Decoupling the $SU(N)_2$-homogeneous Sine-Gordon model

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We provide a systematic construction for all n-particle form factors of the $SU(N)_2/U(1)^{N-1}$-homogeneous Sine-Gordon model in terms of general determinant formulae for a huge class of local operators. The ultraviolet limit is carried out and the corresponding Virasoro central charge together with the conformal dimensions of various operators are identified. The renormalization group flow is studied and we find a precise rule, depending on the relative order of magnitude of the resonance parameters, according to which the theory decouples into new cosets along the flow.

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I. INTRODUCTION

For most integrable quantum field theories in 1+1 space-time dimensions it remains an open challenge to complete the entire bootstrap program, i.e. to compute the exact on-shell S-matrix, closed formulae for the n-particle form factors, identify the entire local operator content and in particular thereafter to compute the related correlation functions. Recently we investigated [1–3] a class of models, the $SU(3)_2/U(1)^2$-homogeneous Sine-Gordon model [4] (HSG), for which this task was completed to a large extend. In particular we provided general formulae for the n-particle form factors related to a huge class of local operators. In order to understand the generic group theoretical structure of the n-particle form factor expressions it is highly desirable to extend that analysis to higher rank as well as to higher level. One of the main purposes of this manuscript is to do the former, that is the investigation of the $SU(N)_2/U(1)^{N-1}$ case. This model may be viewed as the perturbation of a gauged WZNW-coset with Virasoro central charge

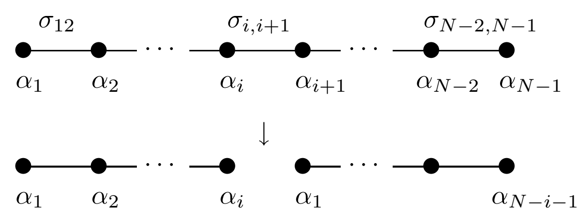
$$c_{SU(N)_2/U(1)^{N-1}} = \frac{N(N-1)}{(N+2)} \tag{1}$$

by an operator of conformal dimension $\Delta = N/(N+2)$. The theory possesses already a fairly rich particle content, namely $N-1$ asymptotically stable particles characterized by a mass scale m_i and $N-2$ unstable particles whose energy scale is characterized by the resonance parameters σ_{ij} $(1 \leq i,j \leq N-1)$. We relate the stable particles in a one-to-one fashion to the vertices of the $SU(N)$-Dynkin diagram and associate to the link between vertex i and j the $N-2$ linearly independent resonance parameters σ_{ij}.

We find that once an unstable particle becomes extremely heavy the original coset decouples into a direct product of two cosets different from the original one

$$\lim_{\sigma_{i,i+1}\to\infty} SU(N)_2/U(1)^{N-1} \equiv \tag{2}$$
$$SU(i+1)_2/U(1)^i \otimes SU(N-i)_2/U(1)^{N-i-1}\,.$$

Equivalently we may summarize the flow along the renormalization group trajectory with increasing RG-parameter r_0 to cutting the related Dynkin diagrams at decreasing values of the σ's. For instance taking $\sigma_{i,i+1}$ to be the largest resonance parameter at some energy scale the following cut takes place:



Using the usual expressions for the coset central charge [5], the decoupled system has the central charge

$$\lim_{\sigma_{i,i+1}\to\infty} c_{SU(N)_2/U(1)^{N-1}} = N-5+\frac{6(N+5)}{(N+2-i)(3+i)}\,. \tag{3}$$

Our manuscript is organized as follows: In section II we present the main characteristics of the HSG-scattering matrix. In section III we systematically construct solutions to the form factor consistency equations, which correspond to a huge class of local operators. In section IV we investigate the renormalization group flow of the Virasoro central charge, reproducing the decoupling (2). In section V we compute the operator content in terms of primary fields of the underlying conformal field theory. In section VI we investigate the RG flow of conformal dimensions. Our conclusions are stated in section VII.

II. THE S-MATRIX

The prerequisite for the computation of form factors and correlation functions thereafter is the knowledge of the exact scattering matrix. The two-particle S-matrix describing the scattering of two stable particles of type i and j, with $1 \leq i,j \leq N-1$, as a function of the rapidity

θ related to this model was proposed in [6]. Adopted to a slightly different notation it may be written as

$$S_{ij}(\theta) = (-1)^{\delta_{ij}} \left[c_i \tanh \frac{1}{2} \left(\theta + \sigma_{ij} - i\frac{\pi}{2} \right) \right]^{I_{ij}} . \qquad (4)$$

The incidence matrix of the $SU(N)$-Dynkin diagram is denoted by I. The parity breaking which is characteristic for the HSG models and manifests itself by the fact that $S_{ij} \neq S_{ji}$, takes place through the resonance parameters $\sigma_{ij} = -\sigma_{ji}$ and the colour value c_i. The latter quantity arises from a partition of the Dynkin diagram into two disjoint sets, which we refer to as "+" and "−". We then associate the values $c_i = \pm 1$ to the vertices i of the Dynkin diagram of $SU(N)$, in such a way that no two vertices related to the same set are linked together. Likewise we could simply divide the particles into odd and even, however, such a division would be specific to $SU(N)$ and the bi-colouration just outlined admits a generalization to other groups as well. The resonance poles in $S_{ij}(\theta)$ at $(\theta_R)_{ij} = -\sigma_{ij} - i\pi/2$ are associated in the usual Breit-Wigner fashion to the $N-2$ unstable particles as explained for instance in [7,6,3]. It is important for us to recall that the mass of the unstable particle $M_{\tilde{c}}$ formed in the scattering between the stable particles i and j behaves as $M_{\tilde{c}} \sim e^{|\sigma_{ij}|/2}$. There are no poles present on the imaginary axis, which indicates that no stable bound states may be formed.

It is clear from the expression of the scattering matrix (4), that whenever a resonance parameter σ_{ij} with $I_{ij} \neq 0$ goes to infinity, we may view the whole system as consisting out of two sets of particles which only interact freely amongst each other. The unstable particle, which was created in interaction process between these two theories before taking the limit, becomes so heavy that it can not be formed anymore at any energy scale.

