IC/98/28
SISSA/EP/98/28

Tricritical Ising Model with a Boundary *

A. De Martino †
International School for Advanced Studies, Via Beirut 2-4, 34014 Trieste, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste

M. Moriconi ‡
The Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11, 34100 Trieste, Italy

Abstract

We study the integrable and supersymmetric massive $\hat{\phi}_{(1,3)}$ deformation of the tricritical Ising model in the presence of a boundary. We use constraints from supersymmetry in order to compute the exact boundary S-matrices, which turn out to depend explicitly on the topological charge of the supersymmetry algebra. We also solve the general boundary Yang-Baxter equation and show that in appropriate limits the general reflection matrices go over the supersymmetry preserving solutions. Finally, we briefly discuss the possible connection between our reflection matrices and boundary perturbations within the framework of perturbed boundary conformal field theory.

*Work done under partial support of the EC TMR Programme *Integrability, non–perturbative effects and symmetry in Quantum Field Theories*, grant FMRX-CT96-0012

†`email: demarti@sissa.it`

‡`email: moriconi@ictp.trieste.it`

1 Introduction

There are several problems in different areas of theoretical physics that involve the study of boundary field theories, such as the Kondo effect, quantum impurities in strongly correlated electron systems, the catalysis of baryon decay in the presence of magnetic monopoles (Callan-Rubakov effect), and even black-hole evaporation, to name a few. Therefore the study of boundary theories is more than an interesting exercise, and we should try to learn as much as possible about them. An especially interesting class of boundary field theories can be obtained by restricting 1+1-dimensional integrable field theories to the half-line while preserving integrability. A remarkable example is the boundary sine-Gordon model [1], which has found very important applications [2] in real physical systems in the past few years.

In this paper we study one of the simplest two-dimensional models which has nonetheless a very rich structure, the tricritical Ising model (TIM), in the presence of a boundary. The TIM provides a useful venue to study many non-trivial aspects of two-dimensional quantum field theory such as superconformal invariance [3, 4], renormalization group flows [5, 6] and exact S-matrices [7, 8].

As a lattice model the TIM can be realized as an Ising model with annealed vacancies [9], with Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \mu \sum_i (\sigma_i)^2, \tag{1.1}$$

where the first sum is performed over nearest neighbors, $\sigma_i = \pm 1\ , 0$ are the spin variables, μ is the chemical potential and J is the energy of a configuration of a pair of unlike spins. This model has a critical point for some value of (J, μ) where three phases can coexist and to which can be associated [10] a conformal field theory with central charge $c = 7/10$, corresponding both to the next simplest minimal model $\mathcal{M}_4$ and to the simplest $N = 1$ superconformal minimal model $\mathcal{SM}_3$. This fact will be extremely useful in the following.

In [11] Zamolodchikov has shown that unitary minimal models can be associated to the infrared fixed point of some particular scalar field theories, having an effective Landau-Ginzburg (LG) description. To the $\mathcal{M}_m$ model, with central charge $c = 1 - 6/m(m+1)$, $m = 3, 4, \ldots$, one can associate the action

$$S_{LG} = \int d^2z \left[\frac{1}{2} (\partial\phi)^2 + \phi^{2m-2} \right] . \tag{1.2}$$

There is also a LG description for the $N = 1$ superconformal unitary minimal series $\mathcal{SM}_n$, with central charge $c = 3/2 - 12/n(n+2)$, $n = 3, 4, \ldots$, given by the action

$$S_{LG}^{N=1} = \int d^2z \, d^2\theta \left[\frac{1}{2} (D\Phi)^2 + \Phi^n \right] , \tag{1.3}$$

where the superfield Φ written in components is $\Phi = \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F$, and the conformal dimensions for the fields ϕ, ψ, $\bar{\psi}$ and F are $(1/10, 1/10)$, $(3/5, 1/10)$, $(1/10, 3/5)$ and $(3/5, 3/5)$ respectively. Therefore the conformal theory associated to the TIM can be studied as the critical point of a bosonic theory with a ϕ^6 potential or as a $N = 1$ supersymmetric theory with a Φ^3 potential.

Any $N = 1$ superconformal field theory (SCFT) allows two different projections onto local field theories [3]. This comes about in the following way. The fields in a SCFT can be divided in two types, Neveu-Schwarz (NS) and Ramond (R) fields, depending on how they behave under rotations around the origin in the punctured plane. NS fields are periodic and R fields are antiperiodic. This means that the operators in the NS sector form a closed algebra under operator product expansions while the ones in the R sector do not. So if we project out the R fields we obtain a consistent local quantum field theory, which will be manifestly supersymmetric. Another way of obtaining a consistent local theory is by projecting out the fermions, which is usually called the GSO projection. This way we obtain what is usually called the spin model associated to the SCFT. The important observation is that for the TIM we can associate each of the LG actions (1.3), (1.2) to each of these local projections.

As it was argued in [12], a minimal model perturbed by the $\hat{\phi}_{(1,3)}$ operator gives an integrable theory. In the bosonic description (1.2) the operator $\hat{\phi}_{(1,3)}$ corresponds to $:\phi^6:$, and in the manifestly supersymmetric description (1.3) to the auxiliary field $F = \int d^2\theta\, \Phi$. In both descriptions the LG action can still be used off criticality as a guide to the solitonic structure of these massive deformations. An S-matrix based on this deformation of (1.2) has been proposed in [7]. In this paper we study the S-matrix proposed in [8], corresponding to the perturbed action

$$S = \int d^2z\, d^2\theta \left[\frac{1}{2}(D\Phi)^2 + \Phi^3 + \lambda\Phi\right] , \tag{1.4}$$

which is manifestly supersymmetric[1]. By looking at (1.4) in terms of components it can be argued that in this model there are solitonic and antisolitonic supersymmetric doublets, which we will denote by (B, F) and $(\bar{B}, \bar{F})$.

Due to the fact that multi-soliton configurations have to alternate solitons and anti-solitons, there is an adjacency condition to be respected by the allowed multi-particle states.

In this paper we study the factorized scattering theory associated to the action (1.4) in the presence of a boundary. Some related work has been done by Chim in [13]; he solved the boundary Yang-Baxter equation (BYBE) for the S-matrix proposed in [7], where supersymmetry acts non-locally. In that case it is difficult to identify which reflection

[1]From now on we will refer to this perturbed model as TIM.

matrix corresponds to boundary supersymmetry preserving interactions. As we will see, in the present formulation this identification is done in a transparent way, due to the fact that we are dealing with a manifestly supersymmetric theory, where the supersymmetric charges act locally.

This paper is organized as follows. In the next section we discuss general aspects of two-dimensional integrable theories. In section 3 we discuss the S-matrix proposed by Fendley to describe the $\hat{\phi}_{(1,3)}$ perturbation of the TIM. In section 4 we find the reflection matrices that preserve boundary supersymmetry. In section 5 we solve the boundary Yang-Baxter equation, and find the general reflection matrices which preserve integrability but not necessarily supersymmetry, and relate these solutions to the solutions of section 4 in the appropriate limits. In section 6 we discuss the possible boundary interactions which preserve integrability and supersymmetry, and connect them with the reflections matrices we have obtained. In the last section we discuss our results and some possible extensions of this work.

