LAPTH

Laboratoire d'Annecy-le-Vieux de Physique Théorique

Spontaneous symmetry breaking in the non-linear Schrödinger hierarchy with defect

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Abstract

We introduce and solve the one-dimensional quantum non-linear Schrödinger (NLS) equation for an N-component field defined on the real line with a defect sitting at the origin. The quantum solution is constructed using the quantum inverse scattering method based on the concept of Reflection-Transmission (RT) algebras recently introduced.
The symmetry of the model is generated by the reflection and transmission defect generators defining a defect subalgebra. We classify all the corresponding reflection and transmission matrices. This provides the possible boundary conditions obeyed by the canonical field and we compute these boundary conditions explicitly.
Finally, we exhibit a phenomenon of spontaneous symmetry breaking induced by the defect and identify the unbroken generators as well as the exact remaining symmetry.

PACS numbers: 02.30.Ik, 03.70.+k, 11.10.Kk

math-ph/0411022
LAPTH-1076/04
November 2004

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1 Introduction

In the framework of quantum integrable systems (qIS), a challenging question which is addressed nowadays, is the treatment of a localized defect (or impurity), sitting at the origin, reflecting and transmitting particles. The ever-increasing progress in handling quantum systems has made it possible to create experiments able to probe the 1D behaviour of quantum gases (see e.g. [1]). In this context, the construction of realistic qIS, such as ones with defects, becomes crucial. This is a very debated issue and essentially, two different approaches were developed, that we briefly present.

Originally, qIS have been studied on the real line. Powerful tools, such as quantum inverse scattering method, have been developed for such a purpose. The framework is now well-known and relies on the possibility to adapt the inverse scattering method [2–4] and the arguments of Shabat and Zakharov [5] to the quantum case, e.g. [6–8]. This subtle construction can then be related to algebraic structures such as Lie algebras, quantum groups or Yangians. Of particular interest is the Zamolodchikov-Faddeev (ZF) algebra [9]. It describes the scattering of quasi-particles and allows to construct a hierarchy (i.e. an infinite set of Hamiltonians in involution). The definition of the ZF algebra relies on an S-matrix, which obeys the Yang-Baxter (YB) equation and a unitarity relation. It is interpreted as the basic two-body scattering matrix of the quasi-particles. This S-matrix is just the usual R-matrix of quantum groups. It has been recently shown that the ZF algebra allows to construct the symmetry of this hierarchy, via the notion of well-bred operators [10]. These operators realize a quantum group defined in the FRT formalism.

As a fundamental example, the non-linear Schrödinger (NLS) equation (for a review see e.g. [11]) has played a central role in the development of QISM, for Lax pairs and ZF algebras, e.g. [12–19]. In particular, the canonical field of the quantum NLS model has been constructed using a ZF algebra based on the $gl(N)$-Yangian R-matrix . Then, it was proved that the Yangian of $gl(N)$ is the symmetry algebra of the NLS hierarchy [20], a realization of which in terms of ZF generators being given by well-bred operators [10].

Later on, the study of qIS defined on the half-line was undertaken. The pioneering work of Cherednik [21] and Sklyanin [22] underlined the importance of the so-called reflection matrix, obeying a reflection or boundary Yang-Baxter equation (RE). This matrix encodes the reflection of particles on the boundary, and the RE ensures the consistency between bulk and boundary scattering.

At the algebraic level, two frameworks were developed. In the first approach [23, 24], the ZF algebra is kept and a new operator Z_{ij}, corresponding to the boundary, is consistently added. From this point of view, the boundary is assimilated to an infinitively heavy particle (an impurity), sitting at the origin, on which the particles reflects, through the reflection matrix. Although efficient for dealing with correlation functions, this first approach is however unable to treat the off-shell properties of the canonical fields. That was the motivation of the second approach, where a (non-central) extension of the ZF algebra, the boundary algebra [25] was defined. It contains a new generator $b_{ij}(k)$ accounting for the reflection of a particle (of impulsion k) on the boundary. From this point of view, the boundary does not appear as a heavy particle anymore, but more as a border of the half-line on which the particles reflects. The value of the boundary generator in a given (boundary algebra) Fock space gives back the reflection matrix [26], making the contact with the previous approach. In this context, the different possible reflection matrices classify the scalar representations of the reflection subalgebra and also the different integrable boundary conditions obeyed by the canonical fields of the qIS defined on the half-line. The second approach also enables the construction of the canonical fields of the theory. Just like for the qIS on the whole line, the boundary algebra allows to construct an infinite hierarchy associated to the models on the half-line. Moreover, the $b_{ij}(k)$ generators, which form a reflection subalgebra, appear to be the symmetry of the hierarchy [27]. In this context, the reflection algebra is a subalgebra of the quantum group which is a symmetry of the corresponding model defined on the line [28]. This second approach has been successively applied on the fundamental examples of scalar and N-component NLS model defined on the half-line [29].

In the 90's, the question of a defect which reflects and transmits particles was studied. The approach followed Cherednik's original point of view to incorporate transmission. This led to the first notion of reflection-transmission algebra, together with reflection and transmission matrices [30–32]. However, it was shown later on [33] that the consistent definition of these algebras together with the assumption of Lorentz invariance implies that either the S-matrix must be trivial, or there must be only reflection or only transmission to maintain the integrability of the system. The case of purely reflecting impurity just reproduces the half-line case in the first approach, while the purely transmitting impurity offers new possibilities (see for instance examples in [34]).

Recently, a new type of reflection-transmission (RT) algebras was introduced [35, 36] along the line of the second approach for the half-line. These algebras are extensions of boundary algebras and contain, apart from ZF-like generators, two new generators $r_{ij}(k)$ and $t_{ij}(k)$ corresponding to the reflection and the transmission of a particle through the defect at the origin. Contrarily to the first reflection-transmission algebras, the RT algebras can have both reflection and transmission and a non-trivial S-matrix (obeying YB and unitarity equations). This crucial point can be related to the fact that RT algebras are more general than the original ones. A detailed study of this fact and the proof that the original reflection-transmission algebras are a special case of RT algebras will be published elsewhere [37]. The RT algebras also allow to construct an infinite hierarchy associated to the models with defect, $r_{ij}(k)$ and $t_{ij}(k)$ generating the symmetry algebra (called the defect subalgebra) of the hierarchy [38]. The defect subalgebra is also a subalgebra of the quantum group symmetry algebra of the corresponding systems on the line [38]. The representations of the defect algebra classify the integrable boundary conditions obeyed by the canonical fields of the theory. RT algebras framework has been worked out by generalizing the study of a free field on the line with a defect. The validity of this approach has been given via the resolution at the quantum level of the scalar NLS model with a defect [39, 40].

Although convincing, the scalar NLS model possesses a scalar S-matrix, so that one may wonder whether the concept of RT algebras is still relevant for systems with internal degrees of freedom, i.e. when the S-matrix is a true matrix. It is this question that we want to address in this paper, through the study of the quantum N-component NLS equation with defect. The article is organized as follows. In section 2 we briefly review some basic notions on RT-algebras, focusing on the NLS systems. Then, in section 3, we show that the RT-algebra allows to solve this system by constructing the canonical fields of the theory. We also show that the hierarchy constructed from the RT algebra is the one associated to the NLS model, and that the defect subalgebra is a symmetry algebra of this hierarchy. Section 4 is devoted to the classification of the scalar representations of the defect algebra and to the presentation of boundary conditions obeyed by the canonical fields for each of the representations. Finally, in section 5, we show that the impurity induces a spontaneous symmetry breaking of the defect algebra, and we compute the unbroken symmetry algebra of the system.