III. FORM FACTORS

We are now in the position to compute the n-particle form factors related to this model, i.e. the matrix elements of a local operator $\mathcal{O}(\vec{x})$ located at the origin between a multiparticle in-state of particles (solitons) of species μ, created by $V_\mu(\theta)$, and the vacuum

$$F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1 \ldots \theta_n) = \langle \mathcal{O}(0) V_{\mu_1}(\theta_1) V_{\mu_2}(\theta_2) \ldots V_{\mu_n}(\theta_n) \rangle . \qquad (5)$$

We proceed in the usual fashion by solving the form factor consistency equations [8,9]. For this purpose we extract explicitly, according to standard procedure, the singularity structure. Since no stable bound states may be formed during the scattering of two stable particles the only poles present are the ones associated to the kinematic residue equations, that is a first order pole for particles of the same type whose rapidities differ by $i\pi$. Therefore, we parameterize the n-particle form factors as

$$F_n^{\mathcal{O}|\mathfrak{M}(l_+,l_-)}(\theta_1 \ldots \theta_n) = H_n^{\mathcal{O}|\mathfrak{M}(l_+,l_-)} Q_n^{\mathcal{O}|\mathfrak{M}(l_+,l_-)}(\mathfrak{X}) \times \prod_{1\leq i<j\leq n} \frac{F_{\min}^{\mu_i\mu_j}(\theta_{ij})}{\left(x_i^{c_{\mu_i}} + x_j^{c_{\mu_j}} \right)^{\delta_{\mu_i\mu_j}}} . \qquad (6)$$

As usual we abbreviate the rapidity difference as $\theta_{ij} = \theta_i - \theta_j$. Aiming towards a universally applicable and concise notation, it is convenient to collect the particle species $\mu_1 \ldots \mu_n$ in form of particular sets

$$\mathfrak{M}_{\mathfrak{i}}(l_i) = \{ \mu \,|\, \mu = i \} \qquad (7)$$

$$\mathfrak{M}_\pm(l_\pm) = \bigcup_{i\in\pm} \mathfrak{M}_{\mathfrak{i}}(l_i) \qquad (8)$$

$$\mathfrak{M}(l_+,l_-) = \mathfrak{M}_+(l_+) \cup \mathfrak{M}_-(l_-). \qquad (9)$$

The number of elements belonging to the sets $\mathfrak{M}_{\mathfrak{i}}$,$\mathfrak{M}_\pm$ is indicated by their arguments $l_i, l_\pm$, respectively. We understand here that inside the sets $\mathfrak{M}_\pm$ the order of the individual sets $\mathfrak{M}_{\mathfrak{i}}$ is arbitrary. This simply reflects the fact that particles of different species but identical colour interact freely. However, $\mathfrak{M}$ is an ordered set since elements of $\mathfrak{M}_+$ and $\mathfrak{M}_-$ do not interact freely and w.l.g. we adopt the convention that particles belonging to the "+"-colour set come first. The H_n are some overall constants and the Q_n are polynomial functions depending on the variables $x_i = \exp\theta_i$ which are collected in the sets $\mathfrak{X}$,$\mathfrak{X}_{\mathfrak{i}}$,$\mathfrak{X}_\pm$ in a one-to-one fashion with respect to the particle species sets $\mathfrak{M}$,$\mathfrak{M}_{\mathfrak{i}}$,$\mathfrak{M}_\pm$. The functions $F_{\min}^{\mu_i\mu_j}(\theta_{ij})$ are the so-called minimal form factors which by construction contain no singularities in the physical sheet and solve Watson's equations [8,9] for two particles. For the $SU(N)_2$-HSG model they are found to be

$$F_{\min}^{ij}(\theta) = \mathcal{N}^{I_{ij}} \left(\sin\frac{\theta}{2i} \right)^{\delta_{ij}} e^{-I_{ij}\int_0^\infty \frac{dt}{t} \frac{\sin^2\left((i\pi-\theta\mp\sigma_{ij})\frac{t}{2\pi} \right)}{\sinh t \cosh t/2}} . \qquad (10)$$

Here $\mathcal{N} = 2^{\frac{1}{4}} \exp\left(i\pi(1-c_i)/4 + c_i\theta/4 - G/4 \right)$ is a normalization function with G being the Catalan constant. It is convenient also to introduce the function $\tilde{F}_{\min}^{ij}(\theta) = (e^{-c_i\theta/4} F_{\min}^{ij}(\theta))^{I_{ij}}$. The minimal form factors obey the functional identities

$$F_{\min}^{ij}(\theta+i\pi) F_{\min}^{ij}(\theta) = \left(-\frac{i}{2}\sinh\theta \right)^{\delta_{ij}} \left(\frac{i^{\frac{2-c_i}{2}}\, e^{\frac{\theta}{2}c_i}}{\cosh\left(\frac{\theta}{2} - \frac{i\pi}{4} \right)} \right)^{I_{ij}} . \qquad (11)$$

Substituting the ansatz (6) into the kinematic residue equation [8,9,1], we obtain with the help of (11) a recursive equation for the overall constants for $\mu_i \in \mathfrak{M}_+$

$$H_{n+2}^{\mathcal{O}|\mathfrak{M}(l_++2,l_-)} = i^{\bar{l}_i} 2^{2l_i-\bar{l}_i+1} e^{I_{ij}\sigma_{ij}l_j/2} H_n^{\mathcal{O}|\mathfrak{M}(l_+,l_-)} . \qquad (12)$$

We introduced here the numbers $\bar{l}_i = \sum_{\mu_j\in\mathfrak{M}_-} I_{ij} l_j$, which count the elements in the neighbouring sets of $\mathfrak{M}_{\mathfrak{i}}$.

The Q-polynomials have to obey the recursive equations

$$Q_{n+2}^{\mathcal{O}|\mathfrak{M}(2+l_+,l_-)}(\mathfrak{X}^{xx}) = \sum_{k=0}^{\bar{s}_i} x^{2s_i-2k+\tau_i+1-\varsigma_i}\sigma_{2k+\varsigma_i}(I_{ij}\hat{\mathfrak{X}}_j) \times(-i)^{2s_i+\tau_i+1}\sigma_{2s_i+\tau_i}(\mathfrak{X}_i)Q^{\mathcal{O}|\mathfrak{M}(l_+,l_-)}(\mathfrak{X}) \qquad (13)$$

For convenience we defined the sets $\mathfrak{X}^{xx} := \{-x,x\}\cup X$, $\hat{\mathfrak{X}} := ie^{\sigma_{i,i+1}}\mathfrak{X}$ and the integers ζ_i which are 0 or 1 depending on whether the sum $\vartheta+\tau_i$ is odd or even, respectively. ϑ is related to the factor of local commutativity $\omega = (-1)^\vartheta = \pm 1$. $\sigma_k(x_1,\ldots,x_m)$ is the k-th elementary symmetric polynomial. Furthermore, we used the sum convention $I_{ij}\hat{\mathfrak{X}}_j \equiv \bigcup_{\mu_j\in\mathfrak{M}} I_{ij}\hat{\mathfrak{X}}_j$ and parameterized $l_i = 2s_i+\tau_i$, $\bar{l}_i = 2\bar{s}_i+\bar{\tau}_i$ in order to distinguish between odd and even particle numbers.

