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Reflection–Transmission Quantum Yang–Baxter Equations

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Abstract

We explore the reflection–transmission quantum Yang–Baxter equations, arising in factorized scattering theory of integrable models with impurities. The physical origin of these equations is clarified and three general families of explicit solutions are described in detail. Explicit representatives of each family are also displayed. These results allow to establish a direct relationship with the different previous works on the subject and make evident the advantages of the reflection–transmission algebra as an universal approach to integrable systems with impurities.

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1 Introduction

Quantum mechanics and quantum field theory with point-like impurities attract recently much attention in relation to the rapid progress of condensed matter physics with defects. Integrable systems with impurities represent in this context a relevant testing ground for the basic theoretical ideas and have also some direct physical applications.

In the present Letter we are concerned with those universal features of integrable systems with impurities in 1+1 space-time dimensions, which are captured by the reflection-transmission quantum Yang-Baxter equations (QYBE's), following from factorized scattering theory. Our main goal below is to describe the origin and the present status of these equations. In the next section we sketch the derivation of the reflection–transmission QYBE's from first principles, namely physical unitarity and the reflection (boundary) QYBE, familiar from the case of purely reflecting boundary. Afterwards we briefly describe the concept of reflection–transmission (RT) algebra, which is based on the reflection–transmission QYBE's. Sections 3 and 4 are devoted to some concrete examples of scattering data, which obey these equations and illustrate the general structure. In the last section we compare the existing two approaches to factorized scattering with impurities, displaying the advantages of the RT algebra framework. This section collects also our conclusions and indicates some further developments in the subject.

2 Reflection-transmission QYBE's

The method of factorized scattering [1] is a powerful tool for studying both the mathematical structure and the physical properties of integrable quantum systems in 1+1 dimensions. The main ingredient of this method is the two–body scattering matrix

$$\left\{ \mathcal{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\chi_1, \chi_2) \ : \ \alpha_1, ..., \beta_2 = 1, ..., N; \ \chi_1, \chi_2 \in \mathbb{R} \right\}. \qquad (2.1)$$

Here N is the number of internal degrees of freedom, whereas $\chi \in \mathbb{R}$ parametrizes the dispersion relation of the asymptotic particles. In order to construct from (2.1) a consistent total scattering operator, the two-body matrix $\mathcal{S}$ must satisfy

$$\mathcal{S}_{12}(\chi_1, \chi_2)\mathcal{S}_{13}(\chi_1, \chi_3)\mathcal{S}_{23}(\chi_2, \chi_3) = \mathcal{S}_{23}(\chi_2, \chi_3)\mathcal{S}_{13}(\chi_1, \chi_3)\mathcal{S}_{12}(\chi_1, \chi_2), \quad (2.2)$$

which is the celebrated quantum Yang-Baxter equation (QYBE). One can associate [2]–[10] with $\mathcal{S}$ an algebra $\mathcal{A}_\mathcal{S}$ with identity $\mathbf{1}$, whose generators $\{a^{*\alpha}(\chi), a_\alpha(\chi)\}$ satisfy:

$$a_{\alpha_1}(\chi_1) \, a_{\alpha_2}(\chi_2) \ - \ \mathcal{S}_{\alpha_2\alpha_1}^{\beta_2\beta_1}(\chi_2, \chi_1) \, a_{\beta_2}(\chi_2) \, a_{\beta_1}(\chi_1) = 0, \qquad (2.3)$$

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$$a^{*\alpha_1}(\chi_1)\,a^{*\alpha_2}(\chi_2) - a^{*\beta_2}(\chi_2)\,a^{*\beta_1}(\chi_1)\,\mathcal{S}^{\alpha_2\alpha_1}_{\beta_2\beta_1}(\chi_2,\chi_1) = 0\,, \tag{2.4}$$

$$a_{\alpha_1}(\chi_1)a^{*\alpha_2}(\chi_2) - a^{*\beta_2}(\chi_2)\mathcal{S}^{\beta_1\alpha_2}_{\alpha_1\beta_2}(\chi_1,\chi_2)a_{\beta_1}(\chi_1) = 2\pi\delta(\chi_1 - \chi_2)\mathbf{1}\,. \tag{2.5}$$

The elements $\{a^{*\alpha}(\chi),\,a_\alpha(\chi)\}$ are interpreted as creators and annihilators of asymptotic particles. The QYBE (2.2) ensures the associativity of $\mathcal{A}_{\mathcal{S}}$ and applying twice (2.3), one deduces the consistency relation

$$\mathcal{S}_{12}(\chi_1,\chi_2)\,\mathcal{S}_{21}(\chi_2,\chi_1) = \mathbb{I}\otimes\mathbb{I}\,, \tag{2.6}$$

known as *unitarity*. Moreover, requiring that the mapping

$$I\ :\ a^{*\alpha}(\chi)\mapsto a_\alpha(\chi)\,,\qquad I\ :\ a_\alpha(\chi)\mapsto a^{*\alpha}(\chi)\,, \tag{2.7}$$

generates an involution in $\mathcal{A}_{\mathcal{S}}$ (i.e. that I extends as an antilinear anti-homomorphism on $\mathcal{A}_{\mathcal{S}}$), one gets the so called *Hermitian analyticity* condition

$$\mathcal{S}^{\dagger}_{12}(\chi_1,\chi_2) = \mathcal{S}_{21}(\chi_2,\chi_1)\,, \tag{2.8}$$

where the dagger stands for the Hermitian conjugation.[5] We stress that combining (2.6) and (2.8), which are assumed throughout the paper, one deduces the *physical* unitarity

$$\mathcal{S}_{12}(\chi_1,\chi_2)\mathcal{S}^{\dagger}_{12}(\chi_1,\chi_2) = \mathbb{I}\otimes\mathbb{I} \tag{2.9}$$

of the two-body scattering matrix. Following the already standard terminology, in what follows we refer to $\mathcal{A}_{\mathcal{S}}$ as Zamolodchikov-Faddeev (ZF) algebra.