2 RT algebras

2.1 Basic structures

Let us recall briefly the ingredients involved in RT algebras [36], using compact tensor notations for convenience. Our starting point is the following $\mathcal{S}$-matrix

$$\mathcal{S}_{12}(k_1,k_2) = \begin{pmatrix} s_{12}(k_1-k_2) & 0 & 0 & 0 \\ 0 & s_{12}(k_1+k_2) & 0 & 0 \\ 0 & 0 & s_{12}(-k_1-k_2) & 0 \\ 0 & 0 & 0 & s_{12}(-k_1+k_2) \end{pmatrix}$$

$$\text{with} \quad s_{12}(k) = \frac{k\,\mathbb{I}_N \otimes \mathbb{I}_N - ig\,P_{12}}{k+ig}\,, \tag{2.1}$$

where P_{12} is the usual flip operator between two auxiliary spaces $End(\mathbb{C}^N)$. They correspond to internal degrees of freedom. One recognizes in $s_{12}(k)$ the S-matrix of the NLS equation on the line, and in the particular form of $\mathcal{S}$ the one used for solving the scalar NLS equation with defect [39]. We would like to stress that the total auxiliary space is $End(\mathbb{C}^2) \otimes End(\mathbb{C}^N)$, as it can be seen from the $\mathcal{S}$-matrix (2.1). The latter satisfies the unitarity and Yang-Baxter equations as required

$$\begin{aligned} \mathcal{S}_{12}(k_1,k_2)\,\mathcal{S}_{21}(k_2,k_1) &= (\mathbb{I}_2 \otimes \mathbb{I}_2) \otimes (\mathbb{I}_N \otimes \mathbb{I}_N)\,, \\ \mathcal{S}_{12}(k_1,k_2)\,\mathcal{S}_{13}(k_1,k_3)\,\mathcal{S}_{23}(k_2,k_3) &= \mathcal{S}_{23}(k_2,k_3)\,\mathcal{S}_{13}(k_1,k_3)\,\mathcal{S}_{12}(k_1,k_2)\,. \end{aligned}$$

We will split the index α corresponding to these auxiliary spaces into $\alpha = (\xi, i)$, with $\xi = \pm$ labelling the part of the half-lines $\mathbb{R}^{\pm}$ where processes take place, and $i = 1, ..., N$ the internal ("isotopic") degrees of freedom.
As usual, we refer to $k \in \mathbb{R}$ as a spectral parameter.

The RT algebra corresponding to $\mathcal{S}(k_1, k_2)$ is an associative algebra with identity element $\mathbf{1}$ and two types of generators, $\{a_\alpha(k),\, a^\dagger_\alpha(k)\}$ and $\{r^\beta_\alpha(k),\, t^\beta_\alpha(k)\}$, called bulk and defect (reflection and transmission) generators, respectively, subject to the following relations

$$\begin{aligned} a_1(k_1)a_2(k_2) &= \mathcal{S}_{21}(k_2,k_1)a_2(k_2)a_1(k_1) && (2.2)\\ a^\dagger_1(k_1)a^\dagger_2(k_2) &= a^\dagger_2(k_2)a^\dagger_1(k_1)\mathcal{S}_{21}(k_2,k_1) && (2.3)\\ a_1(k_1)a^\dagger_2(k_2) &= a^\dagger_2(k_2)\mathcal{S}_{12}(k_1,k_2)a_1(k_1) + \delta(k_1-k_2)\Big(\mathbf{1}+t_1(k_1)\Big)\delta_{12} \\ &\quad +\delta(k_1+k_2)r_1(k_1)\delta_{12} && (2.4) \end{aligned}$$

$$\begin{aligned} a_1(k_1)\,t_2(k_2) &= \mathcal{S}_{21}(k_2,k_1)\,t_2(k_2)\,\mathcal{S}_{12}(k_1,k_2)\,a_1(k_1) && (2.5)\\ a_1(k_1)\,r_2(k_2) &= \mathcal{S}_{21}(k_2,k_1)\,r_2(k_2)\,\mathcal{S}_{12}(k_1,-k_2)\,a_1(k_1) && (2.6)\\ t_1(k_1)\,a^\dagger_2(k_2) &= a^\dagger_2(k_2)\,\mathcal{S}_{12}(k_1,k_2)\,t_1(k_1)\,\mathcal{S}_{21}(k_2,k_1) && (2.7)\\ r_1(k_1)\,a^\dagger_2(k_2) &= a^\dagger_2(k_2)\,\mathcal{S}_{12}(k_1,k_2)\,r_1(k_1)\,\mathcal{S}_{21}(k_2,-k_1) && (2.8) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{12}(k_1,k_2)\,t_1(k_1)\,\mathcal{S}_{21}(k_2,k_1)\,t_2(k_2) &= t_2(k_2)\,\mathcal{S}_{12}(k_1,k_2)\,t_1(k_1)\,\mathcal{S}_{21}(k_2,k_1) && (2.9)\\ \mathcal{S}_{12}(k_1,k_2)\,t_1(k_1)\,\mathcal{S}_{21}(k_2,k_1)\,r_2(k_2) &= r_2(k_2)\,\mathcal{S}_{12}(k_1,\text{-}k_2)\,t_1(k_1)\,\mathcal{S}_{21}(\text{-}k_2,k_1) && (2.10)\\ \mathcal{S}_{12}(k_1,k_2)\,r_1(k_1)\,\mathcal{S}_{21}(k_2,\text{-}k_1)\,r_2(k_2) &= r_2(k_2)\,\mathcal{S}_{12}(k_1,\text{-}k_2)\,r_1(k_1)\,\mathcal{S}_{21}(\text{-}k_2,\text{-}k_1) && (2.11) \end{aligned}$$

where $r(k)$ and $t(k)$ satisfy:

$$\begin{aligned} t(k)t(k) + r(k)r(-k) &= \mathbf{1} && (2.12)\\ t(k)r(k) + r(k)t(-k) &= 0\,. && (2.13) \end{aligned}$$

We refer to [36] for the significance of the RT-algebras, and to [39, 40] for their use in the context of NLS.

The key point for physical applications is the possibility to construct Fock representations of the above algebra. As explained in [36] and explicitly shown in [39,40], this allows to apply the quantum inverse scattering method to get off-shell quantum fields and to compute scattering amplitudes. We know that these representations involve a particular (vacuum) state Ω annihilated by $a(k)$ and are uniquely determined by the two matrices $\mathcal{R}$ and $\mathcal{T}$ defined by

$$r(k)\,\Omega = \mathcal{R}(k)\,\Omega \quad \text{and} \quad t(k)\,\Omega = \mathcal{T}(k)\,\Omega\,. \tag{2.14}$$

In the present context of NLS with defect, inspired by the scalar case [40] we take

$$\mathcal{R}(k) = \begin{pmatrix} R_+(k) & 0 \\ 0 & R_-(k) \end{pmatrix}, \qquad \mathcal{T}(k) = \begin{pmatrix} 0 & T_+(k) \\ T_-(k) & 0 \end{pmatrix}, \tag{2.15}$$

where $R_\pm(k), T_\pm(k)$ are now $N \times N$ matrices. We recall that $\mathcal{R}$ and $\mathcal{T}$ must satisfy

$$\mathcal{R}(k)^\dagger = \mathcal{R}(-k)\,, \quad \mathcal{T}(k)^\dagger = \mathcal{T}(k)\,, \tag{2.16}$$

$$\mathcal{T}(k)\mathcal{T}(k) + \mathcal{R}(k)\mathcal{R}(-k) = \mathbb{I}_N\,, \tag{2.17}$$

$$\mathcal{T}(k)\mathcal{R}(k) + \mathcal{R}(k)\mathcal{T}(-k) = 0\,, \tag{2.18}$$

where † denotes transposition and conjugation.

2.2 Corresponding hierarchies

It was shown in [38] that one can naturally associate a hierarchy of Hamiltonians to any RT algebra as follows

$$\widetilde{H}^{(n)}_{RT} = \int_{\mathbb{R}} dk\, k^n\, a^{\dagger\alpha}(k) a_\alpha(k),\ \ n \in \mathbb{Z}_+\,. \tag{2.19}$$

We remind the reader that these natural Hamiltonians are not all in involution in general

$$[\widetilde{H}^{(m)}_{RT}, \widetilde{H}^{(n)}_{RT}] = [(-1)^m - (-1)^n] \int_{\mathbb{R}} dk\, k^{m+n}\ \ a^{\dagger\alpha}(k) r_\alpha^\beta(k) a_\beta(-k)\,. \tag{2.20}$$

Instead, we propose to consider the Hamiltonians

$$H^{(n)}_{RT} = \int_{\mathbb{R}} dk\, |k|^n\, a^{\dagger\alpha}(k) a_\alpha(k),\ \ n \in \mathbb{Z}_+\,, \tag{2.21}$$

which are all in involution and coincide with the previous ones when n is even:

$$[H^{(m)}_{RT}, H^{(n)}_{RT}] = 0 \quad ; \quad H^{(2m)}_{RT} = \widetilde{H}^{(2m)}_{RT}\,. \tag{2.22}$$

One can also compute

$$[H^{(m)}_{RT}, \widetilde{H}^{(n)}_{RT}] = [1 - (-1)^n] \int_{\mathbb{R}} dk\, |k|^m\, k^n\ \ a^{\dagger\alpha}(k) r_\alpha^\beta(k) a_\beta(-k)\,. \tag{2.23}$$

Interpreting $\widetilde{H}^{(1)}_{RT}$ and $\widetilde{H}^{(2)}_{RT}$ as the momentum and the Hamiltonian of the system, (2.20) shows that translation invariance is broken, as expected from the presence of a defect at the origin. However, $H^{(1)}_{RT}$ is conserved: the modulus of the impulsion is conserved. Remark the noticable exception of purely transmiting systems ($r(k) = 0$), where $H^{(1)}_{RT}$ is conserved, in accordance with the above interpretation.