We will now solve the recursive equations (12) and (13) systematically. The equations for the constants are solved by

$$H_n^{\mathcal{O}|\mathfrak{M}(l_+,l_-)} = \prod_{\mu_i\in\mathfrak{M}_+} i^{s_i\bar{l}_i}\, 2^{s_i(2s_i-\bar{l}_i-1-2\tau_i)} e^{\frac{s_i I_{ij}\sigma_{ij} l_j}{2}} H^{\mathcal{O}|\tau_i,\bar{l}_i}. \qquad (14)$$

The lowest nonvanishing constants $H^{\mathcal{O}|\tau_i,\bar{l}_i}$ are fixed by demanding, similarly as in the $SU(3)_2$-case [1–3], that any form factor which involves only one particle type should correspond to a form factor of the thermally perturbed Ising model. To achieve this we exploit the ambiguity present in (12), that is the fact that we can multiply it by any constant which only depends on the l_--quantum numbers.

As the main building blocks for the construction of the Q-polynomials serve the determinants of the $(t+s)\times(t+s)$-matrix

$$\mathcal{A}_{2s+\tau^+,2t+\tau^-}^{\nu^+,\nu^-}(\mathfrak{X}_+,\hat{\mathfrak{X}}_-)_{ij} = \begin{cases} \sigma_{2(j-i)+\nu^+}(\mathfrak{X}_+), 1\le i\le t \\ \sigma_{2(j-i+t)+\nu^-}(\hat{\mathfrak{X}}_-)\,,\ t< i\le s+t \end{cases} \qquad (15)$$

for $\nu^\pm,\tau^\pm = 0,1$ which were introduced in [2]. The determinant of $\mathcal{A}$ essentially captures the summation in (13) due to the fact that it satisfies the recursive equations

$$\det\mathcal{A}_{2+l,2t+\tau^-}^{\nu^+,\nu^-}(\mathfrak{X}_+^{xx},\hat{\mathfrak{X}}_-) = \left(\sum_{p=0}^{t} x^{2(t-p)}\sigma_{2p+\nu^-}(\hat{\mathfrak{X}}_-)\right) \times\det\mathcal{A}_{l,2t+\tau^-}^{\nu^+,\nu^-}(\mathfrak{X}_+,\hat{\mathfrak{X}}_-) \qquad (16)$$

as was shown in [2]. Analogously to the procedure in [2] we can build up a simple product from elementary symmetric polynomials which takes care of the pre-factor in the recursive equation (13). Defining therefore the polynomials

$$Q_n^{\mathfrak{M}(l_+,l_-)}(\mathfrak{X}_+,\mathfrak{X}_-) = \prod_{\mu_{i/k}\in\mathfrak{M}_{+/-}} \det\mathcal{A}_{2s_i+\tau_i,\bar{l}_i}^{\nu_i,\varsigma_i}(\mathfrak{X}_i,I_{ij}\hat{\mathfrak{X}}_j) \qquad (17)$$
$$\times\sigma_{2s_i+\tau_i}(\mathfrak{X}_i)^{\frac{2s_i+\tau_i-2\bar{s}_i-1-\varsigma_i}{2}}\sigma_{\bar{l}_i}(I_{ij}\hat{\mathfrak{X}}_j)^{\frac{\bar{\nu}_i-1}{2}}\sigma_{l_k}(\hat{\mathfrak{X}}_k))^{\frac{1-l_k}{2}}$$

it follows immediately with the help of property (16) that they obey the recursive equations

$$Q_{2+n}^{\mathfrak{M}(2+l_+,l_-)}(\mathfrak{X}_+^{xx},\mathfrak{X}_-) = Q_n^{\mathfrak{M}(l_+,l_-)}(\mathfrak{X}_+,\mathfrak{X}_-)\sigma_{2s_i+\tau_i}(\mathfrak{X}_i) \times\sum_{p=0}^{\bar{s}_i} x^{2(s_i-p)+\tau_i+1-\varsigma_i}\sigma_{2p+\varsigma_i}(I_{ij}\hat{\mathfrak{X}}_j)\ . \qquad (18)$$

Comparing now the equations (13) and (18) we obtain complete agreement. Notice that the numbers $\bar{\nu}_i$ are not constrained at all at this point of the construction. However, by demanding relativistic invariance, which on the other hand means that the overall power in (6) has to be zero, we obtain the additional constraints

$$\nu_i = 1+\tau_i-\bar{\nu}_i \qquad \text{and} \qquad \tau_i\varsigma_i = \bar{\tau}_i(\bar{\nu}_i-1)\ . \qquad (19)$$

Taking in addition the constraints into account which are needed to derive (16) (see [2]), this is most conveniently written as

$$\tau_i\varsigma_i+\bar{\tau}_i\nu_i = \tau_i\bar{\tau}_i, \qquad 2+\varsigma_i>\bar{\tau}_i, \qquad 2+\nu_i>\tau_i\ . \qquad (20)$$

For each $\mu_i\in\mathfrak{M}_+$ the equations (20) admit the 10 feasible solutions found in [2]. However, one should notice that the individual solutions for different values of i are not all independent of each other. We would like to stress that despite the fact that (17) represents a huge class of independent solutions, it does certainly not exhaust all of them. Nonetheless, many additional solutions, like the energy momentum tensor, may be constructed from (20) by simple manipulations like the multiplication of some CDD-like ambiguity factors or by setting some expressions to zero on the base of asymptotic considerations (see [2]) for more details. For many applications we wish to carry out in the next section we require the form factors for the trace of the energy momentum Θ, the first non-vanishing terms read