2 Generalities about the Scattering Matrix

In this section we briefly review the main aspects of factorized scattering theories that will be needed in the rest of the paper. This section is meant mainly to set the notation. As a general reference for bulk integrable field theories in two-dimensions we refer the reader to [14].

We parameterize asymptotic states in terms of the rapidity variable θ, such that energy and momentum are given by $p_0 = m\cosh\theta$ and $p_1 = m\sinh\theta$, respectively. One-particle states are labeled by $|A_i(\theta)\rangle_{in,out}$, where A_i could be a boson or a fermion. Since we have solitons and anti-solitons, both bosonic and fermionic, we will denote solitons by A_i and antisolitons by $\bar{A}_i$. Multiparticle states are given by $|A_{i_1}(\theta_1)A_{i_2}(\theta_2)\ldots A_{i_n}(\theta_n)\rangle_{in,out}$ such that $\theta_1 > \theta_2 > \ldots > \theta_n$ for in-states, and the other way around for out-states. As a basis for one-particle states we use $\{|B\rangle, |F\rangle, |\bar{B}\rangle, |\bar{F}\rangle\}$ and for two-particle states we use $\{|B\bar{B}\rangle, |F\bar{F}\rangle, |B\bar{F}\rangle, |F\bar{B}\rangle\}$. The S-matrix is defined by

$$|A_{i_1}(\theta_1)A_{i_2}(\theta_2)\rangle_{in} = S_{i_1 i_2}^{j_1 j_2}(\theta_1-\theta_2)|A_{j_2}(\theta_2)A_{j_1}(\theta_1)\rangle_{out} \ , \tag{2.1}$$

and is represented graphically [2] in figure 1.

[2]Inside figures we will denote the one-particle state A_i simply by i.

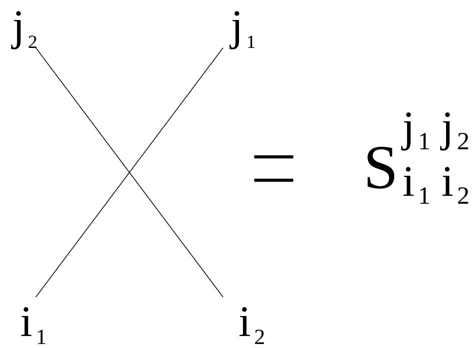

Fig. 1 The S-matrix.

Once we have a bulk S-matrix we can consider the associated problem of "boundarizing" this model [1, 15]. A boundary scattering theory is described in the bulk by the same S-matrix as the bulk model we are studying. In order to have a complete description we have to introduce the boundary scattering matrix which tells us how particles scatter off the boundary. This is the reflection matrix and is defined by

$$|A_i(\theta)\rangle = R_i^j(\theta)|A_j(-\theta)\rangle \tag{2.2}$$

and in a similar fashion as the bulk S-matrix, is represented graphically as shown in figure 2.

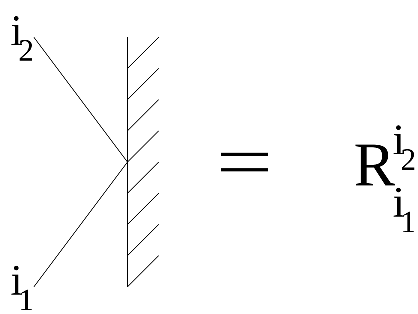

Fig. 2 The Reflection matrix.

The consistency between boundary integrability and bulk integrability is encoded in the boundary Yang-Baxter equation (BYBE) [16] which reads

$$S_{i_1 i_2}^{c_1 c_2}(\theta_{12}) R_{c_1}^{d_1}(\theta_1) S_{c_2 d_1}^{d_2 j_1}(\bar{\theta}_{12}) R_{d_2}^{j_2}(\theta_2) = R_{i_2}^{c_2}(\theta_2) S_{i_1 c_2}^{c_1 d_2}(\bar{\theta}_{12}) R_{c_1}^{d_1}(\theta_1) S_{d_2 d_1}^{j_2 j_1}(\theta_{12}) \;, \tag{2.3}$$

where $\theta_{12} = \theta_1 - \theta_2$ and $\bar{\theta}_{12} = \theta_1 + \theta_2$. The graphic representation of the BYBE is given in figure 3.

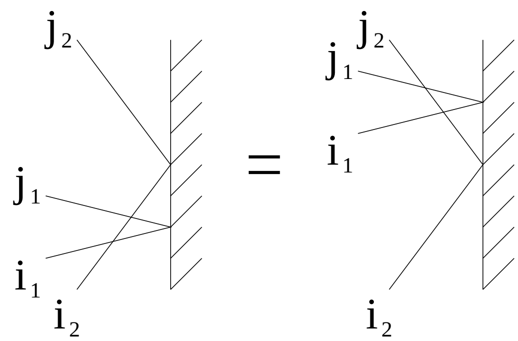

Fig. 3 The BYBE

2.1 Supersymmetry Algebra

The superfield sector of the TIM carries a representation of a $N = 1$ supersymmetry algebra with topological charge, which is given by two supersymmetry generators, Q_+ and Q_-, and a fermion number operator Q_L, whose eigenvalues measure if the state is bosonic ($+1$) or fermionic (-1). The algebra reads explicitly [17]

$$\begin{aligned} {Q_+}^2 = p_0 + p_1 \ , \qquad {Q_-}^2 = p_0 - p_1 \ , \\ \{Q_+, Q_-\} = 2T \ , \\ \{Q_L, Q_\pm\} = 0 \ , \end{aligned} \tag{2.4}$$

and T is the topological charge. In our one-particle basis this algebra has the following realization:

$$Q_+(\theta) = \sqrt{m}\, e^{\frac{\theta}{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad Q_-(\theta) = \sqrt{m}\, e^{-\frac{\theta}{2}} \begin{pmatrix} 0 & \mathrm{e}^{i\alpha} & 0 & 0 \\ \mathrm{e}^{-i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{-i\alpha} \\ 0 & 0 & -\mathrm{e}^{i\alpha} & 0 \end{pmatrix} . \tag{2.5}$$

The topological charge T in (2.4) is given by

$$T = m \cos\alpha \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.6}$$

Notice that in our notation the Bogomolnyi bound $|T| \leq m$ is saturated when $\alpha = 0$, whereas the topological charge vanishes when $\alpha = \pi/2$. Finally, the action of the supercharges on multi-particle states is

$$\widehat{Q}_+(\theta) = \sum_{l=1}^{N} Q_+^l(\theta) \ , \quad \widehat{Q}_-(\theta) = \sum_{l=1}^{N} Q_-^l(\theta) \ , \tag{2.7}$$

where $Q^l_\pm(\theta)$ is defined by

$$\begin{aligned} & Q^l_\pm(\theta)|A_{a_1}(\theta_1)\dots A_{a_N}(\theta_N)\rangle = \\ & = |(Q_L A)_{a_1}(\theta_1)\dots(Q_L A)_{a_{l-1}}(\theta_{l-1})(Q_\pm A)_{a_l}(\theta_l)A_{a_{l+1}}(\theta_{l+1})\dots A_{a_N}(\theta_N)\rangle \ . \end{aligned} \tag{2.8}$$

This is a local realization of the $N = 1$ supersymmetry algebra, that is, the supersymmetric charges act on particle states, in contrast to the non-local realization used in [7].