The existence of an infinite number of Hamiltonians in involution has to be related to the integrability of the models under consideration. For instance, in [39, 40], the authors showed that $H^{(2)}_{RT}$ as defined above was the Hamiltonian of *scalar* NLS with defect, the other Hamiltonians being integrals of motion.

Finally, from (2.5)-(2.8), one easily gets

$$[H^{(n)}_{RT}, t(k)] = 0 \quad \text{and} \quad [H^{(n)}_{RT}, r(k)] = 0\,. \tag{2.24}$$

This shows that the subalgebra $\mathcal{D}_{\mathcal{S}}$ (2.9)-(2.11) generated by $r(k)$ and $t(k)$, which we call the defect algebra, is the symmetry algebra of the hierarchy described by $H^{(n)}_{RT}$. (2.11) shows that the reflection algebra is itself a subalgebra of $\mathcal{D}_{\mathcal{S}}$ and this situation should be compared to the well-known fact that the reflection algebra is symmetry algebra of various integrable systems on the half-line where only reflection occurs.

3 NLS hierarchy with defect

3.1 The $gl(N)$ model

As in the scalar case [39, 40], the quantum NLS model with defect is described by an equation of motion on $\mathbb{R} \setminus \{0\}$ and boundary conditions at $x = 0$ to be satisfied by the field. Here, the field $\Phi_0(x,t)$ is a N-component vector which splits according to

$$\Phi_0(x,t) = \theta(x)\Phi_{+,0}(x,t) + \theta(-x)\Phi_{-,0}(x,t) \tag{3.1}$$

where

$$\Phi_{\xi,0}(x,t) = \sum_{j=1}^{N} \Phi_{\xi,j}(x,t)\; e_j \quad , \quad \xi = \pm \tag{3.2}$$

and e_j is the j-th canonical basis vector of $\mathbb{C}^N$. From this point, using the convenient tensor notations already introduced, the $gl(N)$ model can be dealt with in a natural way similar to the treatment detailed in [39, 40]. The fundamental ingredient is the Fock representation of the RT algebra defined above labelled by the choice

$$R_+(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 + i(a+d)k - c}\, \mathbb{I}_N\,, \qquad T_+(k) = \frac{2i\alpha k}{bk^2 + i(a+d)k - c}\, \mathbb{I}_N\,, \tag{3.3}$$

$$R_-(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 - i(a+d)k - c}\, \mathbb{I}_N\,, \qquad T_-(k) = \frac{-2i\overline{\alpha} k}{bk^2 - i(a+d)k - c}\, \mathbb{I}_N\,, \tag{3.4}$$

where

$$\{a, ..., d \in \mathbb{R},\, \alpha \in \mathbb{C}\, :\, ad - bc = 1,\, \overline{\alpha}\alpha = 1\}\,. \tag{3.5}$$

We further require

$$\left\{ \begin{array}{ll} a + d - \frac{b}{|b|}\sqrt{(a-d)^2 + 4} \leq 0\,, & b \neq 0\,, \\ c(a+d)^{-1} \geq 0\,, & b = 0\,, \end{array} \right. \tag{3.6}$$

to avoid bound states since we concentrate on the scattering theory.

Then the constructions detailed in [36,39] provide us with the elements $\{\mathcal{H}, \mathcal{D}, \Omega, \Phi\}$ necessary for a second quantized theory:

- A Hilbert space $\mathcal{H}$ with positive definite scalar product $\langle \cdot \, , \, \cdot \rangle$, which describes the states of the system;

- An operator valued distribution $\Phi(x,t)$, defined on a dense domain $\mathcal{D} \subset \mathcal{H}$, the finite particle subspace, and satisfying the equation of motion, the boundary conditions in mean value on $\mathcal{D}$ and the equal time canonical commutation relations;

- A distinguished normalizable state $\Omega \in \mathcal{D}$ – the vacuum.

In this context the off-shell field is obtained by the quantum inverse scattering method and reads

$$\Phi_{\xi,0}(x,t) = \sum_{n \geq 0} (-g)^n \Phi^{(n)}_{\xi,0}(x,t) \tag{3.7}$$

where

$$\Phi^{(n)}_{\xi,0}(x,t) = \frac{1}{N^n} tr_{1\ldots n} \left[\Phi^{(n)}_{\xi,01\ldots n}(x,t) \right] \quad , \quad n \geq 1 \tag{3.8}$$

tr_j being the trace over the j-th auxiliary space, and

$$\begin{aligned} \Phi^{(n)}_{\xi,01\ldots n}(x,t) = \int_{\mathbb{R}^{2n+1}} \prod_{\substack{i=1 \\ j=0}}^{n} \frac{dp_i}{2\pi} \frac{dq_j}{2\pi} \, a^\dagger_{\xi,1}(p_1) \ldots a^\dagger_{\xi,n}(p_n) a_{\xi,n}(q_n) \ldots a_{\xi,1}(q_1) a_{\xi,0}(q_0) \\ \times \frac{e^{i \sum\limits_{j=0}^{n} (q_j x - q_j^2 t) - i \sum\limits_{i=1}^{n} (p_i x - p_i^2 t)}}{\prod\limits_{i=1}^{n} (p_i - q_{i-1} - \xi i \epsilon)(p_i - q_i - \xi i \epsilon)} \quad , \quad n \geq 0 \end{aligned} \tag{3.9}$$

The conjugate field $\Phi^\dagger(x,t)$ is constructed in the same way from

$$\begin{aligned} \Phi^{\dagger(n)}_{\xi,01\ldots n}(x,t) = \int_{\mathbb{R}^{2n+1}} \prod_{\substack{i=0 \\ j=1}}^{n} \frac{dp_i}{2\pi} \frac{dq_j}{2\pi} \, a^\dagger_{\xi,0}(p_0) a^\dagger_{\xi,1}(p_1) \ldots a^\dagger_{\xi,n}(p_n) a_{\xi,n}(q_n) \ldots a_{\xi,1}(q_1) \\ \times \frac{e^{i \sum\limits_{j=1}^{n} (q_j x - q_j^2 t) - i \sum\limits_{i=0}^{n} (p_i x - p_i^2 t)}}{\prod\limits_{i=1}^{n} (q_i - p_{i-1} + \xi i \epsilon)(q_i - p_i + \xi i \epsilon)} \quad , \quad n \geq 0 \end{aligned} \tag{3.10}$$

and one can show that they satisfy the equal time canonical commutation relations

$$[\Phi_j(x,t), \Phi_k(y,t)] = \left[\Phi^\dagger_j(x,t), \Phi^\dagger_k(y,t) \right] = 0 \tag{3.11}$$

$$\left[\Phi_j(x,t), \Phi^\dagger_k(y,t) \right] = \delta_{jk} \, \delta(x-y) \, , \tag{3.12}$$

where

$$\Phi_j(x,t)=\theta(x)\Phi_{+,j}(x,t)+\theta(-x)\Phi_{-,j}(x,t)\,,\qquad j=1,\ldots,N\,, \tag{3.13}$$

and $\Phi_{\pm,j}(x,t)$ has been defined in (3.2). As usual for the equation of motion for quantum NLS, the delicate point lies in the cubic term. One needs to specify a normal ordering $:\quad:$ and in our case, the normal ordered cubic term must be compatible with the use of auxiliary spaces in the tensor notation. In this respect, we can show the following

Theorem 3.1 *The quantum field $\Phi_0(x,t)$ is solution of the $gl(N)$ NLS model with defect. It obeys*

$$\forall\varphi,\psi\in\mathcal{D},\quad (i\partial_t+\partial_x^2)\langle\varphi,\Phi_0(x,t)\,\psi\rangle=2g\,\langle\varphi\,,\,tr_1\;:\Phi_0\widetilde{\Phi}_1^\dagger\Phi_1:(x,t)\,\psi\rangle \tag{3.14}$$

and

$$\lim_{x\downarrow 0}\begin{pmatrix}\langle\varphi\,,\,\Phi_0(t,x)\psi\rangle\\ \partial_x\langle\varphi\,,\,\Phi_0(t,x)\psi\rangle\end{pmatrix}=\alpha\begin{pmatrix}a\,\mathbb{I}_N & b\,\mathbb{I}_N\\ c\,\mathbb{I}_N & d\,\mathbb{I}_N\end{pmatrix}\lim_{x\uparrow 0}\begin{pmatrix}\langle\varphi\,,\,\Phi_0(t,x)\psi\rangle\\ \partial_x\langle\varphi\,,\,\Phi_0(t,x)\psi\rangle\end{pmatrix}\,, \tag{3.15}$$

$$\lim_{x\to\pm\infty}\langle\varphi\,,\,\Phi_0(t,x)\psi\rangle=0\,. \tag{3.16}$$

<u>Proof:</u> The proof of this result is just a generalization to the vector case of the proof detailed in [39, 40] once the meaning of $:\quad:$ and $\widetilde{\Phi}^\dagger$ is given.