$$F_2^{\Theta|\mu_i\mu_i} = -2\pi i m^2\sinh(\theta/2)$$
$$F_4^{\Theta|\mu_i\mu_i\mu_j\mu_j} = \frac{\pi m^2(2+\sum_{i<j}\cosh(\theta_{ij}))\prod_{i<j}\tilde{F}_{\min}^{\mu_i\mu_j}(\theta_{ij})}{-2\cosh(\theta_{12}/2)\cosh(\theta_{34}/2)}$$
$$F_6^{\Theta|\mu_i\mu_i\mu_i\mu_i\mu_j\mu_j} = \frac{\pi m^2(3+\sum_{i<j}\cosh(\theta_{ij}))\prod_{i<j}\tilde{F}_{\min}^{\mu_i\mu_j}(\theta_{ij})}{4\prod_{1\le i<j\le 4}\cosh(\theta_{ij}/2)}$$
$$F_6^{\Theta|\mu_i\mu_i\mu_k\mu_k\mu_j\mu_j} = \frac{\pi m^2(3+\sum_{i<j}\cosh(\theta_{ij}))\prod_{i<j}\tilde{F}_{\min}^{\mu_i\mu_j}(\theta_{ij})}{4\cosh(\theta_{12}/2)\cosh(\theta_{34}/2)}$$

for $I_{ij}\neq 0$ and $I_{kj}\neq 0$. When considering the RG flow in the next section, it will be important to note that from $\lim_{\sigma_{i,i+1}\to\infty}F_{\min}^{\mu_i\mu_{i+1}}(\theta)\sim\exp(-\sigma_{i,i+1}/4)$ follows

$$\lim_{\sigma_{i,i+1}\to\infty}F_n^{\Theta|\mu_i\mu_{i+1}\cdots}(\theta) = 0\ . \qquad (21)$$

Having determined the form factors, we are in principle in the position to compute the two-point correlation function between two local operators in the usual way by expanding it in terms of n-particle form factors

$$\langle \mathcal{O}(r)\mathcal{O}'(0)\rangle = \sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n}e^{-r\,E} \tag{22}$$
$$\times\, F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\,\left(F_n^{\mathcal{O}'|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\right)^* .$$

We abbreviated the sum of the on-shell energies as $E=\sum_{i=1}^{n} m_{\mu_i}\cosh\theta_i$. Now we want to evaluate the expression (22) in several different applications in order to compute various quantities of interest.

IV. RENORMALIZATION GROUP FLOW

Renormalization group methods have been developed originally [10] to carry out qualitative analysis of regions of quantum field theories which are not accessible by perturbation theory in the coupling constant. In particular the β-function provides an inside into various possible asymptotic behaviours and especially it allows to identify the fixed points of the theory. We now want to employ this method to check our solutions and at the same time the physical picture advocated for the HSG-models.

For this purpose we want to investigate first of all the renormalization group flow, in a similar spirit as for the $SU(3)_2/U(1)^2$-case in [3], by evaluating the c-theorem [11].

$$c(r_0)=\frac{3}{2}\int_{r_0}^{\infty} dr\, r^3\ \langle\Theta(r)\Theta(0)\rangle\ . \tag{23}$$

In particular for $r_0=0$ the function $c(r_0)$ coincides with $\Delta c = c_{\mathrm{uv}}-c_{\mathrm{ir}}$, i.e. the difference between the ultraviolet and infrared Virasoro central charges. Computing the correlation function for the trace of the energy-momentum tensor Θ in (23) by means of (22) and using the form factor expressions of the previous section the individual n-particle contributions turn out to be

$$\Delta c^{(2)}=(N-1)\cdot 0.5 \tag{24}$$
$$\Delta c^{(4)}=(N-2)\cdot 0.197 \tag{25}$$
$$\Delta c^{(6)}=(N-2)\cdot 0.002+(N-3)\cdot 0.0924 \tag{26}$$
$$\sum_{k=2}^{6}\Delta c^{(k)}=N*0.7914-1.1752\ . \tag{27}$$

Apart from the two particle contribution (24), which is usually quite trivial and in this situation can even be evaluated analytically, we have carried out the multidimensional integrals in (22) by means of a Monte Carlo method. We use this method up to a precision which is higher than the last digit we quote. For convenience we report some explicit numbers in table 1.

N	c	$\Delta c^{(2)}$	$\Delta c^{(4)}$	$\Delta c^{(6)}$	$\sum_{k=2}^{6}\Delta c^{(k)}$
3	1.2	1	0.197	0.002	1.199
4	2	1.5	0.394	0.096	1.990
5	2.857	2	0.591	0.191	2.782
6	3.75	2.5	0.788	0.285	3.573
7	4.6	3	0.985	0.380	4.365
8	5.6	3.5	1.182	0.474	5.156

Table 1: n-particle contributions to the c-theorem versus $SU(N)_2/U(1)^{N-1}$ -WZNW coset model central charge.

The evaluation of (24)-(27) illustrates that the series (22) converges slower and slower for increasing values of N, such that the higher n-particle contributions become more and more important to achieve high accuracy. Our analysis suggests that it is not the functional dependence of the individual form factors which is responsible for this behaviour. Instead this effect is simply due to the fact that the symmetry factor, that is the sum $\sum_{\mu_1\ldots\mu_n}$, resulting from permutations of the particle species increases drastically for larger N.

Having confirmed the expected ultraviolet central charge, we now study the RG-flow by varying r_0 in (23). We expect to find that whenever we reach an energy scale at which an unstable particle can be formed, the model will flow to a different coset. This means following the flow with increasing r_0 we will encounter a situation in which certain $\sigma_{i,i+1}$ are considered to be large and we observe the decoupling into two freely interacting systems in the way described in (2). For instance for the situation $\sigma_{12}>\sigma_{23}>\sigma_{34}>\ldots$ we observe the following decoupling along the flow with increasing r_0:

$$\begin{gathered} SU(N)_2/U(1)^{N-1} \\ \downarrow \\ SU(N-1)_2/U(1)^{N-2}\otimes SU(2)_2/U(1) \\ \downarrow \\ SU(N-2)_2/U(1)^{N-3}\otimes (SU(2)_2/U(1))^2 \\ \downarrow \\ \vdots \\ \downarrow \\ (SU(2)_2/U(1))^{N-1} \end{gathered}$$

We can understand this type of behaviour in a semi-analytical way. The precise difference between the central charges related to (2) is

$$c_{SU(i+1)_2/U(1)^i\otimes SU(N-i)_2/U(1)^{N-i-1}} = \tag{28}$$
$$c_{SU(N)_2/U(1)^{N-1}}-\frac{2i(N+5)(N-i-1)}{(N+2)(i+3)(N-i+2)}.$$

Noting with (21) that we loose at each step all the contributions $F_n^{\Theta|\mu_i\mu_{i+1}\ldots}(\theta)$ to Δc, we may collect the values (24)-(26), which we have determined numerically and find

$$\lim_{\sigma_{i,i+1}\to\infty}\Delta c(\sigma_{i,i+1},\ldots)= \tag{29}$$
$$\Delta c(\sigma_{i,i+1}=0,\ldots)-0.2914 I_{i,i+1}-0.0924 I_{i,i-1}$$

Similarly as for the deep UV-region we find a relatively good agreement between (28) and (29) for small values of N. The difference for larger values is once again due to the convergence behaviour of the series in (22).