The above framework has been successfully generalized [12]-[17] to the case when a purely reflecting boundary is present in the space. Describing the process of particle reflection from the boundary by a reflection matrix $\mathcal{R}^{\beta}_{\alpha}(\chi)$, Cherednik [12] discovered in the early eighties that $\mathcal{R}$ must satisfy the following *reflection* QYBE

$$\mathcal{S}_{12}(\chi_1,\chi_2)\,\mathcal{R}_1(\chi_1)\,\mathcal{S}_{21}(\chi_2,-\chi_1)\,\mathcal{R}_2(\chi_2) = $$
$$\mathcal{R}_2(\chi_2)\,\mathcal{S}_{12}(\chi_1,-\chi_2)\,\mathcal{R}_1(\chi_1)\,\mathcal{S}_{21}(-\chi_2,-\chi_1) \tag{2.10}$$

in order to have consistent factorized scattering. In analogy with (2.6,2.8) one requires also unitarity

$$\mathcal{R}(\chi)\mathcal{R}(-\chi) = \mathbb{I} \tag{2.11}$$

and Hermitian analyticity

$$\mathcal{R}^{\dagger}(\chi) = \mathcal{R}(-\chi)\,, \tag{2.12}$$

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[5]More general involutions in $\mathcal{A}_{\mathcal{S}}$ and the relative Hermitian analyticity conditions have been studied in [11].

which imply the physical unitarity

$$\mathcal{R}(\chi)\mathcal{R}^\dagger(\chi) = \mathbb{I}\,. \tag{2.13}$$

Let us mention in passing that the ZF algebra $\mathcal{A}_\mathcal{S}$ has a counterpart $\mathcal{B}_\mathcal{S}$ in the boundary case [18]. Instead of describing $\mathcal{B}_\mathcal{S}$ now, we will obtain it later on as a special case of the more general structure, discussed below.

It is quite natural at this stage to consider instead of the purely reflecting boundary an impurity (defect), which both reflects and transmits. In addition to $\mathcal{R}_\alpha^\beta(\chi)$, one will have in this case also a transmission matrix $\mathcal{T}_\alpha^\beta(\chi)$. Quantum mechanical potential scattering theory (see e.g. [19]) suggest to substitute (2.11) by

$$\mathcal{T}(\chi)\mathcal{T}(\chi) + \mathcal{R}(\chi)\mathcal{R}(-\chi) = \mathbb{I}\,, \tag{2.14}$$
$$\mathcal{T}(\chi)\mathcal{R}(\chi) + \mathcal{R}(\chi)\mathcal{T}(-\chi) = 0\,, \tag{2.15}$$

where $\mathcal{T}$ satisfies the Hermitian analyticity condition

$$\mathcal{T}(\chi) = \mathcal{T}^\dagger(\chi)\,. \tag{2.16}$$

Due to (2.12), (2.16), $\mathcal{T}(\chi)\mathcal{T}(\chi)$ and $\mathcal{R}(\chi)\mathcal{R}(-\chi)$ are non–negative Hermitian matrices which are simultaneously diagonalizable because of (2.14). The corresponding eigenvalues $\lambda_i(\chi)$ and $\mu_i(\chi)$ satisfy

$$0 \le \lambda_i(\chi) \le 1\,, \quad 0 \le \mu_i(\chi) \le 1\,, \quad \lambda_i(\chi) + \mu_i(\chi) = 1\,, \qquad i = 1, ..., N. \tag{2.17}$$

Obviously, non-trivial transmission occurs if and only if $\mu_i(\chi) < 1$ (or equivalently $\lambda_i(\chi) > 0$) for some $i = 1, ..., N$.

Following [20, 21], our goal now is to uncover the algebraic structure which models the mechanism of reflection and transmission in integrable systems with impurities. It is natural to expect that in addition to Cherednik's reflection QYBE (2.10), $\mathcal{R}$ and $\mathcal{T}$ satisfy some *transmission* and *reflection-transmission* QYBE's as well. The most direct way to derive these equations is to solve the unitarity constraints (2.14, 2.15), expressing $\mathcal{T}$ in terms of $\mathcal{R}$, and use afterwards (2.10). Solving (2.14), we consider below the positive square root $\sqrt{\mathbb{I} - \mathcal{R}(\chi)\mathcal{R}(-\chi)}$ of the non-negative Hermitian matrix $\mathbb{I} - \mathcal{R}(\chi)\mathcal{R}(-\chi)$. More precisely, in the basis in which $\mathbb{I} - \mathcal{R}(\chi)\mathcal{R}(-\chi)$ is diagonal we take diagonal $\sqrt{\mathbb{I} - \mathcal{R}(\chi)\mathcal{R}(-\chi)}$, whose elements are the positive square roots $\sqrt{1 - \lambda_i}$. The latter are well-defined in view of (2.17) and we set

$$\mathcal{T}(\chi) = \tau(\chi)\sqrt{\mathbb{I} - \mathcal{R}(\chi)\mathcal{R}(-\chi)}\,, \tag{2.18}$$

where the τ–function obeys

$$\overline{\tau}(\chi) = \tau(\chi)\,, \qquad \tau(\chi)^2 = 1\,, \qquad \tau(-\chi) = -\tau(\chi)\,, \tag{2.19}$$