First, the normal ordering is chosen in the same way as in [29]: as usual, all creation operators stand to the left of all annihilation operators but with the further requirement that the original order of the creators is preserved while that of two annihilators is conserved if both belong to the same Φ or $\widetilde{\Phi}^\dagger$ and inverted otherwise.

Second, for all $n\geq 1$, we define

$$\widetilde{\Phi}^{\dagger(n)}_{\xi,01\ldots n}(x,t)=\int_{\mathbb{R}^{2n+1}}\prod_{\substack{i=0\\ j=1}}^{n}\frac{dp_i}{2\pi}\frac{dq_j}{2\pi}\,a^\dagger_{\xi,0}(p_0)a^\dagger_{\xi,1}(p_1)\ldots a^\dagger_{\xi,n}(p_n)\mathcal{P}_{01\ldots n}\,a_{\xi,n}(q_n)\ldots a_{\xi,1}(q_1)$$
$$\times\frac{e^{-i\sum\limits_{j=1}^{n}(q_jx-q_j^2t)+i\sum\limits_{i=0}^{n}(p_ix-p_i^2t)}}{\prod\limits_{i=1}^{n}(q_i-p_{i-1}+\xi i\epsilon)(q_i-p_i+\xi i\epsilon)}\,,$$

with $\mathcal{P}_{01\ldots n}=P_{01}P_{02}\cdots P_{0n}$. Then, $\widetilde{\Phi}^\dagger$ is given by

$$\widetilde{\Phi}^\dagger_{\xi,0}(x,t)=\Phi^{\dagger(0)}_{\xi,0}(x,t)+\sum_{n\geq 1}(-g)^n\frac{1}{N^n}tr_{1\ldots n}\left[\widetilde{\Phi}^{\dagger(n)}_{\xi,01\ldots n}(x,t)\right] \tag{3.17}$$

This ensures the compatibility between the normal ordering $:\quad:$ and the use of auxiliary spaces to any order in the cubic term, so that the order by order proof *à* *la Rosales* [15,16] works at the quantum level. ■

The case treated here corresponds to "scalar boundary conditions", i.e. the boundary conditions do not affect the internal degree of freedom ("isospin"). More general boundary conditions will be studied below (see section 4.2).

For completeness, we mention that the RT algebra under consideration is also suitable to compute transition amplitudes between "in" and "out" states. A transition between $|k_1, \xi_1, i_1; \dots; k_n, \xi_n, i_n >^{in}$, $k_\ell \in \mathbb{R}^{-\xi_\ell}$ ordered according to $|k_n| > \dots > |k_1|$ and ${}^{out} < p_1, \nu_1, j_1; \dots; p_m, \nu_m, j_m|$, $p_\ell \in \mathbb{R}^{\nu_\ell}$ ordered according to $|p_1| > \dots > |p_m|$ is computed thanks to the identification (see [39] for more details)

$$\begin{aligned} |k_1, \beta_1; \dots; k_n, \beta_n >^{in} &= a^\dagger_{\beta_1}(k_1) \dots a^\dagger_{\beta_n}(k_n)\Omega \,, \quad \beta_\ell = (\xi_\ell, i_\ell) \\ |p_1, \alpha_1; \dots; p_m, \alpha_m >^{out} &= a^\dagger_{\alpha_1}(p_1) \dots a^\dagger_{\alpha_m}(p_m)\Omega \,, \quad \alpha_\ell = (\nu_\ell, j_\ell) \end{aligned}$$

It is completely determined by the exchange relations encoded in the RT algebra and is made out of the fundamental quantities $\mathcal{S}$, $R_\pm$ and $T_\pm$: this is the well-known factorization of the scattering processes.

3.2 NLS symmetry and Hamiltonian

From the previous section, we see that the quantum NLS equation with a transmitting and reflecting defect is completely solved (in the absence of bound states). This has been done thanks to the concept of RT algebras which substitutes the ZF algebra in the quantum inverse inverse method when a defect is present. One can go further with RT algebras and show that the system is completely integrable by exhibiting a whole hierarchy of Hamiltonians in involution. Finally, one can also identify the symmetry algebra of the hierarchy.

Actually, the algebraic part of this has already been done in section 2.2 and all we have to do is to show that the Hamiltonian of quantum NLS with defect belongs to the hierarchy of $H^{(n)}_{RT}$ in the Fock representation labelled by $\mathcal{S}, \mathcal{R}, \mathcal{T}$ defined by (2.1), (2.15), (3.3) and (3.4).

It is remarkable that the results known for NLS without defect survives in the presence of a defect: the time evolution of the field $\Phi_{\xi,j}(x,t)$ is generated by $H^{(2)}_{RT} = \int_{\mathbb{R}} dk\, k^2\, a^{\dagger\alpha}(k) a_\alpha(k)$ according to

$$\Phi_{\xi,j}(x,t) = e^{iH^{(2)}_{RT} t}\ \Phi_{\xi,j}(x,0)\ e^{-iH^{(2)}_{RT} t} \,. \tag{3.18}$$

It follows immediately from (2.24) that the defect operators $r^\beta_\alpha(k), t^\beta_\alpha(k)$ generate integrals of motion for $gl(N)$-NLS model with defect: the defect algebra $\mathcal{D}_\mathcal{S}$ is the symmetry algebra of our model.

We now turn to the study of $\mathcal{D}_\mathcal{S}$ for the particular representation (2.15).

4 Classifying NLS reflection and transmission matrices

4.1 Classification

Taking the vacuum expectation value of (2.9)-(2.11), one gets the reflection-transmission equations to be satisfied by $\mathcal{R}$ and $\mathcal{T}$. It is possible to classify the solutions of these equations starting from the particular form of (2.15). Actually the problem reduces to solving the reflection equation for each submatrix $R_+(k), R_-(k), T_+(k), T_-(k)$ separately. The problem is further constrained by additional reflection equations involving all the possible pairs $\{R_\pm, R_\mp\}$, $\{T_\pm, T_\mp\}$, $\{R_\pm, T_\mp\}$ and $\{R_\pm, T_\pm\}$. Finally, we will have to take (2.16)-(2.18) into account.

To complete the classification, we will use the following lemma (proved by direct calculation, and valid whatever the $\mathcal{S}$-matrix is):

Lemma 4.1 *The RT-algebra, defined by the relations (2.2)-(2.13), admits as automorphisms the following dilatations:*

$$r_\varepsilon(k) \quad\rightarrow\quad \mu_\varepsilon(k)\, r_\varepsilon(k) \quad \textit{and} \quad t_\pm(k) \quad\rightarrow\quad (\nu_0(k))^{\pm 1}\, t_\pm(k) \quad \textit{with} \quad \varepsilon = \pm \tag{4.1}$$

where $\nu_0(k)$ and $\mu_\varepsilon(k)$ are functions obeying the relations:

$$\mu_\varepsilon(k)\, \mu_\varepsilon(-k) = 1 \quad \textit{and} \quad \nu_0(-k) = \mu_+(-k)\, \mu_-(k)\, \nu_0(k)\,. \tag{4.2}$$

From the classification made in [27] for the reflection equation, we know that we can consider only diagonalizable and triangularizable solutions. In our case, (2.16) rules out the triangular case and imposes that

$$R_\pm(k) = \rho_\pm(k)\frac{\mathbb{I}_N + ia_\pm k\, \mathbb{G}_\pm}{1 + ia_\pm k}\,, \qquad T_\pm(k) = \tau_\pm(k)\frac{\mathbb{I}_N + ib_\pm k\, \mathbb{F}_\pm}{1 + ib_\pm k}\,, \tag{4.3}$$

where $a_\pm, b_\pm \in \mathbb{R}\cup\{\infty\}$, $\rho_\pm$ and $\tau_\pm$ are complex functions, and $\mathbb{G}_\pm$ and $\mathbb{F}_\pm$ are $N\times N$ matrices obeying $(\mathbb{G}_\pm)^2 = \mathbb{I}_N = (\mathbb{F}_\pm)^2$. Note that the latter condition ensures that these matrices are diagonalizable.

