the other. From the braiding relations (2.3) and (2.4), we obtain with the help of the unitarity relations (2.6) and (2.7)

$$\begin{pmatrix} Z_{\alpha_1}\ldots Z_{\alpha_n}Z_i(\theta) \\ Z_{\alpha_1}\ldots Z_{\alpha_n}Z_i(-\theta)\end{pmatrix} = \left(\prod_{k=1}^n \mathcal{M}_{\alpha_k}^i(\theta)\right)\begin{pmatrix} Z_i(\theta)Z_{\alpha_1}\ldots Z_{\alpha_n} \\ Z_i(-\theta)Z_{\alpha_1}\ldots Z_{\alpha_n}\end{pmatrix}, \tag{2.25}$$

with

$$\mathcal{M}_{\alpha_k}^i(\theta) = \begin{pmatrix} T_i^{\alpha_k}(\theta)^{-1} & -R_i^{\alpha_k}(\theta)T_i^{\alpha_k}(\theta)^{-1} \\ -R_i^{\alpha_k}(-\theta)T_i^{\alpha_k}(-\theta)^{-1} & T_i^{\alpha_k}(-\theta)^{-1}\end{pmatrix}. \tag{2.26}$$

This means alternatively to the recursive way (2.23) and (2.24), we can also compute the multi-defect transmission and reflection amplitudes as

$$T_i^{\boldsymbol{\alpha}}(\theta) = \left(\prod_{k=1}^{n} \mathcal{M}_{\alpha_k}^i(\theta)\right)_{11}^{-1}, \qquad R_i^{\boldsymbol{\alpha}}(\theta) = -\left(\prod_{k=1}^{n} \mathcal{M}_{\alpha_k}^i(\theta)\right)_{12} \left(\prod_{k=1}^{n} \mathcal{M}_{\alpha_k}^i(\theta)\right)_{11}^{-1}. \tag{2.27}$$

One may convince oneself that this formulation is indeed the same as (2.23) and (2.24). It has, however, the virtue that it allows for a more elegant computation of the band structures. In particular, it is most suitable for numerical computations, since it just involves matrix multiplications rather than recurrence operations.

Let us now consider the case in which all the defects are of the same type α, equidistantly separated by an amount y and send $n \to \infty$. First of all we have to include now explicitly the dependence of the defect on its position into the discussion. We assume

$$\prod_{l=1}^{n} \mathcal{M}_\alpha^i(x = ly) = \prod_{l=1}^{n} \left[Q_y \mathcal{M}_\alpha^i(x=0)\right]^l Q_{ny}^{-1}, \quad Q_y = \begin{pmatrix} e^{iky} & 0 \\ 0 & e^{-iky} \end{pmatrix}, \tag{2.28}$$

where k corresponds to the wavevector of the lattice. Taking then $n \to \infty$ this accommodates Bloch's theorem (e.g., [33]) for the relativistic set-up. The simple requirement, that the product of transmission matrices $\lim_{n\to\infty} \prod_{l=1}^{n} \mathcal{M}_\alpha^i(x = ly)$ remains finite, leads now in the usual way to a restriction for the allowed energies, that is to band structures. To see when this is the case we can exploit the r.h.s. of the first equation in (2.28) and diagonalize the matrix $Q_y \mathcal{M}_\alpha^i(x = 0)$. Then it is clear that the limit $n \to \infty$ only remains finite when the eigenvalues of this matrix are not real

$$\lambda^{i,\alpha} \notin \mathbb{R}. \tag{2.29}$$

The eigenvalues are computed to

$$\lambda_{1,2}^{i,\alpha} = \chi_i^\alpha(\theta) \pm \sqrt{\chi_i^\alpha(\theta)^2 - \tilde{T}_i^\alpha(-\theta)/T_i^\alpha(-\theta)}, \tag{2.30}$$

$$\chi_i^\alpha(\theta) = \frac{[e^{iky} T_i^\alpha(\theta)^{-1} + e^{-iky} (\tilde{T}_i^\alpha(\theta)^*)^{-1}]}{2}. \tag{2.31}$$

In the parity invariant case the criterium (2.29) becomes simpler. From (2.30) and (2.31) follows in that case that the allowed energies in the infinite lattice have to respect

$$\chi_i^\alpha(\theta) = \mathrm{Re}[e^{iky} T_i^\alpha(\theta)^{-1}] < 1, \qquad \text{for } T = \tilde{T}. \tag{2.32}$$

In other words particles are only allowed to travel in the system with rapidities for which the inequality (2.32) holds. In conclusion, this means the determination of the transmission amplitudes for a single defect is sufficient to determine multiple defects and the energy band structure. Let us illustrate the working of this general formulae with a concrete example.

2.3 Free Fermion with defects

The continuous version of the 1+1 dimensional free Fermion (Ising model) with a line of defect was first treated in [35]. Thereafter it has also been considered in [29, 25] and [30] from a different point of view. In [35, 29, 25] the defect line has the form of the energy operator and in [30] also a perturbation in form of a single Fermion has been considered. In this manuscript we want to enlarge the class of perturbations having in mind to obtain various different kinds of structural and physical behaviours.

Let us consider the Lagrangian density for a complex free Fermion ψ with ℓ defects*

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \sum_{n=0}^{\ell-1} \delta(x - x_n)\mathcal{D}^{\alpha_n}(\bar{\psi}, \psi)\,, \tag{2.33}$$

where we describe the defect by the functions $\mathcal{D}^{\alpha_n}(\bar{\psi}, \psi)$, which we assume to be linear in the Fermi fields $\bar{\psi}$ and ψ. In the following we will restrict ourselves mainly to the case of equidistantly distributed defects of the same type, i.e. $x_n = ny$ and $\mathcal{D}^{\alpha_n}(\bar{\psi}, \psi) = \mathcal{D}(\bar{\psi}, \psi)$ for $n \in \{0, \ell - 1\}$.

2.3.1 Transmission and reflection amplitudes

Unfortunately, it follows from the arguments outlined in section 2.1, that when one is seeking a situation with simultaneously occurring reflection and transmission the constraining equations for diagonal bulk scattering matrices reduce simply to unitarity and crossing. These equations are, however, not restrictive enough by themselves to fix $R/\tilde{R}$ and $T/\tilde{T}$ and therefore one has to resort to alternative arguments. For instance one may proceed in analogy to standard quantum mechanical potential scattering theory (see also [29, 25, 30]) and construct the amplitudes by adequate matching conditions on the field. We consider now a single defect at the origin which suffices, since multiple defect amplitudes can be constructed from the single defect ones, according to the arguments of the previous section. We decompose the fields of the bulk theory as $\psi(x) = \Theta(x)\ \psi_+(x) + \Theta(-x)\ \psi_-(x)$, with $\Theta(x)$ being the Heavyside step function, and substitute this ansatz into the equations of motion. This way we obtain the constraints

$$i\gamma^1(\psi_+(x) - \psi_-(x))|_{x=0} = \left.\frac{\partial \mathcal{D}(\bar{\psi}(x), \psi(x))}{\partial \bar{\psi}(x)}\right|_{x=0}\,. \tag{2.34}$$

Using here for the left $(-)$ and right $(+)$ parts of ψ the Fourier decomposition of the free field

$$\psi_j(x) = \int \frac{dp_j^1}{\sqrt{4\pi} p_j^0}\left(a_j(p)u_j(p)e^{-ip_j\cdot x} + a_{\bar{j}}^\dagger(p)v_j(p)e^{ip_j\cdot x}\right)\,, \tag{2.35}$$

with $\sqrt{m_j^2 + p_j^2} = p_j^0$ and the Weyl spinors

$$u_j(p) = \sqrt{\frac{m_j}{2}}\begin{pmatrix} e^{-\theta/2} \\ e^{\theta/2} \end{pmatrix} \qquad \text{and} \qquad v_j(p) = i\sqrt{\frac{m_j}{2}}\begin{pmatrix} e^{-\theta/2} \\ -e^{\theta/2} \end{pmatrix}\,, \tag{2.36}$$

*Throughout the paper we use the following conventions:

$$x^\mu = (x^0, x^1), \qquad p^\mu = (m\cosh\theta, m\sinh\theta), \quad g^{00} = -g^{11} = \varepsilon^{01} = -\varepsilon^{10} = 1,$$
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1, \quad \psi_\alpha = \begin{pmatrix} \psi_\alpha^{(1)} \\ \psi_\alpha^{(2)} \end{pmatrix}, \quad \bar{\psi}_\alpha = \psi_\alpha^\dagger \gamma^0\,.$$

we can substitute them into the constraint (2.34). Treating the equations obtained in this manner componentwise, stripping off the integrals, we can bring them thereafter into the form

$$\begin{pmatrix} a^{\dagger}_{\bar{\jmath},-}(\theta) \\ a^{\dagger}_{\bar{\jmath},+}(-\theta) \end{pmatrix} = \begin{pmatrix} R_{\bar{\jmath}}(\theta)^{*} & T_{\bar{\jmath}}(\theta)^{*} \\ \tilde{T}_{\bar{\jmath}}(\theta)^{*} & \tilde{R}_{\bar{\jmath}}(\theta)^{*} \end{pmatrix} \begin{pmatrix} a^{\dagger}_{\bar{\jmath},-}(-\theta) \\ a^{\dagger}_{\bar{\jmath},+}(\theta) \end{pmatrix}, \tag{2.37}$$

$$\begin{pmatrix} a_{j,-}(\theta) \\ a_{j,+}(-\theta) \end{pmatrix} = \begin{pmatrix} R_{j}(\theta) & T_{j}(\theta) \\ \tilde{T}_{j}(\theta) & \tilde{R}_{j}(\theta) \end{pmatrix} \begin{pmatrix} a_{j,-}(-\theta) \\ a_{j,+}(\theta) \end{pmatrix}. \tag{2.38}$$

The creation and annihilation operators $a_i(\theta)$, $a_i^{\dagger}(\theta)$ play in (2.1) and (2.2) the role of the ZF-algebra generators in view of the usual fermionic anti-commutation relations $\{a_i(\theta_1), a_j(\theta_2)\} = 0$, $\{a_i(\theta_1), a_j^{\dagger}(\theta_2)\} = 2\pi\delta_{ij}\delta(\theta_{12})$. When including the defect operator in the equations (2.37) and (2.38), on the right/left for $-/+$-subscript, they acquire precisely the form of the extended ZF-algebra (2.3)-(2.4), such that one can read off directly the reflection and transmission amplitudes. One may convince oneself that the expressions found this way indeed satisfy the consistency equations like crossing (2.16), (2.17), unitarity (2.6), (2.7) and respect (2.18). In order to find the explicit expressions, we have to consider some concrete defects. Let us first concentrate on the energy perturbation.

2.3.2 The energy operator defect $\mathcal{D}^{\alpha}(\bar{\psi}, \psi) = g\bar{\psi}\psi$

The defect $\mathcal{D}^{\alpha}(\bar{\psi}, \psi) = g\bar{\psi}\psi$ has received already some amount of consideration, for the reason that it possesses a well studied [36] discrete counterpart. Taking the continuum limit of these lattice models the defect term in there acquires the form of the energy operator $\varepsilon(x) = g\bar{\psi}\psi(x)$, with g being a coupling constant. According to (2.34), (2.37) and (2.38) we compute

$$\tilde{R}^{\alpha}_{j}(\theta, y) = R^{\alpha}_{\bar{\jmath}}(\theta, y) = R^{\alpha}_{j}(\theta, -y) = \tilde{R}^{\alpha}_{\bar{\jmath}}(\theta, -y) = \frac{\sin B \cosh\theta}{i\sinh\theta - \sin B} e^{2iym\sinh\theta}, \tag{2.39}$$

$$T^{\alpha}_{j}(\theta) = \tilde{T}^{\alpha}_{j}(\theta) = T^{\alpha}_{\bar{\jmath}}(\theta) = \tilde{T}^{\alpha}_{\bar{\jmath}}(\theta) = \frac{\cos B \sinh\theta}{\sinh\theta + i\sin B}, \tag{2.40}$$

where we used a common and convenient parameterization in this context†

$$\sin B = -\frac{4g}{4+g^2}, \qquad -\frac{\pi}{2} \leq B \leq 0. \tag{2.41}$$

Note, that there is no explicit y-dependence in $T/\tilde{T}$ and that (2.39)-(2.40) satisfy the "unitarity" relations (2.6)-(2.7) and the crossing-hermiticity relations (2.16)-(2.17) when the defect is situated at the origin. The residues are constrained as in (2.18). The expressions $R^{\alpha}_{j}(\theta, B)$ and $T^{\alpha}_{j}(\theta, B)$ coincide with the solutions found in [25], which, however, in general does not correspond to taking our particles simply to be self-conjugated, since we use Dirac Fermions. Having obtained these amplitudes, we can easily compute the corresponding quantities associated to multiple defects by means of (2.23), (2.24) or (2.27). The explicit formulae are

†This is suggestive since many bulk theories admit such a relation between the bare and effective coupling. One may equate some combinations of R and T with some well known bulk scattering matrices. For instance, we identify the sinh-Gordon S-matrix $S_{SG}(\theta, B_{SG}) = T_j(\theta, B\pi/2)/T_j(-\theta, B\pi/2)$, with the indicated relation amongst the effective defect coupling constants.

obvious and since they are quite cumbersome we will not report them here. Instead, we will depict them as functions of $\cosh\theta$ in figure 1 for various parameters in order to emphasize some of their characteristics.

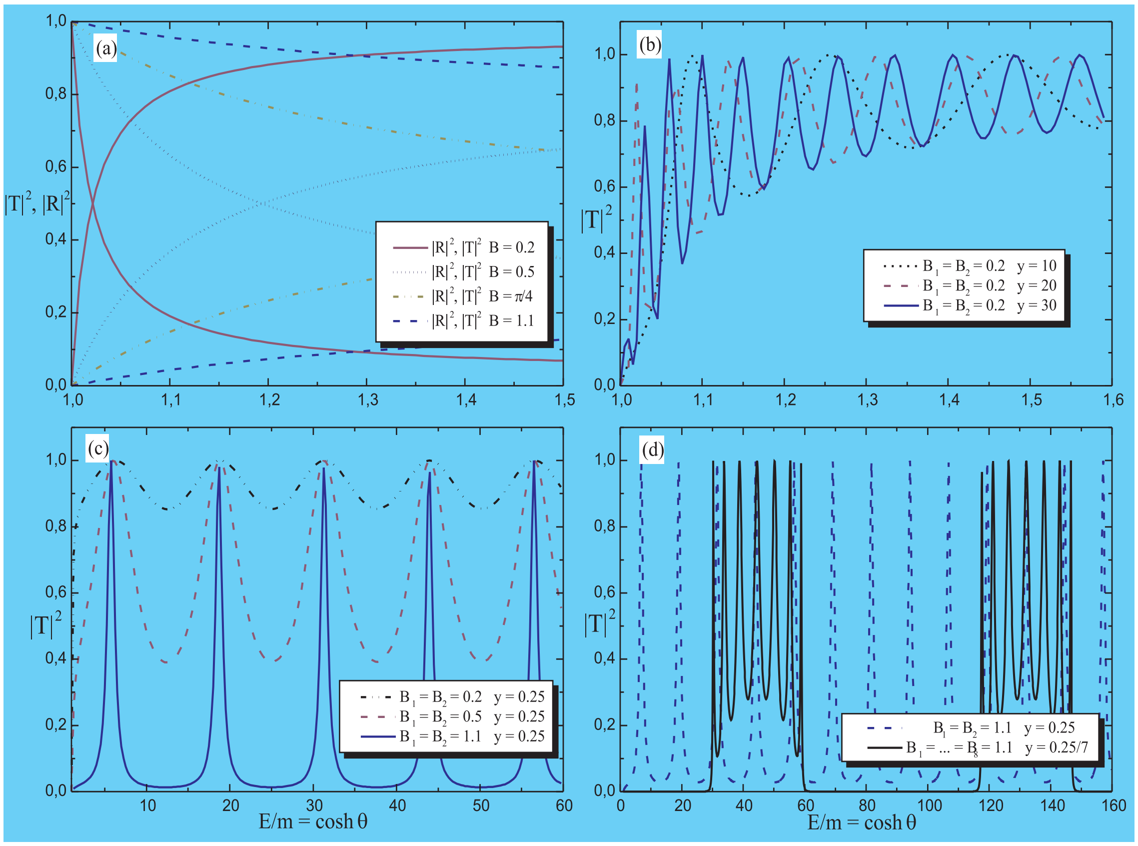

Figure 1: (a) Single defect with varying coupling constant. $|T|^2$ and $|R|^2$ correspond to curves starting at 0 and 1 of the same line type, respectively. (b) Double defect with varying distance y. (c) Double defect with varying effective coupling constant B. (d) Double defect $\equiv$ dotted line, eight defects $\equiv$ solid line.

Part (a) of figure 1 confirms the unitarity relation (2.6) where we used $R_j^*(\theta, B) = R_j(-\theta, B)$ and $T_j^*(\theta, B) = T_j(-\theta, B)$. Part (b) and (c) show the typical resonances of a double defect, which become stretched out and pronounced with respect to the energy when the distance becomes smaller and the coupling constant increases, respectively. Part (d) exhibits a general feature which extends to an even number of higher multiple defects, say $2n$, when keeping the distance y between the two most separated defects fixed: The resonances accumulate at the position around the $(2n-1)$-th resonances of the double defect. For increasing n they become very dense in that region such that one may speak of energy bands.