3 Fendley's S-matrix

In this section we briefly review the S-matrix proposed in [8] to describe the TIM, and we refer the reader to that paper for a more detailed discussion.

In the two-particle basis given in section 2, the S-matrix has the following form

$$S(\theta) = \begin{pmatrix} a(\theta) & -b(\theta) & 0 & 0 \\ b(\theta) & a(\theta) & 0 & 0 \\ 0 & 0 & d(\theta) & c(\theta) \\ 0 & 0 & -c(\theta) & d(\theta) \end{pmatrix} \ . \tag{3.1}$$

The scattering theory turns out to be different if the Bogomolnyi bound is saturated ($\alpha = 0$) or not ($\alpha \neq 0$). In this paper we will refer to these cases simply as saturated and non-saturated. The S-matrix does not depend explicitly on α, the difference between the two cases being that in the former the amplitude $c(\theta)$ does not vanish, whereas in the latter $c(\theta) = 0$. This is due to the fact that the S-matrix is determined by local interactions, while the topological charge is a global property of the theory.

Commutativity with the supercharges implies that the amplitudes in the S-matrix are related by

$$b(\theta) = a(\theta)\sinh\frac{\theta}{2} + c(\theta)\cosh\frac{\theta}{2} \ , \tag{3.2}$$

$$d(\theta) = a(\theta)\cosh\frac{\theta}{2} + c(\theta)\sinh\frac{\theta}{2} \ . \tag{3.3}$$

From this expression we see immediately that this S-matrix satisfies the free-fermion condition $a^2+b^2 = c^2+d^2$, which is extremely important for thermodynamical calculations [8, 18, 19, 20]. Crossing-symmetry requires $a(i\pi - \theta) = a(\theta)$ and $c(i\pi - \theta) = -c(\theta)$[3], and from (3.3) one finds $b(i\pi - \theta) = id(\theta)$ and $d(i\pi - \theta) = -ib(\theta)$.

[3] Notice that the minus sign in the crossing relation for $c(\theta)$ is related to the fact that this is an amplitude involving one fermion in the in-state.

The Yang-Baxter equation (YBE) yields

$$\frac{c}{a} = \epsilon \tanh\theta \ , \tag{3.4}$$

where $\epsilon = \pm 1$, and one finds

$$\frac{b}{a} = \frac{\sinh p\theta}{\cosh\theta} \ , \quad \frac{d}{a}(\theta) = \frac{\cosh p\theta}{\cosh\theta} \ , \tag{3.5}$$

where $p = 3/2$ when $\epsilon = 1$ and $p = -1/2$ when $\epsilon = -1$. The YBE implies also that scattering amplitudes should be the same when exchanging solitons $\leftrightarrow$ antisolitons. Finally, by solving unitarity it is found for the saturated case

$$\begin{aligned} a(\theta) \;&=\; 2\sinh^2\left(\frac{i\pi}{4}+\frac{\theta}{2}\right)\prod_{l=0}^{\infty}\frac{\Gamma(\frac{3}{2}+l)\Gamma(\frac{1}{2}+l-\frac{i\theta}{2\pi})\Gamma(1+l+\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2}+l)\Gamma(\frac{3}{2}+l+\frac{i\theta}{2\pi})\Gamma(1+l-\frac{i\theta}{2\pi})}\times \\ &\times\frac{\Gamma(\frac{1}{2}+2ql+2q)\Gamma(\frac{1}{2}+2ql-\frac{iq\theta}{\pi})\Gamma(\frac{1}{2}+2ql+q+\frac{iq\theta}{\pi})}{\Gamma(\frac{1}{2}+2ql)\Gamma(\frac{1}{2}+2ql+2q+\frac{iq\theta}{\pi})\Gamma(\frac{1}{2}+2ql+q-\frac{iq\theta}{\pi})} \ , \end{aligned} \tag{3.6}$$

where $q = 1 + |p|$. Notice that the formula in [8] differs from (3.6) by a factor of $\tanh^2(\frac{i\pi}{4}+\frac{\theta}{2})$. The reason for this change is that $a(\theta)$ in [8] has a zero in the physical strip, and therefore should not be taken as the minimal solution. The integral representation for (3.6) is

$$a(\theta) = \exp\left[-\int_0^\infty \frac{dt}{t}\, h(t)\frac{\sin t\theta \sin t(i\pi-\theta)}{\cosh \pi t}\right] \ , \tag{3.7}$$

where

$$h(t) = \frac{2}{\sinh \pi t} - \frac{1}{\sinh 2\pi t} - \frac{1}{\sinh \frac{\pi t}{q}} \ . \tag{3.8}$$

As we have seen, the non-saturated case is gotten by setting $c(\theta) = 0$. This changes the unitarity equation and as a consequence the minimal solution is

$$a(\theta) = \prod_{j=0}^{\infty}\left[\frac{\Gamma(\frac{3}{2}+j)\Gamma(1+j+\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+j-\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2}+j)\Gamma(1+j-\frac{i\theta}{2\pi})\Gamma(\frac{3}{2}+j+\frac{i\theta}{2\pi})}\right]^2 \ . \tag{3.9}$$

The integral representation is given by (3.7) with

$$h(t) = \frac{2}{\sinh 2\pi t} \ . \tag{3.10}$$

In the next sections we discuss the boundary scattering associated to this S-matrix. Initially we will consider the reflection matrices for supersymmetry-preserving boundary interactions and later more general solutions.

4 Supersymmetry Preserving Reflection Matrices

The introduction of a boundary will in general destroy some of the conserved charges. In particular we can not preserve the whole supersymmetry of the bulk model, the best we can do being to preserve "half" of supersymmetry [21]. In this section we study which combination of the supercharges can be preserved in the presence of a boundary and the corresponding reflection matrices.

In the one-particle basis of section 2 we see that all possible reflection processes can be encoded in a 4×4 matrix. This reflection matrix can be written in general as

$$\mathcal{R}(\theta) = \begin{pmatrix} R & U \\ V & \bar{R} \end{pmatrix} , \tag{4.1}$$

where R, $\bar{R}$, U and V are 2×2 matrices. Our convention is that rows label in-states and columns out-states. Matrices R and $\bar{R}$ describe topological charge preserving processes, while U and V describe reflections of solitons into antisolitons and vice-versa. Since the bulk S-matrix satisfies an adjacency condition we should set $U = V = 0$. The reflection matrix has, therefore, block-diagonal form.