Now the mixed relation severely restrict the freedom in (4.3) and one finds actually

$$R_\pm(k) = \rho_\pm(k)\, M\, \frac{\mathbb{I}_N \pm iak\, \mathbb{E}}{1 \pm iak}\, M^{-1}\,, \qquad T_\pm(k) = \tau_\pm(k)\, M\, \frac{\mathbb{I}_N \pm iak\, \mathbb{E}}{1 \pm iak}\, M^{-1}\,, \tag{4.4}$$

where $a \in \mathbb{R}\cup\{\infty\}$, $\mathbb{E}$ is a diagonal matrix which squares to $\mathbb{I}_N$ and M is a (constant) unitary diagonalization matrix. Finally, the functions $\rho_\pm, \tau_\pm$ are constrained by (2.16)-(2.18)

$$\overline{\rho}_\pm(k) = \rho_\pm(-k)\,, \qquad \overline{\tau}_-(k) = \tau_+(k) \equiv \tau_0(k)\,, \tag{4.5}$$

$$|\rho_\pm(k)|^2 + |\tau_0(k)|^2 = 1\,, \tag{4.6}$$

$$\rho_\mp(k)\tau_\pm(k) + \rho_\pm(k)\tau_\pm(-k) = 0\,. \tag{4.7}$$

The general solution to these equation takes the form:

$$\rho_\pm(k) \;=\; \varepsilon_\pm \,\frac{A(k)-i}{A(k)+i}\,\frac{C(k)\mp i}{C(k)\pm i}\,\cos\theta(k) \tag{4.8}$$

$$\tau_\pm(k) \;=\; \frac{B(k)\mp i}{B(k)\pm i}\,\frac{C(k)\mp i}{C(k)\pm i}\,\sin\theta(k) \tag{4.9}$$

where $\varepsilon_+,\varepsilon_- = \pm 1$ and A, B, D, θ are real-valued functions obeying

$$A(-k) = -A(k)\,,\ B(-k) = B(k)\,,\ C(-k) = -C(k) \quad \text{and} \quad \theta(-k) = -\varepsilon_+\,\varepsilon_-\,\theta(k)\,.$$

Of course, when $\theta(k) = 0$, one recovers the classification for reflection matrices given in [27], once one has noticed the product law

$$\frac{A(k)-i}{A(k)+i}\,\frac{C(k)-i}{C(k)+i} = \frac{\widetilde{A}(k)-i}{\widetilde{A}(k)+i} \quad \text{with} \quad \widetilde{A}(k) = \frac{A(k)+C(k)}{1-A(k)C(k)}\,.$$

Using the invariance given in lemma 4.1, one gets the following reduced parametrization

$$\rho_+(k) \;=\; \cos\theta(k) \equiv \frac{1-\omega(k)^2}{1+\omega(k)^2}\,, \tag{4.10}$$

$$\rho_-(k) \;=\; \varepsilon_+\varepsilon_-\,\cos\theta(k) \equiv \varepsilon_+\varepsilon_-\,\frac{1-\omega(k)^2}{1+\omega(k)^2}\,, \tag{4.11}$$

$$\tau_\pm(k) \;=\; \sin\theta(k) \equiv \frac{2\omega(k)}{1+\omega(k)^2} \quad \text{with} \quad \omega(k) = \tan\frac{\theta(k)}{2} = -\varepsilon_-\varepsilon_+\,\omega(-k)\,. \tag{4.12}$$

4.2 New boundary conditions

4.2.1 Generalities

When the reflection and transmission matrices take the particular values (3.3) and (3.4), we know that the field $\Phi(x,t)$ satisfy the boundary conditions (3.15). Thus, one can think of using the previous general parametrization to implement more general boundary conditions on the field.

Let us begin by defining the differential operator D_A associated to a function $A(k)$:

$$(D_A f)(y) = \int dx\ \hat{A}(x)\,f(x+y)\ ,\ \forall f \quad \mathbb{C}\text{-function} \tag{4.13}$$

where $\hat{A}$ is the inverse Fourier transform of A. In particular, for $A(k) = k$ and for $A(k) = a$ (constant), one has

$$(D_{(k)} f)(y) = -i\frac{d}{dy} f(y) \quad \text{and} \quad (D_a f)(y) = a\,f(y)\,. \tag{4.14}$$

This is easily extended to a matrix $A_{ij}(k),\ i,j=1,\dots,N$

$$(D_A f)_i(y) = \int dx\ \sum_{j=1}^{N} \hat{A}_{ij}(x)\, f_j(x+y)\ ,\ \forall\, f_i \quad \mathbb{C}\text{-function} \quad ,\ i=1,\dots,N\,. \tag{4.15}$$

Remark that, because of the property $\widehat{AB} = \widehat{A} * \widehat{B}$, where $*$ denotes the convolution, we have, for any matrices A and B,

$$D_{AB} = D_A D_B\,. \tag{4.16}$$

Hence, the mapping $A(k) \to D_A$ is a morphism.

Now, for given reflection and transmission matrices $R_\pm(k), T_\pm(k)$, we look for matrix-valued differential operators such that

$$\lim_{x\downarrow 0} \begin{pmatrix} D_{u_1}\,\mathbb{I}_N & 0 \\ 0 & D_{u_2}\,\mathbb{I}_N \end{pmatrix} \begin{pmatrix} \langle \varphi\,,\,\Phi(t,x)\psi\rangle \\ \partial_x \langle \varphi\,,\,\Phi(t,x)\psi\rangle \end{pmatrix} = \lim_{x\uparrow 0} \begin{pmatrix} D_{V_{11}} & D_{V_{12}} \\ D_{V_{21}} & D_{V_{22}} \end{pmatrix} \begin{pmatrix} \langle \varphi\,,\,\Phi(t,x)\psi\rangle \\ \partial_x \langle \varphi\,,\,\Phi(t,x)\psi\rangle \end{pmatrix} \tag{4.17}$$

where the matrix-valued differential operators $D_{V_{kl}},\ k,l=1,2$ are defined as in (4.15), while the scalar differential operators D_{u_1} and D_{u_2} are defined by (4.13). As a notation, we will use

$$U(k) = \begin{pmatrix} u_1(k)\,\mathbb{I}_N & 0 \\ 0 & u_2(k)\,\mathbb{I}_N \end{pmatrix} \quad \text{and} \quad V(k) = \begin{pmatrix} V_{11}(k) & V_{12}(k) \\ V_{21}(k) & V_{22}(k) \end{pmatrix}\,. \tag{4.18}$$

For convenience, we will loosely write the boundary conditions (4.17) as

$$D_{u_1}\Phi(+0) = D_{V_{11}}\Phi(-0) + D_{V_{12}}\,\partial_x\Phi(-0)\,, \tag{4.19}$$
$$D_{u_2}\partial_x\Phi(+0) = D_{V_{21}}\Phi(-0) + D_{V_{22}}\,\partial_x\Phi(-0)\,, \tag{4.20}$$

keeping in mind that they must hold for any time, although we have omitted t.

Note that matrices (4.18) are not the most general ones: we merely aim at finding (simple) boundary conditions for any solution to the RT equations. We do not try to classify all the possible boundary conditions leading to a given solution.

Following the argument of [29], one can write

$$\langle \varphi\,,\,\Phi_i(t,x)\psi\rangle = \int \frac{dk}{2\pi} e^{-ik^2 t} \left(\Psi^{+,j}_{k,i}(x)\chi^+_j(k) + \Psi^{-,j}_{k,i}(x)\chi^-_j(k)\right) \tag{4.21}$$

where

$$\Psi^\pm_k(x) = \theta(\mp k)\left[\theta(\mp x)T_\mp(k)e^{ikx} + \theta(\pm x)(\mathbb{I}_N e^{ikx} + R_\pm(-k)e^{-ikx})\right] \tag{4.22}$$

are the solutions of the free problem ($g=0$) and $\chi^\pm_j(k)$ are $2N$ wave-packets.

Then, the problem (4.17) reduces to two copies of the following functional equations

$$\begin{cases} X(k) + X(-k)R_+(-k) - Y(k)T_-(k) = 0\,, \\ Y(k) + Y(-k)R_-(-k) - X(k)T_+(k) = 0\,, \end{cases} \tag{4.23}$$

where

$$\begin{cases} X(k) = u_1(k)\mathbb{I}_N \\ Y(k) = V_{11}(k) + ikV_{12}(k) \end{cases} \quad \text{or} \quad \begin{cases} X(k) = ik\,u_2(k)\mathbb{I}_N \\ Y(k) = V_{21}(k) + ikV_{22}(k) \end{cases} \tag{4.24}$$

In the following, we can take $\varepsilon_+ = 1$ without loss of generality.