For $r_0 = 0$ qualitatively a similar kind of behaviour was previously observed, for the two particle contribution only, in the context of the roaming Sinh-Gordon model [12]. Nonetheless, there is a slight difference between the two situations. Instead of a decoupling into different cosets in these type of models the entire S-matrix takes on the value -1, when the resonance parameter goes to infinity. The resulting effect, i.e. a depletion of Δc, is the same. However, we do not comply with the interpretation put forward in [12], namely that such a behaviour should constitute a "violation of the c-theorem". The observed effect is precisely what one expects from the physical point of view and the c-theorem.

We present our full numerical results in figure 1, which confirm the outlined flow for various values of N.

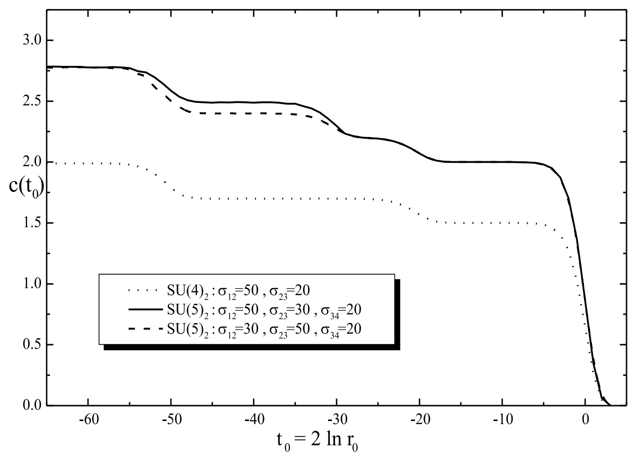


Figure 1: RG flow for the Virasoro central charge.

We observe that the c-function remains constant, at a value corresponding to the new coset, in some finite interval of r_0. In particular, we observe the non-equivalence of the flows when the relative order of magnitude amongst the different resonance parameters is changed. For $N = 5$ we confirm (we omit here the $U(1)$-factors and report the corresponding central charges as superscripts on the last factor)

$$\begin{array}{ccc} \boxed{\sigma_{12} > \sigma_{23}} & SU(5)_2^{\frac{20}{7}} & \boxed{\sigma_{23} > \sigma_{12}} \\ \swarrow & & \searrow \\ SU(4)_2 \otimes SU(2)_2^{\frac{5}{2}} & & SU(3)_2 \otimes SU(3)_2^{\frac{12}{5}} \\ \searrow & & \swarrow \\ & SU(3)_2 \otimes SU(2)_2 \otimes SU(2)_2^{\frac{11}{5}} & \\ & \downarrow & \\ & SU(2)_2 \otimes SU(2)_2 \otimes SU(2)_2 \otimes SU(2)_2^2 & \end{array}$$

The precise difference in the central charges is explained with (29), since the contribution $0.0924 I_{i,i-1}$ only occurs for $i = 1$.

To establish more clearly that the plateaus admit indeed an interpretation as fixed points and extract the definite values of the corresponding Virasoro central charge we can also, following [11], determine a β type function from $c(r)$. The β-function should obey the Callan-Symanzik equation [13]

$$r \frac{d}{dr} g = \beta(g) \ . \tag{30}$$

The "coupling constant" $g := c_{\mathrm{uv}} - c(r)$ is normalized in such a way that it vanishes at the ultraviolet fixed point. Whenever we find $\beta(\tilde{g}) = 0$, we can identify $\tilde{c} = c_{\mathrm{uv}} - \tilde{g}$ as the Virasoro central charge of the corresponding conformal field theory. Hence, taking the data obtained from (23), we compute β as a function of g by means of (30). Our results for various values of N are depicted in figure 2, which allow a definite identification of the fixed points corresponding to the coset models expected from the decoupling (2).

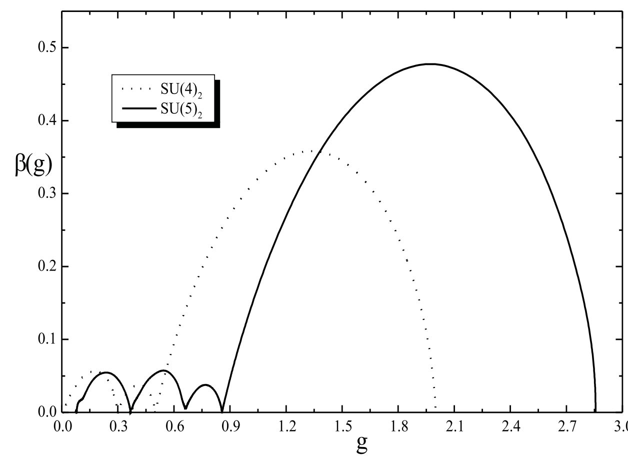


Figure 2: The β-function.

For $SU(4)_2$ we clearly identify from figure 2 the four fixed points $\tilde{g} = 0, 0.3, 0.5, 2$ with high accuracy. The five fixed points $\tilde{g} = 0, 0.357, 0.657, 0.857, 2.857$, which we expect to find for $SU(5)_2$ are all slightly shifted due to the absence of the higher order contributions.

V. OPERATOR CONTENT OF $SU(N)_2/U(1)^{N-1}$

We now want to identify the operator content of our theory by carrying out the ultraviolet limit and matching the conformal dimension of each operator with the one in the $SU(N)_2/U(1)^{N-1}$-WZNW-coset model. For this purpose we have to determine first of all the entire operator content of the conformal field theory.