It is interesting to compare these bands with those obtained from the criterium (2.32), which translates in this case into

$$\sinh\theta(\cos ky - \cos B) < \sin ky \sin B, \qquad k = m \sinh\theta\,. \tag{2.42}$$

Figure 1 part (d) shows that when taking $2n$ defects separated by a distance $y/(2n-1)$ one obtains for large n an energy spectrum which resembles a band structure. Analyzing instead the function $\chi_i^\alpha(\theta)$ in (2.31) we obtain the same band structure from the criterium (2.32). The two computations show that the positions as well as the width of the bands in the two figures 1(d) and 2 coincide quite well. Remarkably, even for the double defect the criterium (2.31) yields energy regions, see figure 2(b), which are in good agreement with the exact computation as presented in figure 1(d).

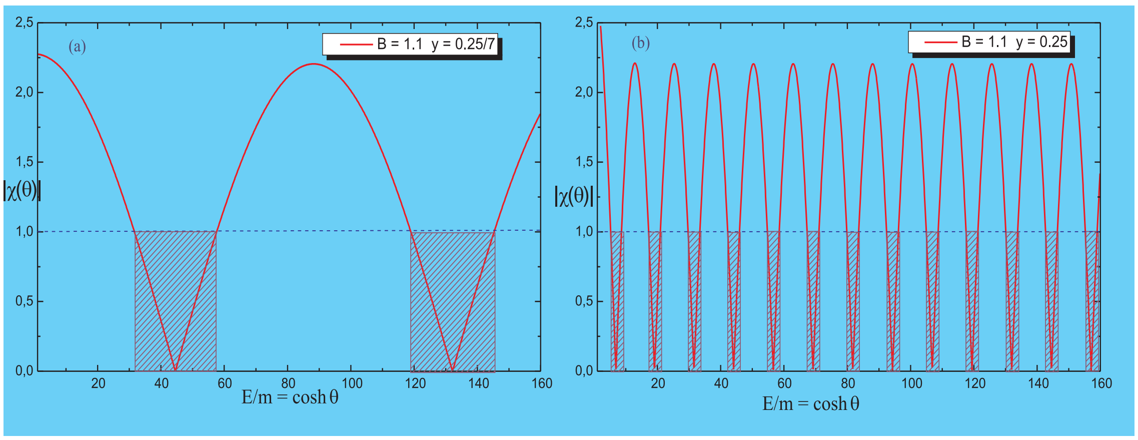

Figure 2: Band structures according to the criterium (2.32). The non-shaded regions are forbidden. (a) eight defects with $B = B_1 = \ldots = B_8 = 1.1$ equidistant by $y = 0.25/7$ (b) double defect with $B = B_1 = B_2 = 1.1$ distanced by $y = 0.25$.

Very often we will not be able to perform certain computations analytically, but instead we can carry them out in the massless limit. The prescription for taking this limit was originally introduced in [37]. It consists of replacing in every rapidity dependent expression θ by $\theta \pm \sigma$, where an additional auxiliary parameter σ has been introduced. Thereafter one should take the limit $\sigma \to \infty$, $m \to 0$ while keeping the quantity $\hat{m} = m/2\exp(\sigma)$ finite. For instance, carrying out this prescription for the momentum yields $p_\pm = \pm\hat{m}\exp(\pm\theta)$, such that one may view the model as splitted into its two chiral sectors and one can speak naturally of left (L) and right (R) movers. In this way the expressions (2.39)-(2.40) become

$$R^\alpha_{j,L/R}(\theta, B) = \pm i \sin B \, e^{\pm 2iy_\alpha \hat{m} e^\theta} \qquad \text{and} \qquad T^\alpha_{j,L/R}(\theta, B) = \cos B. \tag{2.43}$$

Similarly we compute the expression involving two and four defects for later purposes

$$T^{\alpha_1\alpha_2}_{j,L/R}(\theta, B) = \tilde{T}^{\alpha_1\alpha_2}_{j,L/R}(\theta, B) = \frac{\cos^2 B}{1 + \sin^2 B \exp[\mp 2i\hat{m}(y_{\alpha_1} - y_{\alpha_2})e^\theta]}, \tag{2.44}$$

$$R^{\alpha_1\alpha_2}_{j,L/R}(\theta, B) = \pm i \sin B \, e^{-2iy_{\alpha_1}\hat{m}e^\theta} \pm \frac{i \sin B \cos^2 B e^{-2iy_{\alpha_2}\hat{m}e^\theta}}{1 + \sin^2 B \exp[\mp 2i\hat{m}(y_{\alpha_1} - y_{\alpha_2})e^\theta]}, \tag{2.45}$$

$$\tilde{R}^{\alpha_1\alpha_2}_{j,L/R}(\theta, B) = \pm i \sin B \, e^{2iy_{\alpha_2}\hat{m}e^\theta} \pm \frac{i \sin B \cos^2 B e^{-2iy_{\alpha_1}\hat{m}e^\theta}}{1 + \sin^2 B \exp[\mp 2i\hat{m}(y_{\alpha_1} - y_{\alpha_2})e^\theta]}, \tag{2.46}$$

$$T^{\alpha_1\alpha_2\alpha_3\alpha_4}_{j,L/R}(\theta, B) = \tilde{T}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{j,L/R}(\theta, B) = \frac{T^{\alpha_1\alpha_2}_{j,L/R}(\theta, B) T^{\alpha_3\alpha_4}_{j,L/R}(\theta, B)}{1 - \tilde{R}^{\alpha_1\alpha_2}_{j,L/R}(\theta, B) R^{\alpha_3\alpha_4}_{j,L/R}(\theta, B)}. \tag{2.47}$$

The remaining amplitudes can be obtained analogously. The expressions of physical quantities, e.g., the conductance, in the massless limit should not depend on the parameter $\hat{m}$, such that the amplitudes (2.43)-(2.47) should in fact always appear in combination with other functions in order to make the prescription meaningful.

Having discussed this type of defect in some detail we will now compute $R/\tilde{R}$ and $T/\tilde{T}$ for various other defects in order to illustrate several types of physical behaviours.

2.3.3 Transparent defects, $\mathcal{D}^0(\bar{\psi},\psi) = 0$, $\mathcal{D}^\beta(\bar{\psi},\psi) = g\bar{\psi}\gamma^1\psi$

The examples which can be handled most easily in later considerations are defects which behave physically as if they were transparent ones, i.e., as $|T^{\boldsymbol{\alpha}}| = 1$. Note that this does not necessarily mean the absence of the defect. For instance considering the defect $\mathcal{D}^\beta(\bar{\psi},\psi) = g\bar{\psi}\gamma^1\psi$, we compute with the method outlined above

$$R_j^\beta(\theta,B) \;=\; \tilde{R}_j^\beta(\theta,B) = R_{\bar{j}}^\beta(\theta,B) = \tilde{R}_{\bar{j}}^\beta(\theta,B) = 0\,, \tag{2.48}$$

$$T_j^\beta(\theta,-B) \;=\; T_{\bar{j}}^\beta(\theta,B) = \tilde{T}_{\bar{j}}^\beta(\theta,-B) = \tilde{T}_j^\beta(\theta,B) = e^{iB}\,, \tag{2.49}$$

for this defect. The coupling constant is parameterized as in (2.41). Evidently the "unitarity" (2.6)-(2.7) and the crossing relations (2.16)-(2.17) are satisfied. Note that this is also an example for a defect which breaks parity invariance, i.e., the left and right transmission amplitudes are not identical. In the infinite lattice limit, i.e. when the number of defects tends to infinity, we find

$$\chi_{j/\bar{j}}^\beta(\theta) = \cos(ky \mp B) \qquad \Rightarrow \qquad \lambda_{j/\bar{j}}^\beta(\theta) \notin \mathbb{R},\, \forall\, \theta, B\,, \tag{2.50}$$

which means that according to (2.29) there are no forbidden energy regimes.

2.3.4 Energy insensitive defects, $\mathcal{D}^\gamma(\bar{\psi},\psi) = g\bar{\psi}\gamma^5\psi$, $\mathcal{D}^{\delta_\pm}(\bar{\psi},\psi) = g\bar{\psi}(\gamma^1 \pm \gamma^5)\psi$

In comparison with the transparent defects the next complication arises when the defect causes a phase shift independent of the energy of the incoming particle. For $\mathcal{D}^\gamma(\bar{\psi},\psi) = g\bar{\psi}\gamma^5\psi$ we compute

$$R_j^\gamma(\theta,B,-y) \;=\; \tilde{R}_j^\gamma(\theta,-B,y) = R_{\bar{j}}^\gamma(\theta,B,y) = \tilde{R}_{\bar{j}}^\gamma(\theta,-B,-y) = ie^{2iym\sinh\theta}\tan B, \tag{2.51}$$

$$T_j^\gamma(B) \;=\; T_{\bar{j}}^\gamma(B) = \tilde{T}_{\bar{j}}^\gamma(B) = \tilde{T}_j^\gamma(B) = \cos^{-1}(B)\,. \tag{2.52}$$

In this case we observe that parity is broken for the reflection amplitudes, i.e. $R \neq \tilde{R}$. The relations (2.6)-(2.7) and (2.16)-(2.17) for $y = 0$ are satisfied. For $y = 0$ none of the amplitudes depend on the rapidities. In the infinite lattice limit we find

$$\chi_j^\gamma(\theta) = \chi_{\bar{j}}^\gamma(\theta) = \cos ky \cos B < 1 \qquad \forall\, \theta, B\,, \tag{2.53}$$

such that according to (2.32) there are no forbidden energy regimes.

For $\mathcal{D}^{\delta_\pm}(\bar{\psi},\psi) = g\bar{\psi}(\gamma^1 \pm \gamma^5)\psi$ we compute

$$R_j^{\delta_\pm}(\theta,B,-y) \;=\; \tilde{R}_j^{\delta_\pm}(\theta,-B,y) = R_{\bar{j}}^{\delta_\pm}(\theta,B,y) = \tilde{R}_{\bar{j}}^{\delta_\pm}(\theta,-B,-y) = \pm i\,\tan\frac{B}{2}e^{2iym\sinh\theta}, \tag{2.54}$$

$$T_j^{\delta_\pm}(B) \;=\; T_{\bar{j}}^{\delta_\pm}(-B) = \tilde{T}_{\bar{j}}^{\delta_\pm}(B) = \tilde{T}_j^{\delta_\pm}(-B) = 1 - 2i\tan\frac{B}{2}\,. \tag{2.55}$$

These are examples in which parity is broken for the reflection as well as for the transmission amplitudes. Again the relations (2.6)-(2.7) and (2.16)-(2.17) are satisfied when the defect is placed at the origin and, as for $\mathcal{D}^{\gamma}$, when $y = 0$ none of the amplitudes depends on the rapidities. In this case we find in the infinite lattice limit

$$\chi^{\delta_+}_{j/\bar{\jmath}}(\theta) = \chi^{\delta_-}_{j/\bar{\jmath}}(\theta) = \frac{\cos ky}{1 \mp 2i \tan \frac{B}{2}} \qquad \Rightarrow \qquad \lambda^{\delta_\pm}_{j/\bar{\jmath}}(\theta) \notin \mathbb{R}, \forall\, \theta, B\,, \tag{2.56}$$

such that according to (2.32) there are no forbidden energy regions.

2.3.5 Luttinger liquid type $\mathcal{D}^{\varepsilon}(\bar{\psi}, \psi) = \bar{\psi}(g_1 + g_2\gamma^0)\psi$

When taking the conformal limit of a defect of the type $\mathcal{D}^{\varepsilon}(\bar{\psi}, \psi) = \bar{\psi}(g_1 + g_2\gamma^0)\psi$ one obtains an impurity which played a role in the context of Luttinger liquids [38] when setting the bosonic number counting operator to zero, see e.g. [39]. Besides $\mathcal{D}^{\alpha}(\bar{\psi}, \psi)$ this is also an example of a defect for which the potential is real. With (2.34), (2.37) and (2.38) we compute the related transmission and reflection amplitudes

$$R^{\varepsilon}_{j}(\theta, g_1, g_2, -y) \;=\; \tilde{R}^{\varepsilon}_{j}(\theta, g_1, g_2, y) = \frac{4i(g_2 + g_1 \cosh\theta)e^{2iym\sinh\theta}}{(4 + g_1^2 - g_2^2)\sinh\theta - 4i(g_1 + g_2\cosh\theta)}\,, \tag{2.57}$$

$$R^{\varepsilon}_{\bar{\jmath}}(\theta, g_1, g_2, -y) \;=\; \tilde{R}^{\varepsilon}_{\bar{\jmath}}(\theta, g_1, g_2, y) = \frac{4i(g_1 - g_2 \cosh\theta)e^{-2iym\sinh\theta}}{(4 + g_1^2 - g_2^2)\sinh\theta - 4i(g_1 - g_2\cosh\theta)}\,, \tag{2.58}$$

$$T^{\varepsilon}_{j}(\theta, g_1, g_2) \;=\; \tilde{T}^{\varepsilon}_{j}(\theta, g_1, g_2) = \frac{(4 + g_2^2 - g_1^2)\sinh\theta}{(4 + g_1^2 - g_2^2)\sinh\theta - 4i(g_1 + g_2\cosh\theta)}\,, \tag{2.59}$$

$$T^{\varepsilon}_{\bar{\jmath}}(\theta, g_1, g_2) \;=\; \tilde{T}^{\varepsilon}_{\bar{\jmath}}(\theta, g_1, g_2) = \frac{(4 + g_2^2 - g_1^2)\sinh\theta}{(4 + g_1^2 - g_2^2)\sinh\theta - 4i(g_1 - g_2\cosh\theta)}\,. \tag{2.60}$$

As we expect, since $\lim_{g_2\to 0} \mathcal{D}^{\varepsilon}(\bar{\psi}, \psi) = \mathcal{D}^{\alpha}(\bar{\psi}, \psi)$, we recover the related results also for the $T/\tilde{T}$'s and $R/\tilde{R}$'s in (2.39)-(2.40). On the other hand, for $g_1 \to 0$ we obtain the defect $\mathcal{D}^{\zeta}(\bar{\psi}, \psi) = g_2\bar{\psi}\gamma^0\psi$ for which the expressions simplify to

$$R^{\zeta}_{j}(\theta, B, -y) \;=\; \tilde{R}^{\zeta}_{j}(\theta, B, y) = \frac{-ie^{2iym\sinh\theta}\sin B}{\cos B \sinh\theta + i \sin B \cosh\theta}, \tag{2.61}$$

$$R^{\zeta}_{\bar{\jmath}}(\theta, B, y) \;=\; \tilde{R}^{\zeta}_{\bar{\jmath}}(\theta, B, -y) = \frac{ie^{2iym\sinh\theta}\sin B}{\cos B \sinh\theta - i \sin B \cosh\theta}, \tag{2.62}$$

$$T^{\zeta}_{j}(\theta, B) \;=\; \tilde{T}^{\zeta}_{j}(\theta, B) = T^{\zeta}_{\bar{\jmath}}(\theta, -B) = \tilde{T}^{\zeta}_{\bar{\jmath}}(\theta, -B) = \frac{\sinh\theta}{\cos B \sinh\theta + i \sin B \cosh\theta}\,. \tag{2.63}$$

Where the effective coupling B is given by (2.41) with $g \to g_2$. The relations (2.6)-(2.7) and (2.16)-(2.17) may be verified once again for $y = 0$. In this case the infinite lattice limit leads to forbidden energy regimes, since according to (2.32), the rapidities have to respect the inequality

$$(4 + g_1^2 - g_2^2)\sinh\theta \cos ky + \sin ky(g_1 \pm g_2\cosh\theta) < (4 + g_2^2 - g_1^2)\sinh\theta \quad \text{for } j, \bar{\jmath}\,, \tag{2.64}$$

which possess non trivial solutions.

2.3.6 The defects $\mathcal{D}^{\eta_\pm}(\bar{\psi},\psi) = g\bar{\psi}(1\pm\gamma^5)/2\psi$

For this case we compute now

$$
\begin{aligned}
R_j^{\eta_\pm}(\theta,B,y) &= R_{\bar{j}}^{\eta_\pm}(\theta,B,-y) = \frac{e^{\mp\theta}e^{-2iym\sinh\theta}}{i\cot(B/2)\sinh\theta-1}\,, && (2.65)\\
\tilde{R}_{\bar{j}}^{\eta_\pm}(\theta,B,-y) &= \tilde{R}_j^{\eta_\pm}(\theta,B,y)= \frac{e^{\pm\theta}e^{2iym\sinh\theta}}{i\cot(B/2)\sinh\theta-1}\,, && (2.66)\\
T_j^{\eta_\pm}(\theta,B) &= T_{\bar{j}}^{\eta_\pm}(\theta,B) = \tilde{T}_{\bar{j}}^{\eta_\pm}(\theta,B) = \tilde{T}_j^{\eta_\pm}(\theta,B) = \frac{1}{1\mp i\tan^{-1}(B/2)\sinh^{-1}(\theta)}\,, && (2.67)
\end{aligned}
$$

which is once again in agreement with (2.6)-(2.7) and (2.16)-(2.17) for $y=0$. In the infinite lattice limit we obtain also in this case forbidden energy regimes. The criterium (2.32) gives

$$
\pm\cos ky/2 < \sinh\theta\tan B/2\sin ky/2\,, \qquad (2.68)
$$

which has non-trivial solutions for the rapidities.