4.2.2 Case $R_\pm(k) = \rho_\pm(k)\mathbb{I}_N$ and $T_\pm(k) = \tau_\pm(k)\mathbb{I}_N$

In that case we can assume $u_1(k) = iku_2(k) \equiv u(k)$ and $V(k)$ diagonal matrix with $V_{11} = v_1(k)\mathbb{I}_N$ and $V_{22} = v_2(k)\mathbb{I}_N$. Solving the systems in u, v_1 and v_2 thanks to the parametrization (4.8), (4.9) and plugging back the result into X, Y, one finds a solution which can be recasted as:

$$U(k) = 2(A(k)+i)(B(k)+i)(C(k)+i)\omega(k)\,\mathbb{I}_{2N}\,, \tag{4.25}$$

$$V(k) = (A(k)+i)(B(k)-i)(C(k)-i)\,V_0(k)\,, \tag{4.26}$$

$$V_0(k) = (\omega(k)^2+1)\begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N \end{pmatrix} + \varepsilon_-(\omega(k)^2-1)\begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & -\mathbb{I}_N \end{pmatrix}. \tag{4.27}$$

This corresponds, using the property (4.16), to the following boundary conditions:

$$2D_f D_\omega \Phi(+0) = \big(1 - \varepsilon_- + (1+\varepsilon_-)D_\omega^2\big)\, D_g\Phi(-0) \tag{4.28}$$

$$2D_f D_\omega \partial_x\Phi(+0) = \big(1 + \varepsilon_- + (1-\varepsilon_-)D_\omega^2\big)\, D_g\partial_x\Phi(-0) \tag{4.29}$$

where

$$\begin{aligned} f(k) &= (A(k)+i)(B(k)+i)(C(k)+i) \\ g(k) &= (A(k)+i)(B(k)-i)(C(k)-i)\,. \end{aligned} \tag{4.30}$$

As examples, let us use the reduced parametrization (4.10)-(4.12) (*i.e.* $f = g \sim 1$) and two particular choices for ω: $w(k) = 1$ (so that $\varepsilon_- = -1$) or $w(k) = k$ (then $\varepsilon_- = 1$). They lead respectively to the following boundary conditions:

$$\Phi(+0) = \Phi(-0) \quad \text{and} \quad \partial_x\Phi(+0) = \partial_x\Phi(-0) \qquad (\omega(k) = 1)$$

$$i\partial_x\Phi(+0) = \partial_x^2\Phi(-0) \quad \text{and} \quad \partial_x^2\Phi(+0) = i\partial_x\Phi(-0)\,. \qquad (\omega(k) = k)$$

Remark that the first case corresponds to a purely transmitting system (because $\omega(k) = 1$): the reduced parametrization (4.10)-(4.12) then leads naturally to a trivial boundary condition (*i.e.* no "real" defect at the origin). More involved boundary conditions can be obtained using different matrices $U(k)$ and $V(k)$ (see for instance section 4.2.4).

4.2.3 Case $R_\pm(k) = \rho_\pm(k)\mathbb{E}$ and $T_\pm(k) = \tau_\pm(k)\mathbb{E}$

We take for $Y(k)$ the following form: $Y(k) = y_0(k)\mathbb{I}_N + y_1(k)\mathbb{E}$. We find

$$\begin{aligned} U(k) &= 2f(k)\omega(k)\,\mathbb{I}_{2N} \quad \text{and} \quad V(k) = g(k)\,V_0(k) \quad \text{with} && (4.31)\\ V_0(k) &= (\omega(k)^2+1)\begin{pmatrix}\mathbb{E} & 0\\ 0 & \mathbb{E}\end{pmatrix} + \varepsilon_-(\omega(k)^2-1)\begin{pmatrix}\mathbb{I}_N & 0\\ 0 & -\mathbb{I}_N\end{pmatrix} && (4.32)\end{aligned}$$

where $f(k)$ and $g(k)$ are defined in (4.30). It leads to the following boundary conditions:

$$D_f D_\omega \Phi(+0) = \varepsilon_-\left(\frac{\mathbb{I}_N+\varepsilon_-\mathbb{E}}{2}D_\omega^2 - \frac{\mathbb{I}_N-\varepsilon_-\mathbb{E}}{2}\right)D_g\Phi(-0) \qquad (4.33)$$

$$D_f D_\omega \partial_x\Phi(+0) = \varepsilon_-\left(\frac{\mathbb{I}_N+\varepsilon_-\mathbb{E}}{2} - \frac{\mathbb{I}_N-\varepsilon_-\mathbb{E}}{2}D_\omega^2\right)D_g\partial_x\Phi(-0) \qquad (4.34)$$

which shows that the defect affects the isotopic degrees of freedom. Indeed, these boundary conditions suggest the introduction of the "right-handed" and "left-handed" parts of Φ defined by

$$\Phi_L(t,x) = \frac{\mathbb{I}_N - \mathbb{E}}{2}\,\Phi(t,x), \quad \Phi_R(t,x) = \frac{\mathbb{I}_N + \mathbb{E}}{2}\,\Phi(t,x)\,. \qquad (4.35)$$

For instance, taking the particular examples given in the previous section, the conditions read:

$$\begin{cases}\Phi_R(+0) = \Phi_R(t,-0) \quad \text{and} \quad \partial_x\Phi_R(+0) = \partial_x\Phi_R(t,-0)\,,\\ \Phi_L(+0) = -\Phi_L(t,-0) \quad \text{and} \quad \partial_x\Phi_L(+0) = -\partial_x\Phi_L(t,-0)\,.\end{cases} \qquad (\omega(k)=1)$$

$$\begin{cases}i\partial_x\Phi_R(+0) = \partial_x^2\Phi_R(-0) \quad \text{and} \quad -i\partial_x^2\Phi_R(+0) = \partial_x\Phi_R(-0)\,,\\ i\partial_x\Phi_L(+0) = \Phi_L(-0) \quad \text{and} \quad -i\partial_x^2\Phi_L(+0) = \partial_x^3\Phi_L(-0)\,,\end{cases} \qquad (\omega(k)=k)$$

One sees that the right and left-handed parts of the field do not satisfy the same boundary conditions: the defect scatters differently the left and right-handed parts of the field.

4.2.4 Case $R_\pm(k) = \rho_\pm(k)\dfrac{\mathbb{I}_N \pm iak\,\mathbb{E}}{1\pm iak}$ and $T_\pm(k) = \tau_\pm(k)\dfrac{\mathbb{I}_N \pm iak\,\mathbb{E}}{1\pm iak}$

Again, one can take $Y(k) = y_0(k)\mathbb{I}_N + y_1(k)\mathbb{E}$ to find a solution of the form

$$\begin{aligned} U(k) &= 2(1+iak)f(k)\omega(k)\,\mathbb{I}_{2N} \quad \text{and} \quad V(k) = g(k)\,V_0(k) \quad \text{with}\\ V_0(k) &= (\omega(k)^2+1)\begin{pmatrix}\mathbb{I}_N + aki\,\mathbb{E} & 0\\ ik\mathbb{I}_N & aik\,\mathbb{E}\end{pmatrix} + \varepsilon_-(\omega(k)^2-1)\begin{pmatrix}(1-iak)\mathbb{I}_N & 0\\ -ik\mathbb{I}_N & aik\,\mathbb{I}_N\end{pmatrix}\end{aligned}$$

This corresponds to the following boundary conditions:

$$\begin{aligned}
D_f D_\omega(1+a\partial_x)\Phi(+0) &= \left(\frac{1-\varepsilon_-}{2}+\frac{1+\varepsilon_-}{2}D_\omega^2\right)D_g\Phi(-0)\\
&\quad +a\ \varepsilon_-\left(\frac{\mathbb{I}_N+\varepsilon_-\mathbb{E}}{2}-\frac{\mathbb{I}_N-\varepsilon_-\mathbb{E}}{2}D_\omega^2\right)D_g\partial_x\Phi(-0)\\
D_f D_\omega(1+a\partial_x)\partial_x\Phi(+0) &= \left(\frac{1+\varepsilon_-}{2}+\frac{1-\varepsilon_-}{2}D_\omega^2\right)D_g\partial_x\Phi(-0)\\
&\quad -a\ \varepsilon_-\left(\frac{\mathbb{I}_N-\varepsilon_-\mathbb{E}}{2}-\frac{\mathbb{I}_N+\varepsilon_-\mathbb{E}}{2}D_\omega^2\right)D_g\partial_x^2\Phi(-0)
\end{aligned}$$

Taking once more the two particular examples, we get

$$\begin{cases}(1+a\partial_x)\Phi(+0)=\Phi(-0)+a\partial_x\left(\Phi_R(-0)-\Phi_L(-0)\right)\\ (1+a\partial_x)\partial_x\Phi(+0)=\partial_x\Phi(-0)+a\partial_x^2\left(\Phi_R(-0)-\Phi_L(-0)\right)\end{cases}\qquad (\omega(k)=1)$$

$$\begin{cases}i(1+a\partial_x)\partial_x\Phi(+0)=\partial_x^2\Phi(-0)-a\partial_x\Phi_R(-0)-a\partial_x^3\Phi_L(-0)\\ i(1+a\partial_x)\partial_x^2\Phi(+0)=-\partial_x\Phi(-0)+a\partial_x^2\Phi_L(-0)+a\partial_x^4\Phi_R(-0)\end{cases}\qquad (\omega(k)=k)$$

Note that the first case corresponds to a purely transmitting system with non-trivial boundary conditions (for $a\neq 0$). Again, the defect is sensitive to the "isospin".