According to [14] the conformal dimensions of the parafermionic vertex operators are given by

$$\Delta(\Lambda, \lambda) = \frac{(\Lambda \cdot (\Lambda + 2\rho))}{(4 + 2N)} - \frac{(\lambda \cdot \lambda)}{4} \ . \tag{31}$$

Here Λ is a highest dominant weight of level smaller or equal to 2 and $\rho = 1/2 \sum_{\alpha>0} \alpha$ is the Weyl vector, i.e. half the sum of all positive roots. The λ's are the corresponding lower weights, which may be constructed in the usual fashion (see e.g. [15]): Consider a complete weight string $\lambda + n\alpha, \ldots, \lambda, \ldots, \lambda - m\alpha$, that is all the weights obtained by successive additions (subtractions) of a root α from the weight λ, such that $\lambda + (n+1)\alpha$ $(\lambda - (m+1)\alpha)$ is not a weight anymore. It is then a well known fact that the difference between the two integers m, n is $m - n = \lambda \cdot \alpha$ for simply laced Lie algebras. This means starting with the highest weight Λ, we can work our way downwards by deciding after each subtraction of a simple root α_i whether the new vector, say χ, is a weight or not from the criterion $m_i = n_i + \chi \cdot \alpha_i > 0$. With the procedure just outlined we obtain all possible weights of the theory. Nonetheless, it may happen that a weight corresponds to more than one linear independent weight vector, such that the weight space may be more than one dimensional. The dimension of each weight vector n_λ^Λ is computed by means of

$$n_\lambda^\Lambda = \frac{\sum_{\alpha>0} \sum_{l=1}^{\infty} 2\, n_{\lambda+l\alpha}((\lambda + l\alpha)\cdot\alpha)}{((\Lambda + \lambda + 2\rho)\cdot(\Lambda - \lambda))} \quad . \tag{32}$$

For consistency it is useful to compare the sum of all these multiplicities with the dimension of the highest weight representation computed directly from the Weyl dimensionality formula (see e.g. [15])

$$\sum_\lambda n_\lambda^\Lambda = \dim \Lambda = \prod_{\alpha>0} \frac{((\Lambda + \rho)\cdot\alpha)}{(\rho\cdot\alpha)} \quad . \tag{33}$$

To compute all the conformal dimensions $\Delta(\Lambda, \lambda)$ according to (31) in general is a formidable task and therefore we concentrate on a few distinct ones for generic N and only compute the entire content for $N = 4$.

Noting that $\lambda_i \cdot \lambda_j = K_{ij}^{-1}$, with K being the Cartan matrix, we can obtain relative concrete formulae from (31). For instance

$$\Delta(\lambda_i, \lambda_i) = \frac{4\sum_{l=1}^{N-1} K_{il}^{-1} - N K_{ii}^{-1}}{8 + 2N} \quad . \tag{34}$$

Similarly we may compute $\Delta(\lambda_i + \lambda_j, \lambda_i + \lambda_j)$, etc. in terms of components of the inverse Cartan matrix. Even more explicit formulae are obtainable when we express the simple roots α_i and fundamental weights λ_i of $SU(N)$ in terms of a concrete basis. For instance we may choose an orthonormal basis $\{\varepsilon_i\}$ in $\mathbb{R}^N$ (see e.g. [16]), i.e. $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \;\; \lambda_i = \sum_{j=1}^{i} \varepsilon_j - \frac{i}{N}\sum_{j=1}^{N} \varepsilon_j, \;\; i = 1, \ldots N-1\,.$$

Noting further that the set of positive roots is given by $\{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq N\}$, we can evaluate (31), (32) and (33) explicitly. This way we obtain for instance

$$\Delta(\lambda_i, \lambda_i) = \frac{i(N-i)}{8+4N} \qquad \text{and} \qquad \Delta(2\lambda_i, 2\lambda_i) = 0. \tag{35}$$

Of special physical interest is the dimension of the perturbing operator. As was already argued in [3], it corresponds to $\Delta(\psi, 0)$, with ψ being the highest root, and moreover it is unique. Noting that for $SU(N)$ we have $\psi = \lambda_1 + \lambda_{N-1}$, we confirm once more

$$\Delta(\psi = \lambda_1 + \lambda_{N-1}, 0) = \frac{N}{N+2} \quad . \tag{36}$$

Other dimensions may be computed similarly.

A. The $SU(4)/U(1)^3$ example

For $SU(4)/U(1)^3$ we present the result of the computation of the entire operator content in table 2. In case the multiplicity of a weight vector is bigger than one, we indicate this by a superscript on the conformal dimension.

$\lambda \backslash \Lambda$	λ_1	λ_2	$2\lambda_1$	$2\lambda_2$	$\lambda_1+\lambda_2$	$\lambda_1+\lambda_3$
dim Λ	4	6	10	20	20	15
Λ	1/8	1/6	0	0	1/8	1/6
$\Lambda-\alpha_1$	1/8		1/2		1/8	1/6
$\Lambda-\alpha_2$		1/6		1/2	1/8	
$\Lambda-\alpha_3$						1/6
$\Lambda-\alpha_1-\alpha_2$	1/8	1/6	1/2	1/2	$5/8^2$	1/6
$\Lambda-\alpha_2-\alpha_3$		1/6		1/2	1/8	1/6
$\Lambda-\alpha_1-\alpha_3$						1/6
$\Lambda-2\alpha_1$			0			
$\Lambda-2\alpha_2$				0		
$\Lambda-2\alpha_1-2\alpha_2$			0	0		
$\Lambda-2\alpha_2-2\alpha_3$				0		
$\Lambda-2\alpha_1-\alpha_2$			1/2		1/8	
$\Lambda-\alpha_1-2\alpha_2$				1/2	1/8	
$\Lambda-2\alpha_2-\alpha_3$				1/2		
$\Lambda-\alpha_1-\alpha_2-\alpha_3$	1/8	1/6	1/2	1/2	$5/8^2$	$2/3^3$
$\Lambda-2\alpha_1-\alpha_2-\alpha_3$			1/2		1/8	1/6
$\Lambda-\alpha_1-2\alpha_2-\alpha_3$		1/6		1^2	$5/8^2$	1/6
$\Lambda-\alpha_1-\alpha_2-2\alpha_3$						1/6
$\Lambda-2\alpha_1-2\alpha_2-\alpha_3$			1/2	1/2	$5/8^2$	1/6
$\Lambda-\alpha_1-2\alpha_2-2\alpha_3$				1/2	1/8	1/6
$\Lambda-2\alpha_1-2\alpha_2$					1/8	
$\Lambda-2\alpha_1-2\alpha_2-2\alpha_3$			0	0	1/8	1/6
$\Lambda-\alpha_1-3\alpha_2-2\alpha_3$				1/2		
$\Lambda-\alpha_1-3\alpha_2-\alpha_3$				1/2		
$\Lambda-2\alpha_1-3\alpha_2-\alpha_3$				1/2	1/8	
$\Lambda-2\alpha_1-3\alpha_2-2\alpha_3$				1/2	1/8	
$\Lambda-2\alpha_1-4\alpha_2-2\alpha_3$				0		

Table 2: Conformal dimensions for $\mathcal{O}^{\Delta(\Lambda,\lambda)}$ in the $SU(4)_2/U(1)^3$ -WZNW coset model.