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following from (2.14–2.16). If one assumes in addition that τ is continuous for $\chi \neq 0$, one finds

$$\tau(\chi) = \pm\varepsilon(\chi)\,, \qquad (2.20)$$

ε being the sign function. Adopting the representation (2.18) and the reflection QYBE (2.10), one can prove [21] the following statement.

Proposition: *Let $\mathcal{R}$ and $\mathcal{T}$ satisfy hermitian analyticity (2.12,2.16), unitarity (2.14, 2.15) and the reflection QYBE (2.10). Then $\mathcal{T}$ defined by (2.18) and $\mathcal{R}$ obey the transmission QYBE*

$$\begin{aligned}
\mathcal{S}_{12}(\chi_1,\chi_2)\,\mathcal{T}_1(\chi_1)\,\mathcal{S}_{21}(\chi_2,\chi_1)\,\mathcal{T}_2(\chi_2) = \\
\mathcal{T}_2(\chi_2)\,\mathcal{S}_{12}(\chi_1,\chi_2)\,\mathcal{T}_1(\chi_1)\,\mathcal{S}_{21}(\chi_2,\chi_1)
\end{aligned} \qquad (2.21)$$

and the reflection-transmission QYBE

$$\begin{aligned}
\mathcal{S}_{12}(\chi_1,\chi_2)\,\mathcal{T}_1(\chi_1)\,\mathcal{S}_{21}(\chi_2,\chi_1)\,\mathcal{R}_2(\chi_2) = \\
\mathcal{R}_2(\chi_2)\,\mathcal{S}_{12}(\chi_1,-\chi_2)\,\mathcal{T}_1(\chi_1)\,\mathcal{S}_{21}(-\chi_2,\chi_1)
\end{aligned} \qquad (2.22)$$

as well.

Remark: The precise determination of the square root we are taking in (2.18) is essential. The point is that $\mathbb{I} - \mathcal{R}(\chi)\mathcal{R}(-\chi)$ has in general infinitely many square roots - a standard phenomenon [22, 23] in matrix theory. By investigating some concrete examples we have seen that instead of eqs. (2.21, 2.22), some of the roots obey more complicated "twisted" versions of these equations in which $\mathcal{T}$ is substituted by $A\mathcal{T}A^{-1}$ with an invertible matrix A. Postponing the study of the latter case to the future, we focus in this paper on the subset of roots satisfying (2.21, 2.22). The above proposition simply states that $\mathcal{T}$, corresponding to the positive square root of $\mathbb{I} - \mathcal{R}(\chi)\mathcal{R}(-\chi)$, belongs to this subset.

Equations (2.21, 2.22) have the same structure as (2.10), but for some sign-changes in the arguments of $\mathcal{S}$ with obvious kinematical interpretation. They are essential for the reconstruction [21] of the reflection-transmission (RT) algebra $\mathcal{C}_{\mathcal{S}}$ - the analog of $\mathcal{A}_{\mathcal{S}}$ when impurities are present. It turns out that in addition to $\{a^{*\alpha}(\chi),\, a_\alpha(\chi)\}$, $\mathcal{C}_{\mathcal{S}}$ involves $2N^2$ defect generators $\{t_\alpha^\beta(\chi),\, r_\alpha^\beta(\chi)\}$, which describe the particle interaction with the impurity and modify the right hand side of the exchange relation (2.5) as follows

$$\begin{aligned}
a_{\alpha_1}(\chi_1)\,a^{*\alpha_2}(\chi_2) \;-\; a^{*\beta_2}(\chi_2)\,\mathcal{S}_{\alpha_1\beta_2}^{\beta_1\alpha_2}(\chi_1,\chi_2)\,a_{\beta_1}(\chi_1) = \\
2\pi\,\delta(\chi_1-\chi_2)\left[\delta_{\alpha_1}^{\alpha_2}\,\mathbf{1} + t_{\alpha_1}^{\beta_2}(\chi_1)\right] + 2\pi\,\delta(\chi_1+\chi_2)\,r_{\alpha_1}^{\alpha_2}(\chi_1)\,.
\end{aligned} \qquad (2.23)$$

The relations (2.3, 2.4) remain invariant. Finally, for defining $\mathcal{C}_{\mathcal{S}}$ one must add the exchange relations among $\{t_\alpha^\beta(\chi),\, r_\alpha^\beta(\chi)\}$ themselves and with

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$\{a^{*\alpha}(\chi),\, a_\alpha(\chi)\}$. Since these relations will be of no use in this paper, we omit them, referring for the explicit form of the complete set of constraints imposed on the generators of $\mathcal{C}_\mathcal{S}$ to [21]. The boundary algebra $\mathcal{B}_\mathcal{S}$, mentioned above, is obtained from $\mathcal{C}_\mathcal{S}$ by setting $t_\alpha^\beta(\chi) = 0$. The interplay between the three algebras $\mathcal{A}_\mathcal{S}$, $\mathcal{B}_\mathcal{S}$ and $\mathcal{C}_\mathcal{S}$ has been investigated in [24].