R and $\bar{R}$ can be written quite generally as

$$R(\theta) = \begin{pmatrix} R_b & P \\ Q & R_f \end{pmatrix} , \quad \bar{R}(\theta) = \begin{pmatrix} \bar{R}_b & \bar{P} \\ \bar{Q} & \bar{R}_f \end{pmatrix} . \tag{4.2}$$

Non-diagonal amplitudes in (4.2) correspond to fermion-number changing processes. We will see later that R and $\bar{R}$ are connected by a simple transformation.

4.1 Boundary Supersymmetry

We start by assuming that the boundary action[4] preserves both integrability and supersymmetry. As explained in [21, 22] only a linear combination of the supersymmetric charges can survive in the presence of a boundary. It is easy to see that the only candidates are $\widetilde{\mathcal{Q}}_{\pm} = Q_+ \pm Q_-$, since when squared these are the only linear combinations which do not depend on linear momentum. We then require that the reflection matrix "commutes" with this new charge, that is

$$\widetilde{\mathcal{Q}}_{\pm}(\theta)\mathcal{R}(\theta) = \mathcal{R}(\theta)\widetilde{\mathcal{Q}}_{\pm}(-\theta) . \tag{4.3}$$

From this equation it is easy to see that $R(\theta)$ should be of the form

$$R_{\pm}(\theta) = R_0(\theta) \begin{pmatrix} \cosh(\frac{i\alpha}{2}^{\pm} - \frac{\theta}{2}) & e^{\frac{i\alpha}{2}^{\pm}} p(\theta) \\ e^{-\frac{i\alpha}{2}^{\pm}} p(\theta) & \cosh(\frac{i\alpha}{2}^{\pm} + \frac{\theta}{2}) \end{pmatrix} , \tag{4.4}$$

[4]We speak of a symbolic action.

where $\alpha^+ = \alpha$ and $\alpha^- = \alpha + \pi$, and $\bar{R}_\pm(\theta)$ of the form

$$\bar{R}_\pm(\theta) = \bar{R}_0(\theta) \begin{pmatrix} \cosh(\frac{i\alpha}{2}^\mp + \frac{\theta}{2}) & e^{-\frac{i\alpha}{2}^\mp} p(\theta) \\ e^{\frac{i\alpha}{2}^\mp} p(\theta) & \cosh(\frac{i\alpha}{2}^\mp - \frac{\theta}{2}) \end{pmatrix} . \tag{4.5}$$

For convenience we have denoted by $R_\pm(\theta)$ the reflection matrix which commute with the combinations $\widetilde{\mathcal{Q}}_\pm(\theta)$ and we will adhere to this convention in the following. It is evident that $\bar{R}$ can be obtained from R by the simple substitution

$$\alpha^\pm \to -\alpha^\mp , \tag{4.6}$$

as it can also be seen directly from the structure of the supercharges. From now on we will concentrate only on R.

As a result, we see that the requirement of commutativity (4.3) determines the ratios between the diagonal elements, R_f/R_b, and between the off-diagonal ones, $Y = Q/P$, and in addition, it implies a precise relation between the reflection amplitudes in the solitonic and in the anti-solitonic sector. In order to fix the last unknown ratio $p \equiv e^{-i\alpha^\pm/2} P/R_0$, we use the BYBE. The relevant equation is the one corresponding to $B\bar{B} \to F\bar{B}$, and it reads explicitly

$$\begin{aligned} &P(\theta_1)\frac{R_b(\theta_2)}{R_b(\theta_1)}\left(x(\bar{\theta}_{12}) - x(\theta_{12})\right) + Q(\theta_1)\frac{R_b(\theta_2)}{R_b(\theta_1)}\left(y(\theta_{12})v(\bar{\theta}_{12}) - v(\theta_{12})y(\bar{\theta}_{12}))\right) = \\ &= P(\theta_2)\left(y(\theta_{12})x(\bar{\theta}_{12}) + \frac{R_f(\theta_1)}{R_b(\theta_1)}x(\theta_{12})y(\bar{\theta}_{12})\right) + Q(\theta_2)\left(v(\bar{\theta}_{12}) + \frac{R_f(\theta_1)}{R_b(\theta_1)}v(\theta_{12})\right) \end{aligned} \tag{4.7}$$

where $v = b/a$, $x = c/a$, and $y = d/a$. To make the discussion clear let us treat the saturated and non-saturated cases separately.

- *The Saturated Case* ($c(\theta) \neq 0$)

  We find

$$\begin{aligned} R_+(\theta) &= R_0(\theta) \begin{pmatrix} 1 & A\sinh q\theta \\ A\sinh q\theta & 1 \end{pmatrix}, && (4.8) \\ R_-(\theta) &= R_0(\theta) \begin{pmatrix} 1 & A\cosh q\theta \\ -A\cosh q\theta & -1 \end{pmatrix}, && (4.9) \end{aligned}$$

  where A is a constant (which could be zero, of course).

- *The Non-Saturated Case* ($c(\theta) = 0$)

  We find

$$R_\pm(\theta) = R_0(\theta) \begin{pmatrix} \cosh(\frac{i\alpha}{2}^\pm - \frac{\theta}{2}) & e^{\frac{i\alpha}{2}^\pm} A\sinh\theta \\ e^{-\frac{i\alpha}{2}^\pm} A\sinh\theta & \cosh(\frac{i\alpha}{2}^\pm + \frac{\theta}{2}) \end{pmatrix} . \tag{4.10}$$

The reflection matrices depend explicitly on the topological charge, as expected, since the introduction of a boundary brings global properties into the local description of scattering theory. All we will have to do now is to fix the overall prefactor by requiring unitarity and boundary crossing-unitarity. We will treat these requirements in the next subsection.

4.2 Unitarity and Boundary Crossing-Symmetry

In this section we fix the prefactor $R_0(\theta)$ in the following way. As customary [1], we write $R_0(\theta) = Z_1(\theta)Z_2(\theta)$ where $Z_1(\theta)$ solves unitarity and does nothing to boundary crossing-unitarity and $Z_2(\theta)$ solves boundary crossing-unitarity and does nothing to unitarity. We restrict ourselves to the minimal solutions, with no poles in the physical strip. In the following we will also give integral representations for these prefactors, since they are very useful for thermodynamical computations.

The unitarity requirement for the reflection matrix is given by $R(\theta)R(-\theta) = 1$, which in our case implies the following four equations:

$$
\begin{aligned}
R_b(\theta)R_b(-\theta) + P(\theta)Q(-\theta) &= 1 \ , \\
R_b(\theta)P(-\theta) + P(\theta)R_f(-\theta) &= 0 \ , \\
R_f(\theta)R_f(-\theta) + Q(\theta)P(-\theta) &= 1 \ , \\
R_f(\theta)Q(-\theta) + Q(\theta)R_b(-\theta) &= 0 \ .
\end{aligned}
$$

It turns out that the the second and the fourth equation are automatically satisfied by (4.8), (4.9) and (4.10), whereas the first and the third equations are non-trivial. Let us discuss the saturated and non-saturated cases separately.