2.3.7 The defects $\mathcal{D}^{\lambda_\pm}(\bar{\psi},\psi) = g\bar{\psi}(\gamma^0\pm\gamma^1)/2\psi$

For this case we compute now

$$
\begin{aligned}
R_j^{\lambda_\pm}(\theta,B,y) &= R_{\bar{j}}^{\lambda_\pm}(\theta,B,-y) = \frac{e^{-2iym\sinh\theta}\tan\frac{B}{2}}{i\sinh\theta-\tan\frac{B}{2}\cosh\theta}\,, && (2.69)\\
\tilde{R}_{\bar{j}}^{\lambda_\pm}(\theta,B,y) &= \tilde{R}_j^{\lambda_\pm}(\theta,B,-y) = \frac{-e^{-2iym\sinh\theta}\tan\frac{B}{2}}{i\sinh\theta+\tan\frac{B}{2}\cosh\theta}\,, && (2.70)\\
T_j^{\lambda_\pm}(\theta,B) &= T_{\bar{j}}^{\lambda_\pm}(\theta,-B) = \tilde{T}_{\bar{j}}^{\lambda_\pm}(-\theta,B) = \tilde{T}_j^{\lambda_\pm}(-\theta,-B) = \frac{(i\pm\tan\frac{B}{2})\sinh\theta}{i\sinh\theta-\tan\frac{B}{2}\cosh\theta}. && (2.71)
\end{aligned}
$$

The crossing-hermiticity and unitarity relations hold for $y=0$.

In principle we could of course prolong this list of defects and construct their corresponding R's and T's. However, the main purpose of this exercise was to review how the transmission and reflection amplitudes for a defect may be obtained and also to give a brief account of some of their characteristic features. Important to note is that indeed all variations of possible parity breaking occur and one should keep therefore the discussion generic in that sense. Note that when the defect is real, namely $\mathcal{D}^{\alpha}(\bar{\psi},\psi)$, $\mathcal{D}^{\varepsilon}(\bar{\psi},\psi)$, parity invariance is preserved, which is a well known fact from quantum mechanics (see e.g. [34]). Complex potentials might look at first sight somewhat unphysical from the energy spectrum point of view. However, as is well-known for some bulk theories, such as for instance affine Toda field theories with purely complex coupling constants, one can still associate well defined quantum field theories to such Lagrangians and construct even classically soliton solutions with real energies and momenta [40].

A classification scheme for possible defects which maintain integrability is highly desirable. It is interesting to note that in the conformal limit, as outlined before equation (2.43), some of the defects, namely $\mathcal{D}^{\zeta}(\bar{\psi},\psi)$ and $\mathcal{D}^{\lambda_\pm}(\bar{\psi},\psi)$, become purely transmitting. Therefore, in contrast to first impression, the folding idea [17] could be employed. We have now enough examples at hand to use them in the following to determine the conductance in a multiparticle system, which we shall do in two alternative ways.

3 Conductance from the Landauer formula

3.1 Conductance through an impurity

The most intuitive way to compute the conductance is via Landauer transport theory [1]. Let us consider a set up as depicted in figure 3, that is we place a defect in the middle of a rigid bulk wire, where the two halves might be at different temperatures.

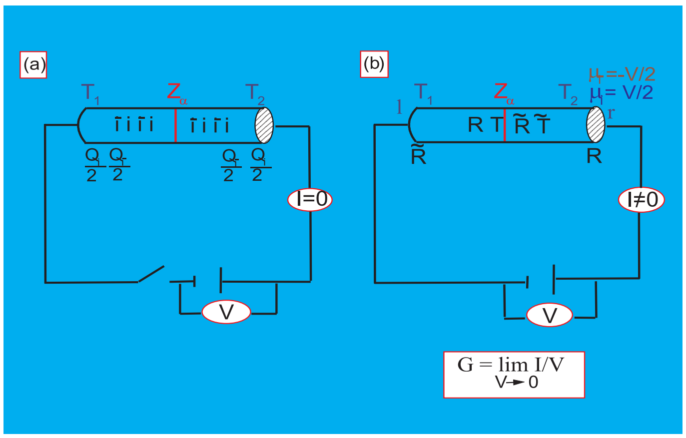

Figure 3: A conductance measurement. Part (a) represents the initial condition with no current flowing, i.e., I=0 and part (b), $I \neq 0$. The defect is placed in the middle of the wire and the left and right half are assumed to be at temperatures T_1 and T_2, respectively.

The direct current I through such a quantum wire can be computed simply by determining the difference between the static charge distributions at the right and left constriction of the wire, i.e. $I = Q_r - Q_l$. This is based on the assumption [13, 16], that $Q(t) \sim (Q_r - Q_l)t \sim (\rho_r - \rho_l)t$, where the ρs are the corresponding density distribution functions. Placing an impurity in the middle of the wire, we have to quantify the overall balance of particles of type i and anti-particles $\bar{\imath}$ carrying opposite charges $q_i = -q_{\bar{\imath}}$ at the end of the wire at different potentials. This information is of course encoded in the density distribution function $\rho_i^r(\theta, T, \mu_i)$. In the described set up half of the particles of one type are already at the same potential at one of the ends of the wire and the probability for them to reach the other is determined by the transmission and reflection amplitudes through the impurity. We assume that there is no effect coming from the constrictions of the wire, i.e. they are purely transmitting surfaces with $T = \tilde{T} = 1$. One could, however, also consider a situation in which those constrictions act as boundaries, namely purely reflecting surfaces. The situation could be described with the same transport theory picture, see e.g. [13, 14, 41], but then the conductance can only be non-vanishing if the reflection amplitudes in the constrictions are non-diagonal in the particle degrees of freedom, such as for instance for sine-Gordon [29], that is in general affine Toda field theories with purely imaginary coupling constant or in the massless limit folded purely reflecting (transmitting) diagonal bulk theories.

According to the Landauer transport theory the direct current (DC) along the wire is given by

$$I^{\boldsymbol{\alpha}} = \sum_i I_i^{\boldsymbol{\alpha}}(r, \mu_i^l, \mu_i^r) = \sum_i \frac{q_i}{2} \int_{-\infty}^{\infty} d\theta \left[\rho_i^r(\theta, r, \mu_i^r) |T_i^{\boldsymbol{\alpha}}(\theta)|^2 - \rho_i^r(\theta, r, \mu_i^l) |\tilde{T}_i^{\boldsymbol{\alpha}}(\theta)|^2 \right], \quad (3.1)$$

$$= \quad I_B - \sum_i \frac{q_i}{2} \int_{-\infty}^{\infty} d\theta \left[\rho_i^r(\theta, r, \mu_i^r) |R_i^{\boldsymbol{\alpha}}(\theta)|^2 - \rho_i^r(\theta, r, \mu_i^l) |\tilde{R}_i^{\boldsymbol{\alpha}}(\theta)|^2 \right] \,, \tag{3.2}$$

where we assume here $T_1 = T_2$. The relation (3.2) is obtained from (3.1) simply by making use of the fact that $|R|^2 + |T|^2 = 1$. Equation (3.2) has the virtue that it extracts explicitly the bulk contribution to the current which we refer to as I_B. There are some obvious limits, namely a transparent and an impenetrable defect

$$\lim_{|T^{\boldsymbol{\alpha}}| \to 1} I^{\boldsymbol{\alpha}} = I_B \qquad \text{and} \qquad \lim_{|T^{\boldsymbol{\alpha}}| \to 0} I^{\boldsymbol{\alpha}} = 0 \,, \tag{3.3}$$

respectively. A short comment is needed on the validity of (3.1). Apparently it suggests that when the parity between left and right scattering is broken, there is the possibility of a net current even when an external source is absent. In this picture we have of course not taken into account that charged particles moving through the defect will alter the potential, such that we did in fact not describe a perpetuum mobile. Thus the limitation of our analysis is that $\mu_i^l - \mu_i^r$ has to be much larger than the change in the potential induced by the moving particles.

Finally we want to compute the conductance from the DC current, which by definition is obtained from

$$G^{\boldsymbol{\alpha}}(r) = \sum\nolimits_i G_i^{\boldsymbol{\alpha}}(r) = \sum\nolimits_i \lim_{(\mu_i^l - \mu_i^r) \to 0} I_i^{\boldsymbol{\alpha}}(r, \mu_i^l, \mu_i^r) / (\mu_i^l - \mu_i^r) \tag{3.4}$$

and is of course a property of the material itself and a function of the temperature. In general the expressions in (3.1) tend to zero for vanishing chemical potential difference such that the limit in (3.4) is non-trivial.

Thus from the knowledge of the transmission matrix and the density distribution function we can compute the conductance. Having already described how $T_i^{\boldsymbol{\alpha}}(\theta)$ can be determined, we will now explain how $\rho_i^r(\theta, r, \mu_i)$ can be evaluated by exploiting the integrability of the theory.

3.2 Defect TBA equations

The thermodynamic Bethe ansatz is a powerful tool which may be used to compute various thermodynamic properties of multi-particle systems which interact via a factorizing scattering matrix [9] and some chosen statistics. In addition, it allows to check the theory for consistency and to extract some distinct structural quantities such as the Virasoro central in the ultraviolet limit. The original bulk formulation has been accommodated to a situation which includes a purely transmitting defect [42] and a boundary [44]. In this section we want to propose a new formulation which is valid for a situation not treated before in this context, namely when reflection and transmission occur simultaneously.

As usual we obtain the Bethe ansatz equation by dragging a particle along the world line of length L. We introduce for convenience the following shorthand notation for the product of various particle $Z_i(\theta)$ and defect operators Z_α

$$Z^{\mu_1 \ldots \mu_N}_{k_1, \alpha_1; k_2, \alpha_2 \ldots k_n, \alpha_n} := Z_{\mu_1}(\theta_{\mu_1}) \ldots Z_{\mu_{k_1}}(\theta_{\mu_{k_1}}) Z_{\alpha_1} \ldots Z_{\mu_{k_n}}(\theta_{\mu_{k_n}}) Z_{\alpha_n} \ldots Z_{\mu_N}(\theta_{\mu_N}). \tag{3.5}$$

Then we compute the braiding of a particle operator of type i and the product of N further particles $Z_{\mu_1} \ldots Z_{\mu_N}$ with one defect Z_α situated on the right of the particle Z_{μ_k} by using the ZF-algebra (2.3) and (2.4)

$$Z_i(\theta_i) Z_{k,\alpha}^{\mu_1 \cdots \mu_N} = Z_{k,\alpha}^{\mu_1 \cdots \mu_N} Z_i(\theta_i) \tilde{F}_{i\alpha} - Z_{k,\alpha}^{\mu_1 \cdots \mu_N} Z_i(-\theta_i) \tilde{G}_{i\alpha} , \tag{3.6}$$

$$Z_{k,\alpha}^{\mu_1 \cdots \mu_N} Z_i(\theta_i) = Z_i(\theta_i) Z_{k,\alpha}^{\mu_1 \cdots \mu_N} F_{i\alpha} - Z_i(-\theta_i) Z_{k,\alpha}^{\mu_1 \cdots \mu_N} G_{i\alpha} . \tag{3.7}$$

We abbreviated here

$$\tilde{F}_i^\alpha = \frac{1}{\tilde{T}_i^\alpha(-\theta_i)} \prod_{l=1}^{N} S_{i\mu_l}(\theta_{i\mu_l}) , \quad \tilde{G}_i^\alpha = \frac{\tilde{R}_i^\alpha(-\theta_i)}{\tilde{T}_i^\alpha(-\theta_i)} \prod_{l=1}^{k} S_{i\mu_l}(\theta_{i\mu_l}) \prod_{l=k+1}^{N} S_{i\mu_l}(-\hat{\theta}_{i\mu_l}) , \tag{3.8}$$

$$F_i^\alpha = \frac{1}{T_i^\alpha(\theta_i)} \prod_{l=1}^{N} S_{\mu_l i}(\theta_{\mu_l i}) , \qquad G_i^\alpha = \frac{R_i^\alpha(\theta_i)}{T_i^\alpha(\theta_i)} \prod_{l=1}^{k} S_{\mu_l i}(\hat{\theta}_{\mu_l i}) \prod_{l=k+1}^{N} S_{\mu_l i}(\theta_{\mu_l i}) . \tag{3.9}$$

Being on a circle of length L, we can make the usual assumption on the Bethe wavefunction, see e.g. [9], which is captured in the requirement

$$Z_i(\theta) Z_{k,\alpha}^{\mu_1 \cdots \mu_N} = Z_{k,\alpha}^{\mu_1 \cdots \mu_N} Z_i(\theta) \exp(-iLm_i \sinh\theta) . \tag{3.10}$$

Using this monodromy property together with the braiding relations (3.6), (3.7) and the unitarity relation (2.6), we obtain

$$\prod_{l=1}^{N} \frac{S_{li}(\hat{\theta}_{li})}{S_{li}(\theta_{li})} \left(\prod_{l=1}^{N} S_{li}(\theta_{li}) - \frac{e^{iLm_i \sinh\theta_i}}{\tilde{T}_i^\alpha(-\theta_i)} \right) = \frac{T_i^\alpha(-\theta_i)}{\tilde{T}_i^\alpha(-\theta_i)} \left(\frac{e^{-iLm_i \sinh\theta_i}}{T_i^\alpha(\theta_i)} - \prod_{l=1}^{N} S_{il}(\theta_{il}) \right) . \tag{3.11}$$

Viewing the subscripts as entire spaces rather than components, equation (3.11) corresponds to the Bethe ansatz equation with simultaneously occurring transmission and reflection amplitudes for the generic, that is also the non-diagonal, case. We restrict it here to the diagonal case and can therefore use the constraints (2.15), such that the bulk scattering matrix becomes rapidity independent and the relation (3.11) may be re-written as

$$1 = e^{iLm_i \sinh\theta} D_{i\alpha}^{\pm}(\theta) \prod\nolimits_{l=1}^{N} S_{il} \tag{3.12}$$

where

$$D_{i\alpha}^{\pm}(\theta) = \frac{\tilde{T}_i^\alpha(\theta) + T_i^\alpha(\theta) \prod_{l=1}^{N} S_{il}^2}{2} \pm \frac{1}{2} \left[\left(\tilde{T}_i^\alpha(\theta) + T_i^\alpha(\theta) \prod_{l=1}^{N} S_{il}^2 \right)^2 - \frac{4T_i^\alpha(\theta) \prod_{l=1}^{N} S_{il}^2}{T_i^\alpha(-\theta)} \right]^{\frac{1}{2}} . \tag{3.13}$$

For consistency reasons it is instructive to consider the limit when the reflection amplitude tends to zero. In that case we can employ the relations (2.5)-(2.7) and may take the square root in (3.13), such that we obtain from (3.12) the two equations

$$R, \tilde{R} \to 0 : \qquad 1 = e^{iLm_i \sinh\theta} \tilde{T}_i^\alpha(\theta) \prod_{l=1}^{N} S_{il} , \quad 1 = e^{-iLm_i \sinh\theta} T_i^\alpha(\theta) \prod_{l=1}^{N} S_{li} . \tag{3.14}$$

This means we recover the Bethe ansatz equations for a purely transmitting defect, which were originally proposed by Martins in [42]. The two signs in (3.13) capture the breaking of parity invariance in the limiting case, i.e. the two equations in (3.14) correspond to taking the particle either clockwise or anti-clockwise around the world line as formulated for the parity breaking case for the first time in [43]. We do not expect to recover from here the equations for a purely reflecting boundary which were suggested in [44], since the equations (3.6) and (3.7) do not make sense in the limit $T, \tilde{T} \to 0$. For $\prod_{l=1}^{N} S_{il}^2 = 1$, i.e. the free Boson and Fermion, we can exploit the fact that (3.12) with (3.13) look formally precisely like the Bethe ansatz equations for a purely transmitting defect. If we want to exploit this analogy we should of course be concerned about the question whether $D_{j\alpha}^{\pm}(\theta)$ is a meromorphic function. Assuming parity invariance, we may take the square root

$$D_{j\alpha}^{\pm}(\theta) = T_j^{\alpha}(\theta) \pm R_j^{\alpha}(\theta) \qquad \text{for} \quad T = \tilde{T}, R = \tilde{R}\,. \tag{3.15}$$

The matrix $D_{j\alpha}^{\pm}(\theta)$ has now the usual properties, namely it is unitarity in the sense that $D_{j\alpha}^{\pm}(\theta) D_{j\alpha}^{\pm}(-\theta) = 1$. It follows further from (3.15), (2.16) and (2.17) that the hermiticity relation $D_{j\alpha}^{\pm}(\theta) = D_{j\alpha}^{\pm}(-\theta)^*$ and the crossing relations $D_{\bar{j}\alpha}^{\pm}(\theta) = D_{j\alpha}^{\mp}(i\pi - \theta)$ and $D_{\bar{j}\alpha}^{\pm}(\theta) = D_{j\alpha}^{\pm}(i\pi - \theta)$ hold for the free Fermion and Bosons, respectively.