$\mathcal{C}_\mathcal{S}$ is an infinite algebra and from its formal definition it is not obvious at all that it has an operator realization. For this reason the Fock representation $\mathcal{F}(\mathcal{C}_\mathcal{S})$ has been constructed in [21] explicitly in terms of (generally unbounded) operators acting on a suitable dense domain of a Hilbert space. In the construction of $\mathcal{F}(\mathcal{C}_\mathcal{S})$ one needs an involution in $\mathcal{C}_\mathcal{S}$, which is obtained by extending (2.7) to the reflection and transmission generators according to

$$I\,:\, r_\alpha^\beta(\chi) \mapsto r_\beta^\alpha(-\chi)\,, \qquad\qquad I\,:\, t_\alpha^\beta(\chi) \mapsto t_\beta^\alpha(\chi)\,. \qquad (2.24)$$

Hermitian analyticity (2.12, 2.16) ensures the consistency of this extension. We would like to recall also that $\{t_\alpha^\beta(\chi),\, r_\alpha^\beta(\chi)\}$ condense in the vacuum state $\Omega \in \mathcal{F}(\mathcal{C}_\mathcal{S})$, i.e.

$$\langle t_\alpha^\beta(\chi)\rangle_\Omega = \mathcal{T}_\alpha^\beta(\chi)\,, \qquad \langle r_\alpha^\beta(\chi)\rangle_\Omega = \mathcal{R}_\alpha^\beta(\chi)\,. \qquad (2.25)$$

This direct relationship between reflection–transmission generators and amplitudes is crucial for the physical interpretation.

The Fock representation $\mathcal{F}(\mathcal{C}_\mathcal{S})$ is useful in several respects. From the scattering data $\{\mathcal{S},\, \mathcal{R},\, \mathcal{T}\}$ one can reconstruct [21] in $\mathcal{F}(\mathcal{C}_\mathcal{S})$ the total scattering operator, which is unitary as expected. Remarkably enough, $\mathcal{F}(\mathcal{C}_\mathcal{S})$ applies also in the construction of off–shell interacting quantum fields. An instructive example in this respect is the non–linear Schrödinger (NLS) model (the non–relativistic φ^4–theory) with a point–like impurity in 1+1 dimensions. This system is investigated in [25, 26], where the exact off–shell operator solution is constructed in terms of the representation $\mathcal{F}(\mathcal{C}_\mathcal{S})$ of an appropriate RT algebra $\mathcal{C}_\mathcal{S}$. Recently, this solution has been generalized [27] to the case of the $GL(N)$–invariant NLS model, where inequivalent Fock representations $\mathcal{F}(\mathcal{C}_\mathcal{S})$ implement a mechanism of spontaneous symmetry breaking.

Let us mention finally that $\mathcal{C}_\mathcal{S}$ admits also finite temperature representations. In contrast to the Fock vacuum, the cyclic state of such representations is not annihilated by $a_\alpha(\chi)$ and models a thermal bath, keeping the system in equilibrium at fixed (inverse) temperature β. A finite temperature representation of $\mathcal{C}_\mathcal{S}$ is introduced in [28], where also some applications to the statistical mechanics of systems with impurities are discussed.

Summarizing, the RT algebra admissible scattering data $\{\mathcal{S},\, \mathcal{R},\, \mathcal{T}\}$ must satisfy the QYBE's (2.2, 2.10, 2.21, 2.22), unitarity (2.6, 2.14, 2.15) and hermitian analyticity (2.8, 2.12, 2.16). The relevant issue at this point is the classification of admissible triplets. This is a hard theoretical problem, which has not been yet solved even in the case of the QYBE (2.2) alone.

Fortunately however, some classes of admissible scattering data $\{\mathcal{S}, \mathcal{R}, \mathcal{T}\}$ are known. They are the subject of our discussion in the next section.

3 RT algebra admissible data $\{\mathcal{S}, \mathcal{R}, \mathcal{T}\}$

Let us fix first of all the physical setting. We will study a system with a single impurity in $\mathbb{R}$. Since the impurity divides $\mathbb{R}$ in two disconnected parts $\mathbb{R}_\pm$, we take $N = 2n$ and split the index α as follows $\alpha = (\xi, i)$. Here $\xi = \pm$ indicates the half–line where the asymptotic particle is created or annihilated and $i = 1, ..., n$ labels the "isotopic" type. We emphasize that the results of the present paper are valid for any dispersion relation

$$E = E(\chi), \qquad p = p(\chi), \tag{3.1}$$

between the particle energy E and momentum p. In particular, it is instructive to keep in mind the conventional relativistic

$$E(\chi) = m\cosh(\chi), \qquad p(\chi) = m\sinh(\chi), \tag{3.2}$$

and non-relativistic

$$E(\chi) = \frac{m\chi^2}{2} + U, \qquad p(\chi) = m\chi, \tag{3.3}$$

relations, m being the particle mass and U some constant. We observe that Lorentz and Galilean transformations of the vector (E, p) are implemented in both (3.2) and (3.3) by translations $\chi \mapsto \chi + \alpha$. Therefore the scattering matrix $\mathcal{S}$ is Lorentz (Galilean) invariant provided that it depends on χ_1 and χ_2 only through the difference $\chi_1 - \chi_2$.