- *The Saturated Case* ($c(\theta) \neq 0$)

  The unitarity equation can be written as follows:

$$
Z_1(\theta)Z_1(-\theta) = \frac{A^{-2}}{\sinh(\kappa - q\theta)\sinh(\kappa + q\theta)} \ , \tag{4.11}
$$

  where the parameter κ is defined by $\sinh\kappa = 1/A$ for the "+" combination of supercharges and $\cosh\kappa = 1/A$ for the "−" combination. The minimal solutions are $Z_1(\theta) = \sigma(x,\theta)$ and $Z_1(\theta) = \sigma(x,\theta)/\tanh^2\kappa$, respectively, $x = \frac{\pi}{2} + i\kappa$. The explicit expression of σ as an infinite product of gamma functions [1] is

$$
\begin{aligned}
\sigma(x,\theta) &= \frac{\Pi(x,\frac{\pi}{2}+i\theta)\Pi(-x,\frac{\pi}{2}+i\theta)\Pi(x,-\frac{\pi}{2}-i\theta)\Pi(-x,-\frac{\pi}{2}-i\theta)}{\Pi(x,\frac{\pi}{2})\Pi(-x,\frac{\pi}{2})\Pi(x,-\frac{\pi}{2})\Pi(-x,-\frac{\pi}{2})} \ , \\
\Pi(x,\theta) &= \prod_{l=0}^{\infty} \frac{\Gamma(\frac{1}{2}+(2l+\frac{1}{2})q+\frac{x}{\pi}+iq\frac{\theta}{\pi})\Gamma(\frac{1}{2}+(2l+\frac{3}{2})q+\frac{x}{\pi})}{\Gamma(\frac{1}{2}+(2l+\frac{3}{2})q+\frac{x}{\pi}+iq\frac{\theta}{\pi})\Gamma(\frac{1}{2}+(2l+\frac{1}{2})q+\frac{x}{\pi})} \ .
\end{aligned} \tag{4.12}
$$

The integral representation is given by

$$Z_1(\theta) = \exp\left[-\frac{1}{2}\int_0^\infty \frac{dt}{t}\frac{\cosh(\frac{\pi}{2}+i\kappa)\frac{2t}{q}}{\cosh \pi t \sinh\frac{\pi t}{q}} \sin t\theta \sin t(i\pi-\theta)\right] . \tag{4.13}$$

The last thing to be done is to impose boundary crossing-unitarity and we will have the complete (minimal) reflection matrix. The boundary crossing-unitarity was introduced in [1], and we should note that their formula (3.35) assumes that one is dealing with a parity preserving, neutral theory. Since in our case we have fermions, we have to pay attention to possible minus signs and charge conjugation phases (we refer the reader to the appendix for a discussion on these issues). The crossing-unitarity equation turns out to be

$$K^{ab}(\theta) = S^{ba}_{a'b'}(2\theta)K^{a'b'}(-\theta) , \tag{4.14}$$

and in our case it reads

$$Z_2(\frac{i\pi}{2}-\theta) = Z_2(\frac{i\pi}{2}+\theta)\frac{2a(2\theta)}{\cosh 2\theta}\cosh(\frac{\eta\pi i}{4}+\frac{\theta}{2})\cosh(\frac{\eta\pi i}{4}\epsilon+q\theta) , \tag{4.15}$$

and η is the sign of the charge combination in (4.3). The minimal solution can be found by elementary methods and is given by

$$Z_2(\theta) = \frac{\cos(\frac{\pi}{8}-\frac{i\theta}{2})}{\cos(\frac{\pi}{8}+\frac{i\theta}{2})}\Omega_1(\theta)\Omega_2(\theta) , \tag{4.16}$$

where

$$\begin{aligned}\Omega_1(\theta) = \prod_{k=0}^{\infty} & \frac{\Gamma(\frac{3}{4}+k-\frac{\eta}{4}+\frac{i\theta}{2\pi})\Gamma(\frac{5}{4}+k+\frac{\eta}{4}+\frac{i\theta}{2\pi})}{\Gamma(\frac{3}{4}+k-\frac{\eta}{4}-\frac{i\theta}{2\pi})\Gamma(\frac{5}{4}+k+\frac{\eta}{4}-\frac{i\theta}{2\pi})}\times \\ & \frac{\Gamma(1+k+\frac{\eta}{4}-\frac{i\theta}{2\pi})\Gamma(1+k-\frac{\eta}{4}-\frac{i\theta}{2\pi})}{\Gamma(1+k+\frac{\eta}{4}+\frac{i\theta}{2\pi})\Gamma(1+k-\frac{\eta}{4}+\frac{i\theta}{2\pi})} ,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\Omega_1(\theta) = \prod_{k=0}^{\infty} & \frac{\Gamma(\frac{1}{2}+2q(k+\frac{3}{4})+\frac{\eta\epsilon}{4}+\frac{iq\theta}{\pi})\Gamma(\frac{1}{2}+2q(k+\frac{1}{4})-\frac{\eta\epsilon}{4}+\frac{iq\theta}{\pi})}{\Gamma(\frac{1}{2}+2q(k+\frac{3}{4})+\frac{\eta\epsilon}{4}-\frac{iq\theta}{\pi})\Gamma(\frac{1}{2}+2q(k+\frac{1}{4})-\frac{\eta\epsilon}{4}-\frac{iq\theta}{\pi})}\times \\ & \frac{\Gamma(\frac{1}{2}+2q(k+\frac{1}{2})+\frac{\eta\epsilon}{4}-\frac{iq\theta}{\pi})\Gamma(\frac{1}{2}+2q(k+\frac{1}{2})-\frac{\eta\epsilon}{4}-\frac{iq\theta}{\pi})}{\Gamma(\frac{1}{2}+2q(k+\frac{1}{2})+\frac{\eta\epsilon}{4}+\frac{iq\theta}{\pi})\Gamma(\frac{1}{2}+2q(k+\frac{1}{2})-\frac{\eta\epsilon}{4}+\frac{iq\theta}{\pi})} .\end{aligned} \tag{4.18}$$

The integral expression for $Z_2(\theta)$ is

$$Z_2(\theta) = \exp\left[-\frac{i}{2}\int_0^\infty \frac{dt}{t}\frac{\sin\theta t}{\cosh\frac{\pi t}{4}\cosh\frac{\pi t}{2}}\left(-1+A(t)+B(t)\right)\right] , \tag{4.19}$$

where the functions $A(t)$ and $B(t)$ are given by

$$A(t) = \frac{\sinh\frac{\pi t}{4}(1+2\eta)}{\sinh\pi\theta} , \quad B(t) = \frac{\sinh\frac{\pi t}{4q}(q+\eta\epsilon)}{\sinh\frac{\pi t}{2q}} . \tag{4.20}$$

The prefactor $R_0(\theta)$ can be easily written now. This concludes the discussion for the saturated case.