5 Spontaneous symmetry breaking

5.1 General discussion

It is known [36] that the matrices $\mathcal{R},\mathcal{T}$ in (2.14) fully determine the different Fock representations of the RT algebra. For convenience we denote by $\langle\ \rangle$ the vacuum expectation value of any RT generator on Ω and by construction one has

$$\langle t(k)\rangle=\mathcal{T}(k),\quad \langle r(k)\rangle=\mathcal{R}(k)\,. \tag{5.1}$$

Thus, under an expansion in power of k^{-1}, one can identify those generators of the defect algebra whose vacuum expectation value is nonzero. As we showed, the defect algebra is the symmetry of the NLS hierarchy so that this mechanism is a spontaneous symmetry breaking of the defect algebra. To make this more quantitative, let us introduce some notations. We write series expansions for the generators $r_\pm(k)$, $t_\pm(k)$ of the defect algebra as follows

$$r_\pm(k)=r_\pm^{(0)}+\sum_{n=1}^{\infty}r_\pm^{(n)}k^{-n},\quad t_\pm(k)=t_\pm^{(0)}+\sum_{n=1}^{\infty}t_\pm^{(n)}k^{-n}\,. \tag{5.2}$$

The generators corresponding to the physical symmetries are $r_\pm^{(n)},t_\pm^{(n)},\ n\geq 1$. Thus, there is spontaneous symmetry breaking if $R^\pm(k),T^\pm(k)$ are not constant matrices.

The problem then is to identify the remaining exact symmetry *i.e.* one has to find the unbroken generators and the algebra they satisfy. In other words, we look for generators $\widehat{r}_\pm^{(n)}, \widehat{t}_\pm^{(n)}$, for some $n \geq 1$, such that

$$\langle \widehat{r}_\pm^{(n)} \rangle = 0, \quad \langle \widehat{t}_\pm^{(n)} \rangle = 0\,. \tag{5.3}$$

This is the question we address in the case of NLS in the next paragraph.

5.2 Symmetry breaking for NLS

In this case, we know the general form of $R_\pm(k), T_\pm(k)$:

$$R_\pm(k) = \rho_\pm(k)\, M\, \Lambda(\pm k)\, M^{-1}, \quad T_\pm(k) = \tau_\pm(k)\, M\, \Lambda(\pm k)\, M^{-1}, \quad \Lambda(k) = \frac{\mathbb{I}_N + iak\,\mathbb{E}}{1+iak}\,.$$

As these are not constant matrices in general, the corresponding symmetry is broken. To identify the remaining symmetry algebra, we introduce

$$\lambda(k) = \frac{1}{2}\left[\left(1 + \frac{1-iak}{\sqrt{1+(ak)^2}}\right)\mathbb{I}_N + \left(1 - \frac{1-iak}{\sqrt{1+(ak)^2}}\right)\mathbb{E}\right], \tag{5.4}$$

which obeys

$$\lambda(k)^2 = \Lambda(k) \quad \text{and} \quad \lambda(k)\lambda(-k) = \mathbb{I}_N\,. \tag{5.5}$$

We also introduce the generators

$$\widetilde{r}_\pm(k) = \lambda(\mp k)\; M^{-1}\, r_\pm(k)\, M\;\, \lambda(\mp k), \quad \widetilde{t}_\pm(k) = \lambda(\mp k)\; M^{-1}\, t_\pm(k)\, M^{-1}\; \lambda(\mp k) \tag{5.6}$$

which we gather into

$$\widetilde{r}(k) = \begin{pmatrix} \widetilde{r}_+(k) & 0 \\ 0 & \widetilde{r}_-(k) \end{pmatrix}, \qquad \widetilde{t}(k) = \begin{pmatrix} 0 & \widetilde{t}_+(k) \\ \widetilde{t}_-(k) & 0 \end{pmatrix}. \tag{5.7}$$

After some algebra, one finds that the new generators satisfy the following relations:

$$\begin{aligned}
\widetilde{\mathcal{S}}_{12}(k_1,k_2)\,\widetilde{t}_1(k_1)\,\widetilde{\mathcal{S}}_{21}(k_2,k_1)\,\widetilde{t}_2(k_2) &= \widetilde{t}_2(k_2)\,\widetilde{\mathcal{S}}_{12}(k_1,k_2)\,\widetilde{t}_1(k_1)\,\widetilde{\mathcal{S}}_{21}(k_2,k_1)\\
\widetilde{\mathcal{S}}_{12}(k_1,k_2)\,\widetilde{t}_1(k_1)\,\widetilde{\mathcal{S}}_{21}(k_2,k_1)\,\widetilde{r}_2(k_2) &= \widetilde{r}_2(k_2)\,\widetilde{\mathcal{S}}_{12}(k_1,-k_2)\,\widetilde{t}_1(k_1)\,\widetilde{\mathcal{S}}_{21}(-k_2,k_1)\\
\widetilde{\mathcal{S}}_{12}(k_1,k_2)\,\widetilde{r}_1(k_1)\,\widetilde{\mathcal{S}}_{21}(k_2,-k_1)\,\widetilde{r}_2(k_2) &= \widetilde{r}_2(k_2)\,\widetilde{\mathcal{S}}_{12}(k_1,-k_2)\,\widetilde{r}_1(k_1)\,\widetilde{\mathcal{S}}_{21}(-k_2,-k_1)\\
\widetilde{t}(k)\,\widetilde{t}(k) + \widetilde{r}(k)\,\widetilde{r}(-k) = \mathbf{1} \quad &\text{and} \quad \widetilde{t}(k)\,\widetilde{r}(k) + \widetilde{r}(k)\,\widetilde{t}(-k) = 0
\end{aligned}$$

where the new matrix $\widetilde{\mathcal{S}}$ defined by

$$\widetilde{\mathcal{S}}_{12}(k_1,k_2) = \begin{pmatrix} \widetilde{s}_{12}(k_1,k_2) & 0 & 0 & 0 \\ 0 & \widetilde{s}_{12}(k_1,-k_2) & 0 & 0 \\ 0 & 0 & \widetilde{s}_{12}(-k_1,k_2) & 0 \\ 0 & 0 & 0 & \widetilde{s}_{12}(-k_1,-k_2) \end{pmatrix} \tag{5.8}$$

$$\widetilde{s}_{12}(k_1,k_2) = \lambda_1(-k_1)\,\lambda_2(-k_2)\,s_{12}(k_1-k_2)\,\lambda_1(k_1)\,\lambda_2(k_2) \tag{5.9}$$

satisfies the unitarity and Yang-Baxter equations

$$\begin{aligned}\widetilde{\mathcal{S}}_{12}(k_1,k_2)\,\widetilde{\mathcal{S}}_{21}(k_2,k_1) &= (\mathbb{I}_2\otimes\mathbb{I}_2)\otimes(\mathbb{I}_N\otimes\mathbb{I}_N)\\ \widetilde{\mathcal{S}}_{12}(k_1,k_2)\,\widetilde{\mathcal{S}}_{13}(k_1,k_3)\,\widetilde{\mathcal{S}}_{23}(k_2,k_3) &= \widetilde{\mathcal{S}}_{23}(k_2,k_3)\,\widetilde{\mathcal{S}}_{13}(k_1,k_3)\,\widetilde{\mathcal{S}}_{12}(k_1,k_2)\,.\end{aligned}$$

Thus, the generators $\widetilde{r}(k)$ and $\widetilde{t}(k)$, which by construction belong to the original defect algebra $\mathcal{D}_{\mathcal{S}}$, generates a $\mathcal{D}_{\mathcal{S}}$-subalgebra. This subalgebra appears to be itself a $\mathcal{D}_{\widetilde{\mathcal{S}}}$ defect algebra*.