We focus in this section on block–diagonal $\mathcal{S}$-matrices

$$\mathcal{S}(\chi_1, \chi_2) = \begin{pmatrix} S^{++}(\chi_1, \chi_2) & 0 & 0 & 0 \\ 0 & S^{+-}(\chi_1, \chi_2) & 0 & 0 \\ 0 & 0 & S^{-+}(\chi_1, \chi_2) & 0 \\ 0 & 0 & 0 & S^{--}(\chi_1, \chi_2) \end{pmatrix}, \tag{3.4}$$

each block being a $n^2 \times n^2$ matrix in the isotopic space. The QYBE for $\mathcal{S}$ generates six equations for its blocks. One has

$$S_{12}^{++}(\chi_1, \chi_2)\, S_{13}^{++}(\chi_1, \chi_3)\, S_{23}^{++}(\chi_2, \chi_3) = S_{23}^{++}(\chi_2, \chi_3)\, S_{13}^{++}(\chi_1, \chi_3)\, S_{12}^{++}(\chi_1, \chi_2), \tag{3.5}$$

$$S_{12}^{++}(\chi_1, \chi_2)\, S_{13}^{+-}(\chi_1, \chi_3)\, S_{23}^{+-}(\chi_2, \chi_3) = S_{23}^{+-}(\chi_2, \chi_3)\, S_{13}^{+-}(\chi_1, \chi_3)\, S_{12}^{++}(\chi_1, \chi_2), \tag{3.6}$$

$$S_{12}^{+-}(\chi_1,\chi_2)\, S_{13}^{-+}(\chi_1,\chi_3)\, S_{23}^{-+}(\chi_2,\chi_3) = S_{23}^{-+}(\chi_2,\chi_3)\, S_{13}^{-+}(\chi_1,\chi_3)\, S_{12}^{+-}(\chi_1,\chi_2)\,,$$
$$\tag{3.7}$$

$$S_{12}^{+-}(\chi_1,\chi_2)\, S_{13}^{+-}(\chi_1,\chi_3)\, S_{23}^{--}(\chi_2,\chi_3) = S_{23}^{--}(\chi_2,\chi_3)\, S_{13}^{+-}(\chi_1,\chi_3)\, S_{12}^{+-}(\chi_1,\chi_2)\,,$$
$$\tag{3.8}$$

the remaining four equations following from (3.5-3.8) with the exchange $+ \leftrightarrow -$. We will call the latter the mirror counterparts of (3.5-3.8). All of these equations have the structure of a QYBE. We stress however that apart from (3.5) and its mirror counterpart, the other four equations involve different matrices in the isotopic space and are therefore not genuine QYBE's. Similar equations appear in the context of the quartic algebras introduced by Freidel and Maillet [29].

Taking $\mathcal{R}$ and $\mathcal{T}$ of the form

$$\mathcal{R}(\chi) = \begin{pmatrix} R^+(\chi) & 0 \\ 0 & R^-(\chi) \end{pmatrix}, \qquad \mathcal{T}(\chi) = \begin{pmatrix} 0 & T^+(\chi) \\ T^-(\chi) & 0 \end{pmatrix}, \tag{3.9}$$

Hermitian analyticity (2.12, 2.16) and unitarity (2.14, 2.15) imply

$$\left[R^\pm\right]^\dagger(\chi) = R^\pm(-\chi)\,, \qquad \left[T^+\right]^\dagger(\chi) = T^-(\chi)\,, \tag{3.10}$$

and

$$T^\pm(\chi)T^\mp(\chi) + R^\pm(\chi)R^\pm(-\chi) = \mathbb{I}\,, \tag{3.11}$$
$$T^\pm(\chi)R^\mp(\chi) + R^\pm(\chi)T^\pm(-\chi) = 0\,, \tag{3.12}$$

respectively. Moreover, from the QYBE's (2.10, 2.21, 2.22) one infers

$$S_{12}^{++}(\chi_1,\chi_2)\, R_1^+(\chi_1)\, S_{21}^{++}(\chi_2,-\chi_1)\, R_2^+(\chi_2) =$$
$$R_2^+(\chi_2)\, S_{12}^{++}(\chi_1,-\chi_2)\, R_1^+(\chi_1)\, S_{21}^{++}(-\chi_2,-\chi_1)\,, \tag{3.13}$$

$$S_{12}^{+-}(\chi_1,\chi_2)\, R_1^+(\chi_1)\, S_{21}^{-+}(\chi_2,-\chi_1)\, R_2^-(\chi_2) =$$
$$R_2^-(\chi_2)\, S_{12}^{+-}(\chi_1,-\chi_2)\, R_1^+(\chi_1)\, S_{21}^{-+}(-\chi_2,-\chi_1)\,, \tag{3.14}$$