Let us now carry out the thermodynamic limit in the usual way, namely by increasing the particle number and the system size in such a way that their mutual ratio remains finite. The amount of defects is kept constant in this limit, such that there is no contribution to the TBA-equations from the defect in that situation, see also [42] where the same argument was employed. Hence this means that essentially we can employ the usual bulk TBA analysis when the considerations are carried out not per unit length.

Let us therefore recall the main equations of the TBA analysis. For more details on the derivation see [9] and in particular for the introduction of the chemical potential see [10]. The main input into the entire analysis is the dynamical interaction, which enters via the logarithmic derivative of the scattering matrix $\varphi_{ij}(\theta) = -id \ln S_{ij}(\theta)/d\theta$ and the assumption on the statistical interaction, which we take to be fermionic. As usual [9, 10], we take the logarithmic derivative of the Bethe ansatz equation (3.12) and relate the density of states $\rho_i(\theta, r)$ for particles of type i as a function of the inverse temperature $r = 1/T$ to the density of occupied states $\rho_i^r(\theta, r)$

$$\rho_i(\theta, r) = \frac{m_i}{2\pi} \cosh\theta + \sum\nolimits_j [\varphi_{ij} * \rho_i^r](\theta)\,. \tag{3.16}$$

By $(f * g)(\theta) := 1/(2\pi) \int d\theta' f(\theta - \theta') g(\theta')$ we denote as usual the convolution of two functions. The mutual ratio of the densities serves as the definition of the so-called pseudo-energies $\varepsilon_i(\theta, r)$

$$\frac{\rho_i^r(\theta, r)}{\rho_i(\theta, r)} = \frac{e^{-\varepsilon_i(\theta, r)}}{1 + e^{-\varepsilon_i(\theta, r)}}\,, \tag{3.17}$$

which have to be positive and real. At thermodynamic equilibrium one obtains then the TBA-equations, which read in these variables and in the presence of a chemical potential μ_i

$$r m_i \cosh\theta = \varepsilon_i(\theta, r, \mu_i) + r\mu_i + \sum\nolimits_j [\varphi_{ij} * \ln(1 + e^{-\varepsilon_j})](\theta)\,, \tag{3.18}$$

where $r = m/T$, $m_l \to m_l/m$, $\mu_i \to \mu_i/m$, with m being the mass of the lightest particle in the model. It is important to note that μ_i is restricted to be smaller than 1. This follows immediately from (3.18) by recalling that $\varepsilon_i \geq 0$ and that for r large $\varepsilon_i(\theta, r, \mu_i)$ tends to infinity. As pointed out already in [9] (here just with the small modification of a chemical potential), the comparison between (3.18) and (3.16) leads to the useful relation

$$\rho_i(\theta, r, \mu_i) = \frac{1}{2\pi}\left(\frac{d\varepsilon_i(\theta, r, \mu_i)}{dr} + \mu_i\right) . \tag{3.19}$$

The main task is therefore first to solve (3.18) for the pseudo-energies from which then all densities can be reconstructed.

3.3 Thermodynamic quantities

Treating the equations (3.12) and (3.13) in the mentioned analogy we can also construct various thermodynamic quantities. It should be stressed that these quantities are computed per unit length. Similarly as the expression found in [42] for a purely transmitting defect the free energy is

$$F(r) = -\frac{1}{\pi r}\sum_{l,\alpha} \hat{m}_l \int_0^\infty d\theta \left[\cosh\theta + m^{-1}\varphi_{l\alpha}(\theta)\right] \ln[1 + \exp(-rm\cosh\theta)] . \tag{3.20}$$

It is made up of two parts, one coming from the bulk and one including the data of the defect in form of $\varphi_{l\alpha}(\theta) = -id\ln D_{l\alpha}(\theta)/d\theta$. From equation (3.20) we also see that when taking the mass scale to be large in comparison to the dominating scale in the defect, the latter contribution to the scaling function becomes negligible with regard to the bulk and vice versa.

3.4 The high temperature regime

Since the physical quantities require a solution of the TBA-equations, which up to now, due to their non-linear nature, can only be solved numerically, we have to resort in general to a numerical analysis to obtain the conductance for some concrete theories. However, there exist various approximations for different special situations, such as the high temperature regime. For large rapidities and small r, it is known [9] (here we only need the small modification of the introduction of a chemical potential μ_i) that the density of states can be approximated by

$$\rho_i(\theta, r, \mu_i) \sim \frac{m_i}{4\pi}e^{|\theta|} \sim \frac{1}{2\pi r}\epsilon(\theta)\frac{d\varepsilon_i(\theta, r, \mu_i)}{d\theta} , \tag{3.21}$$

where $\epsilon(\theta) = \Theta(\theta) - \Theta(-\theta)$ is the step function, i.e. $\epsilon(\theta) = 1$ for $\theta > 0$ and $\epsilon(\theta) = -1$ for $\theta < 0$. In equation (3.17), we assume that in the large rapidity regime $\rho_i^r(\theta, r, \mu_i)$ is dominated by (3.21) and in the small rapidity regime by the Fermi distribution function. Therefore

$$\rho_i^r(\theta, r, \mu_i) \sim \frac{1}{2\pi r}\epsilon(\theta)\frac{d}{d\theta}\ln\left[1 + \exp(-\varepsilon_i(\theta, r, \mu_i))\right] . \tag{3.22}$$

Using this expression in equation (3.1), we approximate the direct current in the ultraviolet by

$$\lim_{r\to 0} I_i^{\boldsymbol{\alpha}}(r, \mu_i) \sim \frac{q_i}{4\pi r}\int_{-\infty}^{\infty} d\theta \ln\left[\frac{1 + \exp(-\varepsilon_i(\theta, r, \mu_i^l))}{1 + \exp(-\varepsilon_i(\theta, r, \mu_i^r))}\right] \frac{d\left[\epsilon(\theta)\left|T_i^{\boldsymbol{\alpha}}(\theta)\right|^2\right]}{d\theta} , \tag{3.23}$$

after a partial integration. For simplicity we also assumed here parity invariance, that is $|T_i^{\boldsymbol{\alpha}}(\theta)| = |\tilde{T}_i^{\boldsymbol{\alpha}}(\theta)|$. The derivation of the analogue to (3.23) for the situation when parity is broken is of course similar. Taking now the potentials at the end of the wire to be $\mu_i^r = -\mu_i^l = V/2$, the conductance reads in this approximation

$$\lim_{r\to 0} G_i^{\boldsymbol{\alpha}}(r) \sim \frac{q_i}{2\pi r}\int_{-\infty}^{\infty} d\theta \frac{1}{1+\exp[\varepsilon_i(\theta,r,0)]}\left.\frac{d\varepsilon_i(\theta,r,V/2)}{dV}\right|_{V=0}\frac{d\left[\epsilon(\theta)\,|T_i^{\boldsymbol{\alpha}}(\theta)|^2\right]}{d\theta}\,. \tag{3.24}$$

In order to evaluate these expressions further, we need to know explicitly the precise form of the transmission matrix, i.e. the concrete form of the defect. An interesting situation occurs when the defect is transparent or rapidity independent, that is $|T_i^{\boldsymbol{\alpha}}(\theta)| \to |T_i^{\boldsymbol{\alpha}}|$, in which case we can pursue the analysis further. Noting that $d\epsilon(\theta)/d\theta = 2\delta(\theta)$, we obtain

$$\lim_{r\to 0} G_i^{\boldsymbol{\alpha}}(r) \sim \frac{q_i}{\pi r}\frac{|T_i^{\boldsymbol{\alpha}}|^2}{1+\exp\varepsilon_i(0,r,0)}\left.\frac{d\varepsilon_i(0,r,V/2)}{dV}\right|_{V=0}\,. \tag{3.25}$$

The derivative $d\varepsilon_i(0,r,V/2)/dV$ can be obtained by solving recursively

$$\frac{d\varepsilon_i(0,r,V/2)}{dV} = -\frac{r}{2} - \sum_j N_{ij}\frac{1}{1+\exp\varepsilon_j(0,r,V/2)]}\frac{d\varepsilon_j(0,r,V/2)}{dV}\,, \tag{3.26}$$

which results form a computation similar to a standard one in this context [9] leading to the so-called constant TBA-equations. Here only the asymptotic phases of the scattering matrix enter via $N_{ij} = \lim_{\theta\to\infty}[\ln[S_{ij}(-\theta)/S_{ij}(\theta)]]/2\pi i$. The values of $\varepsilon_i(0,r,0)$ needed in (3.25) can be obtained for small r in the usual way from the standard constant TBA-equations.

3.5 Free Fermion with defects

Let us exemplify the general formulae once more with the free Fermion. First of all we note that in this case in the TBA-equations (3.18) the kernel φ_{ij} is vanishing and the equation is simply solved by

$$\varepsilon_i(\theta,r,\mu_i) = rm_i\cosh\theta - r\mu_i\,. \tag{3.27}$$

Therefore, we have explicit functions for the densities with (3.19) and (3.17)

$$\rho_i(\theta,r,\mu_i) = \frac{1}{2\pi}m_i\cosh\theta \qquad \text{and} \qquad \rho_i^r(\theta,r,\mu_i) = \frac{m_i\cosh\theta/2\pi}{1+\exp(rm_i\cosh\theta - r\mu_i)}\,. \tag{3.28}$$

According to (3.1) the direct current reads

$$\begin{aligned} I^{\boldsymbol{\alpha}}(r,V) \;\; = \;\; & \frac{q_i}{2}\int_{-\infty}^{\infty} d\theta \left[\rho_i^r(\theta,r,V/2)\,|T_i^{\boldsymbol{\alpha}}(\theta)|^2 - \rho_i^r(\theta,r,-V/2)\,|T_i^{\boldsymbol{\alpha}}(\theta)|^2\right. \\ & \left. -\rho_i^r(\theta,r,-V/2)\,|\tilde{T}_i^{\boldsymbol{\alpha}}(\theta)|^2 + \rho_i^r(\theta,r,V/2)\,|\tilde{T}_i^{\boldsymbol{\alpha}}(\theta)|^2\right]\,. \end{aligned} \tag{3.29}$$

Using atomic units $m_e = e = \hbar = m_i = q_i = 1$, we obtain explicitly with (3.28)

$$I^{\boldsymbol{\alpha}}(r,V) = \frac{1}{\pi}\int_0^{\infty} d\theta \frac{\cosh\theta\sinh(rV/2)\,|T^{\boldsymbol{\alpha}}(\theta)|^2}{\cosh(r\cosh\theta)+\cosh(rV/2)}, \tag{3.30}$$

for $|T_{\bar{i}}^{\boldsymbol{\alpha}}(\theta)| = |T_{i}^{\boldsymbol{\alpha}}(\theta)| = |\tilde{T}_{\bar{i}}^{\boldsymbol{\alpha}}(\theta)| = |\tilde{T}_{i}^{\boldsymbol{\alpha}}(\theta)| = |T^{\boldsymbol{\alpha}}(\theta)|$. Then by (3.4) the conductance results to

$$G^{\boldsymbol{\alpha}}(r) = rm\frac{e^2}{h}\int_0^\infty d\theta \frac{\cosh\theta\, |T^{\boldsymbol{\alpha}}(\theta)|^2}{1+\cosh(rm\cosh\theta)} \tag{3.31}$$

in this case. We have re-introduced dimensional quantities instead of atomic units to be able to match with some standard results from the literature. The most characteristic features can actually be captured when we carry out the massless limit as indicated in section 2.3.2, which can be done even analytically. Substituting $t = e^\theta$, we obtain

$$\lim_{m\to 0} G^{\boldsymbol{\alpha}}(r) \sim \frac{e^2}{h}\int_0^\infty dt \frac{|T_{L/R}^{\boldsymbol{\alpha}}(t\,y/r)|^2}{1+\cosh(t)} = \frac{e^2}{h}\begin{cases} \overline{|T_{L/R}^{\boldsymbol{\alpha}}(t\,y/r)|^2} & \text{for } y \gg r \\ |T_{L/R}^{\boldsymbol{\alpha}}(y/r=0)|^2 & \text{for } y \ll r \end{cases} . \tag{3.32}$$

We have identified here two distinct regions. When $y \ll r$ we can replace the left/right transmission amplitudes by their values at $y/r = 0$. When $y \gg r$ the transmission amplitudes enter the expression as a strongly oscillatory function in which y/r plays the role of the frequency. It is then a good approximation to replace this function by its means value as indicated by the overbar. It is straightforward to extend the expression (3.32) to the case when the assumption on $T^{\boldsymbol{\alpha}}$ in (3.30) is relaxed and to the case with different values of y. To proceed further we need to specify the defect.

3.5.1 Energy insensitive defects, $\mathcal{D}^0(\bar{\psi},\psi) = 0$, $\mathcal{D}^\beta(\bar{\psi},\psi)$, $\mathcal{D}^\gamma(\bar{\psi},\psi)$, $\mathcal{D}^{\delta_\pm}(\bar{\psi},\psi)$

Let us first consider the easiest example, which supports the general working of the method. When the defect is transparent, i.e., $|T^{\boldsymbol{\alpha}}| = 1$, we can compute the expression for the conductance (3.31) directly in the large temperature limit and obtain the well known behaviour [45]

$$\lim_{r\to 0, |T^{\boldsymbol{\alpha}}|\to 1} G^{\boldsymbol{\alpha}}(r) \sim \frac{e^2}{h}(1 - \frac{rm}{2}) . \tag{3.33}$$

Alternatively we obtain the expression (3.33) also from equation (3.25) and (3.27). In the massless limit of (3.32) we obtain e^2/h which coincides with the result in [13]. However, we should stress that we consider here purely massive cases and the massless limit only serves as a benchmark. Note that a transparent defect in this context does not necessarily mean the absence of the defect. Considering for instance the defect $\mathcal{D}^\beta(\bar{\psi},\psi)$, we compute with (2.48) and (2.49) the same conductance as if there was no defect at all. Similarly simple are the computations for the defects $\mathcal{D}^\gamma(\bar{\psi},\psi)$, $\mathcal{D}^{\delta_\pm}(\bar{\psi},\psi)$. We simply get

$$G^0(r) = G^\beta(r) = G^\gamma(r)\cos^2 B = G^{\delta_\pm}(r)/(1+4\tan^2 B/2) \;= \frac{e^2}{h}. \tag{3.34}$$

Since the amplitudes do not depend on the rapidities, the TBA-kernel is zero and there is no contribution from this defect to the free energy, even unit length.

3.5.2 The energy operator defect $\mathcal{D}^\alpha(\bar{\psi},\psi) = g\bar{\psi}\psi$

For this defect the computation of the conductance according to (3.31) is more involved. The results of our numerical analysis of the expression (3.31) are depicted in figure 4.

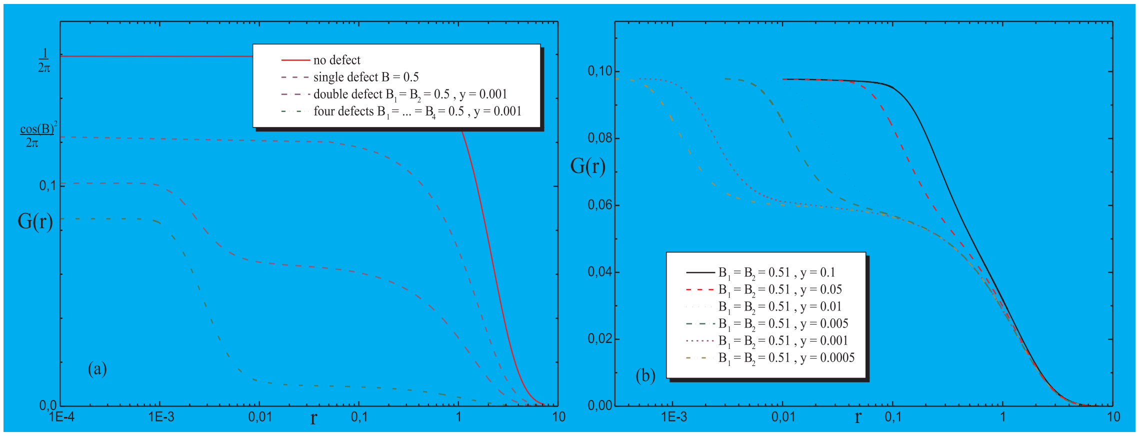

Figure 4: Conductance $G(r)$ for the complex free Fermion with the energy operator defects as a function of the inverse temperature r, for fixed effective coupling constant B and (a) for varying amounts of defects $\ell = 0, 1, 2, 4$. (b) for $\ell = 2$ for varying distances y.