- *The Non-Saturated Case* ($c(\theta) = 0$)

  The unitarity condition is

$$Z_1(\theta)Z_1(-\theta) = \left[\cosh(\frac{i\alpha^\pm}{2} - \frac{\theta}{2})\cosh(\frac{i\alpha^\pm}{2} + \frac{\theta}{2}) - A^2 \sinh^2\theta\right]^{-1} . \tag{4.21}$$

  The solution can be expressed in terms of the function $\sigma(x,\theta)$ defined in (4.12) with $q = 1/2$ and it is given by

$$Z_1(\theta) = \frac{\sqrt{\cosh 2\phi - \cosh 2\xi}}{\sqrt{2}\sinh\phi\cosh\xi}\sigma(\frac{\pi}{2} + i\phi, \theta)\sigma(i\xi, \theta) . \tag{4.22}$$

  where the parameter ξ, ϕ are defined by

$$\cosh 2\phi - \cosh 2\xi = \frac{1}{2A^2} , \quad \cosh 2\phi \cosh 2\xi = 1 + \frac{\cosh i\alpha^\pm}{2A^2} . \tag{4.23}$$

  Finally boundary crossing-symmetry yields the following equation for $Z_2(\theta)$:

$$Z_2(\frac{i\pi}{2} - \theta) = Z_2(\frac{i\pi}{2} + \theta)a(2\theta)\cosh\theta , \tag{4.24}$$

  whose minimal solution is given by

$$Z_2(\theta) = \prod_{l=0}^{\infty} \frac{\Gamma(1 + l + \frac{i\theta}{\pi})\Gamma^2(\frac{3}{2} + 2l - \frac{i\theta}{\pi})}{\Gamma(1 + l - \frac{i\theta}{\pi})\Gamma^2(\frac{3}{2} + 2l + \frac{i\theta}{\pi})} . \tag{4.25}$$

  This has a simple integral expression which we quote below:

$$Z_2(\theta) = \exp\left[-i\int_0^\infty \frac{dt}{t}\frac{\sin t\theta \sinh\frac{\pi t}{4}}{\sinh \pi t \cosh\frac{\pi t}{4}}\right] . \tag{4.26}$$

The complete minimal solution for the prefactor $R_0(\theta)$ can be easily written now. This concludes our description of the supersymmetry preserving boundary reflection matrices.

5 General Reflection Matrices

In the previous section we have computed the reflection amplitudes assuming that the underlying boundary interaction preserves both integrability and supersymmetry. This last requirement simplified computations since the constraint (4.3) severely restricts the form of $R(\theta)$. In order to study the interplay between integrability and supersymmetry in the presence of a boundary [5] it is also interesting to invert the logic of the previous section: first require only boundary integrability, namely solve the full BYBE, and then try to

[5]See [23, 24] for related discussions in the Lagrangian approach.

understand in what limits, if any, supersymmetry can be restored. This is what we do this section. As an initial remark, notice that since the bulk S-matrix does not change under the substitution soliton $\leftrightarrow$ anti-soliton, the reflection amplitudes in the soliton sector and in the anti-soliton sector will satisfy the same BYBEs. As a consequence, the functional form of the amplitude ratios are the same in the two sectors, but depending on two different sets of free parameters. This means that we can again concentrate only on $R(\theta)$. Clearly enough, we expect that the two sets of parameters have to be related somehow, in order to recover boundary supersymmetry, as in (4.6).

Let us look initially at the BYBE corresponding to the factorization of $B\bar{B} \to F\bar{F}$ reflection process

$$\frac{R_f(\theta_2)}{R_b(\theta_2)}v(\bar{\theta}_{12}) + \frac{R_f(\theta_1)}{R_b(\theta_1)}\frac{R_f(\theta_2)}{R_b(\theta_2)}v(\theta_{12}) = v(\theta_{12}) + \frac{R_f(\theta_1)}{R_b(\theta_1)}v(\bar{\theta}_{12}) \ , \tag{5.1}$$

This equation has the interesting (and simplifying) feature of being the same whether we impose that there are non-diagonal processes or not. The simplest solution of (5.1) is $R_f(\theta)/R_b(\theta) = \pm 1$, which is a solution for any $v(\theta)$. In order to obtain other solutions for (5.1) we convert it into a differential equation for $f(\theta) = R_f(\theta)/R_b(\theta)$:

$$\frac{\dot{f}(\theta)}{1 - f^2(\theta)} = -\frac{\dot{v}(0)}{v(2\theta)} \ . \tag{5.2}$$

The solution of (5.2) is

$$f(\theta) = \frac{R_f(\theta)}{R_b(\theta)} = -\tanh\left(F(\theta) + k\right) \ , \tag{5.3}$$

where $F(\theta) = \int^{\theta} d\theta' \, \dot{v}(0)/v(2\theta')$ and k is an integration constant. In the non-saturated case this gives

$$\frac{R_f(\theta)}{R_b(\theta)} = \frac{\cosh(\frac{i\alpha'}{2} + \frac{\theta}{2})}{\cosh(\frac{i\alpha'}{2} - \frac{\theta}{2})} \ , \tag{5.4}$$

where α' is such that

$$\tanh\frac{i\alpha'}{2} = -e^{2k} \ . \tag{5.5}$$

In the saturated case the solutions of (5.2) have an “unusual” functional form and we will not treat them in the following. Nonetheless it would be an interesting problem to analyze their physical meaning.

The ratio $r(\theta) = Q(\theta)/P(\theta)$ is fixed by the BYBEs corresponding to the factorization of $B\bar{B} \to B\bar{B}$ and of $B\bar{F} \to B\bar{F}$:

$$\begin{aligned}
(r(\theta_2) - r(\theta_1))y(\bar{\theta}_{12}) &= (1 - r(\theta_1)r(\theta_2))x(\bar{\theta}_{12})v(\theta_{12}) \ , & (5.6)\\
(r(\theta_2) - r(\theta_1))y(\theta_{12}) &= (1 - r(\theta_1)r(\theta_2))x(\theta_{12})v(\bar{\theta}_{12}) \ . & (5.7)
\end{aligned}$$

These equations imply that $r(\theta) = \pm 1$ in the saturated case and $r(\theta) = \gamma =$ constant, in the non-saturated case. Since the solution of unitarity and boundary crossing-unitarity is closely related to the one in section 3.1 we will simply quote the results in the following.

- *The Saturated Case* ($c(\theta) \neq 0$)

  The ratio between the diagonal elements, as well as the ratio between off-diagonal ones, is fixed to be ± 1. From (4.7) we find

$$R(\theta) = R_0(\theta) \begin{pmatrix} 1 & A \sinh(1 \pm \epsilon p)\theta \\ \pm A \sinh(1 \pm \epsilon p)\theta & 1 \end{pmatrix} , \tag{5.8}$$

  and

$$R(\theta) = R_0(\theta) \begin{pmatrix} 1 & A \cosh(1 \mp \epsilon p)\theta \\ \pm A \cosh(1 \mp \epsilon p)\theta & -1 \end{pmatrix} , \tag{5.9}$$

  where ϵ is the sign of $c(\theta)$ in (3.4). The prefactor in this case is easily seen to be the same as in (4.11), (4.19) with the substitution $q \to \tilde{q} = 1 \pm \epsilon p$ and $q \to \tilde{q} = 1 \mp \epsilon p$, respectively. Notice that out of four possible sign combinations, two coincide exactly with (4.8, (4.9).