$$S_{12}^{++}(\chi_1,\chi_2)\, T_1^+(\chi_1)\, S_{21}^{+-}(\chi_2,\chi_1)\, T_2^+(\chi_2) =$$
$$T_2^+(\chi_2)\, S_{12}^{+-}(\chi_1,\chi_2)\, T_1^+(\chi_1)\, S_{21}^{--}(\chi_2,\chi_1)\,, \tag{3.15}$$

$$S_{12}^{+-}(\chi_1,\chi_2)\, T_1^+(\chi_1)\, S_{21}^{--}(\chi_2,\chi_1)\, T_2^-(\chi_2) =$$
$$T_2^-(\chi_2)\, S_{12}^{++}(\chi_1,\chi_2)\, T_1^+(\chi_1)\, S_{21}^{+-}(\chi_2,\chi_1)\,, \tag{3.16}$$

$$S_{12}^{++}(\chi_1,\chi_2)\, T_1^+(\chi_1)\, S_{21}^{+-}(\chi_2,\chi_1)\, R_2^+(\chi_2) =$$
$$R_2^+(\chi_2)\, S_{12}^{++}(\chi_1,-\chi_2)\, T_1^+(\chi_1)\, S_{21}^{+-}(-\chi_2,\chi_1)\,, \tag{3.17}$$

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$$S_{12}^{+-}(\chi_1, \chi_2)\, T_1^+(\chi_1)\, S_{21}^{--}(\chi_2, \chi_1)\, R_2^-(\chi_2) =$$
$$R_2^-(\chi_2)\, S_{12}^{+-}(\chi_1, -\chi_2)\, T_1^+(\chi_1)\, S_{21}^{--}(-\chi_2, \chi_1) \qquad (3.18)$$

and their mirror analogs obtained by $+ \leftrightarrow -$ in (3.13–3.18). Some solutions of the above equations are described in the next section.

In order to keep the discussion as simple as possible, we concentrated above on triplets $\{\mathcal{S},\, \mathcal{R},\, \mathcal{T}\}$ of the form (3.4, 3.9), but more general admissible scattering data can be analyzed along the same lines.

4 Some families of scattering data

Different classes of solutions of eqs. (3.5-3.8) give rise to different families of admissible scattering data. We concentrate on three of them, which proved to be fundamental in the understanding of the physical content of RT algebras. Indeed, the type I family answers the long-standing question of the relationship between RT algebras and the approach to impurities in integrable systems followed in [30]-[32]. It shows that the RT algebra framework is more general and reproduces the content of [30]-[32] as a very special case. This is at the heart of the discussion of the next section. Our concern with the type II family is to exhibit a case where one can implement Lorentz invariance in the bulk scattering matrix, keeping non-trivial transmission. After presenting the general family, we show an explicit example where this is realized. Finally, the type III family is an important class of data since it involves the first example of interacting, exactly solvable and integrable quantum field model with impurity [25]-[27].

We turn now to the detailed describtion of the families I-III.

(i) **Type I** scattering data are determined by:

$$S^{+-}(\chi_1, \chi_2) = S^{-+}(\chi_1, \chi_2) = \mathbb{I}\,,$$
$$S^{++}(\chi_1, \chi_2) = S(\chi_1, \chi_2)\,, \qquad S^{--}(\chi_1, \chi_2) = \widetilde{S}(\chi_1, \chi_2)\,, \qquad (4.1)$$

where S and $\widetilde{S}$ are in general two different solutions of the QYBE. Bulk scattering matrices of the type (4.1) appear in [33], where the so called "folding trick" for describing reflecting and transmitting impurities is attempted. It is instructive to see what are the implications of eqs. (3.13–3.18) and their mirror counterparts on $\mathcal{R}$ and $\mathcal{T}$. One finds the following reflection

$$S_{12}(\chi_1, \chi_2)\, R_1^+(\chi_1)\, S_{21}(\chi_2, -\chi_1)\, R_2^+(\chi_2) =$$
$$R_2^+(\chi_2)\, S_{12}(\chi_1, -\chi_2)\, R_1^+(\chi_1)\, S_{21}(-\chi_2, -\chi_1)\,, \qquad (4.2)$$

$$\widetilde{S}_{12}(\chi_1, \chi_2)\, R_1^-(\chi_1)\, \widetilde{S}_{21}(\chi_2, -\chi_1)\, R_2^-(\chi_2) =$$
$$R_2^-(\chi_2)\, \widetilde{S}_{12}(\chi_1, -\chi_2)\, R_1^-(\chi_1)\, \widetilde{S}_{21}(-\chi_2, -\chi_1) \,, \qquad (4.3)$$

transmission

$$T_1^+(\chi_1)\, \widetilde{S}_{21}(\chi_2, \chi_1)\, T_2^-(\chi_2) = T_2^-(\chi_2)\, S_{12}(\chi_1, \chi_2)\, T_1^+(\chi_1) \,, \qquad (4.4)$$

$$S_{12}(\chi_1, \chi_2)\, T_1^+(\chi_1)\, T_2^+(\chi_2) = T_2^+(\chi_2)\, T_1^+(\chi_1)\, \widetilde{S}_{21}(\chi_2, \chi_1) \,, \qquad (4.5)$$

$$\widetilde{S}_{12}(\chi_1, \chi_2)\, T_1^-(\chi_1)\, T_2^-(\chi_2) = T_2^-(\chi_2)\, T_1^-(\chi_1)\, S_{21}(\chi_2, \chi_1) \,, \qquad (4.6)$$