We observe several distinct features. First of all it is naturally to be expected that when we increase the number of defects the resistance will grow. This is confirmed, as for fixed temperature and increasing number of defects, the conductance decreases. Second we see several well extended plateaux. They can be reproduced with the analytical expressions obtained in the massless limit (3.32). To be able to compare with (3.31) we re-introduce atomic units for convenience, i.e. $e^2/h \to 1/2\pi$. For a single defect there is only one plateaux and from (3.32) we obtain with (2.43)

$$G^{\alpha}(r) \sim \frac{\cos^2 B}{2\pi}\,. \tag{3.35}$$

For $B = 0.5$ the value 0.1226 is well reproduced in figure 4(a). The lower lying plateaux correspond to the region when $y \ll r$. In that case we obtain from (3.32) together with the expressions (2.44)-(2.47) for a double and four defects

$$G^{\alpha_1\alpha_2}(r) \sim \frac{1}{2\pi}\left(\frac{\cos^2 B}{1+\sin^2 B}\right)^2 \qquad \text{for } y \ll r\,, \tag{3.36}$$

$$G^{\alpha_1\alpha_2\alpha_3\alpha_4}(r) \sim \frac{1}{2\pi}\left(\frac{\cos^4 B}{\cos^4 B - 2(1+\sin^2 B)^2}\right)^2 \qquad \text{for } y \ll r. \tag{3.37}$$

For $B = 0.5$ the values 0.0624 and 0.0095 are well reproduced in figure 4(a) for $\ell = 2$ and $\ell = 4$, respectively. The plateaux extending to the ultraviolet regime result from (3.32) and by taking in (2.44)-(2.47) the mean values

$$G^{\alpha_1\alpha_2}(r) \sim \frac{2}{\pi}\frac{1+\sin^4 B}{(\cos^2(2B)-3)^2}\,, \qquad \text{for } y \gg r\,, \tag{3.38}$$

$$G^{\alpha_1\alpha_2\alpha_3\alpha_4}(r) \sim \frac{1}{4\pi} + \frac{\cos^8 B}{4\pi[\cos^4 B - 2(1+\sin^2 B)^2]^2}, \qquad \text{for } y \gg r. \tag{3.39}$$

Also in this case the values for $B = 0.5$, i.e., 0.110784 and 0.084311 for $\ell = 2$ and $\ell = 4$, respectively, match very well with the numerical analysis. Finally we have to explain the reason for the increase from one to the next plateaux and why the curves are shifted precisely in the way as indicated in figure 4(b) when we change the distance between the defects. This phenomenon is attributed to resonances as we shall discuss in more detail in the next subsection.

3.5.3 Resonances versus unstable particles

In [46] we demonstrated that resonances may be described similarly as unstable particles. The latter provide an intuitively very clear picture which explains the relatively sharp onset of the conductance with increasing temperature. The temperature at which this onset occurs, say T_C can be attributed directly to the energy scale at which the unstable particle is formed, since then it starts to participate in the conducting process. The Breit-Wigner formula [47] constitutes in this case a relation for the mass M and the decay width Γ of an unstable particle. Supposing that in the scattering process between particles of type i and j an unstable particle can be formed, this is reflected by a pole in $S_{ij}(\theta)$ at $\theta = \sigma - i\bar{\sigma}$. Then, for large values of the resonance parameter σ one may approximate

$$M^2 \approx 1/2 m_i m_j (1 + \cos\bar{\sigma}) \exp|\sigma| \qquad \text{and} \qquad \Gamma^2 \approx 2 m_i m_j (1 - \cos\bar{\sigma}) \exp|\sigma| \,. \tag{3.40}$$

Since a renormalization group flow is provided by $M \to r\,M$, one should observe that the quantities $M \sim r_1 e^{\sigma_1/2} = r_2 e^{\sigma_2/2}$ and $\Gamma \sim r_1 e^{\sigma_1/2} = r_2 e^{\sigma_2/2}$ remain invariant. Accordingly, this creation of the unstable particle should be reflected in the conductance as

$$G(r_1, \sigma_1) = G(r_2, \sigma_2) \quad \text{for} \quad r_1 e^{\sigma_1/2} = r_2 e^{\sigma_2/2}. \tag{3.41}$$

This means we can control the position of the onset in the conductance by M and its extension in the temperature direction by Γ. For a model which possesses unstable particles we found indeed such a behaviour [46]. From the data of the previous subsections we find that the conductance scales as $G(r_1, y_1) = G(r_2, y_2)$ for $r_2 y_1 = r_1 y_2$. Then the comparison with (3.41) suggests that we can relate the distance between the two defects to the resonance parameter as $\sigma = 2\ln(\text{const}/y)$. From the maxima in $|T(\theta)|$ we may identify various σs and in fact in this case the net result can be attributed to two resonances [46].

3.5.4 Multiple plateaux

Up to now, we have observed that we always obtain essentially two plateaux in the conductance, no matter how many (≥ 2) and what type of defects we implement. The natural question arising at this point is whether it is possible to have a set up which leads to a more involved plateaux structure? It is clear that if we had many defects in a row separated far enough from each other such that the relaxation time of the passing particles is so large that they could be treated as single rather than multiple defects, then any desired type of multiple plateau structure could be obtained. In this case the conductance is simply the sum of the expressions one has for each defect independently. Recalling the origin of the different plateaux, there is another slightly less obvious option. The density distribution function ρ^r is a peaked function of the rapidity and if the resonances in $T^{\boldsymbol{\alpha}}(\theta)$ would be separated far enough, such that they are resolved by ρ^r, we would also get a multiple plateaux pattern. However, tuning the distance between the defects or the coupling constant will merely translate the position of the resonances in the rapidity variable or change their amplitudes, respectively (see section 2).

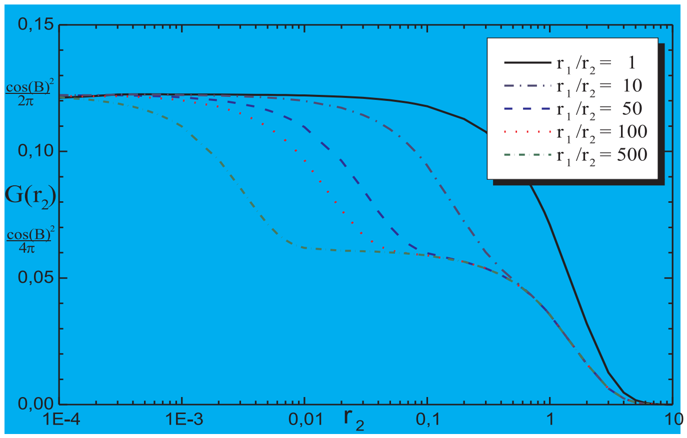

Figure 5: Conductance $G(r_2)$ for the complex free Fermion with the energy operator defects as a function of the inverse temperature r_2, for fixed effective coupling constant $B = 0.5$ and varying temperature ratios in the two halves of the wire.

Therefore the last option left is to change the ρ^rs, which is possible by varying the temperature. Choosing now a configuration as in figure 3 with different temperatures T_1 and T_2, one can "create" a second plateau at half the height of the original one. The reason for this is simply that the cooled half of the wire will cease to contribute to the conductance as can be directly deduced from (3.31). We depict the results of our computations in figure 5.
From this it also obvious that if we only cool the fraction x of the wire, the lowest plateau will be positioned at the height x times the height of the upper plateau. Thus, by combining these different configurations, i.e., different temperatures or defects, we could produce any desired plateau structure.

4 Conductance from the Kubo formula

Having computed the DC conductance by means of a TBA analysis, we want to proceed now by introducing an alternative method for the acquisition of the same quantity, that is the evaluation of the celebrated Kubo formula[‡] [3]

$$G(T) = -\lim_{\omega\to 0} \frac{1}{2\omega\pi^2} \int_{-\infty}^{\infty} dt\, e^{i\omega t} \; \langle J(t)\, J(0)\rangle_{T,m} \,. \tag{4.1}$$

The key quantity needed for the explicit computation of (4.1) is the occurrence of the current-current correlation function $\langle J(r)\, J(0)\rangle_{T,m}$. In the latter, the subscripts $(T,\, m)$ indicate that, in general, one is interested in a situation when both, the mass scale of the particles in the quantum field theory and the temperature, are non-vanishing. This is precisely the same regime in which we have carried out the TBA analysis in the previous section and ultimately

[‡]For a model independent derivation in the context of dynamical respose theory see, e.g., [4].

we want to compare the outcome of both computations. So far, formula (4.1) still refers to a situation in which no defect is present in the theory. Later on we will see how the Kubo formula can also be generalized in order to incorporate the presence of defects.

As a consequence of the central role played by the two-point function of the current operator in (4.1), we will devote an important part of this section to recall the key features of a concrete method which will allow for the computation of such a quantity that is, the form factor bootstrap approach [11]. To carry out this program one needs essentially as the only input the scattering matrix and it is then in principle possible to compute form factors associated to various local operators of the quantum field theory under investigation. Form factors are defined as matrix elements of some local operator $\mathcal{O}(\vec{x})$ located at the origin between a multiparticle in-state and the vacuum,

$$F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1, \theta_2 \ldots, \theta_n) := \langle 0|\mathcal{O}(0)|Z_{\mu_1}^\dagger(\theta_1) Z_{\mu_2}^\dagger(\theta_2) \ldots Z_{\mu_n}^\dagger(\theta_n)\rangle. \tag{4.2}$$

They can be obtained by a direct computation once a representation for the operator involved is known or as solution to a certain set of physically motivated consistency equations [11, 12, 48, 49], in a similar fashion as one can determine exact scattering matrices or transmission and reflection amplitudes for 1+1 dimensional integrable systems as discussed in section 2.

In the zero-temperature regime, the latter fact is well-known since the original works [11] and has lead successfully to the computation of correlation functions for many models, albeit in most cases only approximately. It is easy to show that once the corresponding form factors associated to two local operators $\mathcal{O}$ and $\mathcal{O}'$ are known, the computation of their two-point function is reduced to the task, still non-trivial, of evaluating the following series

$$\begin{aligned}\left\langle \mathcal{O}(r)\mathcal{O}'(0)\right\rangle_{T=0,m} = \sum_{n=1}^{\infty} \sum_{\mu_1\cdots\mu_n} \int \frac{d\theta_1 \cdots d\theta_n}{n!(2\pi)^n} \prod_{i=1}^{n} e^{-rm_i \cosh\theta_i} \\ \times F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1, \ldots, \theta_n) \left[F_n^{\mathcal{O}'|\mu_1\ldots\mu_n}(\theta_1, \ldots, \theta_n)\right]^*,\end{aligned} \tag{4.3}$$

with $x^\mu = (-ir, 0)$. The previous expression is simply obtained by introducing a sum over a complete set of states in between the two operators involved in the correlation function and by shifting the operator $\mathcal{O}(r)$ to the origin thereafter. However, as indicated by the subscripts, this formula applies only to the zero temperature regime. Obviously, when setting $\mathcal{O} = \mathcal{O}' = J$, the correlation function (4.3) is precisely the quantity entering the Kubo formula (4.1), although for $T = 0$. It is therefore necessary to find a generalization of the expansion (4.3) to the ($T \neq 0$)-regime. Such type of generalization was first suggested in [50] for the Ising model. It appears, however, to be difficult to generalize this to dynamically interacting models [51] and since by now this has not been achieved we shall concentrate on the zero temperature regime in this paper.

4.1 Conductance through an impurity

With the help of (4.3) we could in principle compute the current-current correlation function and therefore evaluate the Kubo formula when there are no boundaries or defects present. With regard to the inclusion of boundaries, the first examples in which the Kubo formula was generalized in order to accommodate that situation were provided in [16]. In there the expression (4.1) was evaluated for the sinh-Gordon and sine-Gordon model in the $m = T = 0$ limit and in the presence of a boundary. This was done by replacing the vacuum state $|0\rangle$ with a boundary state $|B\rangle$ in the current-current correlation function as follows

$$\langle J(r)\, J(0)\rangle_{T,m} \to \langle J(r)\, J(0) B\rangle_{T,m} \equiv \langle 0|\, J(r)\, J(0)\mathrm{B}\, |0\rangle_{T,m} \,. \tag{4.4}$$

The boundary state $|B\rangle :=\mathrm{B}|0\rangle$ is understood as the action of a boundary operator B on the vacuum state $|0\rangle$ [29]. Following [29], one exchanges usually the roles of space and time, such that the correlation functions are radially rather than time ordered. This is the reason why the boundary operator B can only enter at the very right or left, since one formulates such theories in half space. In contrast to the boundary, a defect can also enter in-between the operators. Therefore, in order to include a defect in (4.1), one has to consider terms of the form

$$\langle J(r)\, J(0)\rangle_{T,m} \to \langle J(r)\, Z_{\boldsymbol{\alpha}} J(0)\rangle_{T,m}\,, \quad \langle J(r)\, J(0) Z_{\boldsymbol{\alpha}}\rangle_{T,m}\,, \quad \langle Z_{\boldsymbol{\alpha}} J(r)\, J(0)\rangle_{T,m}\,, \tag{4.5}$$

where $Z_{\boldsymbol{\alpha}}$ represents the defect operator.

As a consequence of (4.5), the evaluation of the defect Kubo formula will require the computation of matrix elements involving the operators $Z_{\boldsymbol{\alpha}}$

$$G^{\boldsymbol{\alpha}}(T) = -\lim_{\omega\to 0} \frac{1}{2\omega\pi^2} \int_{-\infty}^{\infty} dr\; e^{i\omega r}\;\; \langle J(r)\, Z_{\alpha_1}\cdots Z_{\alpha_n} J(0)\rangle_{T,m}\,. \tag{4.6}$$

Equation (4.6) expresses the conductance for a situation in which n generic defects $Z_{\alpha_1}\cdots Z_{\alpha_n}$ are present in the theory and located at positions $y_{\alpha_1}\cdots y_{\alpha_n}$ in space. The defect degrees of freedom are encoded into the vector $\boldsymbol{\alpha} =\{\alpha_1,\cdots,\alpha_n\}$, as done in previous sections. In order to compare with the TBA results we would like, of course, to compute the conductance in the massive, finite temperature regime. As mentioned the evaluation of temperature dependent correlation functions is still poorly understood, even for the simplest models. In addition, the presence of the limit in the parameter ω, together with the introduction of the defect operator $Z_{\boldsymbol{\alpha}}$ in the current-current two-point function makes the generic evaluation of (4.6) fairly involved and constitutes a problem which in general can not be solved analytically. This is specially cumbersome when double defects are considered, since the expressions for the reflection and transmission amplitudes (2.21), (2.22) are, in general, quite messy to handle. For these reasons it is interesting to start with a more simplified situation, in which some analytical calculations can still be performed, that is the $T = 0$ regime. One may now view (4.5) as a three-point function and extend the expansion (4.3) to the case when three operators enter the correlation function. This will only require the inclusion of one more set of complete states, such that (4.5) is expanded in terms of the form factors of the three operators involved

$$\langle J(r) Z_{\boldsymbol{\alpha}} J(0)\rangle_{T=0,m} = \sum_{n,m=1}^{\infty} \sum_{\substack{\mu_1\cdots\mu_n\\ \nu_1\cdots\nu_m}} \int \frac{d\theta_1\cdots d\theta_n d\tilde\theta_1\cdots d\tilde\theta_m}{m!n!(2\pi)^{n+m}} F_n^{J|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) \tag{4.7}$$

$$\times \left\langle Z_{\mu_n}(\theta_n)\ldots Z_{\mu_1}(\theta_1)|Z_{\boldsymbol{\alpha}}|Z_{\nu_1}(\tilde\theta_1)\ldots Z_{\nu_m}(\tilde\theta_m)\right\rangle F_m^{J|\nu_1\ldots\nu_m}(\tilde\theta_1,\ldots,\tilde\theta_m)^* e^{-r\sum_{i=1}^{n} m_i\cosh\theta_i}.$$

We will now restrict ourselves further and consider the massless version of (4.7). In this limit, the results obtained for the conductance should agree with the UV-limit of the conductance computed by means of (3.1), (3.31). Such a limit can be carried out by exploiting the massless prescription suggested originally in [37] and already introduced in the paragraph before equation (2.43). For the form factors in (4.7) the massless limit yields

$$\lim_{\sigma\to\infty} F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1+\eta_1\sigma,\ldots,\theta_n+\eta_n\sigma) = F_{\nu_1\cdots\nu_n}^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n), \tag{4.8}$$

with $\eta_i = \pm 1$ and $\nu_i = R$ for $\eta_i = +$ and $\nu_i = L$ for $\eta_i = -$. Namely, in the massless limit every massive n-particle form factor is mapped into 2^n massless form factors. Using these expressions, performing a Wick rotation and introducing the variable $E = \sum_{i=1}^n \hat{m}_i e^{\theta_i}$, we obtain from (4.7)

$$\begin{aligned}\langle J(r) Z_{\boldsymbol{\alpha}} J(0)\rangle_{T=m=0} =& \sum_{n,m=1}^{\infty} \sum_{\substack{\mu_1\cdots\mu_n\\ \nu_1\cdots\nu_m}} \int \frac{d\theta_1\cdots d\theta_n d\tilde{\theta}_1\cdots d\tilde{\theta}_m}{m!n!(2\pi)^{n+m}} F_{R_1\ldots R_n}^{J|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) \\ &\times \left\langle Z_{\mu_n}^R(\theta_n)\ldots Z_{\mu_1}^R(\theta_1)|Z_{\boldsymbol{\alpha}}|Z_{\nu_1}^R(\tilde{\theta}_1)\ldots Z_{\nu_m}^R(\tilde{\theta}_m)\right\rangle F_{R_1\ldots R_m}^{J|\nu_1\ldots\nu_m}(\tilde{\theta}_1,\ldots,\tilde{\theta}_m)^* e^{-irE}.\end{aligned} \tag{4.9}$$