- *The Non-Saturated Case* ($c(\theta) = 0$)

  The ratio of the diagonal elements is exactly the same as the boundary supersymmetry-preserving one, with an arbitrary parameter α' which is not necessarily related to the topological charge. The solution of (5.6), (5.7) fixes the ratio between the off-diagonal elements of the reflection matrix to be a constant. We get

$$R(\theta) = R_0(\theta) \begin{pmatrix} \cosh(\frac{i\alpha'}{2} - \frac{\theta}{2}) & P(\theta) \\ \gamma P(\theta) & \cosh(\frac{i\alpha'}{2} + \frac{\theta}{2}) \end{pmatrix} . \tag{5.10}$$

  Only when $\gamma = \exp(-i\alpha')$ we recover the supersymmetric solution. Therefore we can think of γ as a parameter that "measures" how far we are from a supersymmetry preserving boundary interaction. To fix $P(\theta)$ we use (4.7) obtaining $P(\theta) = A \sinh\theta$. Finally, the prefactor $R_0(\theta)$ is fixed to be the same as in (4.22) and (4.26) with parameters ξ and ϕ defined now by

$$\cosh 2\phi - \cosh 2\xi = \frac{\gamma}{2A^2} , \quad \cosh 2\phi \cosh 2\xi = 1 + \frac{\gamma \cosh i\alpha'}{2A^2} . \tag{5.11}$$

This concludes the analysis of the reflection matrices for the TIM. In the next section we discuss the possible boundary perturbations connected to these solutions.

6 Boundary Perturbations

One of the main problems in boundary factorized scattering theory is that it is very difficult, in general, to relate solutions of the BYBE to specific boundary perturbations or, in other words, to connect the parameters appearing in the reflection matrices with the actual boundary coupling constants in a Lagrangian description, if the model admits one [6]. In this section we connect our reflection matrices to specific boundary perturbations, within the formalism of deformed boundary conformal field theory (BCFT) [1]. A microscopic analysis of conformal boundary conditions for the TIM has been performed in [13], where the correspondence with A_4 RSOS model has been used. In that formulation it is difficult to analyze supersymmetric boundary conditions, whereas in the present case it is quite natural.

Let us notice initially that in the massless bulk limit the topological charge in the supersymmetry algebra vanishes [21], and as a consequence we do not expect any difference between the saturated and non-saturated case from the point of view of deformed CFT .

After this preliminary remark, recall that in the bulk our model can be formulated as the massive deformation of the NS sector of $\mathcal{SM}_3$ by the relevant primary operator $\hat{\Phi}_{(1,3)}$. In fact, the chiral NS superfield has two components, the energy operator and the sub-leading (vacancy) operator, $\hat{\Phi}_{(1,3)} = \epsilon + \theta\epsilon'$ The perturbation by ϵ' preserves supersymmetry and the one by ϵ breaks it [26].

A boundary field theory is defined by specifying the conformal boundary conditions (CBCs) and the boundary perturbation. It is well known [27] that in minimal unitary (super)conformal models the CBCs are in one-to-one correspondence with primary operators. This means that in the NS sector of $\mathcal{SM}_3$, the possible CBCs that do not break superconformal invariance correspond to the primary superfields $\hat{\Phi}_{(1,1)}$ and $\hat{\Phi}_{(1,3)}$. The CBCs determine the spectrum of allowed boundary operators [28, 27] and it turns out that in the first case the only boundary operators that can appear are the identity $\mathbf{1}$ and the irrelevant operator ϵ'', whereas in the second ϵ and ϵ'.

Once we know the boundary operator content of a BCFT we can study which ones will preserve boundary integrability. The argument of [1] can be rephrased by saying [21] that a boundary operator preserves integrability if it is in the same representation of the relevant conformal algebra as the bulk perturbation. In our case this means that the perturbation by the boundary superfield $\hat{\Phi}_{(1,3)}$ is integrable; furthermore, the perturbation by ϵ breaks supersymmetry while the perturbation by ϵ' preserves it, as it can be easily verified to first order in conformal perturbation theory. Let us notice, finally, that from Cardy's analysis [28] (see also [29]) it turns out that the free boundary

[6] for related discussions in the case of boundary sine-Gordon see [1], [25].

conditions do not support the boundary operators ϵ and ϵ'. A reasonable proposal is that the reflection matrices we obtained correspond to some sort of fixed boundary conditions perturbed respectively by the operators ϵ' and ϵ.

7 Discussion and Conclusions

In this paper we have found the exact reflection matrices for the S-matrix proposed by Fendley to describe the superfield sector of the tricritical Ising model, where supersymmetry acts locally. Supersymmetry fixes, almost completely, the structure of boundary scattering and predicts a universal ratio for the amplitudes of bosons and fermions scattering diagonally off the boundary. More explicitly, the requirement of boundary supersymmetry alone fixes R_b/R_f, Q/P and establishes the precise relation between R and $\bar{R}$.

We also solved the BYBE in general and we showed that it fixes R_b/R_f, Q/P and that R and $\bar{R}$ should have the same functional form. We were able to connect some of theses solutions to the supersymmetry preserving ones.

As a next step it would be interesting to compute correlation functions in this realization of the TIM by means of the form-factor approach. A first step in the computation of supersymmetric form-factors has been done in the paper [20].

As a last remark we should mention that a thermodynamical Bethe ansatz computation of finite-size effects would be very useful in order to confirm that this description in term of supersymmetric soliton doublet is the correct scattering theory for the massive excitations of the supersymmetric TIM. In any case the necessity of introducing some CDD factors does not change the structure of our reflection matrices.

Acknowledgments

We would like to thank G. Mussardo for a critical reading of the paper and for many valuable suggestions, P. Fendley for useful remarks concerning the S-matrix used in this paper and R. Musto for useful comments. One of us (MM) would like to thank K. Schoutens for discussions on related issues in an early stage of this work and Lis for encouragement.

Appendix

In this appendix we discuss some subtleties in the boundary crossing-unitarity condition that arise in models with fermions. These are well understood, but since we have not found an explicit discussion in the literature about these issues, we will present it here. Recall that the crossing symmetry property of the S-matrix can be written as

$$S_{ab}^{cd}(i\pi-\theta)=C_{bb'}S_{d'a}^{b'c}(\theta)C^{d'd}\ , \tag{7.1}$$

where C_{ab} is the charge conjugation matrix and C^{ab} is its inverse. From the crossing properties of our bulk S-matrix we find that the only non vanishing elements of the charge conjugation matrix can be chosen in the following way:

$$C_{B\bar{B}}=C_{\bar{B}B}=1,\quad C_{F\bar{F}}=C_{\bar{F}F}=i\ , \tag{7.2}$$

As it is well known [1], the reflection amplitude $R(\theta)$ is the analytic continuation of the amplitude $K(\theta)$ to the domain $\mathrm{Im}\theta=\frac{i\pi}{2}$, $\mathrm{Re}\theta<0$. In models that are not invariant under charge conjugation, the appropriate analytic continuation involves the charge conjugation matrix as follows:

$$K^{ab}(\theta)=C^{aa'}R_{a'}^{b}(\frac{i\pi}{2}-\theta)\ . \tag{7.3}$$

In order to write down the boundary crossing-unitarity equation, without assuming we are dealing with a theory invariant under the usual discrete symmetries, we come back to the argument of Ghoshal-Zamolodchikov. The amplitude $K^{ab}(\theta)$ at positive real θ is the coefficient of the two-particle contribution in the expansion of the boundary state in terms of out states,

$$|B\rangle=\left[1+\int_0^\infty d\theta K^{ab}(\theta)A_a(-\theta)A_b(\theta)+\dots\right]|0\rangle, \tag{7.4}$$

whereas at negative real θ it is interpreted as the two-particle contribution in the in states basis,

$$|B\rangle=\left[1+\int_0^\infty d\theta K^{ab}(-\theta)A_a(\theta)A_b(-\theta)+\dots\right]|0\rangle. \tag{7.5}$$