and reflection–transmission equations

$$S_{12}(\chi_1, \chi_2)\, T_1^+(\chi_1)\, R_2^+(\chi_2) = R_2^+(\chi_2)\, S_{12}(\chi_1, -\chi_2)\, T_1^+(\chi_1) \,, \qquad (4.7)$$

$$T_1^+(\chi_1)\, \widetilde{S}_{21}(\chi_2, \chi_1)\, R_2^-(\chi_2) = R_2^-(\chi_2)\, T_1^+(\chi_1)\, \widetilde{S}_{21}(-\chi_2, \chi_1) \,, \qquad (4.8)$$

$$\widetilde{S}_{12}(\chi_1, \chi_2)\, T_1^-(\chi_1)\, R_2^-(\chi_2) = R_2^-(\chi_2)\, \widetilde{S}_{12}(\chi_1, -\chi_2)\, T_1^-(\chi_1) \,, \qquad (4.9)$$

$$T_1^-(\chi_1)\, S_{21}(\chi_2, \chi_1)\, R_2^+(\chi_2) = R_2^+(\chi_2)\, T_1^-(\chi_1)\, S_{21}(-\chi_2, \chi_1) \,. \qquad (4.10)$$

Notice finally that for invertible $T^{\pm}$ eqs. (4.7–4.10) are equivalent to

$$S_{12}(\chi_1, \chi_2)\, R_2^+(\chi_2) = R_2^+(\chi_2)\, S_{12}(\chi_1, -\chi_2) \,, \qquad (4.11)$$

$$\widetilde{S}_{12}(\chi_1, \chi_2)\, R_2^-(\chi_2) = R_2^-(\chi_2)\, \widetilde{S}_{12}(\chi_1, -\chi_2) \,. \qquad (4.12)$$

Type I scattering data are remarkable because eqs. (4.2–4.10) represent a meeting point between the existing approaches to factorized scattering with impurities and allow to compare them (see section 5).

Explicit type I solutions: An example [20] is given by the $GL(n)$–invariant scattering matrices

$$S_{12}(\chi_1, \chi_2) = \widetilde{S}_{21}(\chi_2, \chi_1) = \frac{[s(\chi_1) - s(\chi_2)]\, \mathbb{I} \otimes \mathbb{I} - ig\, P_{12}}{s(\chi_1) - s(\chi_2) + ig} \,, \qquad g \in \mathbb{R} \,, \qquad (4.13)$$

In this case, the complete classification of reflection and transmission matrices is given by

$$R^{\pm}(\chi) = \cos\left[\theta(\chi)\right]\, \exp[ip(\chi) \pm i\, n(\chi)]\, \mathbb{I} \,, \qquad (4.14)$$

$$T^{\pm}(\chi) = \sin\left[\theta(\chi)\right]\, \exp[\pm iq(\chi) \pm i\, n(\chi)]\, U^{\pm 1} \,, \qquad (4.15)$$

where P_{12} is the standard flip operator, U is a unitary matrix ($U^{-1} = U^{\dagger}$), q and s are *even* and θ, p and n are *odd* real–valued functions. The parity of s implies that S is not Lorentz (Galilean) invariant. According to [34], this is a general feature of the type I solutions, following from (4.11, 4.12).

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(ii) **Type II** scattering data are characterized by setting

$$S^{++}(\chi_1, \chi_2) = S^{+-}(\chi_1, \chi_2) = S^{-+}(\chi_1, \chi_2) = S^{--}(\chi_1, \chi_2) = S(\chi_1, \chi_2)\,, \tag{4.16}$$

where S obeys the QYBE. As a consequence, eqs.(3.5-3.8) are also satisfied and one is left with eqs. (3.13-3.18) and their mirror counterparts. A general class of solutions of all these equations is given by the following matrices:

$$R^{\pm}(\chi) = \cos\left[\theta(\chi)\right]\,\exp[ip_{\pm}(\chi)]\,B(\chi)\,, \tag{4.17}$$

$$T^{\pm}(\chi) = \sin\left[\theta(\chi)\right]\,\exp[\pm iq(\chi)]\,\mathbb{I}\,, \tag{4.18}$$

where θ, $p_{\pm}$ and q are odd–real–valued functions and B is a solution of the reflection QYBE relative to S and satisfies eqs. (2.11) and (2.12). This general procedure for deriving RT algebra scattering data $\{\mathcal{S}, \mathcal{R}, \mathcal{T}\}$ from purely reflecting data $\{S, B\}$ is very attractive because of the large amount of existing results concerning the doublet $\{S, B\}$.

Explicit type II solutions: Taking for instance

$$S_{12}(\chi_1, \chi_2) = \frac{(\chi_1 - \chi_2)\,\mathbb{I} \otimes \mathbb{I} - ig\,P_{12}}{\chi_1 - \chi_2 + ig}\,, \qquad g \in \mathbb{R}\,, \tag{4.19}$$

the general solution of (3.13–3.18) can be parametrized according to

$$R^{\pm}(\chi) = \cos\left[\theta(\chi)\right]\,\exp[ip(\chi) \pm i\,n(\chi)]\,\frac{\mathbb{I} + ia\chi\,U\,\mathbb{E}\,U^{\dagger}}{1 + ia\chi}\,, \qquad a \in \mathbb{R}\,, \tag{4.20}$$

$$T^{\pm}(\chi) = \sin\left[\theta(\chi)\right]\,\exp[\pm iq(\chi) \pm i\,n(\chi)]\,\mathbb{I}\,, \tag{4.21}$$

where $\mathbb{E}$ is a diagonal matrix which squares to $\mathbb{I}$ and U, θ, p, q and n have the properties fixed in point (i).