We note that for the massless prescription to work, the matrix element involving the defect $Z_{\boldsymbol{\alpha}}$ can only depend on the rapidity differences, which will indeed be the case as we see below. Performing the variable transformation $\theta_n \to \ln E'/\hat{m}_n - \sum_{i=1}^n \hat{m}_i/\hat{m}_n e^{\theta_i}$, we re-write the r.h.s. of (4.9) as

$$\begin{aligned}&\sum_{n,m=1}^{\infty} \sum_{\substack{\mu_1\cdots\mu_n\\ \nu_1\cdots\nu_m}} \int_0^E dE' \int_{-\infty}^{\ln E'/\hat{m}_n} \frac{d\theta_1\cdots d\theta_{n-1}}{n!(2\pi)^n} \int_{-\infty}^{\infty} \frac{d\tilde{\theta}_1\cdots d\tilde{\theta}_m}{m!(2\pi)^m} F_{R_1\ldots R_n}^{J|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n(E')) \\ &\times \left\langle Z_{\mu_n}^R(\theta_n(E'))\ldots Z_{\mu_1}^R(\theta_1)|Z_{\boldsymbol{\alpha}}|Z_{\nu_1}^R(\tilde{\theta}_1)\ldots Z_{\nu_m}^R(\tilde{\theta}_m)\right\rangle F_{R_1\ldots R_m}^{J|\nu_1\ldots\nu_m}(\tilde{\theta}_1,\ldots,\tilde{\theta}_m)^* e^{-irE'}.\end{aligned} \tag{4.10}$$

We substitute now this correlation function into the Kubo formula, shift all rapidities as $\theta_i \to \theta_i + \ln E'/\hat{m}_n$, $\tilde{\theta}_i \to \tilde{\theta}_i + \ln E'/\hat{m}_n$ use the Lorentz invariance of the form factors§ and carry out the integration in dE'

$$\begin{aligned}G^{\boldsymbol{\alpha}} =& -\lim_{\omega\to 0} \frac{\omega^{2s-2}}{\hat{m}_n^{2s}\pi} \sum_{\substack{\mu_1\cdots\mu_n\\ \nu_1\cdots\nu_m}} \int_{-\infty}^{0} \frac{d\theta_1\cdots d\theta_{n-1}}{n!(2\pi)^n} \int_{-\infty}^{\infty} \frac{d\tilde{\theta}_1\cdots d\tilde{\theta}_m}{m!(2\pi)^m} \frac{1}{1-\sum_{i=1}^{n-1}\hat{m}_i/\hat{m}_n e^{\theta_i}} \\ &\times \left\langle Z_{\mu_n}^R(\ln(1-\textstyle\sum_{i=1}^{n-1}\hat{m}_i/\hat{m}_n e^{\theta_i}))\ldots Z_{\mu_1}^R(\theta_1)|Z_{\boldsymbol{\alpha}}|Z_{\nu_1}^R(\tilde{\theta}_1)\ldots Z_{\nu_m}^R(\tilde{\theta}_m)\right\rangle \\ &\times F_{R_1\ldots R_n}^{J|\mu_1\ldots\mu_n}(\theta_1,\ldots,\ln(1-\sum_{i=1}^{n-1}\hat{m}_i/\hat{m}_n e^{\theta_i})) F_{R_1\ldots R_m}^{J|\nu_1\ldots\nu_m}(\tilde{\theta}_1,\ldots,\tilde{\theta}_m)^* .\end{aligned} \tag{4.11}$$

§Denoting by s the Lorentz spin of the operator $\mathcal{O}$ and λ being a constant, the form factors satisfy

$$F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1+\lambda,\ldots,\theta_n+\lambda) = e^{s\lambda} F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n).$$

We state various observations: Since the matrix element involving the defect only depends on the rapidity difference, it is not affected by the shifts. The Lorentz spin $s = 1$ plays a very special role in (4.11), which makes the current operator especially distinguished. In that case the r.h.s. of (4.11) becomes independent of the frequency ω and the limit is carried out trivially. Furthermore, since the final expression has to be independent of $\hat{m}_n$, we deduce that the form factors have to be linearly dependent on $\hat{m}_n$.

One may now compute the form factors by solving either the associated consistency equations or by using concrete realizations of the operators. For those form factors involving the current operator J, a realization in terms of the ZF-algebra was given in [28] for complex free Fermion type models and used to compute the corresponding matrix elements. We will determine form factors involving the defect operator in the same fashion, which means we require a concrete realization for the operator Z_α.

4.2 Realization of the defect operator

A realization of Z_α can be achieved very much in analogy to a realization of local operators, i.e. as exponentials of bilinears in the ZF-operators [52]. For the case of a boundary a generic model independent realization for the boundary operator B was originally proposed in [29] for the parity invariant case, i.e., $R = \tilde{R}$. This proposal was generalized to the defect operator in [32] with the same restriction and for self-conjugated particles. This realization was used by the authors for the computation of various matrix elements involving the defect operator. Our aim in this section is to extend this realization in order to incorporate the possibility of parity breaking as well as non self-conjugated particles. A non-trivial consistency check for the validity of our proposal will be ultimately provided when exploiting it in the computation of the conductance, obtained before by entirely different means, that is the TBA approach. The realization we want to propose here is a direct generalization of the one presented in [32], namely

$$Z_{\boldsymbol{\alpha}} =: \exp[\frac{1}{4\pi}\int_{-\infty}^{\infty} D_{\boldsymbol{\alpha}}(\theta)\, d\theta] : \ , \tag{4.12}$$

where : : denotes normal ordering and the operator $D_\alpha(\theta)$ has the form

$$\begin{aligned} D_{\boldsymbol{\alpha}}(\theta) \;=\; \sum_i \Big[& K_i^{\boldsymbol{\alpha}}(\theta) Z_i^\dagger(\theta) Z_{\bar{\imath}}^\dagger(-\theta) + \tilde{K}_i^{\boldsymbol{\alpha}}(\theta)^*\, Z_{\bar{\imath}}(-\theta) Z_i(\theta) \\ & + W_i^{\boldsymbol{\alpha}}(\theta) Z_i^\dagger(\theta) Z_i(\theta) + \tilde{W}_i^{\boldsymbol{\alpha}}(\theta)^* Z_i^\dagger(-\theta) Z_i(-\theta) \Big] , \end{aligned} \tag{4.13}$$

with

$$K_i^{\boldsymbol{\alpha}}(\theta) := R_i^{\boldsymbol{\alpha}}(\frac{i\pi}{2} - \theta), \qquad \tilde{K}_i^{\boldsymbol{\alpha}}(\theta) := \tilde{R}_i^{\boldsymbol{\alpha}}(\frac{i\pi}{2} - \theta), \tag{4.14}$$

$$W_i^{\boldsymbol{\alpha}}(\theta) := T_i^{\boldsymbol{\alpha}}(\frac{i\pi}{2} - \theta), \qquad \tilde{W}_i^{\boldsymbol{\alpha}}(\theta) := \tilde{T}_i^{\boldsymbol{\alpha}}(\frac{i\pi}{2} - \theta). \tag{4.15}$$

In comparison with [32] we have used a slightly different normalization factor, since in general we have contributions in the sum over i in (4.13) including both particles and anti-particles, as for the complex free Fermion we shall treat below. Following the arguments given in [29], the operator $D_{\boldsymbol{\alpha}}(\theta)$ depends on the amplitudes $R(\theta)$, $T(\theta)$, $\tilde{R}(\theta)$ and $\tilde{T}(\theta)$ with their arguments shifted according to (4.14)-(4.15), as considered also in [25, 32]. The reason for these shifts is the exchange of the roles played by the space and time coordinates $x^\mu = (t, x) \to (ix, it)$, which was already mentioned after (4.4). Doing this and keeping simultaneously the product $x^\mu \cdot p_\mu$ invariant requires the rapidity shifts in (4.14)-(4.15). In our context, this implies that we must now not only perform the shifts (4.14)-(4.15) in our expressions, but also, with regard to the positions of the defects, the change $y_\alpha \to iy_\alpha$ should be implemented. The latter replacement will play an important role since, similar as in the TBA case, the amplitudes (4.14)-(4.15) will become in this way strongly oscillating functions of y_α. Therefore, we may be able to carry out once more certain analytical calculations, by replacing the mentioned functions with their mean values. In (4.13) we have already specialized to the case when the reflection and transmission amplitudes are diagonal both with respect to the particle and defect degrees of freedom, since that will be the situation for all the examples we want to treat in this paper.

4.3 Defect matrix elements

Having now a concrete generic realization of the defect (4.13), we can compute the defect matrix elements. One way of doing this is to solve a set of consistency equations which relate the lower particle matrix elements to higher particle ones, similar as in the standard form factor program [11]. Such kind of iterative equations were proposed in [25] for a parity invariant defect and for a real free fermionic and bosonic theory. First we note that the operator (4.12) becomes

$$Z_{\boldsymbol{\alpha}} =: \exp[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta \sum_i Z_i^\dagger(\theta) Z_i(\theta)\,] :, \tag{4.16}$$

in the limit $R = \tilde{R} = 0$ and $T = \tilde{T} = 1$. The defect should act in this case as the identity operator and, according to (2.2),

$$\langle Z_i(\theta_1) Z_{\boldsymbol{\alpha}} Z_j^\dagger(\theta_2)\rangle = 2\pi\,\delta(\theta_{12})\delta_{ij}, \tag{4.17}$$

holds, simply by employing Wick's theorem when carrying out the necessary contractions. For two particles we find,

$$\begin{aligned}
\langle Z_{\bar{\imath}}(\theta_1) Z_i(\theta_2) Z_{\boldsymbol{\alpha}}\rangle &= \pi \hat{K}_i^{\boldsymbol{\alpha}}(\theta_2)\delta(\hat{\theta}_{12}), && (4.18)\\
\langle Z_{\boldsymbol{\alpha}} Z_i^\dagger(\theta_1) Z_{\bar{\imath}}^\dagger(\theta_2)\rangle &= \pi\,\hat{K}_i^{\boldsymbol{\alpha}}(\theta_1)^*\delta(\hat{\theta}_{12}), && (4.19)\\
\langle Z_i(\theta_1) Z_{\boldsymbol{\alpha}} Z_j^\dagger(\theta_2)\rangle &= \pi\,\hat{W}_i^{\boldsymbol{\alpha}}(\theta_1)\delta(\theta_{12})\delta_{ij}, && (4.20)
\end{aligned}$$

where we recall from section 2 the notation $\hat{\theta}_{12} = \theta_1 + \theta_2$ and $\theta_{12} = \theta_1 - \theta_2$. For later convenience we have introduced the functions

$$\begin{aligned}
\hat{K}_i^{\boldsymbol{\alpha}}(\theta) &= K_i^{\boldsymbol{\alpha}}(\theta) + S_{\bar{\imath}i}(-2\theta)K_{\bar{\imath}}^{\boldsymbol{\alpha}}(-\theta) = \tilde{K}_i^{\boldsymbol{\alpha}}(\theta) + S_{i\bar{\imath}}(2\theta)\tilde{K}_{\bar{\imath}}^{\boldsymbol{\alpha}}(-\theta), && (4.21)\\
\hat{W}_i^{\boldsymbol{\alpha}}(\theta) &= W_i^{\boldsymbol{\alpha}}(\theta) + \tilde{W}_i^{\boldsymbol{\alpha}}(-\theta)^* = \tilde{W}_{\bar{\imath}}^{\boldsymbol{\alpha}}(-\theta) + W_{\bar{\imath}}^{\boldsymbol{\alpha}}(\theta)^* = \hat{W}_{\bar{\imath}}^{\boldsymbol{\alpha}}(\theta)^*, && (4.22)
\end{aligned}$$

since the $K_i^{\boldsymbol{\alpha}}$, $\tilde{K}_i^{\boldsymbol{\alpha}}$,$W_i^{\boldsymbol{\alpha}}$ and $\tilde{W}_i^{\boldsymbol{\alpha}}$ amplitudes defined by (4.14)-(4.15) will repeatedly appear in the combinations (4.21), (4.22) in what follows. The latter equalities in (4.21), (4.22) follow simply from

$$\tilde{W}_i^{\boldsymbol{\alpha}}(\theta) = W_{\bar{\imath}}^{\boldsymbol{\alpha}}(-\theta) = \tilde{W}_{\bar{\imath}}^{\boldsymbol{\alpha}}(i\pi - \theta)^*,\ \tilde{K}_i^{\boldsymbol{\alpha}}(\theta) = S_{i\bar{\imath}}(2\theta)K_{\bar{\imath}}^{\boldsymbol{\alpha}}(-\theta) = S_{i\bar{\imath}}(2\theta)\tilde{K}_{\bar{\imath}}^{\boldsymbol{\alpha}}(i\pi - \theta)^*, \tag{4.23}$$

which are in turn consequences of the crossing-hermiticity properties (2.16)-(2.17). Having these matrix elements we can construct the ones involving more particles recursively from

$$
\begin{aligned}
&F_{\boldsymbol{\alpha}}^{\mu_m\ldots\mu_1\nu_1\ldots\nu_n}(\theta_m\ldots\theta_1,\theta_1'\ldots\theta_n') := \left\langle Z_{\mu_m}(\theta_m)\ldots Z_{\mu_1}(\theta_1) Z_{\boldsymbol{\alpha}} Z_{\nu_1}^{\dagger}(\theta_1')\ldots Z_{\nu_n}^{\dagger}(\theta_n')\right\rangle = \\
&\quad \pi\sum_{l=2}^{m}\delta_{\mu_1\bar{\mu}_l}\delta(\hat{\theta}_{1l})\hat{K}_{\mu_1}^{\boldsymbol{\alpha}}(\theta_1)\prod_{p=1}^{l-1}S_{\mu_1\mu_p}(\theta_{1p})F_{\boldsymbol{\alpha}}^{\mu_m\ldots\check{\mu}_l\ldots\mu_2\nu_1\ldots\nu_n}(\theta_m\ldots\check{\theta}_l\ldots\theta_2,\theta_1'\ldots\theta_n') \quad (4.24)\\
&+\pi\sum_{l=1}^{n}\delta_{\mu_1\nu_l}\delta(\theta_1-\theta_l')\hat{W}_{\mu_1}^{\boldsymbol{\alpha}}(\theta_1)\prod_{p=1}^{l-1}S_{\mu_1\nu_p}(\theta_{1p})F_{\boldsymbol{\alpha}}^{\mu_m\ldots\mu_2\nu_1\ldots\check{\nu}_l\ldots\nu_n}(\theta_m\ldots\theta_2,\theta_1'\ldots\check{\theta}_l'\ldots\theta_n')
\end{aligned}
$$

$$
\begin{aligned}
&F_{\boldsymbol{\alpha}}^{\mu_m\ldots\mu_1\nu_1\ldots\nu_n}(\theta_m\ldots\theta_1,\theta_1'\ldots\theta_n') = \qquad (4.25)\\
&\quad \pi\sum_{l=2}^{n}\delta_{\nu_1\bar{\nu}_l}\delta(\hat{\theta}_{1l}')\hat{K}_{\nu_1}^{\boldsymbol{\alpha}}(\theta_1')^*\prod_{p=1}^{l-1}S_{\nu_1\mu_p}(\theta_{1p})F_{\boldsymbol{\alpha}}^{\mu_m\ldots\mu_1\nu_2\ldots\check{\nu}_l\ldots\nu_n}(\theta_m\ldots\theta_1,\theta_2'\ldots\check{\theta}_l'\ldots\theta_n')\\
&+\pi\sum_{l=1}^{m}\delta_{\nu_1\mu_l}\delta(\theta_1'-\theta_l)\hat{W}_{\nu_1}^{\boldsymbol{\alpha}}(\theta_1')^*\prod_{p=1}^{l-1}S_{\nu_1\mu_p}(\theta_{1p})F_{\boldsymbol{\alpha}}^{\mu_m\ldots\check{\mu}_l\ldots\mu_1\nu_2\ldots\nu_n}(\theta_m\ldots\check{\theta}_l\ldots\theta_1,\theta_2'\ldots\theta_n')
\end{aligned}
$$

Here we denoted with the check on the rapidities $\check{\theta}$ the absence of the corresponding particle in the matrix element. It is clear from the expressions (4.12) and (4.13) that the only possible non-vanishing matrix elements (4.24) are those when $n+m$ is even. Taking (4.18)-(4.20) as the initial conditions for the recursive equation (4.24)-(4.25), we can now either solve them iteratively or use (4.12) and evaluate the matrix elements directly.

4.4 Free Fermion with defects

Similar as for the TBA we want to exemplify our general formulae with the complex free Fermion. We consider now the particularization of the defect realization (4.13) to this case. Then the generators of the ZF-algebra $Z_i(\theta)$, $Z_i^{\dagger}(\theta)$ are just the usual creation and annihilation operators $a_i(\theta)$, $a_i^{\dagger}(\theta)$ in the free fermionic Fock space and we have to distinguish between particles i and antiparticles $\bar{\imath}$. For the complex free Fermion it is interesting to notice that the realization (4.12) resembles very much the one employed in [52, 28] for a prototype local field.