Computing the vacuum expectation value of these generators, we get

$$\langle\widetilde{r}_\pm(k)\rangle = \rho_\pm(k)\,\mathbb{I}_N,\quad \langle\widetilde{t}_\pm(k)\rangle = \tau_\pm(k)\,\mathbb{I}_N\,, \tag{5.10}$$

so that all the off-diagonal terms of $\widetilde{r}_\pm(k)$ and $\widetilde{t}_\pm(k)$ remain unbroken. Indeed, the form (5.10) shows that only the generators obtained from the k^{-1} expansion of the traces $\mathrm{tr}(\widetilde{r}_\pm(k))$ and $\mathrm{tr}(\widetilde{t}_\pm(k))$ are possibly broken, depending on the exact form of the functions $\rho_\pm(k)$ and $\tau_\pm(k)$. In other words, the use of the generators $\widetilde{r}_\pm(k)$ and $\widetilde{t}_\pm(k)$ takes care of the symmetry breaking induced by the matrix $M\,\Lambda(k)\,M^{-1}$, while the expansion in k^{-1} of the functions $\rho_\pm(k)$ and $\tau_\pm(k)$ will induce a "scalar-like" symmetry breaking (i.e. a breaking of the type $gl(N)\to sl(N)$). These two points will be illustrated in two examples below.

We want to stress that even if one starts with an exchange matrix $S(k)$ depending only on the difference of the parameters, the new matrix $\widetilde{S}$ (which defines the unbroken symmetry algebra) depends *separately* on k_1, k_2. This shows (once again) that a reflection-transmission algebra is naturally (although not compulsorily) associated with an exchange matrix depending on k_1 and k_2 separately.

Let us note finally that this study completes the arguments developed in [27] for NLS on the half-line. Indeed, we know that the reflection algebra is particular case of the defect algebra when one considers $t(k)=0$. Performing the same calculations as above, we deduce that in the case of spontaneous symmetry breaking for NLS on the half-line, the remaining exact symmetry is again a reflection algebra with the exchange matrix $\widetilde{S}(k)$ and unbroken generator $\widetilde{r}(k)$.

5.2.1 Example: $R_\pm(k)=\cos(\theta_0/k)\,\mathbb{I}_N$ and $T_\pm(k)=\sin(\theta_0/k)\,\mathbb{I}_N$

In that case, we have $\widetilde{r}(k)=r(k)$ and $\widetilde{t}(k)=t(k)$, and the symmetry breaking is induced by the series expansion of the sine and cosine functions. One gets

$$< r_\pm^{(2n)} >= (-1)^n\,\theta_0^{2n}\,\mathbb{I}_N \quad ; \quad < r_\pm^{(2n+1)} >= 0 \tag{5.11}$$

$$< t_\pm^{(2n+1)} >= (-1)^n\,\theta_0^{2n+1}\,\mathbb{I}_N \quad ; \quad < t_\pm^{(2n)} >= 0 \tag{5.12}$$

which allow to identify directly the unbroken generators. Let us define

$$r_\pm^{(n)} = \sum_{i,j=1}^N r_\pm^{(n),ij}\;E_{ij} \quad\text{and}\quad t_\pm^{(n)} = \sum_{i,j=1}^N t_\pm^{(n),ij}\;E_{ij}\,. \tag{5.13}$$

*Strictly speaking, we have shown that $\widetilde{r}(k)$ and $\widetilde{t}(k)$ generates a subalgebra of $\mathcal{D}_{\widetilde{\mathcal{S}}}$.

One easily sees that all the generators $r_\pm^{(n),ij}$ and $t_\pm^{(n),ij}$ with $1 \leq i \neq j \leq N$ have a vanishing vacuum expectation value, and hence remain unbroken. It is also the case of the generators $r_\pm^{(2n+1),ii}$ and $t_\pm^{(2n),ii}$. Other unbroken generators are given by the combinations $r_\pm^{(2n),ii} - r_\pm^{(2n),jj}$ and $t_\pm^{(2n+1),ii} - t_\pm^{(2n+1),jj}$, in accordance with the expected "scalar-like" symmetry breaking: only the traces $\sum_{i=1}^N r_\pm^{(2n),ii}$ and $\sum_{i=1}^N t_\pm^{(2n+1),ii}$ are broken.

This fact is easily extended to any functions of k^{-1} (instead of cosine and sine), as long as they are in the class of functions (4.8)-(4.9).

5.2.2 Example: $R_\pm(k) = \pm\cos(\theta_0)\, M\, \Lambda(\pm k)\, M^{-1}$; $T_\pm(k) = \sin(\theta_0)\, M\, \Lambda(\pm k)\, M^{-1}$

We consider the case $N = 2$ and we take the following forms

$$\Lambda(k) = \begin{pmatrix} 1 & 0 \\ 0 & \beta(k) \end{pmatrix} \quad \text{with} \quad \beta(k) = \frac{1 - iak}{1 + iak} \tag{5.14}$$

$$M = \begin{pmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{pmatrix} \quad \text{with} \quad \mu \in [0, \pi] \tag{5.15}$$

A direct computation shows that

$$\langle r_\pm(k) \rangle = \pm\cos\theta_0\ \Gamma(k) \quad , \quad \langle t_\pm(k) \rangle = \sin\theta_0\ \Gamma(k) \tag{5.16}$$

with

$$\Gamma(k) = \begin{pmatrix} \cos^2\mu + \beta(k)\ \sin^2\mu & (\beta(k) - 1)\ \cos\mu\ \sin\mu \\ (\beta(k) - 1)\ \cos\mu\ \sin\mu & \sin^2\mu + \beta(k)\ \cos^2\mu \end{pmatrix} . \tag{5.17}$$

Upon expanding the elements of $\Gamma(k)$ in powers of k^{-1}, one sees that all the generators $r_\pm^{(n),ij}$, $t_\pm^{(n),ij}$, $n > 0$, get non-vanishing vacuum expectation values (for $\mu \neq 0$ and $\mu \neq \pi/2$):

$$\langle r_\pm^{(n),11} \rangle = \pm\cos\theta_0 (-2\sin^2\mu) \left(\frac{\pm i}{a}\right)^n \tag{5.18}$$

$$\langle r_\pm^{(n),22} \rangle = \pm\cos\theta_0 (-2\cos^2\mu) \left(\frac{\pm i}{a}\right)^n \tag{5.19}$$

$$\langle r_\pm^{(n),12} \rangle = \pm\cos\theta_0 (-2\sin\mu\cos\mu) \left(\frac{\pm i}{a}\right)^n = \langle r_\pm^{(n),21} \rangle \tag{5.20}$$

and similar expressions for $t_\pm^{(n),ij}$ replacing $\pm\cos\theta_0$ by $\sin\theta_0$.

However, the following combinations, dictated by (5.6), produce unbroken generators

$$\widetilde{x}_\pm^{11}(k) = x_\pm^{11}\ \cos^2\mu - (x_\pm^{12} + x_\pm^{21})\ \sin\mu\cos\mu + x_\pm^{22}\ \sin^2\mu \tag{5.21}$$

$$\widetilde{x}_\pm^{12}(k) = \frac{1 \pm iak}{\sqrt{1 + a^2k^2}} \left\{ x_\pm^{12}\ \cos^2\mu + (x_\pm^{11} - x_\pm^{22})\ \sin\mu\cos\mu - x_\pm^{21}\ \sin^2\mu \right\} = \widetilde{x}_\pm^{21}(k)$$

$$\widetilde{x}_\pm^{22}(k) = \frac{(1 \pm iak)^2}{1 + a^2k^2} \left\{ x_\pm^{11}\ \sin^2\mu + (x_\pm^{12} + x_\pm^{21})\ \sin\mu\cos\mu + x_\pm^{22}\ \cos^2\mu \right\} \tag{5.22}$$

where $x_\pm$ denote either $r_\pm(k)$ or $t_\pm(k)$.

Indeed, a direct computation yields

$$\langle\widetilde{r}^{11}_\pm(k)\rangle = \langle\widetilde{r}^{22}_\pm(k)\rangle = \pm\cos\theta_0 \quad , \quad \langle\widetilde{t}^{11}_\pm(k)\rangle = \langle\widetilde{t}^{22}_\pm(k)\rangle = \sin\theta_0 \tag{5.23}$$

$$\langle\widetilde{r}^{12}_\pm(k)\rangle = \langle\widetilde{r}^{21}_\pm(k)\rangle = 0 \quad , \quad \langle\widetilde{t}^{12}_\pm(k)\rangle = \langle\widetilde{t}^{21}_\pm(k)\rangle = 0 \tag{5.24}$$

as expected. Similar calculations can be done when N (the size of the matrices) is greater than 2.

Finally, let us note that in this example, we have considered constant functions $\rho_\pm$ and $\tau_\pm$: in the more general case where they do depend on k, the generators $\widetilde{x}_\pm(k)$ will "suffer" from a scalar-like symmetry breaking, as in the previous example. To get true unbroken generators, a second step (similar to the one treated in section 5.2.1) needs to be done.

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