The striking feature of type II data, which is manifest in the example (4.19-4.21), is the coexistence of non-trivial transmission with non–constant Lorentz (Galilean) invariant bulk scattering matrix. This is clearly possible because the data violate (4.7–4.10), in spite of the fact that all (3.5–3.18) are respected.

(iii) **Type III** scattering data are obtained by taking

$$S^{++}(\chi_1, \chi_2) = S^{+-}(\chi_1, -\chi_2) = S^{-+}(-\chi_1, \chi_2) = S^{--}(-\chi_1, -\chi_2) = S(\chi_1, \chi_2)\,, \tag{4.22}$$

where S is a solution of the QYBE. The signs of the arguments in (4.22) are crucial for satisfying (3.6–3.8) and their mirror images. For deriving $\mathcal{R}$ and $\mathcal{T}$ one can proceed following the idea in point (ii). Take a solution B

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of the reflection QYBE relative to S and obeying eqs. (2.11) and (2.12) from the case of pure reflection. Then, using the properties of the previously introduced θ, p and q functions, one can easily verify that

$$R^{\pm}(\chi) = \cos\left[\theta(\chi)\right]\exp[ip(\chi)]\,B(\pm\chi)\,, \tag{4.23}$$

$$T^{\pm}(\chi) = \sin\left[\theta(\chi)\right]\exp[\pm iq(\chi)]\,B(\pm\chi)\,, \tag{4.24}$$

satisfy (2.12, 2.14–2.16), all (3.13–3.18) and their mirror counterparts.

Type III scattering data are inspired by the NLS model with point–like impurity, which has non–trivial bulk scattering but is nevertheless exactly solvable [25]–[27]. For the time being it is unique with these properties and provides therefore a valuable test for the whole RT algebra framework. Since the NLS model is well–known to capture many of the universal properties of integrable system, we strongly believe that type III solutions are relevant also in a more general context.

Explicit type III solutions: As already mentioned, the NLS model provides an instructive example. The relative scattering data are parametrized by (4.22–4.24) with S defined by (4.19) and B given by [35]

$$B(\chi) = \exp[i\,n(\chi)]\,\frac{\mathbb{I} + ia\chi\,U\,\mathbb{E}\,U^{\dagger}}{1 + ia\chi}\,, \qquad a \in \mathbb{R}\,, \tag{4.25}$$

where U and n are defined as above. Because of (4.22), $\mathcal{S}$ is not Lorentz (Galilean) invariant in spite of the fact that S preserves this symmetry. We refer to [25]–[27] for more details.

5   Remarks and conclusions

As already mentioned, the results of the present paper allow to establish a direct relationship between the RT algebra framework and the approach previously developed in [30]-[32]. Extending his idea about purely reflecting boundaries (mirrors in his terminology), Cherednik introduced in [30, 31] the concept of purely transmitting defects (glasses). These ideas have been later independently generalized for impurities which both reflect and transmit by Delfino, Mussardo and Simonetti (DMS) in [32]. The scattering data of the DMS approach consists of a bulk scattering matrix S and right (left) reflection and transmission matrices R^+ (R^-) and T^+ (T^-). As expected, S must satisfy the QYBE. Remarkably enough, the consistency relations imposed in [32] on $R^{\pm}$ and $T^{\pm}$, precisely coincide with the special case (4.2–4.10) of type I scattering data, provided that

$$\widetilde{S}_{12}(\chi_1, \chi_2) = S_{21}(\chi_2, \chi_1)\,, \tag{5.1}$$

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which, as easily verified, solves the QYBE as well. We conclude therefore that the systems discussed in [32] are fully described by type I data in the RT algebra framework. RT algebras are however significantly more general in the sense that they allow for a larger set of scattering data, still leading to a unitary scattering operator [21]. In fact, both type II and type III solutions are not covered by the DMS approach. That is why the physical example of the NLS model with impurity [25]–[27] can not be treated along the lines of [32]: the corresponding scattering data satisfy the RT algebra reflection–transmission QYBE, but do not respect the DMS consistency conditions.

Summarizing, the concept of RT algebra has a richer structure and thus opens new possibilities. Among others, we would like to mention the type II solutions with both Lorentz (Galilean) invariant non-constant bulk scattering matrix and non-vanishing transmission matrix, which are forbidden [34] in the framework of [32]. It will be interesting in this respect to construct exactly solvable models, possessing such scattering data. Some recent developments in model–building with impurities can be found in [36]–[39].

In conclusion, the RT algebra approach to integrable systems with impurities is based on the reflection–transmission QYBE's (2.10, 2.21, 2.22). Keeping in mind that (2.21, 2.22) are deeply related to (2.10) by physical unitarity, this beautiful and compact set of equations is in our opinion extremely natural and deserves further investigation.

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