4.4.1 Defect matrix elements

Let us now use (4.12)-(4.13) in order to evaluate matrix elements involving the defect operator. In what follows, the most relevant matrix elements are those involving four particles, for which we compute

$$\langle a_i(\theta_1)\, a_{\bar{\imath}}(\theta_2) Z_{\boldsymbol{\alpha}}\, a_{\bar{\imath}}^{\dagger}(\theta_3)\, a_i^{\dagger}(\theta_4)\rangle = w_{i\bar{\imath}}^{\boldsymbol{\alpha}}(\theta_1,\theta_2)\delta(\theta_{14})\delta(\theta_{23}) + k_{ii}^{\boldsymbol{\alpha}}(\theta_1,\theta_4)\delta(\hat{\theta}_{12})\delta(\hat{\theta}_{34}), \quad (4.26)$$

$$\langle a_i(\theta_1)\, a_i(\theta_2) Z_{\boldsymbol{\alpha}}\, a_j^{\dagger}(\theta_3)\, a_j^{\dagger}(\theta_4)\rangle = -\pi^2\hat{W}_i^{\boldsymbol{\alpha}}(\theta_1)\hat{W}_i^{\boldsymbol{\alpha}}(\theta_2)\delta(\theta_{13})\delta(\theta_{24})\delta_{ij}, \quad (4.27)$$

$$\langle a_i(\theta_1) a_k(\theta_2) a_i(\theta_3) Z_{\boldsymbol{\alpha}} a_i^{\dagger}(\theta_4)\rangle = \pi^2\hat{W}_i^{\boldsymbol{\alpha}}(\theta_4)\hat{K}_i^{\boldsymbol{\alpha}}(-\theta_2)\left[\delta(\theta_{14})\delta(\hat{\theta}_{23}) - \delta(\hat{\theta}_{12})\delta(\theta_{34})\right]\delta_{i\bar{k}},$$

$$\langle a_i(\theta_1) Z_{\boldsymbol{\alpha}} a_i^{\dagger}(\theta_2) a_k^{\dagger}(\theta_3) a_i^{\dagger}(\theta_4)\rangle = \pi^2\hat{W}_i^{\boldsymbol{\alpha}}(\theta_1)\hat{K}_i^{\boldsymbol{\alpha}}(-\theta_3)^*\left[\delta(\hat{\theta}_{23})\delta(\theta_{14}) - \delta(\theta_{12})\delta(\hat{\theta}_{34})\right]\delta_{i\bar{k}},$$

with the abbreviations

$$w^{\boldsymbol{\alpha}}_{i\bar{i}}(\theta_1,\theta_2)=\pi^2\hat{W}^{\boldsymbol{\alpha}}_i(\theta_1)\hat{W}^{\boldsymbol{\alpha}}_{\bar{i}}(\theta_2) \qquad k^{\boldsymbol{\alpha}}_{ii}(\theta_1,\theta_2)=\pi^2\hat{K}^{\boldsymbol{\alpha}}_i(\theta_1)\hat{K}^{\boldsymbol{\alpha}}_i(\theta_2)^*\,. \tag{4.28}$$

One can also find solutions for all n-particle form factors either from (4.24)-(4.25) or by direct computation. For instance we compute

$$\begin{aligned}
F^{m\times(i\bar{i})\,n\times(\bar{i}i)}_{\boldsymbol{\alpha}}(\theta_{2m}\ldots\theta_1,\theta'_1\ldots\theta'_{2n}) &= \sum_{k=0}^{\min(n,m)}\frac{(-1)^{m+n-2k}\pi^{n+m}}{(m-k)!(n-k)!k!k!}\int_{-\infty}^{\infty}d\beta_1\ldots d\beta_{2n+2m}\\
&\times\det\mathcal{A}^{2n}(\beta_1\ldots\beta_{2n};\theta'_1\ldots\theta'_{2n})\det\mathcal{A}^{2m}(\beta_{2n+1}\ldots\beta_{2n+2m};\theta_1\ldots\theta_{2m})\\
&\times\prod_{p=1}^{k}\hat{W}^{\boldsymbol{\alpha}}_i(\beta_{2p})\hat{W}^{\boldsymbol{\alpha}}_{\bar{i}}(\beta_{2p-1})\delta(\beta_{2p}-\beta_{2n+2p})\delta(\beta_{2p-1}-\beta_{2n+2p-1})\\
&\times\prod_{p=1+k}^{n}\hat{K}^{\boldsymbol{\alpha}}_i(\beta_{2p})^*\delta(\beta_{2p}+\beta_{2p-1})\prod_{p=1+k+n}^{n+m}\hat{K}^{\boldsymbol{\alpha}}_i(\beta_{2p})\delta(\beta_{2p}+\beta_{2p-1})\,,
\end{aligned} \tag{4.29}$$

where $\mathcal{A}^\ell(\theta_1\ldots\theta_\ell;\theta'_1\ldots\theta'_\ell)$ is a rank ℓ matrix whose entries are given by

$$\mathcal{A}^\ell_{ij}=\cos^2[(i-j)\pi/2]\delta(\theta_i-\theta'_j)\,, \qquad 1\leq i,j\leq\ell\,. \tag{4.30}$$

The matrix elements are computed similarly as in [28] and references therein. Likewise we compute

$$\begin{aligned}
F^{n\times i+m\times i}_{\boldsymbol{\alpha}}(\theta_n\ldots\theta_1,\theta'_1\ldots\theta'_m) &= \delta_{n,m}\frac{\pi^n(-1)^{n-1}}{n!}\int_{-\infty}^{\infty}d\beta_1\ldots d\beta_n\prod_{k=1}^{n}\hat{W}^{\boldsymbol{\alpha}}_i(\theta_k)\\
&\times\det\mathcal{B}^n(\theta_n\ldots\theta_1;\beta_1\ldots\beta_n)\det\mathcal{B}^n(\beta_1\ldots\beta_n;\theta'_1\ldots\theta'_n),
\end{aligned} \tag{4.31}$$

where we introduced a new rank ℓ matrix $\mathcal{B}^\ell(\theta_1\ldots\theta_\ell;\theta'_1\ldots\theta'_\ell)$ whose entries are now simply given by

$$\mathcal{B}^\ell_{ij}=\delta(\theta_i-\theta'_j), \qquad 1\leq i,j\leq\ell\,. \tag{4.32}$$

Since (4.31) is simpler than (4.29) we use it to demonstrate explicitly that it satisfies the recurrence relations (4.24) and (4.25). The other cases work the same way. Starting with the expansion of the determinant $\det\mathcal{B}^n(\theta_n\ldots\theta_1;\beta_1\ldots\beta_n)$ with respect to the row involving the variable θ_1 gives

$$\det\mathcal{B}^n(\theta_n\ldots\theta_1;\beta_1\ldots\beta_n)=\sum_{l=1}^{n}(-1)^{n+l+1}\delta(\theta_1-\beta_l)\det\mathcal{B}^{n-1}(\theta_n\ldots\theta_2;\beta_1\ldots\check{\beta}_l\ldots\beta_n). \tag{4.33}$$

Inserting then (4.33) into (4.31), we obtain

$$\begin{aligned}
F^{n\times i+m\times i}_{\boldsymbol{\alpha}}(\theta_n\ldots\theta_1,\theta'_1\ldots\theta'_m) &= \delta_{n,m}\frac{\pi^n}{n!}\sum_{l=1}^{n}\hat{W}^{\boldsymbol{\alpha}}_i(\theta_1)\int_{-\infty}^{\infty}d\beta_1\ldots d\check{\beta}_l\ldots d\beta_n\\
&\times(-1)^l\prod_{k=1}^{l-1}\hat{W}^{\boldsymbol{\alpha}}_i(\theta_k)\prod_{k=l+1}^{n}\hat{W}^{\boldsymbol{\alpha}}_i(\theta_k)\det\mathcal{B}^{n-1}(\theta_n\ldots\theta_2;\beta_1\ldots\check{\beta}_l\ldots\beta_n)\\
&\times\det\mathcal{B}^n(\beta_1\ldots\beta_l\to\theta_1\ldots\beta_n;\theta'_1\ldots\theta'_n).
\end{aligned} \tag{4.34}$$

Expanding now the second determinant in (4.34) with respect to the l-th row, which involves the rapidity θ_1, and using the fact that the βs are just integration variables and therefore, the sum in l gives actually n times the same contribution, we can write

$$
\begin{aligned}
F_{\boldsymbol{\alpha}}^{n\times i+m\times i}(\theta_n\ldots\theta_1,\theta'_1\ldots\theta'_m) &= \delta_{n,m}\frac{\pi^n}{(n-1)!}\sum_{p=1}^{n}\hat{W}_i^{\boldsymbol{\alpha}}(\theta_1)\delta(\theta_1-\theta'_p)\int_{-\infty}^{\infty}d\beta_1\ldots d\beta_{n-1} \\
&\times(-1)^p\prod_{k=1}^{p-1}\hat{W}_i^{\boldsymbol{\alpha}}(\theta_k)\prod_{k=p+1}^{n}\hat{W}_i^{\boldsymbol{\alpha}}(\theta_k)\det\mathcal{B}^{n-1}(\theta_n\ldots\theta_2;\beta_1\ldots\beta_{n-1}) \\
&\times\det\mathcal{B}^{n-1}(\beta_1\ldots\beta_{n-1};\theta'_1\ldots\check{\theta}'_p\ldots\theta'_n). \qquad (4.35)
\end{aligned}
$$

We recognize now the matrix element with two particles less on the l.h.s. of (4.35) and can re-write it as

$$
\begin{aligned}
F_{\boldsymbol{\alpha}}^{n\times i+n\times i}(\theta_n\ldots\theta_1,\theta'_1\ldots\theta'_n) &= \pi\sum_{p=1}^{n}(-1)^{p-1}\hat{W}_i^{\boldsymbol{\alpha}}(\theta_1)\delta(\theta_1-\theta'_p) \qquad (4.36)\\
&\quad\times F_{\boldsymbol{\alpha}}^{(n-1)\times i+(n-1)\times i}(\theta_{n-1}\ldots\theta_2,\theta'_1\ldots\check{\theta}'_p\ldots\theta'_n),
\end{aligned}
$$

which is in complete agreement with (4.24). The validity of (4.25) can be checked similarly.

4.4.2 Conductance in the $T = m = 0$ regime

It is well-known that for a free Fermion theory (also for a single complex free Fermion) the conformal $U(1)$-current-current correlation function is simply

$$
\langle J(r)J(0)\rangle_{T=m=0} = \frac{1}{r^2}. \qquad (4.37)
$$

This expression can also be obtained by using the expansion (4.3), together with the massless prescription outlined before (2.43) (see [51]) and the expressions for the only non-vanishing form factors of the current operator in the complex free Fermion theory

$$
F_2^{J|\bar{\imath}i}(\theta,\tilde{\theta}) = -F_2^{J|i\bar{\imath}}(\theta,\tilde{\theta}) = -i\pi m e^{\frac{\theta+\tilde{\theta}}{2}}\,. \qquad (4.38)
$$

In particular, the massless limit of the previous expressions gives, according to the massless prescription,

$$
\begin{aligned}
F_{RR}^{J|\bar{\imath}i}(\theta,\tilde{\theta}) &= -F_{RR}^{J|i\bar{\imath}}(\theta,\tilde{\theta}) = -2\pi i\,\hat{m}e^{\frac{\theta+\tilde{\theta}}{2}}\,, \qquad (4.39)\\
F_{LL}^{J|\bar{\imath}i}(\theta,\tilde{\theta}) &= F_{LR}^{J|\bar{\imath}i}(\theta,\tilde{\theta}) = F_{RL}^{J|\bar{\imath}i}(\theta,\tilde{\theta}) = 0, \qquad (4.40)\\
F_{LL}^{J|i\bar{\imath}}(\theta,\tilde{\theta}) &= F_{LR}^{J|i\bar{\imath}}(\theta,\tilde{\theta}) = F_{RL}^{J|i\bar{\imath}}(\theta,\tilde{\theta}) = 0\,. \qquad (4.41)
\end{aligned}
$$

Inserting (4.37) into (4.1) reduces the problem of finding the Fourier transform of the function r^{-2} which is given by $\mathcal{P}\int_{-\infty}^{\infty}dr\; e^{i\omega r}r^{-2} = -\pi\omega$ for $\omega > 0$, with $\mathcal{P}$ denoting the principle value. This yields in the absence of a defect $G^0(0) = 1/2\pi$, in complete agreement with the limit (3.34).

Let us now consider a more complicated situation, that is, the evaluation of (4.6) for $T = m = 0$ in the presence of n defects $Z_{\alpha_1} \cdots Z_{\alpha_n}$ located at positions $y_{\alpha_1} \ldots y_{\alpha_n}$ in space. The correlation function (4.5) can now be obtained with the help of (4.3), which has to be generalized for three-point functions. This requires the inclusion of one more sum over a complete set of states in (4.3). Fortunately the only non-vanishing form factors of the current are (4.39), which means the expansion (4.3) will already terminate for two particles. Explicitly, we find

$$\langle J(r) Z_{\alpha_1} \cdots Z_{\alpha_n} J(0) \rangle_{T=m=0} = \sum_i \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2 d\theta_3 d\theta_4}{2(2\pi)^4} F_{RR}^{J|\bar{i}i}(\theta_1, \theta_2) \left[F_{RR}^{J|\bar{i}i}(\theta_3, \theta_4) \right]^*$$
$$\times e^{-r\hat{m}(e^{\theta_1}+e^{\theta_2})} \langle a_i(\theta_1) a_{\bar{\imath}}(\theta_2) Z_{\alpha_1} \cdots Z_{\alpha_n} a_{\bar{\imath}}^\dagger(\theta_3) a_i^\dagger(\theta_4) \rangle_{m=0}, \tag{4.42}$$

In the light of the expressions (4.28), we can re-write (4.42) in a more explicit form without the need of specifying a concrete defect yet. Inserting (4.26) and (4.39) into (4.42), we find

$$\langle J(r) Z_{\alpha_1} \cdots Z_{\alpha_n} J(0) \rangle_{T=m=0} = \frac{\hat{m}^2}{2} \sum_i \left[\int_{-\infty}^{\infty} \frac{d\theta_1}{2} e^{-2r\hat{m}\cosh\theta_1} \hat{K}_i^{\boldsymbol{\alpha}|R}(\theta_1) \int_{-\infty}^{\infty} \frac{d\theta_2}{2} \hat{K}_i^{\boldsymbol{\alpha}|R}(\theta_2)^* \right.$$
$$\left. + \int_{-\infty}^{\infty} \frac{d\theta_1}{2} e^{\theta_1 - r\hat{m}e^{\theta_1}} \hat{W}_i^{\boldsymbol{\alpha}|R}(\theta_1) \int_{-\infty}^{\infty} \frac{d\theta_2}{2} e^{\theta_2 - r\hat{m}e^{\theta_2}} \hat{W}_{\bar{\imath}}^{\boldsymbol{\alpha}|R}(\theta_2) \right], \tag{4.43}$$

where we have exploited the crossing relations stated in (4.23). Here the functions $\hat{W}_i^{\boldsymbol{\alpha}|R}(\theta)$, $\hat{K}_i^{\boldsymbol{\alpha}|R}(\theta)$, ... defined in (4.43) are the massless limits of the corresponding functions $\hat{W}_i^{\boldsymbol{\alpha}}(\theta)$, $\hat{K}_i^{\boldsymbol{\alpha}}(\theta)$, ... For all the defects we will consider below, it turns out that the first contribution to the previous correlation function is actually vanishing, so that (4.43) is considerably simplified. In many of the examples we will treat later, this is due to the fact that the amplitudes $\hat{K}_i^{\boldsymbol{\alpha}}(\theta)$ are vanishing in the first place, as a consequence of the crossing relations (4.23). This will be the case for all energy insensitive defects for which we will present a case-by-case computation of the conductance below. The vanishing of the reflection part in (4.43) also occurs in some cases as a consequence of the parity of the function $\hat{K}_i^{\boldsymbol{\alpha}}(\theta)$. For instance, we find that, for the energy operator defect such function, although initially non-vanishing, satisfies $\hat{K}_i^{\boldsymbol{\alpha}}(\theta) = -\hat{K}_i^{\boldsymbol{\alpha}}(-\theta)$, such that $\lim_{m\to 0} \int_{-\infty}^{\infty} d\theta\, \hat{K}_i^{\boldsymbol{\alpha}}(\theta)^* = 0$.
We can now either use (4.43) in (4.6) to compute the conductance or evaluate the expression (4.11) directly in which the frequency limit is already taken

$$G^{\boldsymbol{\alpha}}(0) = \frac{1}{2(2\pi)^3} \sum_i \int_{-\infty}^{0} d\theta\, e^\theta\, w_{i\bar{\imath}}^{\boldsymbol{\alpha}|RR}[\ln(1-e^\theta), \theta]\,. \tag{4.44}$$

There are, in addition, further generic results which can be obtained independently of the specific defect. We present them at this stage and will confirm their validity below by some specific examples. Specializing to the case in which all ℓ defects are of the same type and equidistantly separated, i.e., $y = y_{\alpha_1} = \cdots = y_{\alpha_n}$. As in the TBA context (3.32), we can identify two distinct regions

$$w_{i\bar{i}}^{\boldsymbol{\alpha}|RR}(\theta_1,\theta_2) = \pi^2 \begin{cases} \overline{\hat{W}_i^{\boldsymbol{\alpha}|R}(\theta_1)\hat{W}_i^{\boldsymbol{\alpha}|R}(\theta_2)^*} & \text{for finite } y \\ |\hat{W}_i^{\boldsymbol{\alpha}|R}|^2 & \text{for } y \to 0 \end{cases} \tag{4.45}$$

where we used in addition (4.22). Supported by our explicit examples below, we find that for $y \to 0$ in (4.45) the amplitudes $\hat{W}_i^{\boldsymbol{\alpha}|R}(\theta)$ become independent functions of the rapidity. As we have already argued above

$$k_{ii}^{\boldsymbol{\alpha}|RR}(\theta_1,\theta_2) = 0. \tag{4.46}$$

The two regions specified in (4.45) are in complete agreement with the regions identified in equation (3.32), since we also consider here the massless limit. When exploiting (4.45), our explicit examples below yield the values of the conductance as those computed from (3.32). In the regime $y \to 0$ this is very apparent, since when $|\hat{W}_i^{\boldsymbol{\alpha}|R}| = const$ it becomes equal to $2|T_R^{\boldsymbol{\alpha}}|$ and the conductance reduces just to the constant factor given by (4.45) times the value $1/2\pi$ obtained when defects are not present in the theory. The vanishing of the first contribution in (4.43) is also quite suggestive with regard to the TBA results, since the conductance obtained in terms of thermodynamic quantities only involves the moduli of the transmission or the reflection amplitudes, but not both simultaneously and, in the light of the previous discussion, that seems to extend also to the form factor computation. The coincidence in the regime for finite value of y between the Kubo formula based on (4.45) and the results from the Landauer formula are less obvious and we support this by some explicit computations for several specific defects, similar as in subsection 3.5.