The remaining dominant weights of level smaller or equal to 2, namely $\Lambda = \lambda_3, 2\lambda_3, \lambda_2 + \lambda_3$, including their multiplicities may be obtained from table 2 simply by the exchange $1 \leftrightarrow 3$, which corresponds to the $\mathbb{Z}_2$-symmetry of the $SU(4)$-Dynkin diagram.

Summing up all the fields corresponding to different lower weights, i.e. not counting the multiplicities, we have the following operator content

$$\mathcal{O}^{2/3}, \mathcal{O}^{1}, 14\times\mathcal{O}^{0}, 8\times\mathcal{O}^{5/8}, 18\times\mathcal{O}^{1/6}, 24\times\mathcal{O}^{1/2}, 32\times\mathcal{O}^{1/8},$$

that is 98 fields.

VI. OPERATOR CONTENT OF HSG

We will now turn to the massive model and evaluate the flow of the conformal dimension [3]

$$\Delta^{\mathcal{O}}(r_0) = -\frac{1}{2\langle\mathcal{O}(0)\rangle}\int_{r_0}^{\infty} dr\, r\ \langle\Theta(r)\mathcal{O}(0)\rangle\ . \qquad (37)$$

Here $\mathcal{O}$ is a local operator which in the conformal limit corresponds to a primary field in the sense of [17]. In particular for $r_0 = 0$, the expression (37) constitutes the delta sum rule [18], which expresses the difference between the ultraviolet and infrared conformal dimension of the operator $\mathcal{O}$.

We start by investigating the operator which in the case when all particles in (5) are of the same type corresponds to the disorder operator μ in the Ising model. Using the fact that we should always be able to reduce to that situation, we consider the solution corresponding to $\tau_i = \bar{\tau}_i = \nu_i = \varsigma_i = 0$ for all i. Then the Δ-sum rule (37) yields for the individual n-particle contributions

$$\Delta^{\mu(2)} = (N-1)\cdot 0.0625 \qquad (38)$$

$$\Delta^{\mu(4)} = (2-N)\cdot 0.0263 \qquad (39)$$

$$\Delta^{\mu(6)} = (N-2)\cdot 0.0017 + (3-N)\cdot 0.0113 \qquad (40)$$

$$\sum_{k=2}^{6}\Delta^{\mu(k)} = 0.0266 + N * 0.0206\ . \qquad (41)$$

We assume that this solution has the conformal dimension $\Delta(\lambda_1,\lambda_1)$ in the ultraviolet limit. For comparison we report a few explicit numbers in table 3.

N	$\Delta(\lambda_1,\lambda_1)$	$\Delta^{\mu(2)}$	$\Delta^{\mu(4)}$	$\Delta^{\mu(6)}$	$\sum_{k=2}^{6}\Delta^{\mu}$
3	0.1	0.125	−0.0263	0.0017	0.1004
4	0.125	0.1875	−0.0526	−0.0079	0.1270
5	0.143	0.25	−0.0789	−0.0175	0.1536
6	0.156	0.3125	−0.1052	−0.0271	0.1802
7	$0.1\bar{6}$	0.375	−0.1315	−0.0367	0.2068
8	0.175	0.4375	−0.1578	−0.0463	0.2334

Table 3: n-particle contributions to the Δ-theorem versus conformal dimensions in the $SU(N)_2/U(1)^{N-1}$ -WZNW coset model.

As we already observed for the c-theorem, the series converges slower for larger values of N. The reason for this behaviour is the same, namely the increasing symmetry factor. Note also that the next contribution is negative.

Following now the RG-flow for the conformal dimension (37) by varying r_0, we assume that the $\Delta(\lambda_1,\lambda_1)$-field flows to the $\Delta(\lambda_1,\lambda_1)$-field in the corresponding new cosets. Similar as for the Virasoro central charge we may compare the exact expression

$$\Delta(\lambda_1,\lambda_1)_{SU(i+1)_2/U(1)^i\otimes SU(N-i)_2/U(1)^{N-i-1}} = \qquad (42)$$
$$\Delta(\lambda_1,\lambda_1)_{SU(N)_2/U(1)^{N-1}} + \frac{i(N+5)(N-i-1)}{4(N+2)(i+3)(N-i+2)}.$$

with the numerical results. The contributions (38)-(40) yield

$$\lim_{\sigma_{i,i+1}\to\infty}\Delta^{\mu}(\sigma_{i,i+1},\ldots) = \qquad (43)$$
$$\Delta^{\mu}(\sigma_{i,i+1}=0,\ldots) + 0.0359 I_{i,i+1} + 0.0113 I_{i,i-1}.$$

Once again we find good agreement between the two computations for small values of N. Our complete numerical results are presented in figure 3, which confirm the outlined flow for various values of N.

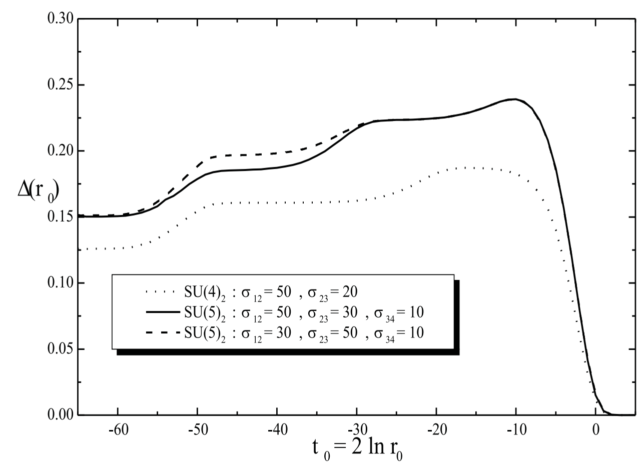


Figure 3: RG flow for the conformal dimension of μ.

Notice by comparing the figures 3 and 1, that, as we expect, the transition from one value for Δ to the one in the decoupled system occurs at the same energy scale t_0 at which the value of the Virasoro central charge flows to the new one.