The boundary cross-unitarity condition is obtained as a consistency condition of these two expressions, using the fact that in and out states are related through the S-matrix:

$$K^{ab}(\theta)=S_{a'b'}^{ba}(2\theta)K^{a'b'}(-\theta). \tag{7.6}$$

Notice the different ordering of indices with respect to equation (3.35) in [1], where invariance under charge conjugation, parity and time reversal were assumed.

References

[1] S. Ghoshal, A Zamolodchikov, *Boundary S-Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory,* Int. Jour. Mod. Phys. **A 9**, (1994) 3841-3886, hep-th/9306002.

[2] P. Fendley, A.W.W. Ludwig, H. Saleur, *Exact Conductance trough Point Contacts in the $\nu = 1/3$ Fractional Quantum Hall Effect*, Phys. Rev. Lett. **74**, (1995) 3005-3008.

[3] D. Friedan, Z. Qiu, S. Shenker, *Superconformal Invariance in Two-Dimensions and the Tricritical Ising Model*, Phys. Lett. **B 151**, (1985) 37-43.

[4] Z. Qiu, *Supersymmetry, Two-Dimensional Critical Phenomena and the Tricritical Ising Model*, Nucl. Phys. **B 270**, (1986) 205-234.

[5] D.A. Kastor, E.J. Martinec, S.H. Shenker, *RG Flow in $N = 1$ Discrete Series*, Nucl. Phys. **B 316**, (1989) 590-608.

[6] A.B. Zamolodchikov, *From Tricritical Ising to Critical Ising Model by Thermodynamic Bethe Ansatz*, Nucl. Phys. **B 358**, (1991) 524-546.

[7] A.B. Zamolodchikov, *Fractional-Spin Integrals of Motion in Perturbed Conformal Field Theory*, in "Fields, Strings and Quantum Gravity", H. Guo, Z. Qiu, H. Tye eds., Gordon and Breach, 1989, 349-372.

[8] P. Fendley, *A Second Supersymmetric S-matrix for the Perturbed Tricritical Ising Model*, Phys. Lett. **B 250**, (1990) 96-101.

[9] M. Blume, V.J. Emery, R.B. Griffiths, *Ising Model for the λ Transition in He^3-He^4 Mixtures*, Phys. Rev. **A 4**, (1971) 1071-1077.

[10] D. Friedan, Z. Qiu, S. Shenker, *Conformal Invariance, Unitarity and Two-Dimensional Critical Exponents*, Phys. Rev. Lett. **52**, (1984) 1575-1578.

[11] A.B. Zamolodchikov, *Conformal Symmetry and Multicritical Points in the Two-Dimensional Quantum Field Theory*, Sov. J. Nucl. Phys. **44**, (1986) 529-533.

[12] A.B. Zamolodchikov, *Integrable Field Theory from Conformal Field Theory*, in "Advanced Studies in Pure Mathematics", vol. 19 (1989), 641-674.

[13] L. Chim, *Boundary S-Matrix for the Tricritical Ising Model*, Int. Jour. Mod. Phys. **A 11**, (1996) 4491-4512, hep-th/9510008.

[14] A.B. Zamolodchikov, A.B. Zamolodchikov, *Factorized S-Matrices in Two Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Theory Models*, Ann. Phys. **120**, (1979) 253-291.

[15] A. Fring, R. Köberle, *Factorized Scattering in the Presence of Reflecting Boundaries*, Nucl. Phys. **B 421**, (1994) 159-172, hep-th/9304141.

[16] I.V. Cherednik, *Factorizing Particles on a Half Line and Root Systems*, Theor. Math. Phys. **61**, (1984) 977-983.

[17] E. Witten, D. Olive, *Supersymmetry Algebras that Include Topological Charges*, Phys. Lett. **B 78**, (1978) 97-101.

[18] M. Moriconi, K. Schoutens, *Thermodynamic Bethe Ansatz for $N = 1$ Supersymmetric Theories*, Nucl. Phys. **B 464**, (1996) 472-491, hep-th/9511008.

[19] P. Fendley, K. Intriligator, *Scattering and Thermodynamics of Fractionally Charged Supersymmetric Solitons*, Nucl. Phys. **B 372**, (1992) 533-588, hep-th/9111014.

[20] C. Ahn, *Thermodynamics and Form Factors of Supersymmetric Integrable Field Theories*, Nucl. Phys. **B 422**, (1994) 449-475, hep-th/9306146.

[21] N.P. Warner, *Supersymmetry in Boundary Integrable Models*, Nucl. Phys. **B 450**, (1995) 663-694, hep-th/9506064.

[22] M. Moriconi, K. Schoutens, *Reflection Matrices for Integrable $N = 1$ Supersymmetric Theories*, Nucl. Phys. **B 487**, (1997) 756-778, hep-th/9605219.

[23] T. Inami, S. Odake, Y.-Z. Zhang, *Supersymmetric Extension of the Sine-Gordon Theory with Integrable Boundary Interactions*, Phys. Lett. **B 359**, (1995) 118-124, hep-th/9506157.

[24] J.N.G.N. Prata, *The Superliouville Equation on the Half Line*, Phys. Lett. **B 405**, (1997) 271-279, hep-th/9704070.

[25] M. Ameduri, R. Konik, A. LeClair, *Boundary Sine-Gordon Interactions at the Free Fermion Point*, Phys. Lett. **B 354**, (1995) 376-382.

[26] D. Depireux, P. Mathieu, *Integrable Supersymmetry Breaking Perturbations of $N = 1$, $N = 2$ Superconformal Minimal Models*, Phys. Lett. **B 308**, (1993) 272-278, hep-th/9301131.

[27] J.L. Cardy, *Boundary Conditions, Fusion Rules and the Verlinde Formula*, Nucl. Phys. **B 324**, (1989) 581-596.

[28] J.L. Cardy, *Effect of Boundary Conditions on the Operator Content of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. **B 275**, (1986) 200-218.

[29] P. Fröjdh, H. Johannesson, *The Two-Dimensional Tricritical Ising Model in Restricted Domains with Free Boundaries*, Nucl. Phys. **B 366**, (1991) 429-466.