4.4.3 Energy insensitive defects, $\mathcal{D}^0(\bar{\psi},\psi) = 0$, $\mathcal{D}^\beta(\bar{\psi},\psi)$, $\mathcal{D}^\gamma(\bar{\psi},\psi)$, $\mathcal{D}^{\delta\pm}(\bar{\psi},\psi)$

A simple example to start with, which at the same time provides a first test of the working of the Kubo formula in the UV-limit is a transparent defect, i.e., purely transmitting. As shown in subsection 2.3.3, examples for this are the absence of a defect $\mathcal{D}^0(\bar{\psi},\psi) = 0$ as well as the defect $\mathcal{D}^\beta(\bar{\psi},\psi) = g\bar{\psi}\gamma^1\psi$, for which the associated reflection and transmission amplitudes are given by (2.48) and (2.49). In this case the observation $\hat{K}_i^{\boldsymbol{\alpha}}(\theta) = 0$ is of course trivial and therefore only the second integral in (4.43) is relevant. The situation in which there is no defect was already commented on in the paragraph after equation (4.41). We found in that case that the Kubo formula leads to entirely consistent results with regard to our TBA analysis, that is $G^0(0) = 1/2\pi$ in the massless limit. Considering now a theory with n defects of the type $\mathcal{D}^{\beta_i}(\bar{\psi},\psi)$, we find

$$\hat{W}_i^{\beta_1\ldots\beta_n}(\theta) = 2e^{-inB}, \qquad \hat{K}_i^{\beta_1\ldots\beta_n}(\theta) = 0\,, \tag{4.47}$$

simply by exploiting the expressions (2.49) for a single defect and the formulae (2.23) and (2.24). From (4.47) it follows that

$$w_{i\bar{i}}^{\beta_1\ldots\beta_n|RR}(\theta_1,\theta_2) = 4\pi^2, \qquad k_{ii}^{\beta_1\ldots\beta_n|RR}(\theta_1,\theta_2) = 0\,, \tag{4.48}$$

just, as in the case in which the defect is absent. Therefore, we recover once more the value

$$G^{\beta_1\ldots\beta_n}(0) = \frac{1}{2\pi}, \tag{4.49}$$

for the conductance at zero mass and temperature.

The next complication arises for defects whose reflection and transmission amplitudes are simultaneously non-vanishing, but at the same time are independent functions of the rapidity. As we have seen in section 3.5, those defects can be very easily handled in the context of a TBA calculation for the conductance, since the modulus of the transmission amplitudes is a constant, depending only on the defect coupling g. Therefore, the conductance is simply given by the constant $|T^{\boldsymbol{\alpha}}|^2$ times the value (4.49). The vanishing of the function $\hat{K}_i^{\boldsymbol{\alpha}}(\theta)$ can be established in those cases by exploiting the crossing properties listened above. Namely, from (2.16)-(2.17) we find $K_i = \tilde{K}_i^* = -\tilde{K}_{\bar{\imath}}$, whenever the reflection amplitudes are independent of the rapidity, and therefore $\hat{K}_i^{\boldsymbol{\alpha}} = K_i + K_i^* = 2\,\mathrm{Re}(K_i)$. The latter quantity is zero for the defects D^γ and $D^{\delta_\pm}$ treated in section 2.3.4, when setting $y = 0$, since the reflection amplitudes are purely complex quantities. Therefore, the Kubo formula computation leads to the same results as found in subsection 3.6.1, since we find

$$w_{i\bar{\imath}}^{\gamma|RR}(\theta_1,\theta_2) = 4\pi^2(1+4\tan^2 B/2), \qquad k_{ii}^{\gamma|RR}(\theta_1,\theta_2) = 0, \tag{4.50}$$

$$w_{i\bar{\imath}}^{\delta\pm|RR}(\theta_1,\theta_2) = \frac{4\pi^2}{\cos^2 B}, \qquad k_{ii}^{\delta\pm|RR}(\theta_1,\theta_2) = 0\,, \tag{4.51}$$

which yields

$$G^\gamma(0) = \frac{(1+4\tan^2 B/2)}{2\pi} \qquad \text{and} \qquad G^{\delta_\pm}(0) = \frac{1}{2\pi\cos^2 B}. \tag{4.52}$$

As expected, for $B = 0$ we recover once more the value (4.49).

4.4.4 The energy operator defect $\mathcal{D}(\bar{\psi},\psi) = g\bar{\psi}\psi$

Let us now treat the energy operator defect, the example which has been most extensively studied in our previous sections. Considering first a theory possessing a single defect of this type, we find

$$\hat{W}_i^\alpha(\theta) = \frac{4\cos B\cosh^2\theta}{\cosh 2\theta + \cos 2B} \qquad \text{and} \qquad \hat{K}_i^\alpha(\theta) = \frac{2i\sin B\sinh\theta}{\sin B - \cosh\theta}. \tag{4.53}$$

Therefore, in this case the amplitude $\hat{K}_i^{\boldsymbol{\alpha}}(\theta)$ is non-vanishing. However, we find that $\hat{K}_i^{\boldsymbol{\alpha}}(\theta) = -\hat{K}_i^{\boldsymbol{\alpha}}(-\theta)$. This means that the integral of this function (or its complex conjugated) is vanishing. Consequently, only the transmission part contributes non-trivially to (4.43). In order to evaluate (4.26) in the massless limit, we are interested in this limit of (4.53) which enters equation (4.43). We obtain

$$w_{i\bar{\imath}}^{\alpha|RR}(\theta_{1,}\theta_2) = 4\pi^2\cos^2 B, \tag{4.54}$$

which, together with (4.39) leads to the result

$$\langle J(r)Z_\alpha J(0)\rangle_{T=m=0} = \frac{\cos^2 B}{r^2} \implies G^\alpha(0) = \frac{\cos^2 B}{2\pi}, \tag{4.55}$$

again in agreement with the corresponding result (3.35) from the Landauer formula.

Let us now proceed to the study of the conductance in the presence of a double defect. Again, we consider first the case $T = m = 0$ and two defects of the energy operator type located at the origin and at a distance y from the origin, respectively. Expression (4.42) again holds for that situation with $n = 2$. As explained in the paragraph after equation (2.22), the Greek indices in the defect operator encode also the space dependence. The reflection and transmission amplitudes are computed according to (2.21) and (2.22) with (4.53). These functions can thereafter be substituted into equation (4.26) in order to determine the explicit form of the functions $w_{i\bar{\imath}}^{\alpha_1\alpha_2|RR}$ and $k_{ii}^{\alpha_1\alpha_2|RR}$ in (4.26) for the double defect system, which depend now on the distance y between the defects and their expressions become very cumbersome. Once more, it is possible to show that the contribution to the conductance depending on $k_{ii}^{\alpha_1\alpha_2|RR}$ is vanishing and therefore only the function $w_{i\bar{\imath}}^{\alpha_1\alpha_2|RR}$ will be of further interest to us. However, it is relatively easy to show that in the massless limit we find

$$
\begin{aligned}
w_{i\bar{\imath}}^{\alpha_1\alpha_2|RR}(\theta_1,\theta_2) \quad &= \quad 4\pi^2\cos^4 B\left[\frac{1+\cos(2\hat{m}ye^{\theta_1})\sin^2 B}{1+2\cos(2\hat{m}ye^{\theta_1})\sin^2 B+\sin^4 B}\right] \\
&\qquad \times\left[\frac{1+\cos(2\hat{m}ye^{\theta_2})\sin^2 B)}{1+2\cos(2\hat{m}ye^{\theta_2})\sin^2 B+\sin^4 B}\right], \qquad (4.56)
\end{aligned}
$$

such that we obtain

$$
\begin{aligned}
\langle J(r)Z_{\alpha_1}Z_{\alpha_2}J(0)\rangle_{T=m=0} \quad &= \quad \frac{\overline{w_{i\bar{\imath}}^{\alpha_1\alpha_2|RR}(\theta_1,\theta_2)}}{4\pi^2 r^2} = \frac{4\left[1+\sin^4 B\right]}{r^2\left[\cos^2(2B)-3\right]^2}, \qquad (4.57)\\
G^{\alpha_1\alpha_2}(0) \quad &= \quad \frac{2}{\pi}\frac{1+\sin^4 B}{\left[3-\cos^2(2B)\right]^2}, \qquad (4.58)
\end{aligned}
$$

which precisely agrees with the corresponding result (3.38) obtained from the Landauer formula. The overbar denotes as before the mean value of the corresponding function.

As explained above, we can also predict the precise position of the second plateau obtained within the TBA analysis given in equation (3.36). This is achieved by considering previously to the UV-limit, the limit when the distance between the defects $y \to 0$. By doing so we find

$$
\lim_{y\to 0} w_{i\bar{\imath}}^{\alpha_1\alpha_2|RR}(\theta_1,\theta_2) = \frac{4\pi^2\cos^4 B}{(1+\sin^2 B)^2}, \qquad (4.59)
$$

which gives

$$
\begin{aligned}
\lim_{y\to 0}\langle J(r)Z_{\alpha_1}Z_{\alpha_2}J(0)\rangle_{T=m=0} \quad &= \quad \frac{1}{r^2}\frac{\cos^4 B}{(1+\sin^2 B)^2}, \qquad (4.60)\\
\lim_{y\to 0} G^{\alpha_1\alpha_2}(0) \quad &= \quad \frac{1}{2\pi}\frac{\cos^4 B}{(1+\sin^2 B)^2}, \qquad (4.61)
\end{aligned}
$$

in agreement with the value (3.36).

Finally, in order to match all the results in subsection 3.6, we would like to address also the case $\ell = 4$ in (4.6), that is, we consider now a complex free Fermion theory with four equidistant defects of the type $\mathcal{D}^{\alpha}(\bar{\psi},\psi)=g\bar{\psi}\psi$. As usual, we denote their mutual distances by y. For the first region in (4.45), that is the UV-limit, we find

$$w_{i\bar{i}}^{\alpha_1\alpha_2\alpha_3\alpha_4|RR}(\theta_1,\theta_2)=\frac{f_1(\theta_1)f_1(\theta_2)}{(f_2(\theta_1)-f_3(\theta_1))(f_2(\theta_2)-f_3(\theta_2))}, \tag{4.62}$$

with

$$\begin{aligned}
f_1(\theta) &= (5-\cos 2B)\cos(4\hat{m}ye^{\theta})+2\cos(6\hat{m}ye^{\theta}))\sin^2 B)-128\pi^2\cos^4 B(2+(6\cos(2\hat{m}ye^{\theta})\\
f_2(\theta) &= 1192\cos 2B-348\cos 4B+24\cos 6B-\cos 8B-995-256\sin^2 B\cos(6\hat{m}ye^{\theta}))\\
f_3(\theta) &= 128\sin^2 B((17-12\cos 2B+\cos 4B)\cos(2\hat{m}ye^{\theta})-4(\cos 2B-2)\cos(4\hat{m}ye^{\theta})\,.
\end{aligned}$$

This expression appears somewhat messy, but when proceeding as indicated in (4.45) it will simplify considerably. Computing the mean value of this function we find

$$\begin{aligned}
\langle J(r)Z_{\alpha_1}Z_{\alpha_2}Z_{\alpha_3}Z_{\alpha_4}J(0)\rangle_{T=m=0} &= \frac{\overline{w_{RR}^{\alpha_1\alpha_2\alpha_3\alpha_4}(\theta_1,\theta_2)}}{4\pi^2r^2}\\
&= \frac{1}{2r^2}\left[1+\frac{\cos^8 B}{[\cos^4 B-2(1+\sin^2 B)^2]^2}\right], &(4.63)\\
G^{\alpha_1\alpha_2\alpha_3\alpha_4}(0) &= \frac{1}{4\pi}\left(1+\frac{\cos^8 B}{[\cos^4 B-2(1+\sin^2 B)^2]^2}\right), &(4.64)
\end{aligned}$$

in complete agreement with the corresponding TBA value (3.39). We can also predict the precise position of the second plateau which, according to (4.45) is expected for the conductance. Once more we find complete agreement with the outcome of our TBA analysis, since in this case

$$\lim_{y\to 0} w_{i\bar{i}}^{\alpha_1\alpha_2\alpha_3\alpha_4}(\theta_1,\theta_2)=\left(\frac{2\pi\cos^4 B}{\cos^4 B-2(1+\sin^2 B)^2}\right)^2, \tag{4.65}$$

which gives

$$\begin{aligned}
\lim_{y\to 0}\langle J(r)Z_{\alpha_1}Z_{\alpha_2}Z_{\alpha_3}Z_{\alpha_4}J(0)\rangle_{T=m=0} &= \frac{1}{r^2}\left(\frac{\cos^4 B}{\cos^4 B-2(1+\sin^2 B)^2}\right)^2, &(4.66)\\
\lim_{y\to 0} G^{\alpha_1\alpha_2\alpha_3\alpha_4}(0) &= \frac{1}{2\pi}\left(\frac{\cos^4 B}{\cos^4 B-2(1+\sin^2 B)^2}\right)^2, &(4.67)
\end{aligned}$$

that is, the same expression as (3.37).

5 Conclusions

We have exploited the special features of 1+1 dimensional integrable quantum field theories in order to compute the DC conductance in an impurity system. For this purpose several non-perturbative techniques have been used. As the main tools we employed the thermodynamic Bethe ansatz in a Landauer transport theory computation and the form factor expansion in the Kubo formula.

The comparison between the Landauer formula (1.1) and the Kubo formula (1.2) yields in particular an identical plateau structure for the DC conductance in the ultraviolet limit.

We have explained to what extend integrability can be exploited in order to determine the reflection and transmission amplitudes through a defect. Unfortunately, for the most interesting situation in this context, namely when $R/\tilde{R}$ and $T/\tilde{T}$ are simultaneously non-vanishing, the Yang-Baxter-bootstrap equations narrow down the possible bulk theories to those which possess rapidity independent scattering matrices [25, 26]. By means of a relativistic potential scattering theory we compute for several types of defects the $R/\tilde{R}$s and $T/\tilde{T}$s, thus enlarging the set of examples available at present. We confirm that for real potentials parity is preserved, but otherwise essentially all possible combinations of parity breaking can occur. From the knowledge of the single defect amplitudes the multiple defect amplitudes, which exhibit the most interesting physical behaviours, can be computed in a standard fashion [33, 34].

We newly formulate the TBA equations for a defect with simultaneously non-vanishing reflection and transmission amplitudes. We indicate how these equations can be used to compute various thermodynamic quantities, which are, however, most interesting only when considered per unit length. By means of the TBA we compute the density distribution functions and use them to evaluate the Landauer conductance formula (1.1) for various defects in a complex free fermionic theory. We predict analytically the most prominent features in the conductance as a function of the temperature, i.e. the plateaux.

We formulate the Kubo formula [3] for a situation in which defects are present (1.2). We evaluate the current-current correlation functions occurring in there by means of another non-perturbative method based on integrability, namely the bootstrap form factor approach [11, 12]. We provide closed formulae which solve explicitly the defect recursive equations involving any arbitrary number of particles. As for the Landauer formula, we also predict in this case the plateaux in the conductance as a function of the temperature analytically.

There are several interesting open issues. Most challenging is to treat in full generality the massive and temperature dependent case of (1.2). Unfortunately, the formulation of non-perturbative methods do not yet cover that situation [51] and it remains to be clarified how the form factor bootstrap program for the computation of two-point functions can be extended to that case. It would be further interesting to compute thermodynamic quantities per unit length by means of the TBA formulated in section 3.3. To classify possible defects more systematically is desirable even for free theories.

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