In analogy to (30) we may now define a function "β'" and demand that it obeys the Callan-Symanzik equation

$$r\frac{d}{dr}g' = \beta'(g')\ . \qquad (44)$$

The "coupling constant" related to β' is normalized in such a way that it vanishes at the ultraviolet fixed point, i.e. $g' := \Delta(r) - \Delta_{\rm uv}$, such that whenever we find $\beta'(\tilde{g}') = 0$, we can identify $\hat{\Delta} = \tilde{g}' - \Delta_{\rm uv}$ as the conformal dimension of the operator under consideration of the corresponding conformal field theory. From our analysis of (37) we may determine β' as a function of g' by means of (44). Our results are presented in figure 4.

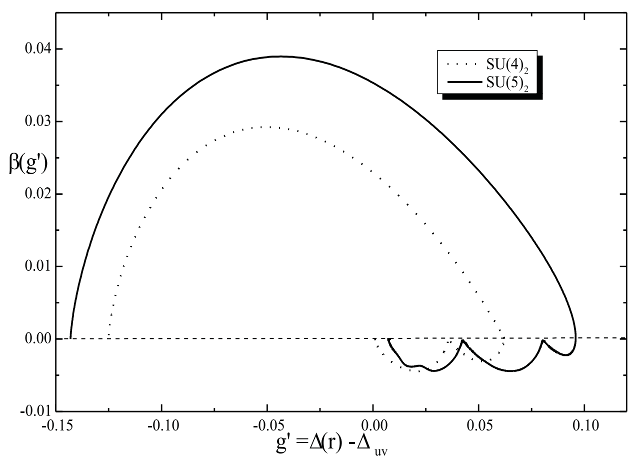


Figure 4: The β'-function .

Once again, for $SU(4)_2$ the accuracy is very high and we clearly read off from figure 4 the expected fixed points $\tilde{g}' = -0.125, 0, 0.0375, 0.0625$. The $SU(5)_2$-fixed points $\tilde{g}' = -0.1429, 0, 0.0446, 0.0821, 0.1071$, are once again slightly shifted.

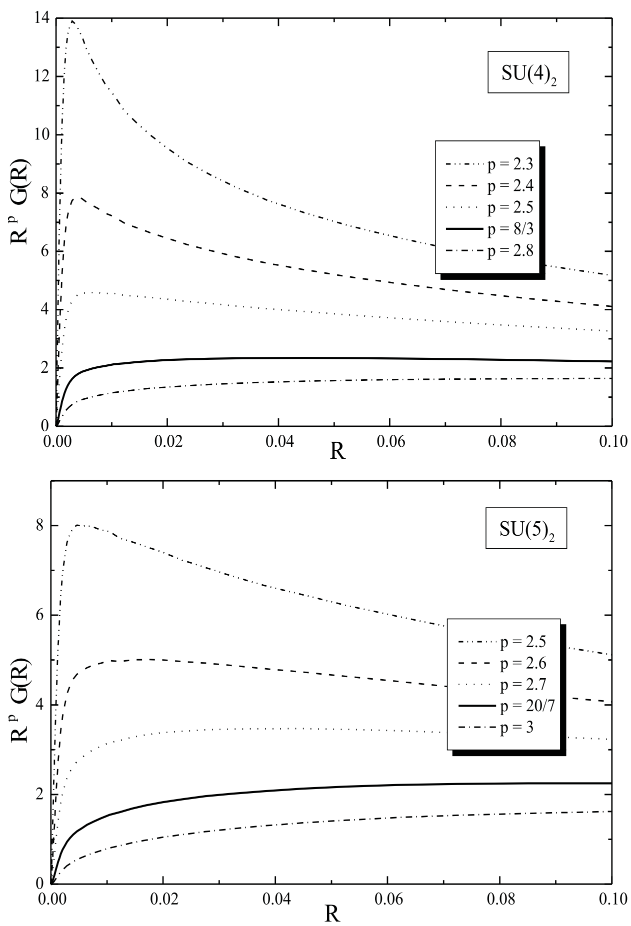


Figure 5: Rescaled correlation function G(R)=$\langle\Theta(R)\Theta(0)\rangle$ as a function of $R = rm$.

Unfortunately, whenever the correlation function between $\mathcal{O}$ and Θ is vanishing, or when we consider an operator which does not flow to a primary field, we can not employ the delta sum rule (37). Alternatively, we may exploit the well known relation

$$\lim_{r\to 0} \langle \mathcal{O}(r)\mathcal{O}(0)\rangle \sim r^{-4\Delta^{\mathcal{O}}}. \tag{45}$$

near the critical point in order to determine the conformal dimension. To achieve consistency with the proposed physical picture we want to identify in particular the conformal dimension of the perturbing operator. Recalling that the trace of the energy momentum tensor is proportional to the perturbing field we analyse $\langle\Theta(R)\Theta(0)\rangle$ for this purpose.

According to (45), we deduce from figure 5 $\Delta = 2/3, 5/7$ for $N = 4, 5$, respectively, which coincides with the expected values.

VII. CONCLUSIONS

One of the main deductions from our analysis is that the scattering matrix proposed in [6] may certainly be associated to the perturbed gauged WZNW-coset. This is based on the fact that we reproduce all the predicted features of this picture, namely the expected ultraviolet Virasoro central charge, various conformal dimensions of local operators and the characteristics of the unstable particle spectrum.

Our construction of general solutions to the form factor consistency equations certainly constitutes a further important step towards a generic group theoretical understanding of the n-particle form factor expressions. The next natural step is to extend the investigation towards higher level algebras [20].

Concerning the computation of correlation functions, our results also indicate that the "folkloristic belief" of the fast convergence of the series expansion of (22) has to be challenged. In fact, for large values of N, this is not true anymore. It would be highly desirable to have more concrete quantitative criteria at hand.

Despite the fact of having identified some part of the operator content, it remains a challenge to perform a definite one-to-one identification between the solutions to the form factor consistency equations and the local operators. It is clear that we require new additional technical tools to do this, since the Δ-sum rule (37) may not be applied in all situations and (45) does not allow a clear cut deduction of Δ.

In comparison with other methods to achieve the same goal, we should note that in principle we could obtain, apart from conformal dimensions different from the one of the perturbing operator, the same qualitative picture from a TBA-analysis [19]. However, in the latter approach the number of coupled non-linear integral equations to be solved increases with N, which means the system becomes extremely complex and cumbersome to solve even numerically. Computing the scaling function with the help of form factors only adds more terms to each n-particle contribution, but is technically not more involved. The price we pay in this setting is, however, the slow convergence of (22).

We conjecture that the "cutting rule" which describes the renormalization group flow also holds for other groups. This is supported by the general structure of the HSG-scattering matrix.

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